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Double space with double line

I. A. Chel'tsov

Abstract. For a singular double cover of $\mathbb{P}^3$ ramified in a sextic with double line, its birational maps into Fano 3-folds with canonical singularities, elliptic fibrations, and fibrations on surfaces of Kodaira dimension zero are described.

Bibliography: 22 titles.

§ 1. Introduction

Varieties[1] with ample anticanonical divisor are called *Fano varieties* (see [1]) or, in the two-dimensional case, *del Pezzo surfaces*. The importance of Fano varieties is largely due to the Minimal Model Program, from which it follows that from the point of view of birational geometry Fano varieties are building blocks of varieties with negative Kodaira dimension. In dimensions 2 and 3 all smooth Fano varieties have been classified and their birational geometry is well studied. In particular, every del Pezzo surface is rational. The majority of smooth Fano 3-folds are also rational. However, there exist non-rational Fano 3-folds, for instance, smooth cubic and quartic 3-folds.

The cube of the anticanonical divisor of a Fano 3-fold, the so-called *degree* of the Fano threefold, is the main biregular invariant of the 3-fold, which determines to a large extent its birational geometry. For instance, it follows from the classification that every smooth Fano 3-fold is rational, provided that its degree is at least 26. The other way around, the smaller the degree of a Fano 3-fold, the more rigid its birational geometry. It is known that the degree of a smooth Fano 3-fold is an even integer and, in particular, cannot be less than 2. Moreover, there exists a unique smooth Fano 3-fold of degree 2, a double cover of $\mathbb{P}^3$ branched over a smooth sextic surface. The birational geometry of this 3-fold is well studied. In particular, every double cover of $\mathbb{P}^3$ branched over a smooth sextic is known to be non-rational, and it cannot be birationally transformed into a fibration of surfaces having negative Kodaira dimension.

By contrast to algebraic surfaces, it is well known that in the 3-dimensional case one must consider singular varieties, because smooth varieties do not suffice for a good birational theory of higher-dimensional varieties. Hence the study of the birational geometry of singular Fano 3-folds comes up in a natural way. However, this problem can be very difficult even in the simplest cases, as many examples show. Thus, it is natural to try to understand in detail the birational geometry of

[1] All varieties are assumed to be projective, normal, and defined over $\mathbb{C}$.

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a double cover of $\mathbb{P}^3$ branched over a singular sextic surface, since it may clarify the problem in general.

Let $\gamma\colon V \to \mathbb{P}^3$ be a double cover ramified in a sextic $S \subset \mathbb{P}^3$ such that S is smooth outside some line $L \subset S$ and S has a singularity of the type $x^2+y^2=0 \subset \mathbb{C}^3$ at the generic point of L. Then V is smooth outside the proper transform $\widetilde{L} \subset V$ of L and has singularities of type $\mathbb{A}_1 \times \mathbb{C}$ at the generic point of $\widetilde{L}$. The variety V is easily seen to be defined by an equation

$$u^2 = x^2 p_4(x,y,z,t) + y^2 q_4(x,y,z,t)$$

in the weighted projective space $\mathbb{P}(1,1,1,3)$, where p_4 and q_4 are homogeneous polynomials of degree 4, while x, y, z, and t are homogeneous coordinate variables of weight 1, and u is a coordinate variable of weight 3. The pencil of planes in $\mathbb{P}^3$ passing through L defines a pencil $\mathcal{P}$ on V such that the normalization of a general surface in $\mathcal{P}$ is a smooth del Pezzo surface of degree 2. Consider now the restriction $f\colon X \to V$ to V of the blowup of the smooth curve $\widetilde{L} \subset \mathbb{P}(1,1,1,3)$.

Remark 1.1. We shall assume in what follows that X is smooth and the equations $p_4(0,0,z,t) = q_4(0,0,z,t) = 0$ have precisely 8 distinct homogeneous solutions defining 8 distinct points $O_i \in \widetilde{L}$ for $i=1,\dots,8$.

The 3-fold X is easily seen to be a double cover of $\operatorname{Proj}\bigl(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\bigr)$ branched over a surface $R_X \sim 4M+2H$, where M and H are the tautological sheaf and the fibre of the projection onto $\mathbb{P}^1$ of $\operatorname{Proj}\bigl(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\bigr)$, respectively. In particular, the smoothness of X ensures the smoothness of the surface R_X. On the other hand R_X is an ample divisor on the 3-fold $\operatorname{Proj}\bigl(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\bigr)$. In combination with Lefschetz's theorem this yields the equality $\operatorname{Pic}(X) = \mathbb{Z} \oplus \mathbb{Z}$. Hence V is a Fano 3-fold with $\mathbb{Q}$-factorial canonical singularities and Picard group $\mathbb{Z}$, the divisor K_V is Cartier, $-K_V^3 = 2$, the linear system $|-K_V|$ is free, $\varphi_{|-K_V|} = \gamma$, $K_X = f^*(K_V)$, the pencil $|-K_X - E|$ defines a fibration $\tau\colon X \to \mathbb{P}^1$ onto a del Pezzo surface of degree 2, f has eight reducible fibres $Z_i = Z_i^0 \cup Z_i^1$ over the points O_i, where E is an f-exceptional divisor, the curves Z_i^0 and Z_i^1 are smooth and intersect transversally at a single point.

Remark 1.2. Blowing up the curve Z_i^k and making the Francia antiflip in the proper transform of the curve Z_i^{1-k}, we obtain a birational map from V into a 3-fold with nef and big anticanonical divisor, which defines a birational map $\rho_{i,j}\colon V \dashrightarrow V_{i,k}$, where $V_{i,k}$ is a Fano 3-fold with canonical singularities and $-K_{V_{i,k}}^3 = \frac{1}{2}$ (see § 4).

The main aim of this paper is the proof of the following results.

Theorem 1.3. *Let $\rho\colon V \dashrightarrow \mathbb{P}^2$ be a dominant rational map whose general fibre is an elliptic curve. Then there exists a rational map $\alpha\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ such that $\alpha \circ \rho = \tau \circ f^{-1}$.*

Theorem 1.4. *The Fano 3-fold V is not birationally isomorphic to any Fano 3-fold with canonical singularities, except V itself and the sixteen 3-folds $V_{i,k}$ with $i=1,\dots,8$ and $k=0,1$.*

We point out that Theorem 1.3 cannot be improved. Indeed, let C be a section of the fibration τ and let $\mathcal{H}_C \subset |-K_X + \tau^*(\mathcal{O}_{\mathbb{P}^1}(n))|$ for $n \gg 0$ be a linear system of surfaces passing through the curve C. Then it is not difficult to show that the general fibre of the rational map $\varphi_{\mathcal{H}_C}: X \dashrightarrow Z_C$ is an elliptic curve, the surface Z_C is rational, the resolution of indeterminacy of the map $\varphi_{\mathcal{H}_C}$ produces an elliptic fibration with section, and there exists a natural projection $\alpha_C\colon Z_C \dashrightarrow \mathbb{P}^1$ such that $\alpha_C \circ \varphi_{\mathcal{H}_C} = \tau$. On the other hand the following result was proved in [1].

Theorem 1.5. *Let Y be a projective variety, $g\colon Y \to R$ a morphism with section onto a smooth curve R, let $r_1, \dots, r_k \in R$ be closed points such that the fibres $Y_i = g^{-1}(r_i)$ are smooth and separably rationally connected, and let $y_i \in Y_i$ be arbitrary closed points. Then there exists a section $C \subset Y$ of g such that $y_i \in C$ for $i = 1, \dots, k$.*

The fibration τ always has a section, in view of the following result (see [2]).

Theorem 1.6. *Let Y be a smooth proper and geometrically irreducible surface over a C_1-field $\mathbb{F}$ such that Y is rational over the algebraic closure of $\mathbb{F}$. Then Y has a point in $\mathbb{F}$.*

Thus, the fibration $\tau\colon X \to \mathbb{P}^1$ has a huge set of sections. Moreover, using the techniques of this paper one can show that for two sufficiently general sections C_1 and C_2 of τ the rational maps $\varphi_{\mathcal{H}_{C_1}}$ and $\varphi_{\mathcal{H}_{C_2}}$ define non-equivalent[2] elliptic fibrations. Hence Theorem 1.3 cannot be improved, but it can be supplemented by the following result from [3], because the structure of the group of birational automorphisms of a del Pezzo surface was described in [4] and [5].

Theorem 1.7. *Let Y be a smooth del Pezzo surface of degree two with* $\mathrm{Pic}(Y) \cong \mathbb{Z}$ *and defined over the field $\mathbb{F}$, let $\rho\colon Y \dashrightarrow U$ be a birational map and $\omega\colon U \to R$ a fibration by elliptic curves. Then there exists a birational automorphism σ of the surface Y and a birational morphism $\alpha\colon W \to Y$ such that $K_W^2 = 0$, $-K_W$ is nef, $|-nK_W|$ is free and $\varphi_{|-nK_W|}$ is an elliptic fibration for some $n \in \mathbb{N}$, and $\varphi_{|-nK_W|} \circ \rho^{-1} \circ \sigma \circ \rho$ is equivalent to ω.*

Birational transformations into elliptic fibrations were used in [6] and [7] in the proof of the potential density[3] of the rational points on smooth Fano 3-folds, where the following result was established.

Theorem 1.8. *The rational points are potentially dense on all smooth Fano 3-folds with the possible exception of the family of double covers of $\mathbb{P}^3$ ramified in a smooth sextic surface.*

The possible exception appears in Theorem 1.8 in view of the following result from [3].

Theorem 1.9. *Let Y be a double cover of $\mathbb{P}^3$ ramified in a smooth sextic. Then the variety Y is not birational to elliptic fibrations or canonical Fano 3-folds other than*

[2]Fibrations $\tau\colon U \to Z$ and $\overline{\tau}\colon \overline{U} \to \overline{Z}$ are *equivalent* if there exist two birational maps $\alpha\colon U \dashrightarrow \overline{U}$ and $\beta\colon Z \dashrightarrow \overline{Z}$ such that $\overline{\tau} \circ \alpha = \beta \circ \tau$ and α induces an isomorphism of the generic fibres of τ and $\overline{\tau}$.

[3]Rational points of a variety X defined over a number field $\mathbb{F}$ are *potentially dense* if for some finite extension $\mathbb{K}/\mathbb{F}$ the set of $\mathbb{K}$-rational points of $X(\mathbb{K})$ is Zariski dense.

itself, and all birational maps of Y into fibrations by surfaces of Kodaira dimension zero are induced by pencils in the linear system $|-K_Y|$.

In fact it follows easily from the explicit classification of smooth Fano 3-folds and Theorem 1.9 that the double cover of $\mathbb{P}^3$ ramified in a smooth sextic is the unique smooth Fano 3-fold that cannot be birationally transformed into an elliptic fibration. Many examples of rationally connected higher-dimensional varieties that cannot be birationally transformed into an elliptic fibration have been described in [3] and [8]–[10]. The following result was proved in [11].

Theorem 1.10. *Let $\theta\colon Y \to \mathbb{P}^3$ be a double cover ramified in a sextic $S_Y \subset \mathbb{P}^3$ such that the surface S_Y is smooth outside a point $O \in S_Y$ and S_Y has a singularity of type $\mathbb{A}_1$ at O. Then Y is not birationally isomorphic to any Fano 3-fold with canonical singularities other than itself, every biholomorphic map of Y into a fibration on surfaces of Kodaira dimension zero is defined by a pencil in $|-K_Y|$, the 3-fold Y is birationally isomorphic to a unique elliptic fibration, which is induced by the projection from O.*

The variety V was studied in [12], where the following result was proved.

Theorem 1.11. *The variety V is not birationally isomorphic to a fibration by rational surfaces distinct from $\tau\colon X \to \mathbb{P}^1$, the sequence $1 \to \Gamma \to \mathrm{Bir}(V) \to \mathbb{G} \to 1$ of groups is exact, where Γ is a free product of Bertini involutions of a generic fibre of τ regarded as a smooth del Pezzo surface of degree two with Picard group $\mathbb{Z}$ over the field $\mathbb{C}(x)$, and $\mathbb{G}$ is the group of fibrewise birational automorphisms of τ acting biregularly on the generic fibre of τ.*

In particular, the 3-fold V is not rational. The following result was proved in [13].

Theorem 1.12. *Suppose that every fibre F of τ is smooth along each curve $C \subset F$ such that $\gamma\circ f(C)$ is a line and $\gamma\circ f\big|_C$ is an isomorphism. Then V is not birationally isomorphic to any Fano 3-fold with terminal $\mathbb{Q}$-factorial singularities and Picard group $\mathbb{Z}$, and there exists an exact sequence $1 \to \Gamma \to \mathrm{Bir}(V) \to \mathrm{Aut}(V) \to 1$ of groups, where Γ is a free product of Bertini involutions of a generic fibre of τ regarded as a smooth del Pezzo surface of degree two with Picard group $\mathbb{Z}$ over the field $\mathbb{C}(x)$.*

The following result was proved in [14].

Theorem 1.13. *Let $\tau_1\colon X_1 \to \mathbb{P}^1$ and $\tau_2\colon X_2 \to \mathbb{P}^1$ be fibrations on del Pezzo surfaces of degree two, let $\alpha\colon X_1 \dashrightarrow X_2$ and $\beta\colon \mathbb{P}^1 \to \mathbb{P}^1$ be birational maps such that $\beta\circ\tau_1 = \tau_2\circ\alpha$, assume that the birational map α induces an isomorphism of the generic fibres of the fibrations τ_1 and τ_2 and that both varieties X_1 and X_2 are smooth. Then α is biregular.*

Corollary 1.14. *There exists an exact sequence $1 \to \Gamma \to \mathrm{Bir}(V) \to \mathrm{Aut}(V) \to 1$ of groups in which Γ is a free product of Bertini involutions of the generic fibre of τ regarded as a smooth del Pezzo surface of degree two with Picard group $\mathbb{Z}$ over the field $\mathbb{C}(x)$.*

We given an independent proof of Corollary 1.14 in § 6, where we also prove the following result.

Theorem 1.15. *Let $E \subset X$ be a smooth surface and let L be the unique line on the sextic S passing through one of the points $\gamma(O_i) \in \mathbb{P}^3$. Let $\rho\colon V \dashrightarrow \mathbb{P}^1$ be a rational map whose general fibre is an irreducible surface of Kodaira dimension zero. Then there exist a birational automorphism σ of the 3-fold V and a pencil $\mathcal{P} \subset |-K_V|$ such that $\rho \circ \sigma = \varphi_{\mathcal{P}}$.*

In the proofs of Theorems 1.3, 1.4, and 1.15 we use the so-called *method of super-maximal singularity* from [15].

Remark 1.16. The linear system $|-(k+1)K_X - kE|$ is free for $k \gg 0$ and defines a double cover of $\operatorname{Proj}\bigl(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\bigr)$ ramified in a surface $R_X \sim 4M + 2H$, where M and H are the tautological sheaf and the fibre of the projection onto $\mathbb{P}^1$ of the variety $\operatorname{Proj}\bigl(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\bigr)$, respectively.

Remark 1.17. The hypersurface $u^2 = x^2(x^4 + z^4 + t^4) + y^2(y^4 + z^4 + 2t^4)$ in $\mathbb{P}(1,1,1,3)$ satisfies all conditions of Theorems 1.3, 1.4, and 1.15, where x, y, z, t are homogeneous coordinates of weight 1 and u is a homogeneous coordinate of weight 3.

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§ 2. Movable log pairs

In this section we consider properties of the movable log pairs introduced in [16].

Defnition 2.1. A *movable log pair* (X, M_X) is a variety X together with a movable boundary M_X, where $M_X = \sum_{i=1}^n a_i \mathcal{M}_i$ is a formal finite linear combination of linear systems $\mathcal{M}_i$ on X without fixed components, where $a_i \in \mathbb{Q}_{\geqslant 0}$.

Every movable log pair can be regarded as an ordinary log pair via the replacement of each linear system by either its general element or an appropriate weighted sum of its general elements. In particular, for a fixed movable log pair (X, M_X) we may treat the movable boundary M_X as an effective divisor and we shall call $K_X + M_X$ the log canonical divisor of the movable log pair (X, M_X). In the rest of this section we shall assume that the log canonical divisors of all the log pairs under consideration are $\mathbb{Q}$-Cartier divisors.

Remark 2.2. For a movable log pair (X, M_X) we can regard M_X^2 as a well-defined effective cycle of codimension 2, provided that X is a $\mathbb{Q}$-factorial variety.

By contrast to the ordinary log pairs, the strict transform of a movable boundary is naturally well defined for every birational map.

Definition 2.3. Movable log pairs (X, M_X) and (Y, M_Y) are *birationally equivalent* if there exists a birational map $\rho\colon X \dashrightarrow Y$ such that $M_Y = \rho(M_X)$.

Discrepancies, terminality, canonicity, log terminality, and log canonicity can be defined for movable log pairs in the same way as for the ordinary case (see [17]).

Remark 2.4. The application of the Log Minimal Model Program to canonical or terminal movable log pairs preserves their canonicity or terminality, respectively.

Every movable log pair is birationally equivalent to a log pair with canonical singularities, and the singularities of a movable log pair coincide with the singularities of the variety outside the base loci of the components of the boundary.

Definition 2.5. A proper irreducible subvariety Y of X is called a *centre of canonical singularities of a movable log pair* (X, M_X) if there exists a birational morphism $f\colon W \to X$ and an f-exceptional divisor $E_1 \subset W$ such that

$$K_W + f^{-1}(M_X) \sim_{\mathbb{Q}} f^*(K_X + M_X) + \sum_{i=1}^{k} a(X, M_X, E_i)E_i,$$

$(X, M_X, E_1) \leqslant 0$, and $f(E_1) = Y$, where $a(X, M_X, E_i) \in \mathbb{Q}$ and E_i is an f-exceptional divisor.

Definition 2.6. We shall denote by $\mathbb{CS}(X, M_X)$ the set of centres of canonical singularities of the movable log pair (X, M_X) and by $\mathrm{CS}(X, M_X)$ the locus of all centres of canonical singularities of the movable log pair (X, M_X), which we regard as a proper subset of X.

In particular, a movable log pair (X, M_X) is terminal $\iff$ $\mathbb{CS}(X, M_X) = \varnothing$.

Definition 2.7. A proper irreducible subvariety Y of X is called a *proper centre of canonical singularities of the movable log pair* (X, M_X) if there exists a birational morphism $f\colon W \to X$ and an f-exceptional divisor $E_1 \subset W$ such that

$$K_W + f^{-1}(M_X) \sim_{\mathbb{Q}} f^*(K_X + M_X) + \sum_{i=1}^{k} a(X, M_X, E_i)E_i,$$

$a(X, M_X, E_1) \leqslant 0$, $f(E_1) = Y$ and the subvariety $Y \subset X$ lies in the base locus of some component of the movable boundary M_X, where $a(X, M_X, E_i) \in \mathbb{Q}$ and E_i is an f-exceptional divisor.

Definition 2.8. By $\overline{\mathbb{CS}}(X, M_X)$ we shall denote the set of all proper centres of canonical singularities of the movable log pair (X, M_X), and by $\overline{\mathrm{CS}}(X, M_X)$ the locus of all proper centres of canonical singularities of (X, M_X) regarded as a subset of X.

Remark 2.9.

$$\mathbb{CS}(X, M_X) = \overline{\mathbb{CS}}(X, M_X) \cup \mathbb{CS}(X, \varnothing),$$

in particular, if X has only terminal singularities, then $\mathbb{CS}(X, M_X) = \overline{\mathbb{CS}}(X, M_X)$.

Definition 2.10. The singularities of (X, M_X) are *semi-terminal* if

$$\overline{\mathbb{CS}}(X, M_X) = \varnothing.$$

Remark 2.11. If the singularities of a movable log pair (X, M_X) are semi-terminal, then for sufficiently small $\varepsilon \in \mathbb{Q}_{>1}$ the singularities of the log pair $(X, \varepsilon M_X)$ are also semi-terminal.

Definition 2.12. The quantity

$$\varkappa(X, M_X) = \begin{cases} \dim\bigl(\varphi_{|nm(K_W+M_W)|}(W)\bigr) & \text{for } n \gg 0 \\ & \text{such that } |n(K_W+M_W)| \neq \varnothing; \\ -\infty & \text{if } \ |nm(K_W+M_W)| = \varnothing \\ & \text{for all positive integers } n, \end{cases}$$

is called the *Kodaira dimension of the movable log pair* (X, M_X), where (W, M_W) is a movable log pair with canonical singularities birationally equivalent to (X, M_X) and $m \in \mathbb{N}$ is an integer such that the divisor $m(K_W + M_W)$ is Cartier.

Lemma 2.13. *The Kodaira dimension of a movable log pair is well defined and does not depend on the choice of the birationally equivalent canonical movable log pair.*

Proof. Assume that movable log pairs (X, M_X) and (Y, M_Y) have canonical singularities and that $M_X = \rho(M_Y)$ for some birational map $\rho\colon Y \dashrightarrow X$. Consider positive integers m_X and m_Y such that the divisors $m(K_X+M_X)$ and $m(K_Y+M_Y)$ are Cartier. For the proof of the claim we must show that either both $|nm_X(K_X+M_X)|$ and $|nm_Y(K_Y+M_Y)|$ are empty for all positive integers n, or that

$$\varphi_{|nm_X(K_X+M_X)|}(X) = \varphi_{|nm_Y(K_Y+M_Y)|}(Y)$$

for sufficiently large integers n. Let $g\colon W \to X$ and $f\colon W \to Y$ be birational morphisms of a smooth variety W and set $\rho = g \circ f^{-1}$. Then

$$K_W + M_W \sim_{\mathbb{Q}} g^*(K_X + M_X) + \Sigma_X \sim_{\mathbb{Q}} f^*(K_Y + M_Y) + \Sigma_Y,$$

where $M_W = g^{-1}(M_X)$, and Σ_X and Σ_Y are exceptional divisors of g and f respectively, and the canonicity of (X, M_X) and (Y, M_Y) means the effectiveness of Σ_X and Σ_Y. Let k be a sufficiently large and sufficiently divisible positive integer. Then it follows by the effectiveness of Σ_X and Σ_Y that the linear systems $|k(K_W+M_W)|$, $|g^*(k(K_X+M_X))|$, and $|f^*(k(K_Y+M_Y))|$ have the same dimension, and if they are non-empty then

$$\varphi_{|k(K_W+M_W)|} = \varphi_{|g^*(k(K_X+M_X))|} = \varphi_{|f^*(k(K_Y+M_Y))|},$$

which produces the required result.

By definition, the Kodaira dimension of a movable log pair is a birational invariant and a non-decreasing function of the coefficients of the movable boundary.

Definition 2.14. A movable log pair (V, M_V) is called a *canonical model of a movable log pair* (X, M_X) if there exists a birational map $\psi\colon X \dashrightarrow V$ such that $M_V = \psi(M_X)$, the log canonical divisor $K_V + M_V$ is ample, and (V, M_V) has canonical singularities.

Theorem 2.15. *A canonical model is unique if it exists.*

Proof. Assume that movable log pairs (X, M_X) and (V, M_V) are canonical models and let $M_X = \rho(M_V)$ for some birational map $\rho\colon V \dashrightarrow X$. Let $g\colon W \to X$ and $f\colon W \to V$ be birational maps such that $\rho = g \circ f^{-1}$. Then

$$K_W + M_W \sim_{\mathbb{Q}} g^*(K_X + M_X) + \Sigma_X \sim_{\mathbb{Q}} f^*(K_V + M_V) + \Sigma_V,$$

where $M_W = g^{-1}(M_X) = f^{-1}(M_V)$ and Σ_X and Σ_V are exceptional divisors of the birational morphisms g and f, respectively. It follows by the canonicity of the log pairs (X, M_X) and (V, M_V) that the divisors Σ_X and Σ_V are effective. Let n be a sufficiently large and sufficiently divisible positive integer such that $n(K_W + M_W)$, $n(K_X + M_X)$, and $n(K_V + M_V)$ are Cartier divisors. Then it follows from the effectiveness of Σ_X and Σ_V that

$$\varphi_{|n(K_W+M_W)|} = \varphi_{|g^*(n(K_X+M_X))|} = \varphi_{|f^*(n(K_V+M_V))|}$$

and that ρ is an isomorphism because $K_X + M_X$ and $K_V + M_V$ are ample.

Note that if a movable log pair has a canonical model, then the Kodaira dimension of the model is equal to the dimension of the variety.

§ 3. Preliminary results

As already mentioned in the previous section, one can regard movable boundaries as effective divisors and treat movable log pairs as ordinary. Hence we can use log pairs containing movable and fixed components alike. In this section we do not impose any restrictions on the coefficients of the boundaries; in particular, boundaries are not necessarily effective unless their effectiveness is explicitly stated. However, we shall assume that log canonical divisors of all log pairs are $\mathbb{Q}$-Cartier.

Definition 3.1. For a log pair (X, B_X) and a birational map $f\colon V \to X$, a log pair (V, B^V) is called a *log pullback of a log pair* (X, B_X) if

$$B^V = f^{-1}(B_X) - \sum_{i=1}^{n} a(X, B_X, E_i)E_i$$

and

$$K_V + B^V \sim_{\mathbb{Q}} f^*(K_X + B_X),$$

where $a(X, B_X, E_i) \in \mathbb{Q}$ and E_i is an exceptional divisor of f.

Definition 3.2. A proper irreducible subvariety Y of X is called a *centre of log canonical singularities of* (X, B_X) if there exist a birational morphism $f\colon W \to X$ and a divisor $E \subset W$ such that E lies in the support of the effective part of the divisor $\lfloor B^Y \rfloor$.

Definition 3.3. We shall denote by $\mathbb{LCS}(X, B_X)$ the set of centres of log canonical singularities of the log pair (X, B_X), and by $\mathrm{LCS}(X, B_X)$ the locus of all centres of log canonical singularities of the log pair (X, B_X) regarded as a proper subset of X.

Consider a log pair (X, B_X), where $B_X = \sum_{i=1}^{k} a_i B_i$, the B_i are effective and prime divisors, and $a_i \in \mathbb{Q}$. We choose a birational morphism $f\colon Y \to X$ such that Y is smooth and the union of all divisors $f^{-1}(B_i)$ and all f-exceptional divisors is a divisor with simple normal crossings. The morphism f is called a *log resolution of the log pair* (X, B_X), and we have

$$K_Y + B^Y \sim_{\mathbb{Q}} f^*(K_X + B_X)$$

for the log pullback (Y, B^Y) of (X, B_X).

Definition 3.4. The subscheme associated with the sheaf $\mathcal{I}(X, B_X) = f_*(\lceil -B^Y \rceil)$ of ideals is called the *log canonical singularities subscheme* of the log pair (X, B_X); we shall denote it by $\mathcal{L}(X, B_X)$.

The support of the subscheme $\mathcal{L}(X, B_X)$ is precisely the locus $\mathrm{LCS}(X, B_X) \subset X$. The following result is the famous Shokurov vanishing theorem.

Theorem 3.5. *Let (X, B_X) be a log pair, B_X an effective boundary, H a nef and big divisor on X such that $D = K_X + B_X + H$ is Cartier. Then*

$$H^i(X, \mathcal{I}(X, B_X) \otimes D) = 0 \quad \textit{for } i > 0.$$

Proof. By the relative Kawamata–Viehweg vanishing theorem,

$$R^i f_*(f^*(K_X + B_X + H) + \lceil -B^W \rceil) = 0$$

for $i > 0$ (see [17]). The degeneration of the local-to-global spectral sequence and the equality

$$R^0 f_*\big(f^*(K_X + B_X + H) + \lceil -B^W \rceil\big) = \mathcal{I}(X, B_X) \otimes D$$

yield that for all $i \geqslant 0$,

$$H^i(X, \mathcal{I}(X, B_X) \otimes D) = H^i(W, f^*(K_X + B_X + H) + \lceil -B^W \rceil),$$

whereas

$$H^i(W, f^*(K_X + B_X + H) + \lceil -B^W \rceil) = 0$$

for $i > 0$ by the Kawamata–Viehweg vanishing theorem.

For a Cartier divisor D on X we have the exact sequence

$$0 \to \mathcal{I}(X, B_X) \otimes D \to \mathcal{O}_X(D) \to \mathcal{O}_{\mathcal{L}(X, B_X)}(D) \to 0,$$

and Theorem 3.5 yields the following two connectedness results of Shokurov.

Theorem 3.6. *Let (X, B_X) be a log pair, B_X an effective boundary, and $-(K_X + B_X)$ a nef and big divisor. Then* $\mathrm{LCS}(X, B_X)$ *is connected.*

Theorem 3.7. *Let (X, B_X) be a log pair, B_X an effective boundary, and $-(K_X + B_X)$ a g-nef and g-big divisor for some morphism $g\colon X \to Z$ with connected fibres. Then* $\mathrm{LCS}(X, B_X)$ *is connected in the neighbourhood of every fibre of g.*

The following result is Theorem 17.4 of [18].

Theorem 3.8. *Let $g\colon X \to Z$ be a morphism, $D_X = \sum_{i\in I} d_i D_i$ a divisor on the variety X, and $h\colon V \to X$ a resolution of the singularities of X such that $g_*(\mathcal{O}_X) = \mathcal{O}_Z$, the divisor $-(K_X + D_X)$ is g-nef and g-big, the codimension of each subvariety $g(D_i)$ of Z is at least 2 if $d_i < 0$, and the union of all divisors $h^{-1}(D_i)$ and all h-exceptional divisors is a divisor with simple normal crossings. Then the locus $\bigcup_{a_E\leqslant -1} E$ is connected in a neighbourhood of every fibre of the morphism $g\circ h$, where the rational numbers a_E are defined by means of the $\mathbb{Q}$-rational equivalence $K_V \sim_{\mathbb{Q}} f^*(K_X + D_X) + \sum_{E\subset V} a_E E$.*

Proof. Set $f = g\circ h$, $A = \sum_{a_E > -1} E$, and $B = \sum_{a_E\leqslant -1} E$. Then

$$\lceil A\rceil - \lfloor B\rfloor \sim_{\mathbb{Q}} K_V - h^*(K_X + D_X) + \{-A\} + \{B\}$$

and $R^1 f_*\mathcal{O}_V(\lceil A\rceil - \lfloor B\rfloor) = 0$ by the relative Kawamata–Viehweg vanishing theorem (see [17]). Hence the map $f_*\mathcal{O}_V(\lceil A\rceil) \to f_*\mathcal{O}_{\lfloor B\rfloor}(\lceil A\rceil)$ is surjective. On the other hand, each irreducible component of the divisor $\lceil A\rceil$ is either h-exceptional or is a proper transform of some divisor D_j with $d_j < 0$. Consequently, $h_*(\lceil A\rceil)$ is g-exceptional and

$$f_*\mathcal{O}_V(\lceil A\rceil) = \mathcal{O}_Z.$$

Thus, the map $\mathcal{O}_Z \to f_*\mathcal{O}_{\lfloor B\rfloor}(\lceil A\rceil)$ is a surjection, and therefore $\lfloor B\rfloor$ is connected in a neighbourhood of each fibre of f because the divisor $\lceil A\rceil$ is effective and has no components in common with $\lfloor B\rfloor$.

In the previous section we defined in Definitions 2.5 and 2.6 centres of canonical singularities of a movable log pair and several related objects, but in none of these concepts did we actually use the movability of the boundary. Still, this redundant assumption of the movability was justified because these concepts are mostly used for movable log pairs and arise naturally in certain constructions related primarily to movable log pairs. However, in certain cases these concepts can be conveniently used also in the case of ordinary log pairs: mostly, for inductive relations to their log analogues. We have already mentioned that such a use is perfectly consistent.

Theorem 3.9. *Suppose that (X, B_X) is a log pair, B_X an effective boundary, $Z\in\mathbb{CS}(X, B_X)$, and let H be an effective irreducible Cartier divisor on X such that $Z\subset H$, H is not a component of B_X, and H is smooth at the generic point in Z. Then*

$$Z \in \mathbb{LCS}\big(H, B_X\big|_H\big).$$

Proof. Let $f\colon W\to X$ be the log resolution of $(X, B_X + H)$, and set $\widehat{H} = f^{-1}(H)$. Then

$$K_W + \widehat{H} \sim_{\mathbb{Q}} f^*(K_X + B_X + H) + \sum_{E\neq\widehat{H}} a(X, B_X + H, E)E$$

and by assumption $\{Z, H\} \subset \mathrm{LCS}(X, B_X + H)$. It follows by the application of Theorem 3.8 to the log pullback of $(X, B_X + H)$ on W that $\widehat{H} \cap E \neq \varnothing$ for some f-exceptional divisor E on W such that $f(E) = Z$ and $a(X, B_X, E) \leqslant -1$. The equivalence

$$K_{\widehat{H}} \sim (K_W + \widehat{H})\big|_{\widehat{H}} \sim_{\mathbb{Q}} f\big|_{\widehat{H}}^{*}\big(K_H + B_X\big|_H\big) + \sum_{E \neq \widehat{H}} a(X, B_X + H, E)E\big|_{\widehat{H}}$$

now yields the required result.

The next result is Theorem 3.1 of [19]; we expound the proof as given in [19] without serious modifications since, as of now, it is the simplest way to prove one very important 3-dimensional result, which we present below.

Theorem 3.10. *Let H be a surface, O a smooth point on H, M_H an effective movable boundary on the surface H, a_1 and a_2 non-negative rational numbers, and Δ_1 and Δ_2 irreducible and reduced curves on H intersecting normally at the point O. Then the inclusion $O \in \mathbb{LCS}(H, (1-a_1)\Delta_1 + (1-a_2)\Delta_2 + M_H)$ yields*

$$\operatorname{mult}_O(M_H^2) \geqslant \begin{cases} 4a_1a_2 & \text{if } a_1 \leqslant 1 \text{ or } a_2 \leqslant 1; \\ 4(a_1 + a_2 - 1) & \text{if } a_1 > 1 \text{ and } a_2 > 1, \end{cases}$$

and this inequality is strict if the log pair $(H, (1-a_1)\Delta_1 + (1-a_2)\Delta_2 + M_H)$ is not log canonical at O.

Proof. Set $D = (1-a_1)\Delta_1 + (1-a_2)\Delta_2 + M_H$ and let $f\colon S \to H$ be a birational morphism from a smooth surface S such that

$$K_S + f^{-1}(D) \sim_{\mathbb{Q}} f^*(K_H + D) + \sum_{i=1}^{k} a(H, D, E_i)E_i,$$

where E_i is an f-exceptional curve, $a(H, D, E_i) \in \mathbb{Q}$, and $a(H, D, E_1) \leqslant -1$. Then f is a composite of k blowups of smooth points.

Assume that we have proved the result in the case when $a_1 \leqslant 1$ or $a_2 \leqslant 1$. That is, we can assume that $a_1 > 1$ and $a_2 > 1$. Define rational numbers $a(H, E_i)$, $m(H, M_H, E_i)$ and $m(H, \Delta_j, E_i)$ by the relations $\sum_{i=1}^{k} a(H, E_i)E_i \sim_{\mathbb{Q}} K_S - f^*(K_H)$, $\sum_{i=1}^{k} m(H, M_H, E_i)E_i \sim_{\mathbb{Q}} f^{-1}(M_H) - f^*(M_H)$, and $\sum_{i=1}^{k} m(H, \Delta_j, E_i)E_i \sim_{\mathbb{Q}} f^{-1}(\Delta_j) - f^*(\Delta_j)$. Then

$$\begin{aligned} a(H, D, E_i) = a(H, E_i) - m(H, M_H, E_i) + m(H, \Delta_1, E_i)(a_1 - 1) \\ + m(H, \Delta_2, E_i)(a_2 - 1), \end{aligned}$$

and we can assume that $m(H, \Delta_1, E_1) \geqslant m(H, \Delta_2, E_1)$. Hence

$$-1 \geqslant a(H, D, E_1) \geqslant a(H, E_i) - m(H, M_H, E_i) + m(H, \Delta_2, E_i)(a_1 + a_2 - 2)$$

and $O \in \mathbb{LCS}(H, (2 - a_1 - a_2)\Delta_2 + M_H)$. This shows that $\operatorname{mult}_O(M_H^2) \geqslant 4(a_1 + a_2 - 1)$ because we have assumed that the theorem holds for the log pair $(H, (2 - a_1 - a_2)\Delta_2 + M_H)$.

We can now assume that $a_1 \leqslant 1$. Let $h\colon T \to H$ be a blowup of the point O and E an h-exceptional curve. Then $f = g \circ h$ for some birational morphism $g\colon S \to T$ that is a composite of $k-1$ blowups of smooth points. Then

$$K_T + (1-a_1)\overline{\Delta}_1 + (1-a_2)\overline{\Delta}_2 + (1-a_1-a_2+m)E + M_T \sim_{\mathbb{Q}} h^*(K_H+D),$$

where $\overline{\Delta}_j = h^{-1}(\Delta_j)$, $m = \operatorname{mult}_O(M_H)$ and $M_T = h^{-1}(M_H)$.

For $k=1$ we have $S=T$, $E_1 = E$, and $a(H,D,E_1) = a_1 + a_2 - m - 1 \leqslant -1$. Thus,

$$\operatorname{mult}_O(M_H^2) \geqslant m^2 \geqslant (a_1+a_2)^2 \geqslant 4a_1a_2,$$

and we are done. We shall assume therefore that $k>1$ and $P = g(E_1)$ is a point in E.

By construction, $P \in \mathbb{LCS}\bigl(T, (1-a_1)\overline{\Delta}_1 + (1-a_2)\overline{\Delta}_2 + (1-a_1-a_2+m)E + M_T\bigr)$, and three cases are possible: $P \in E \cap \overline{\Delta}_1$, $P \in E \cap \overline{\Delta}_2$, and $P \notin \overline{\Delta}_1 \cup \overline{\Delta}_2$. We can assume that the claim holds for the log pair $\bigl(T, (1-a_1)\overline{\Delta}_1 + (1-a_1-a_2+m)E + M_T\bigr)$ in the case when $P \in E \cap \overline{\Delta}_1$; for the log pair $\bigl(T, (1-a_2)\overline{\Delta}_2 + (1-a_1-a_2+m)E + M_T\bigr)$ in the case when $P \in E \cap \overline{\Delta}_2$; and for the log pair $\bigl(T, (1-a_1-a_2+m)E + M_T\bigr)$ in the case when $P \notin \overline{\Delta}_1 \cup \overline{\Delta}_2$, because all assumptions of the theorem hold in each of these cases and the birational morphism g consists of $k-1$ blowups of smooth points. Moreover, $\operatorname{mult}_O(M_H^2) \geqslant m^2 + \operatorname{mult}_P(M_T^2)$.

Consider the case $P \in E \cap \overline{\Delta}_1$. Then by induction

$$\operatorname{mult}_O(M_H^2) \geqslant m^2 + 4a_1(a_1+a_2-m) = (2a_1-m)^2 + 4a_1a_2 \geqslant 4a_1 4a_2.$$

Consider the case $P \in E \cap \overline{\Delta}_2$. If $a_2 \leqslant 1$ or $a_1+a_2-m \leqslant 1$, then we can proceed as in the previous case. Thus, we can assume that $a_2 < 1$ and $a_1+a_2-m<1$. Then

$$\operatorname{mult}_O(M_H^2) \geqslant m^2 + 4(a_1+2a_2-m-1) > 4a_2 \geqslant 4a_1 4a_2.$$

Consider now the case $P \notin \overline{\Delta}_1 \cup \overline{\Delta}_2$. Then by induction

$$\operatorname{mult}_O(M_H^2) \geqslant m^2 + 4(a_1+a_2-m) > m^2 + 4a_1(a_1+a_2-m) \geqslant 4a_1 4a_2,$$

which completes the proof.

Most applications use the following special case of Theorem 3.10.

Lemma 3.11. *Let H be a smooth surface, O a point on H, M_H an effective movable boundary on H, and suppose that $O \in \mathbb{LCS}(H, M_H)$. Then* $\operatorname{mult}_O(M_H^2) \geqslant 4$ *and equality here yields the equality* $\operatorname{mult}_O(M_H) = 2$.

The following result is Corollary 7.3 in [20].

Theorem 3.12. *Let X be a 3-fold, M_X an effective movable boundary on X, and O a smooth point on X such that $O \in \mathbb{CS}(X, M_X)$. Then* $\operatorname{mult}_O(M_X^2) \geqslant 4$ *and equality here yields the equality* $\operatorname{mult}_O(M_H) = 2$.

Proof. Let H be a general hyperplane section on X passing through O. Then O is a centre of log canonical singularities of the log pair $\bigl(H, M_X\big|_H\bigr)$, by Theorem 3.9. On the other hand,

$$\operatorname{mult}_O(M_X) = \operatorname{mult}_O\bigl(M_X\big|_H\bigr), \qquad \operatorname{mult}_O(M_X^2) = \operatorname{mult}_O\Bigl(\bigl(M_X\big|_H\bigr)^2\Bigr)$$

and the required result follows from Lemma 3.11.

Actually, the statement of Theorem 3.12 can be explained in a much more geometric way.

Lemma 3.13. *Let O be a point on a smooth 3-fold X such that $O \in \mathbb{CS}(X, M_X)$, where M_X is an effective movable boundary on X and the singularities of the canonical log pair (X, M_X) are canonical. Then there exists a birational map $f\colon V \to X$ such that V has only terminal $\mathbb{Q}$-factorial singularities, the morphism f contracts a unique exceptional divisor E to the point O, and $K_V + M_V \sim_{\mathbb{Q}} f^*(K_X + M_X)$, where $M_V = f^{-1}(M_X)$.*

Proof. The number of distinct divisorial valuations ν of the field of rational functions on X of which the centre on X is at O and the discrepancy $a(X, M_X, \nu)$ is non-positive is finite, in view of the canonicity of the log pair (X, M_X). Hence we can consider a birational morphism $g\colon W \to X$ such that the 3-fold W is smooth, g contracts k exceptional divisors

$$K_W + M_W \sim_{\mathbb{Q}} g^*(K_X + M_X) + \sum_{i=1}^{k} a_i E_i,$$

the log pair (W, M_W) has canonical singularities, and $\mathbb{CS}(W, M_W)$ does not contain subvarieties of $\bigcup_{i=1}^{k} E_i$, where $M_W = g^{-1}(M_X)$, $g(E_i) = O$, and $a_i \in \mathbb{Q}$. Applying the relative version of the Log Minimal Model Program to the movable log pair (W, M_W) over X, we can assume that W has terminal $\mathbb{Q}$-factorial singularities and $K_W + M_W \sim_{\mathbb{Q}} g^*(K_X + M_X)$ by the canonicity of (X, M_X). Application of the relative version of the Log Minimal Model Program to W over X proves the required result.

The following well-known result was initially conjectured in [21] and proved in [22].

Theorem 3.14. *Let X be a smooth 3-fold, O a point on X, and $f\colon V \to X$ a birational morphism such that V has terminal $\mathbb{Q}$-factorial singularities and the morphism f contracts a unique exceptional divisor E to the point O. Then f is a weighted blowup of O with weights $(1, K, N)$ in a suitable local coordinate system on the 3-fold X, where K and N are coprime positive integers.*

Moreover, Theorem 3.12 was proved in the following way in [21], assuming the result of Theorem 3.14; it explains the geometric nature of the inequality in Theorem 3.12.

Proposition 3.15. *Let X be a smooth 3-fold, O a point on X, and M_X an effective movable boundary on X such that $O \in \mathbb{CS}(X, M_X)$, and let $f\colon V \to X$ be a weighted blowup of O with weights $(1, K, N)$ in suitable local coordinates on X such that $K_V + M_V \sim_{\mathbb{Q}} f^*(K_X + M_X)$, where K and N are coprime positive integers and $M_V = f^{-1}(M_X)$. Then the following result holds*:

$$\mathrm{mult}_O(M_X^2) \geqslant \frac{(K+N)^2}{KN} = 4 + \frac{(K-N)^2}{KN} \geqslant 4,$$

and if $K = N$, then f is an ordinary blowup of O and $\mathrm{mult}_O(M_X) = 2$.

Proof. We have $K_V \sim_{\mathbb{Q}} f^*(K_X) + (N+K)E$ and $M_V \sim_{\mathbb{Q}} f^*(M_X) + mE$ for an f-exceptional divisor E and some $m \in \mathbb{Q}_{>0}$. Thus, $m = K + N$ and $\mathrm{mult}_O(M_X^2) \geqslant m^2E^3 = (K+N)^2/(KN)$.

§ 4. Construction of the birational map $\rho_{i,k}$

In the notation of § 1, let $g\colon W \to X$ be a blowup of the smooth curve Z_1^0, let G be a g-exceptional divisor, and $\overline{Z}_1^1$ the proper transform of the smooth rational curve Z_1^1 on W. Then $K_W \cdot \overline{Z}_1^1 = 1$.

Lemma 4.1. $\mathcal{N}_{\overline{Z}_1^1/W} \cong \mathcal{O}_{\overline{Z}_1^1}(-1) \oplus \mathcal{O}_{\overline{Z}_1^1}(-2)$.

Proof. Suppose that $\mathcal{N}_{\overline{Z}_1^1/W} \cong \mathcal{O}_{\overline{Z}_1^1}(m) \oplus \mathcal{O}_{\overline{Z}_1^1}(n)$ for $m \geqslant n$, and let H be a sufficiently general surface in the anticanonical linear system $|-K_W|$. Then H passes through the curve $\overline{Z}_1^1$ because $-K_W \cdot \overline{Z}_1^1 = -1$. However, $\overline{Z}_1^1$ is the unique base curve of the linear system $|-K_W|$. In particular, H is smooth outside $\overline{Z}_1^1$, by Bertini's theorem. On the other hand the generality of our choice of H and the equation of V demonstrate that the surface $f \circ g(H)$ is smooth outside O_1. Moreover, at O_1 this surface has a Du Val singularity of type $\mathbb{A}_2$. Now, from the explicit equation of X in the neighbourhood of the curves Z_1^0 and Z_1^1 one sees that the surface $g(H)$ is smooth along the curves Z_1^0 and Z_1^1. In particular, the restriction of the birational morphism f to $g(H)$ contracts the two (-2)-curves Z_1^0 and Z_1^1 to the point O_1. The smoothness of $g(H)$ means the smoothness of H along $\overline{Z}_1^1$. Thus, H is a smooth K3 surface, $m+n = -K_W \cdot \overline{Z}_1^1 - 2 = -3$, and the exact sequence

$$0 \to \mathcal{N}_{\overline{Z}_1^1/H} \to \mathcal{N}_{\overline{Z}_1^1/W} \to \mathcal{N}_{H/W}\big|_{\overline{Z}_1^1} \to 0$$

yields $n \geqslant -2$. Hence $n = -2$ and $m = -1$.

Let $h\colon U \to W$ be a blowup of the curve $\overline{Z}_1^1$, let F be an h-exceptional divisor, and G_U a proper transform of the divisor G on the 3-fold U. Then $F \cong \mathbb{F}_1$, $K_U \sim (g \circ h)^*(K_X) + F + G_U$, the anticanonical linear system $|-K_U|$ is free, and the morphism $\varphi_{|-K_U|}\colon U \to \mathbb{P}^2$ is an elliptic fibration with quasi-sections F and G_U.

Remark 4.2. The morphism $\gamma \circ f \circ g \circ h\colon W \to \mathbb{P}^3$ takes the fibres of the elliptic fibration $\varphi_{|-K_U|}$ to lines in $\mathbb{P}^3$ passing through O_1, and $\varphi_{|-K_U|}$ is induced by the projection from O_1.

Let $C \subset U$ be an exceptional section of the surface $F \cong \mathbb{F}_1$. Then the morphism $\varphi_{|-K_U|}\colon U \to \mathbb{P}^2$ contracts C to a point.

Lemma 4.3. $\mathcal{N}_{C/U} \cong \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$.

Proof. Suppose that $\mathcal{N}_{C/U} \cong \mathcal{O}_C(m) \oplus \mathcal{O}_C(n)$ for $m \geqslant n$. Then $m+n = -K_U \cdot C - 2$, while the exact sequence

$$0 \to \mathcal{N}_{C/F} \to \mathcal{N}_{C/U} \to \mathcal{N}_{F/U}\big|_C \to 0$$

yields $n \geqslant -1$. Hence $m = n = -1$.

Let $\widehat{p}\colon \widehat{U} \to U$ be a blowup of the smooth curve C, R a $\widehat{p}$-exceptional divisor, and $\check{p}\colon \widehat{U} \to \check{U}$ a blowdown of $R \cong \mathbb{P}^1 \times \mathbb{P}^1$ to a curve $\check{C}$ such that the birational map $\check{p} \circ \widehat{p}^{-1}$ is not biregular. Then $\check{p} \circ \widehat{p}^{-1}$ is a flop in the curve C, the anticanonical linear system $|-K_{\check{U}}|$ is free, and $\varphi_{|-K_{\check{U}}|} \circ \check{p} = \varphi_{|-K_U|} \circ \widehat{p}$.

Remark 4.4. The proper transform $\check{F} \subset \check{U}$ of the surface F is isomorphic to $\mathbb{P}^2$ with normal bundle $\mathcal{O}_{\mathbb{P}^2}(-2)$.

We can contract $(\check{h}\colon \check{U} \to \check{W})$ the surface $\check{F}$ to the singular point $P \cong \frac{1}{2}(1,1,1)$.

Lemma 4.5. *The variety $\check{W}$ is projective.*

Proof. The antiflip $\check{h}\circ\check{p}\circ\widehat{p}^{-1}\circ h^{-1}$ is the log flip for the log terminal log pair $(W,\varepsilon|-K_W|)$ for a sufficiently small rational $\varepsilon>1$, and this yields the required result.

Lemma 4.6. *The base locus of $|-K_{\check{W}}|$ consists of just the point P, and $-K_{\check{W}}^3=\frac{1}{2}$.*

Proof. The anticanonical linear system $|-K_{\check{W}}|$ is the proper transform of the linear system $|-K_{\check{U}}|$, which shows that the base locus of $|-K_{\check{W}}|$ consists of P alone. The relations $K_{\check{U}}^3=0$, $\check{F}^3=4$, and $K_{\check{U}}\sim_{\mathbb{Q}}\check{h}^*(K_{\check{W}})+\frac{1}{2}\check{F}$ now yield that $-K_{\check{W}}^3=\frac{1}{2}$.

Hence the divisor $-K_{\check{W}}$ is nef and big, the linear system $|-nK_{\check{W}}|$ is free for $n\gg 0$, the morphism $\varphi_{|-nK_{\check{W}}|}\colon\check{W}\to V_{1,0}$ is birational, and the variety $V_{1,0}$ is normal. The resulting map

$$\rho_{1,0}=\varphi_{|-nK_{\check{W}}|}\circ\check{h}\circ\check{p}\circ\widehat{p}^{-1}\circ h^{-1}\circ g^{-1}\circ f^{-1}\colon V\dashrightarrow V_{1,0}$$

takes V birationally to a Fano 3-fold with canonical singularities $V_{1,0}$ such that $-K_{V_{1,0}}^3=\frac{1}{2}$.

Remark 4.7. The singularities of $V_{1,0}$ are terminal and not $\mathbb{Q}$-factorial.

The constructions of the maps $\rho_{i,k}\colon V\dashrightarrow V_{i,k}$ are identical to the above construction.

§ 5. Proof of Theorem 1.3

In the notation and assumptions of § 1, assume now the existence of a birational transformation $\beta\colon V\dashrightarrow Y$ and a fibration $\pi\colon Y\to\mathbb{P}^2$ whose general fibre is a connected smooth elliptic curve. Assume also that there exists no rational map $\alpha\colon\mathbb{P}^2\dashrightarrow\mathbb{P}^1$ such that $\alpha\circ\rho=\tau\circ f^{-1}$. In the rest of this section we shall derive a contradiction to these assumptions. Set $\rho=\pi\circ\beta$ and consider the linear system $|\pi^*(D)|$, where D is a very ample divisor on $\mathbb{P}^2$. Set $\mathcal{D}_V=\beta^{-1}(|\pi^*(D)|)$. Then $\mathcal{D}_V\subset|-nK_V|$ for some positive integer n. Consider the movable boundary $\frac{1}{n}\mathcal{D}_V$.

Lemma 5.1. $\overline{\mathbb{CS}}\bigl(V,\frac{1}{n}\mathcal{D}_V\bigr)\neq\varnothing$.

Proof. Assume that $\bigl(V,\frac{1}{n}\mathcal{D}_V\bigr)$ is semi-terminal. Then for some $\varepsilon\in\mathbb{Q}_{>1/n}$ the log pair $(V,\varepsilon\mathcal{D}_V)$ is a canonical model and $\varkappa(V,\varepsilon\mathcal{D}_V)=3$; however, $\varkappa(V,\varepsilon\mathcal{D}_V)\leqslant 2$ for every rational ε, by the construction of $\mathcal{D}_V$.

Therefore, $\overline{\mathbb{CS}}\bigl(V,\frac{1}{n}\mathcal{D}_V\bigr)$ contains either a point or a curve on V.

Remark 5.2. By construction, $\mathcal{D}_V$ does not lie in fibres of any dominant map $\chi\colon V\dashrightarrow\mathbb{P}^1$.

Lemma 5.3. *The set $\mathbb{CS}\bigl(V,\frac{1}{n}\mathcal{D}_V\bigr)$ does not contain smooth points of V.*

Proof. Assume that $\mathbb{CS}\bigl(V,\frac{1}{n}\mathcal{D}_V\bigr)$ contains a point $O\notin\widetilde{L}$. Consider a general surface $H_O\in|-K_V|$ passing through O. Then $2n^2=H_O\cdot\mathcal{D}_V^2\geqslant n^2\operatorname{mult}_O(\mathcal{D}_V^2)\geqslant 4n^2$ by Theorem 3.12.

Lemma 5.4. *Suppose that* $\mathbb{CS}\big(V, \frac{1}{n}\mathcal{D}_V\big)$ *contains a curve* $C \neq \widetilde{L}$. *Then* $-K_V \cdot C = 1$.

Proof. The inequality $\mathrm{mult}_C(\mathcal{D}_V) \geqslant n$ is equivalent to the fact that C lies in $\mathbb{CS}\big(V, \frac{1}{n}\mathcal{D}_V\big)$. Hence for a sufficiently general divisor H in the free linear system $|-K_V|$, the inequalities

$$2n^2 = H \cdot \mathcal{D}_V^2 \geqslant \mathrm{mult}_C(\mathcal{D}_V^2) H \cdot C \geqslant \mathrm{mult}_C^2(\mathcal{D}_V) H \cdot C \geqslant n^2 H \cdot C$$

yield that $-K_V \cdot C \leqslant 2$. Suppose that $-K_V \cdot C = 2$. Then $\mathrm{mult}_C(\mathcal{D}_V^2) = \mathrm{mult}_C^2(\mathcal{D}_V) = n^2$ and the support of the one-dimensional cycle $\mathcal{D}_V^2$ consists of the single point C. By construction, $\varphi_{\mathcal{D}_V}(V) = \mathbb{P}^2$ and there exists a point P in V such that $P \notin C$ and the linear subsystem $\mathcal{D}_P$ of $\mathcal{D}_V$ consisting of surfaces in the linear system $\mathcal{D}_V$ passing through P has no fixed components. Hence $P \in \mathcal{D}_P^2 \subset \mathcal{D}_V^2 = C$ in the set-theoretic sense, which contradicts our assumption that $P \notin C$.

Lemma 5.5. *Suppose that* $\mathbb{CS}\big(V, \frac{1}{n}\mathcal{D}_V\big)$ *contains a curve* C. *Then* $C \cap \widetilde{L} \neq \varnothing$.

Proof. Assume that $C \cap \widetilde{L} = \varnothing$. Then $\gamma(C) \subset \mathbb{P}^3$ is a line and $\gamma\big|_C$ is an isomorphism by Lemma 5.4.

Assume that $\gamma(C) \not\subset S$. Let $\mathcal{H}_C \subset |-K_V|$ be the linear system consisting of surfaces passing through the curve C. Let H_C be a sufficiently general surface in the linear system $\mathcal{H}_C$ and $\widetilde{C}$ a smooth rational curve on V such that $\widetilde{C} \neq C$ and $\gamma(\widetilde{C}) = \gamma(C)$. By construction, H_C is a smooth K3 surface containing C and $\widetilde{C}$, and we have $C^2 = \widetilde{C}^2 = -2$ and $C \cdot \widetilde{C} = 3$ on H_C. The base locus of the linear system $\mathcal{H}_C$ is $C \cup \widetilde{C}$, so that

$$\mathcal{D}_V\big|_{H_C} = \mathrm{mult}_C(\mathcal{D}_V) C + \mathrm{mult}_{\widetilde{C}}(\mathcal{D}_V) \widetilde{C} + R_{H_C},$$

where R_{H_C} is a movable boundary on H_C. Hence the equalities

$$n = \mathcal{D}_V\big|_{H_C} \cdot \widetilde{C} = 3\,\mathrm{mult}_C(\mathcal{D}_V) - 2\,\mathrm{mult}_{\widetilde{C}}(\mathcal{D}_V) + R_{H_C} \cdot \widetilde{C}$$

yield that $\mathrm{mult}_{\widetilde{C}}(\mathcal{D}_V) = \mathrm{mult}_C(\mathcal{D}_V) = n$ and $R_{H_C} = \varnothing$. In particular, $\mathbb{CS}\big(V, \frac{1}{n}\mathcal{D}_V\big)$ contains also the curve $\widetilde{C}$. On the other hand, since $R_{H_C} = \varnothing$, it follows that $\mathcal{D}_V$ lies in fibres of $\varphi_{\mathcal{H}_C}$, which is impossible.

Assume now that $\gamma(C) \subset S$. Let $g\colon W \to V$ be a blowup of C, let G be a g-exceptional divisor, and set $\mathcal{D}_W = g^{-1}(\mathcal{D}_V)$. It is easy to see that the anticanonical linear system $|-K_W|$ is a pencil and its base locus is a smooth rational curve $\overline{C}$, which is a section of the ruled surface $g\big|_G\colon G \to C$. Consider a sufficiently general divisor $H_{\overline{C}}$ in $|-K_W|$. Then $H_{\overline{C}}$ is a smooth K3 surface and

$$\mathcal{D}_W\big|_{H_{\overline{C}}} = \mathrm{mult}_{\overline{C}}(\mathcal{D}_W) \overline{C} + R_{H_{\overline{C}}},$$

where $R_{H_{\overline{C}}}$ is a movable boundary on $H_{\overline{C}}$. On the surface $H_{\overline{C}}$ we have $\overline{C}^2 = -2$ and

$$\mathcal{D}_W\big|_{H_{\overline{C}}} \sim_{\mathbb{Q}} n\overline{C} + (n - \mathrm{mult}_C(\mathcal{D}_V)) G\big|_{H_{\overline{C}}},$$

which yields that $\mathrm{mult}_{\overline{C}}(\mathcal{D}_W) = \mathrm{mult}_C(\mathcal{D}_V) = n$ and $R_{H_{\overline{C}}} = \varnothing$. Now, as in the previous case, since $R_{H_{\overline{C}}}$ is empty, it follows that the linear system $\mathcal{D}_W$ lies in fibres of the map $\varphi_{|-K_W|}\colon W \dashrightarrow \mathbb{P}^1$, which is impossible by the construction of $\mathcal{D}_V$.

Lemma 5.6. *Suppose that $\widetilde{L}\notin\overline{\mathbb{CS}}\big(V,\frac{1}{n}\mathcal{D}_V\big)$. Then $\overline{\mathbb{CS}}\big(V,\frac{1}{n}\mathcal{D}_V\big)$ does not contain curves.*

Proof. The curve $\widetilde{L}$ does not lie in the base locus of $\mathcal{D}_V$ because $\overline{\mathbb{CS}}\big(V,\frac{1}{n}\mathcal{D}_V\big)$ does not contain $\widetilde{L}$. In particular, the proper transform $\mathcal{D}_X$ of the linear system $\mathcal{D}_V$ on X is a linear subsystem of $|-nK_X|$. For the proof of the required result it is sufficient to show that $\mathbb{CS}\big(X,\frac{1}{n}\mathcal{D}_X\big)$ does not contain curves on the 3-fold X that do not lie in the exceptional divisor E. In fact, assume that $\mathbb{CS}\big(X,\frac{1}{n}\mathcal{D}_X\big)$ does contain a curve $C\subset X$ not contained in E. Then by Lemmas 5.4 and 5.5 the curve $\gamma\circ f(C)\subset\mathbb{P}^3$ is a line, $\gamma\circ f\big|_C$ is an isomorphism, and $C\cap E\neq\varnothing$.

Assume that $\gamma\circ f(C)\not\subset S$. Let $\mathcal{H}_C\subset|-K_X|$ be a linear subsystem consisting of surfaces passing through C. Let H_C be a sufficiently general surface in $\mathcal{H}_C$, Z the fibre of f over the point $f(C)\cap\widetilde{L}$, and let $\widetilde{C}$ be a smooth curve on X such that $\widetilde{C}\neq C$ and $\gamma\circ f(\widetilde{C})=\gamma\circ f(C)$. Then H_C is a smooth K3 surface containing the curves $C\cup\widetilde{C}\cup Z$, where $Z=Z_i^0\cup Z_i^1$ if $f(C)\cap\widetilde{L}=O_i$ and Z is irreducible otherwise. On the surface H_C we have $Z^2=C^2=\widetilde{C}^2=-2$, $Z\cdot C=Z\cdot\widetilde{C}=1$, and $C\cdot\widetilde{C}=2$;

$$\mathcal{D}_X\big|_{H_C}=\operatorname{mult}_C(\mathcal{D}_X)C+\operatorname{mult}_{\widetilde{C}}(\mathcal{D}_X)\widetilde{C}+\Delta_Z+R_{H_C}$$

because the base locus of the pencil $\mathcal{H}_C$ consists of $Z\cup C\cup\widetilde{C}$, where Δ_Z is an effective divisor with support Z, and R_{H_C} is a movable boundary on H_C. In addition, on the surface H_C we have

$$R_{H_C}\sim_{\mathbb{Q}}(n-\operatorname{mult}_C(\mathcal{D}_X))C+(n-\operatorname{mult}_{\widetilde{C}}(\mathcal{D}_X))\widetilde{C}+nZ-\Delta_Z,$$

and the intersection form of curves in $Z\cup\widetilde{C}$ is negative definite. Hence $R_{H_C}=\varnothing$ and $\operatorname{mult}_{\widetilde{C}}(\mathcal{D}_X)=\operatorname{mult}_{\widetilde{Z}}(\mathcal{D}_X)=n$. On the other hand, since R_{H_C} is empty, it follows that the linear system $\mathcal{D}_X$ lies in fibres of $\varphi_{\mathcal{H}_C}$, which is impossible by the construction of $\mathcal{D}_V$.

Now suppose that $\gamma\circ f(C)\subset S$. Let $g\colon W\to X$ be a blowup of the smooth curve C, G a g-exceptional divisor, and set $\mathcal{D}_W=g^{-1}(\mathcal{D}_X)$. Then the linear system $|-K_W|$ is a pencil and its base locus consists of $\overline{Z}$ and $\overline{C}$, where $\overline{C}$ is a section of the ruled surface $g\big|_G\colon G\to C$, and $\overline{Z}$ is the proper transform on W of the fibre of f over the point $f(C)\cap\widetilde{L}$, which consists of two smooth rational curves intersecting transversally at a single point in the case when $f(C)\cap\widetilde{L}=O_i$, and of a smooth rational curve otherwise. Consider a general surface $H_{\overline{C}}\in|-K_W|$. It is smooth and

$$\mathcal{D}_W\big|_{H_{\overline{C}}}=\operatorname{mult}_{\overline{C}}(\mathcal{D}_W)\overline{C}+\Delta_{\overline{Z}}+R_{H_{\overline{C}}},$$

where $\Delta_{\overline{Z}}$ is an effective divisor with support $\overline{Z}$ and $R_{H_{\overline{C}}}$ is a movable boundary on the surface $H_{\overline{C}}$. Moreover, on $H_{\overline{C}}$ we have

$$\mathcal{D}_W\big|_{H_{\overline{C}}}\sim_{\mathbb{Q}}n\overline{Z}+n\overline{C}+(n-\operatorname{mult}_C(\mathcal{D}_X))G\big|_{H_{\overline{C}}},$$

and in all possible cases the intersection form of curves in $\overline{C}\cup\overline{Z}$ is negative definite, which immediately shows that $R_{H_{\overline{C}}}$ is empty and $\operatorname{mult}_{\overline{Z}}(\mathcal{D}_W)=\operatorname{mult}_{\overline{C}}(\mathcal{D}_W)=\operatorname{mult}_C(\mathcal{D}_X)=n$. In particular, since $R_{H_{\overline{C}}}$ is empty, it follows that the linear system $\mathcal{D}_W$ lies in fibres of the map $\varphi_{|-K_W|}\colon W \dashrightarrow \mathbb{P}^1$, which is impossible by the construction of $\mathcal{D}_V$.

Lemma 5.7. *Suppose that* $\overline{\mathbb{CS}}\big(V, \frac{1}{n}\mathcal{D}_V\big)$ *does not contain* $\widetilde{L}$. *Then* $\overline{\mathbb{CS}}\big(V, \frac{1}{n}\mathcal{D}_V\big)$ *contains no point in the* 3*-fold* V *other than the* O_i.

Proof. Let O be a point on V distinct from the O_i and lying in $\overline{\mathbb{CS}}\big(V, \frac{1}{n}\mathcal{D}_V\big)$. Then $O \in \widetilde{L}$ by Lemma 5.3; on the other hand $\widetilde{L}$ does not belong to the base locus of the linear system $\mathcal{D}_V$ since $\widetilde{L} \notin \overline{\mathbb{CS}}\big(V, \frac{1}{n}\mathcal{D}_V\big)$. In particular, the proper transform $\mathcal{D}_X$ of the linear system $\mathcal{D}_V$ on X is a linear subsystem of $|-nK_X|$, and $\big(X, \frac{1}{n}\mathcal{D}_X\big)$ is a log pullback of the movable log pair $\big(V, \frac{1}{n}\mathcal{D}_V\big)$. Hence there exists an irreducible fibre Z of the morphism $f\big|_E\colon E \to \widetilde{L}$ such that $\mathbb{CS}\big(X, \frac{1}{n}\mathcal{D}_X\big)$ contains either Z or a point in it.

Assume that $\mathbb{CS}\big(X, \frac{1}{n}\mathcal{D}_X\big)$ contains the curve Z. Let $g\colon W \to X$ be a blowup of Z, $\mathcal{D}_W$ the proper transform of $\mathcal{D}_X$ on W, and G a g-exceptional divisor. Then

$$\mathcal{D}_W \sim g^*(-nK_X) - \operatorname{mult}_Z(\mathcal{D}_X)G$$

and $\operatorname{mult}_Z(\mathcal{D}_X) \geqslant n$. On the other hand, the anticanonical linear system $|-K_W|$ is free and defines an elliptic fibration $\varphi_{|-K_W|}\colon W \to \mathbb{P}^2$. Let C be a sufficiently general fibre of $\varphi_{|-K_W|}$. Then it is easy to see that $\mathcal{D}_W \cdot C = 2n - 2\operatorname{mult}_Z(\mathcal{D}_X)$. In particular, $\operatorname{mult}_Z(\mathcal{D}_X) = n$ and the linear system $\mathcal{D}_W$ lies in the fibres of the elliptic fibration $\varphi_{|-K_W|}$, which is not yet a contradiction in itself, but which demonstrates that the fibrations $\varphi_{|-K_W|}$ and π are equivalent. By construction, there exists a projection $\alpha\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ from a point such that $\alpha \circ \varphi_{|-K_W|} \circ (f \circ g)^{-1} = \tau \circ f^{-1}$, which contradicts our initial assumption.

Assume now that $\mathbb{CS}\big(X, \frac{1}{n}\mathcal{D}_X\big)$ contains a point $P \in Z$. In that case we set $\mathcal{D}_X^2 = \operatorname{mult}_Z(\mathcal{D}_X^2)Z + C_X$, where the support of the effective 1-cycle C_X does not contain the irreducible curve Z. Then

$$\operatorname{mult}_P(\mathcal{D}_X^2) = \operatorname{mult}_Z(\mathcal{D}_X^2) + \operatorname{mult}_P(C_X) \geqslant 4n^2,$$

by Theorem 3.12. On the other hand, $Z \cdot F = 2$ and $\mathcal{D}_X^2 \cdot F = 2n^2$, where F is a fibre of the fibration τ by del Pezzo surfaces of degree 2. In particular, $\operatorname{mult}_Z(\mathcal{D}_X^2) \leqslant n^2$ and $\operatorname{mult}_P(C_X) \geqslant 3n^2$. Let $H \in |-K_X|$ be a sufficiently general divisor not containing Z. Then H contains no irreducible components of C_X and

$$2n^2 = H \cdot C_X \geqslant \operatorname{mult}_P(C_X) \geqslant 3n^2,$$

which proves the required result.

Let $\mathcal{D}_X$ be the proper transform of the linear system $\mathcal{D}_V$ on the smooth 3-fold X, and let F be a fibre of the del Pezzo fibration $\tau\colon X \to \mathbb{P}^1$. Then

$$K_X \sim f^*(K_V), \qquad F \sim -K_X - E, \qquad \mathcal{D}_X \sim f^*(-nK_V) - mE,$$

where $m = \operatorname{mult}_{\widetilde{L}}(\mathcal{D}_V)$.

Lemma 5.8. $m < n$.

Proof. It follows from the inequality $m \geqslant n$ that $\varphi_{\mathcal{D}_X} = \tau$, which is impossible.

Set $\mu = 1/(n-m)$ and $r = m/(n-m)$. Then $K_X + \mu\mathcal{D}_X \sim_{\mathbb{Q}} rF$.

Remark 5.9. The following equivalences hold:

$$r > 0 \quad \iff \quad m > 0 \quad \iff \quad \widetilde{L} \in \overline{\mathbb{CS}}\biggl(V, \frac{1}{n}\mathcal{D}_V\biggr).$$

Lemma 5.10. $\mathbb{CS}(X, \mu\mathcal{D}_X) \neq \varnothing$.

Proof. Assume that the log pair $(X, \mu\mathcal{D}_X)$ is terminal. If $r > 0$, then $(X, \varepsilon\mathcal{D}_X)$ is a canonical model for some $\varepsilon \in \mathbb{Q}_{>\mu}$ and, in particular, $\varkappa(X, \varepsilon\mathcal{D}_X) = 3$; however, $\varkappa(X, \varepsilon\mathcal{D}_X) \leqslant 2$ by the construction of $\mathcal{D}_V$. If $r = 0$, then it follows from our assumptions that the movable log pair $\bigl(V, \frac{1}{n}\mathcal{D}_V\bigr)$ is semi-terminal, which contradicts Lemma 5.1.

The following result is well known (see [15]).

Lemma 5.11. *Let C be a curve in $\mathbb{CS}(X, \mu\mathcal{D}_X)$ lying in fibres of the del Pezzo fibration τ. Then the log pair $(X, \mu\mathcal{D}_X)$ is canonical at the generic point of C.*

Proof. Let F be a fibre of τ containing C, and L_F a sufficiently general curve in F such that $\gamma \circ f(L_F)$ is a line in $\mathbb{P}^3$ lying in the plane $\gamma \circ f(F) \supset L$. Then

$$2(n-m) = \mathcal{D}_X \cdot L_F \geqslant \operatorname{mult}_C(\mathcal{D}_X)(L_F \cdot C)_F \geqslant (n-m)(L_F \cdot C)_F,$$

which immediately shows that either $\gamma \circ f(C)$ is also a line in $\mathbb{P}^3$ and $\gamma \circ f\big|_C \colon C \to \gamma \circ f(C)$ is an isomorphism, or else $\operatorname{mult}_C(\mathcal{D}_X) \leqslant n-m$ and, in particular, the log pair $(X, \mu\mathcal{D}_X)$ is canonical at the generic point of C. Hence we can assume that $\gamma \circ f(C)$ is a line in $\mathbb{P}^3$ and $\gamma \circ f\big|_C$ is an isomorphism.

Assume that F has a singular point $O \in C$. Let L_O be a sufficiently general curve in F such that $\gamma \circ f(L_O)$ is a line in $\mathbb{P}^3$ passing through the point $\gamma \circ f(O)$ and lying in the plane $\gamma \circ f(F)$ passing through the line L. Then L_O is singular at O and

$$2(n-m) = \mathcal{D}_X \cdot L_O \geqslant \operatorname{mult}_C(\mathcal{D}_X)\operatorname{mult}_O(L_O) \geqslant 2(n-m),$$

which means that $\operatorname{mult}_C(\mathcal{D}_X) = n-m$ and $(X, \mu\mathcal{D}_X)$ is canonical at the generic point of C.

Assume that the surface F is non-singular along C. Let $\widetilde{C}$ be a smooth rational curve in the fibre F such that $\widetilde{C} \neq C$ and $\gamma \circ f(\widetilde{C}) = \gamma \circ f(C)$. Then

$$\mathcal{D}_X\big|_F = m_C C + m_{\widetilde{C}}\widetilde{C} + R_F$$

for integers $m_C \geqslant \operatorname{mult}_C(\mathcal{D}_X)$ and $m_{\widetilde{C}} \geqslant \operatorname{mult}_{\widetilde{C}}(\mathcal{D}_X)$ such that R_F is an effective divisor on F with support not containing C or $\widetilde{C}$. On F we have the equivalence

$$\mathcal{D}_X\big|_F \sim (n-m)C + (n-m)\widetilde{C},$$

the equality $C^2 = \widetilde{C}^2 = -1$, and the inequalities $R_F \cdot C \geqslant 0$ and $R_F \cdot \widetilde{C} \geqslant 0$, which show that $m_C \leqslant n-m$, $m_{\widetilde{C}} \leqslant n-m$, and therefore the log pair $(X, \mu\mathcal{D}_X)$ is canonical at the generic point of C.

The following result is classical and follows by [4] and [5].

Lemma 5.12. *There exists a composite σ of Bertini involutions of the generic fibre of τ such that the log pair $(X, \mu_\sigma\sigma(\mathcal{D}_X))$ has canonical singularities at the generic points of curves on X, where $\mu_\sigma \in \mathbb{Q}_{>0}$ is defined by the relation $K_X + \mu_\sigma\sigma(\mathcal{D}_X) \sim_{\mathbb{Q}} r_\sigma F$ for some non-negative rational number r_σ.*

Proof. Assume that $\operatorname{mult}_C(\mathcal{D}_X) > n - m$ for some curve C on X. This curve does not lie in fibres of τ, by Lemma 5.11. Let F be a sufficiently general fibre of τ. Then

$$2(n-m)^2 = \mathcal{D}_X^2 \cdot F \geqslant \operatorname{mult}_C(\mathcal{D}_X^2) C \cdot F \geqslant \operatorname{mult}_C^2(\mathcal{D}_X) C \cdot F > (n-m)^2 C \cdot F,$$

which shows that $C \cdot F = 1$. Therefore, we can assume for now that X is a smooth del Pezzo surface of degree 2 with Picard group $\mathbb{Z}$ defined over a field $\mathbb{F} = \mathbb{C}(x)$, and we can regard C as an $\mathbb{F}$-point of X and $\mathcal{D}_X$ as a linear system without fixed curves on X such that $\mathcal{D}_X \subset |(m-n)K_X|$ and $\operatorname{mult}_C(\mathcal{D}_X) > n - m$.

The linear system $|-K_X|$ contains no curve Z singular at C because

$$2(n-m) = Z \cdot \mathcal{D}_X > \operatorname{mult}_C(Z)(n-m);$$

in particular, the $\mathbb{F}$-point C cannot lie in the ramification divisor of the double cover $\varphi_{|-K_X|}\colon X \to \mathbb{P}^2$. Let $g\colon W \to X$ be a blowup of C, and E an exceptional divisor of g. Then $\varphi_{|-2K_W|}$ is well known to be a morphism and a double cover. Consider the involution χ of W interchanging the fibres of the double cover $\varphi_{|-2K_W|}$. Easy calculations show that

$$\chi^*(g^*(-K_X)) \sim 3g^*(-K_X) - 4E, \qquad \chi^*(E) \sim 2g^*(-K_X) - 3E,$$

and it is incidentally well known that the biregular involution χ induces a birational and non-biregular involution ψ_C of X, known as a Bertini involution.

Let $\mu_{\psi_C} \in \mathbb{Q}_{>0}$ be a quantity such that $K_X + \mu_{\psi_C}\psi_C(\mathcal{D}_X) \sim_{\mathbb{Q}} 0$. Then

$$\mu_{\psi_C} = \frac{\mu}{3 - 2\operatorname{mult}_C(\mathcal{D}_X)\mu},$$

which shows that $\mu_{\psi_C} > \mu = 1/(n-m)$. We can now repeat the above-described construction of the Bertini involution in the case of the log pair $(X, \mu_{\psi_C}\psi_C(\mathcal{D}_X))$, provided that its singularities are not canonical. Thus, considering a composite σ of finitely many Bertini involutions of the del Pezzo surface X of degree 2 we obtain a movable log pair $(X, \mu_\sigma\sigma(\mathcal{D}_X))$ such that $K_X + \mu_\sigma\sigma(\mathcal{D}_X) \sim_{\mathbb{Q}} 0$ and $1/\mu_\sigma \in \mathbb{N}$. Hence we can assume that μ_σ is the smallest rational number possible and that $(X, \mu_\sigma\sigma(\mathcal{D}_X))$ is canonical.

We again regard X as a smooth 3-fold. The above construction produces a composite σ of finitely many Bertini involutions of the generic fibre of τ such that the log pair $(X, \mu_\sigma\sigma(\mathcal{D}_X))$ has only canonical singularities at generic points of curves not lying in fibres of τ, where $\mu_\sigma \in \mathbb{Q}_{>0}$ and $K_X + \mu_\sigma\sigma(\mathcal{D}_X) \sim_{\mathbb{Q}} r_\sigma F$ for some rational r_σ. We can now apply Lemmas 5.1–5.6 and Lemmas 5.10, 5.11 to the linear system $f \circ \sigma(\mathcal{D}_X)$ in place of $\mathcal{D}_V$ to obtain the inequality $r_\sigma \geqslant 0$ and to prove that the singularities of $(X, \mu_\sigma\sigma(\mathcal{D}_X))$ are canonical at generic points of all curves on X.

The birational automorphism σ commutes with τ, therefore we can replace the birational map β by $\beta \circ \sigma^{-1}$ and assume that the singularities of the log pair $(X, \mu\mathcal{D}_X)$ are canonical at the generic points of curves on X.

Lemma 5.13. $r > 0$.

Proof. Suppose that $r = 0$. Then $\widetilde{L} \notin \overline{\mathbb{CS}}\big(V, \frac{1}{n}\mathcal{D}_V\big)$ and $\overline{\mathbb{CS}}\big(V, \frac{1}{n}\mathcal{D}_V\big)$ contains some point O_i, by Lemma 5.7. We can assume that $i = 1$. Hence $\mathbb{CS}\big(X, \frac{1}{n}\mathcal{D}_X\big)$ contains either a point $P \in Z_1$ or a component of $Z_1 = Z_1^0 \cup Z_1^1$, where Z_1 is the reducible fibre of the birational morphism f over the point O_1. Let D_1 and D_2 be sufficiently general surfaces in the linear system $\mathcal{D}_X$. Then

$$n^2(Z_1 + 2C) \equiv n^2 K_X^2 \equiv D_1 \cdot D_2 = \operatorname{mult}_{Z_1^0}(\mathcal{D}_X^2)Z_1^0 + \operatorname{mult}_{Z_1^1}(\mathcal{D}_X^2)Z_1^1 + C_X + R_X,$$

where C is a curve in the fibres of τ with $-K_X \cdot C = 1$, C_X is an effective 1-cycle on X lying in fibres of τ, and R_X is an effective 1-cycle on X with components not lying in fibres of τ.

Assume that $\mathbb{CS}\big(X, \frac{1}{n}\mathcal{D}_X\big)$ contains some point $P \in Z_1$. By Theorem 3.12, we have $\operatorname{mult}_P(D_1 \cdot D_2) \geqslant 4n^2$, while $Z_1^0 \cdot F = Z_1^1 \cdot F = 1$ and

$$2n^2 = D_1 \cdot D_2 \cdot F = \operatorname{mult}_{Z_1^0}(\mathcal{D}_X^2) + \operatorname{mult}_{Z_1^1}(\mathcal{D}_X^2) + R_X \cdot F,$$

where F is a fibre of τ. In particular, $\operatorname{mult}_{Z_1^0}(\mathcal{D}_X^2) + \operatorname{mult}_{Z_1^1}(\mathcal{D}_X^2) \leqslant 2n^2$, and equality holds for $R_X = \varnothing$. Let H_{Z_1} be a sufficiently general surface in $|-K_X|$ passing through Z_1. By our choice we can assume that H_{Z_1} contains no irreducible components of the cycles C_X and R_X. Thus,

$$\begin{aligned} H_{Z_1} \cdot (C_X + R_X) &\geqslant \operatorname{mult}_P(C_X) + \operatorname{mult}_P(R_X) \\ &\geqslant 4n^2 - \operatorname{mult}_{Z_1^0}(\mathcal{D}_X^2)\delta_P^0 + \operatorname{mult}_{Z_1^1}(\mathcal{D}_X^2)\delta_P^1, \end{aligned}$$

where $\delta_P^i = \operatorname{mult}_P(Z_1^i)$. However, $H_{Z_1} \cdot (C_X + R_X) = 2n^2$. Hence the cycle R_X is empty, $\operatorname{mult}_P(C_X) = H_{Z_1} \cdot C_X = 2n^2$, and either P is an intersection point of the curves Z_1^0 and Z_1^1 and

$$\operatorname{mult}_{Z_1^0}(\mathcal{D}_X^2) + \operatorname{mult}_{Z_1^1}(\mathcal{D}_X^2) = 2n^2,$$

or else $P \in Z_1^k$, $\operatorname{mult}_{Z_1^k}(\mathcal{D}_X^2) = 2n^2$, $\operatorname{mult}_{Z_1^{1-k}}(\mathcal{D}_X^2) = 0$. The multiplicities $\operatorname{mult}_{Z_1^0}(\mathcal{D}_X^2)$ and $\operatorname{mult}_{Z_1^1}(\mathcal{D}_X^2)$ are positive because $D_1 \cdot Z_1^0 = D_1 \cdot Z_1^1 = 0$, so that $P = Z_1^0 \cap Z_1^1$ and

$$\operatorname{mult}_{Z_1^0}(\mathcal{D}_X^2) + \operatorname{mult}_{Z_1^1}(\mathcal{D}_X^2) = 2n^2,$$

and it follows by the equalities $\operatorname{mult}_P(C_X) = H_{Z_1} \cdot C_X = 2n^2$ that the support of C_X lies in the fibre F_P of τ passing through P. On the other hand we have $\varphi_{\mathcal{D}_X}(X) = \mathbb{P}^2$ by construction, and there exists a point $O \in X$ not belonging to $F_P \cup E$ such that the linear subsystem $\mathcal{D}_O \subset \mathcal{D}_X$ consisting of surfaces in $\mathcal{D}_X$ passing through O has no fixed components. Hence

$$O \in \mathcal{D}_O^2 \subset \mathcal{D}_V^2 \subset F_P \cup E$$

in the set-theoretic sense, which contradicts our choice of O.

Assume first that we have the simple case and that $\mathbb{CS}\bigl(X, \frac{1}{n}\mathcal{D}_X\bigr)$ contains both curves Z_1^0 and Z_1^1. Let $g\colon W \to X$ be a blowup of Z_1^0, $h\colon U \to W$ a blowup of the proper transform of Z_1^1, and $\mathcal{D}_U$ the proper transform of the movable linear system $\mathcal{D}_X$ on U. Then $|-K_U|$ is free and the induced morphism $\varphi_{|-K_U|}\colon U \to \mathbb{P}^2$ is an elliptic fibration. Let C be a sufficiently general fibre of $\varphi_{|-K_U|}$. Then

$$\mathcal{D}_U \cdot C = 2n - \operatorname{mult}_{Z_1^0}(\mathcal{D}_X) - \operatorname{mult}_{Z_1^1}(\mathcal{D}_X),$$

so that the linear system $\mathcal{D}_U$ lies in the fibres of the elliptic fibration $\varphi_{|-K_U|}$, which is equivalent to the initial elliptic fibration π. Moreover, by our construction there exists a rational map $\alpha\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ such that $\alpha \circ \varphi_{|-K_U|} \circ (f \circ h \circ g)^{-1} = \tau \circ f^{-1}$, which contradicts our initial assumptions.

Assume now that $\mathbb{CS}\bigl(X, \frac{1}{n}\mathcal{D}_X\bigr)$ contains either Z_1^0 or Z_1^1. We shall also assume without loss of generality that $\mathbb{CS}\bigl(X, \frac{1}{n}\mathcal{D}_X\bigr)$ contains Z_1^0 and does not contain Z_1^1. Let $g\colon W \to X$ be a blowup of Z_1^0, G an exceptional divisor of g, and $\mathcal{D}_W$ the proper transform of the linear system $\mathcal{D}_X$ on W. Then the log pair $\bigl(W, \frac{1}{n}\mathcal{D}_W\bigr)$ is the log pullback of the log pair $\bigl(X, \frac{1}{n}\mathcal{D}_X\bigr)$. In particular, the singularities of the log pair $\bigl(W, \frac{1}{n}\mathcal{D}_W\bigr)$ are also canonical. Moreover, in the case when the singularities of the log pair $\bigl(W, \frac{1}{n}\mathcal{D}_W\bigr)$ are not terminal there exists a curve $C_W \subset W$ such that $C_W \in \mathbb{CS}\bigl(W, \frac{1}{n}\mathcal{D}_W\bigr)$ and C_W is not contracted to a point by g, because $\mathbb{CS}\bigl(V, \frac{1}{n}\mathcal{D}_V\bigr)$ contains no smooth points of V by Lemma 5.3, $\mathbb{CS}\bigl(X, \frac{1}{n}\mathcal{D}_X\bigr)$ contains points in irreducible fibres of the birational morphism f by Lemma 5.7, and we have already proved that $\mathbb{CS}\bigl(X, \frac{1}{n}\mathcal{D}_X\bigr)$ contains no points in the reducible fibres Z_i of f. Moreover, $\mathbb{CS}\bigl(X, \frac{1}{n}\mathcal{D}_X\bigr)$ does not contain curves other than the Z_i^k, by Lemmas 5.6 and 5.7, and by our assumption C_W cannot be the proper transform of Z_1^1. Thus, either C_W dominates the curve Z_1^0, or $C_W = Z_{j\neq 1}^k$. On the other hand $D_1\big|_{H_{Z_1}} \cdot Z_1^1 = 0$, where H_{Z_1} is a general surface in $|-K_X|$ passing through Z_1, so that $\operatorname{mult}_{Z_1^1}(\mathcal{D}_X^2) > 0$. Hence

$$2n^2 = D_1 \cdot D_2 \cdot F \geqslant \operatorname{mult}^2_{C_W}(\mathcal{D}_W) + \operatorname{mult}^2_{Z_1^0}(\mathcal{D}_X) + \operatorname{mult}^2_{Z_1^1}(\mathcal{D}_X) > 2n^2,$$

which is impossible, and therefore the log pair $\bigl(W, \frac{1}{n}\mathcal{D}_W\bigr)$ is terminal. Now let $\check{h} \circ \check{p} \circ \widehat{p}^{-1} \circ h^{-1}\colon W \dashrightarrow \check{W}$ be an antiflip in the smooth curve Z_1^1 in the notation of § 4, and $\mathcal{D}_{\check{W}}$ the proper transform of $\mathcal{D}_W$ on $\check{W}$. Then the singularities of the log pair $\bigl(\check{W}, \frac{1}{n}\mathcal{D}_{\check{W}}\bigr)$ are terminal because $\bigl(K_W + \frac{1}{n}\mathcal{D}_W\bigr) \cdot Z_1^1 = 0$ and the antiflip in Z_1^1 is a log flop for $\bigl(W, \frac{1}{n}\mathcal{D}_W\bigr)$. Thus, for sufficiently small $\varepsilon \in \mathbb{Q}_{>1/n}$ the singularities of the log pair $(\check{W}, \varepsilon\mathcal{D}_{\check{W}})$ are terminal and

$$K_{\check{W}} + \varepsilon\mathcal{D}_{\check{W}} \sim_{\mathbb{Q}} \left(\frac{1}{n} - \varepsilon\right) K_{\check{W}},$$

where the divisor $-K_{\check{W}}$ is nef and big, by Lemma 4.6. Moreover, the linear system $|-nK_{\check{W}}|$ defines a birational morphism $\varphi_{|-nK_{\check{W}}|}\colon \check{W} \to V_{1,0}$ for some $n \gg 0$, where $V_{1,0}$ is a Fano 3-fold with canonical singularities and $-K^3_{V_{1,0}} = \frac{1}{2}$. The morphism $\varphi_{|-nK_{\check{W}}|}$ contracts only curves on $\check{W}$ having trivial intersection with $-K_{\check{W}}$.

In particular, every curve contracted by $\varphi_{|-nK_{\check{W}}|}$ has trivial intersection with the divisor $K_{\check{W}}+\varepsilon\mathcal{D}_{\check{W}}$, and the log pair $(\check{W},\varepsilon\mathcal{D}_{\check{W}})$ is the log pullback of $(V_{1,0},\varepsilon\mathcal{D}_{V_{1,0}})$, where $\mathcal{D}_{V_{1,0}}$ is the proper transform of the linear system $\mathcal{D}_{\check{W}}$ on $V_{1,0}$. Hence the log pair $(V_{1,0},\varepsilon\mathcal{D}_{V_{1,0}})$ is canonical and $\varkappa(V_{1,0},\varepsilon\mathcal{D}_{V_{1,0}})=3$; however, we have $\varkappa(V_{1,0},\varepsilon\mathcal{D}_{V_{1,0}})\leqslant 2$ by the construction of $\mathcal{D}_V$.

Let $h\colon U\to X$ be a birational morphism with smooth U and regular $\beta\circ f\circ h$. Consider the relation

$$K_U+\mu\mathcal{D}_U\sim_{\mathbb{Q}} h^*(rF)+\sum_{i=1}^{k}a_iE_i,$$

where $\mathcal{D}_U$ is the proper transform of the linear system $\mathcal{D}_X$ on U, the divisor E_i is h-exceptional, and $a_i\in\mathbb{Q}$. Consider a finite or empty point subset $\mathcal{J}$ of $\mathbb{P}^1$ such that for at least one divisor E_i with $a_i<0$, its image on X is a point in the fibre of τ over a point in $\mathcal{J}$. For all $\lambda\in\mathcal{J}$ we have

$$h^*(F_\lambda)\sim h^{-1}(F_\lambda)+\sum_{j=1}^{k_\lambda}b_jE_j,$$

where F_λ is a fibre of τ over λ and $b_i\in\mathbb{N}$. Finally, we set $\mathcal{I}=\bigcup_{\lambda\in\mathcal{J}}\mathcal{I}_\lambda$, where for each $\lambda\in\mathcal{J}$ we define $\mathcal{I}_\lambda\subset\{1,\dots,k\}$ as follows: $i\in\mathcal{I}_\lambda$ if and only if $h(E_i)$ is a point in the fibre F_λ and $a_i<0$.

The following result asserts the existence of a *super-maximal singularity* as in [15].

Proposition 5.14.

$$r+\sum_{\lambda\in\mathcal{J}}\min\biggl\{\frac{a_i}{b_i}\biggm| h(E_i)\in F_\lambda\quad\textit{and}\quad a_i<0\biggr\}\leqslant 0.$$

Proof. Assume that the result fails. Then there exist positive rational numbers ε and c_λ such that $r=\varepsilon+\sum_{\lambda\in\mathcal{J}}c_\lambda$ and $c_\lambda+\min\{a_i/b_i\mid h(E_i)\in F_\lambda \text{ and } a_i<0\}>0$. In particular,

$$K_U+\mu\mathcal{D}_U\sim_{\mathbb{Q}} h^*(\varepsilon F)+\sum_{\lambda\in\mathcal{J}}\biggl(h^*(c_\lambda F_\lambda)+\sum_{i\in\mathcal{I}_\lambda}a_iE_i\biggr)+\sum_{i\notin\mathcal{I}}a_iE_i,$$

while for each $\lambda\in\mathcal{J}$ the divisor $h^*(c_\lambda F_\lambda)+\sum_{i\in\mathcal{I}_\lambda}a_iE_i$ is effective by our choice of the positive rational number c_λ, and the divisor $\sum_{i\notin\mathcal{I}}a_iE_i$ is effective because by assumption $(X,\mu\mathcal{D}_X)$ is canonical at the generic points of curves on X. Let C be a sufficiently general curve lying in fibres of $\rho\circ f\circ h$. Then $K_U\cdot C=0$ because C is an elliptic curve and $\mathcal{D}_U\cdot C=0$ by the construction of the linear system $\mathcal{D}_V$. Thus,

$$(K_U+\mu\mathcal{D}_U)\cdot C=h^*(\varepsilon F)\cdot C+\sum_{\lambda\in\mathcal{J}}\biggl(h^*(c_\lambda F_\lambda)+\sum_{i\in\mathcal{I}_\lambda}a_iE_i\biggr)\cdot C+\sum_{i\notin\mathcal{I}}a_iE_i\cdot C=0$$

and, in particular, $h^*(\varepsilon F)\cdot C=0$ due to the generality of C. Hence there exists a rational map $\alpha\colon\mathbb{P}^2\dashrightarrow\mathbb{P}^1$ such that $\alpha\circ\rho=\tau\circ f^{-1}$, which is impossible by our assumption.

Corollary 5.15. *The singularities of the log pair $(X, \mu\mathcal{D}_X)$ are not canonical.*

Remark 5.16. The geometric meaning of Proposition 5.14 is simple: for arbitrary positive rational numbers $\{c_\lambda\}_{\lambda\in\mathcal{J}}$ such that $r = \sum_{\lambda\in\mathcal{J}} c_\lambda$, the set $\mathbb{CS}\bigl(X, \mu\mathcal{D}_X - \sum_{\lambda\in\mathcal{J}} c_\lambda F_\lambda\bigr)$ of centres of canonical singularities is non-empty.

Let Z be a smooth irreducible curve that is a general fibre of the morphism $f\big|_E\colon E \to \widetilde{L}$, and C a smooth curve in a general fibre of the del Pezzo fibration τ such that $-K_X \cdot C = 1$.

Remark 5.17. $\overline{\mathbb{NE}}(X) = \mathbb{R}_{\geqslant 0} Z \oplus \mathbb{R}_{\geqslant 0} C$, $F \cdot Z = E \cdot C = 2$, $K_X^2 \equiv Z + 2C$.

Note that the $\mathbb{Q}$-rational equivalence $\mu\mathcal{D}_X \sim_{\mathbb{Q}} -K_X + rF$ yields the numerical equivalence $\mu^2\mathcal{D}_X^2 \equiv Z + (2+4r)C$ of 1-dimensional cycles. For two sufficiently general divisors D_1 and D_2 in the linear system $\mathcal{D}_X$, consider the effective cycle

$$T_0 = \mu^2 D_1 \cdot D_2 = Z_X + \sum_{\lambda\in\mathbb{P}^1} C_\lambda,$$

where no component of the effective 1-cycle Z_X lies in the fibres of τ, and every component of the effective 1-dimensional cycle C_λ lies in the fibre F_λ of τ over the point $\lambda \in \mathbb{P}^1$.

Remark 5.18.

$$-K_X \cdot \sum_{\lambda\in\mathbb{P}^1} C_\lambda = 2 + 4r \leqslant 2 + 4\sum_{\lambda\in\mathcal{J}} \max\biggl\{\frac{-a_i}{b_i} \biggm| h(E_i) \in F_\lambda \ \text{ and } \ a_i < 0\biggr\}.$$

Corollary 5.19. *There exist a point $\omega \in \mathbb{P}^1$ and an h-exceptional divisor E_t such that $h(E_t)$ is a point O in the fibre F_ω of τ over ω, $-K_X \cdot C_\omega \leqslant 2 - 4a_t/b_t$, and $a_t < 0$, where $a_t/b_t = \min\{a_i/b_i \mid h(E_i) \in F_\omega$ and $a_i < 0\}$.*

Remark 5.20. We have $-K_X \cdot C_\omega \leqslant 2/p - 4a_t/b_t$, where p is the cardinality of $\mathcal{J}$.

Remark 5.21. The log pair $(X, \mu\mathcal{D}_X + (a_t/b_t)F_\lambda)$ is canonical at O, and in the relation

$$K_U + \mu\mathcal{D}_U + \frac{a_t}{b_t}\, h^{-1}(F_\omega) \sim_{\mathbb{Q}} h^*\biggl(\biggl(r + \frac{a_t}{b_t}\biggr)F\biggr) + \sum_{i=1}^{k} a\biggl(X, \mu\mathcal{D}_X + \frac{a_t}{b_t}F_\lambda, E_i\biggr)E_i$$

the log discrepancy $a(X, \mu\mathcal{D}_X + (a_t/b_t)F_\lambda, E_t)$ vanishes, so that O belongs to $\mathbb{CS}(X, \mu\mathcal{D}_X + (a_t/b_t)F_\lambda)$ and there exist only finitely many divisorial discrete valuations with centre at O on X such that the corresponding discrepancies vanish; they are all associated with h-exceptional divisors E_j such that $a_j/b_j = a_t/b_t$, and we can always take any one of them in place of E_t in Corollary 5.19.

We thus have a point ω on $\mathbb{P}^1$, a fibre F_ω over ω of the fibration $\tau\colon X \to \mathbb{P}^1$ by del Pezzo surfaces of degree 2, a smooth manifold U and a birational morphism $h\colon U \to X$, an h-exceptional divisor E_t whose image $h(E_t)$ on X is a point O on the fibre F_ω, and we have the inequalities $-K_X \cdot C_\omega \leqslant 2 - 4a_t/b_t$ and $a_t < 0$, where the 1-dimensional effective cycle C_ω is a component of the cycle $T_0 = \mu^2 D_1 \cdot D_2$ lying in F_ω, D_1 and D_2 are sufficiently general divisors in $\mathcal{D}_X$, the rational number a_t is defined by the $\mathbb{Q}$-rational equivalence

$$K_U + \mu\mathcal{D}_U \sim_{\mathbb{Q}} h^*(rF) + \sum_{i=1}^{k} a_i E_i,$$

and the positive integer b_t is defined by the rational equivalence

$$h^*(F_\omega) \sim h^{-1}(F_\omega) + \sum_{i=1}^{k} b_i E_i,$$

where $\mathcal{D}_U = h^{-1}(\mathcal{D}_X)$, the divisor E_i is h-exceptional, $a_i \in \mathbb{Q}$, and $b_i \in \mathbb{N}$.

Lemma 5.22. *The fibre F_ω is non-singular at the point O.*

Proof. Assume that F_ω is singular at O. Let L_O be a sufficiently general curve on the surface F_ω such that $\gamma \circ f(L_O)$ is a line in $\mathbb{P}^3$ passing through $\gamma \circ f(O)$. Then L_O is singular at O and

$$2(n-m) = \mathcal{D}_X \cdot L_O \geqslant \operatorname{mult}_O(\mathcal{D}_X) \operatorname{mult}_O(L_O) \geqslant 2 \operatorname{mult}_O(\mathcal{D}_X),$$

whereas the inequality $a_t < 0$ yields $\operatorname{mult}_O(\mathcal{D}_X) > \frac{1}{\mu} = n-m$.

Lemma 5.23. $\operatorname{mult}_O(\mathcal{D}_X) \leqslant \frac{2}{\mu} = 2(n-m)$.

Proof. Let L_O be a sufficiently general curve on the surface F_ω containing O such that $\gamma \circ f(L_O)$ is a line in $\mathbb{P}^3$. Then

$$2(n-m) = \mathcal{D}_X \cdot L_O \geqslant \operatorname{mult}_O(\mathcal{D}_X) \operatorname{mult}_O(L_O) \geqslant \operatorname{mult}_O(\mathcal{D}_X),$$

which proves the required result.

The quantities a_t and b_t depend only on the divisorial discrete valuation associated with the divisor E_t: they are the log discrepancy of the log pair $(X, \mu\mathcal{D}_X)$ and the multiplicity of F_ω, respectively. In particular, a_t and b_t do not depend on h, therefore we can choose h to be a composite $\psi_{1,0} \circ \cdots \circ \psi_{N,N-1}$ of birational morphisms $\psi_{i,i-1} \colon X_i \to X_{i-1}$, $i = 1, \dots, N$, $X = X_0$, having the following natural properties: $\psi_{1,0}$ is a blowup of the point $P_0 = O$; $\psi_{i,i-1}$ is a blowup of a point $P_{i-1} \in G_{i-1}$, $i = 2, \dots, K$, where $G_i = \psi_{i,i-1}^{-1}(P_{i-1})$; $\psi_{K+1,K}$ is the blowup of a curve $P_K \subset G_K$; $\psi_{i,i-1}$ is a blowup of a curve $P_{i-1} \subset G_{i-1}$, $i = K+2, \dots, N$, not lying in the fibres of $\psi_{i-1,i-2}\big|_{G_{i-1}}$; the divisorial discrete valuation associated with the divisor G_N coincides with the divisorial discrete valuation associated with E_t.

We set $\psi_{j,i} = \psi_{i+1,i} \circ \cdots \circ \psi_{j,j-1} \colon X_j \to X_i$, $j > i$, $\psi_{j,j} = \mathrm{id}_{X_j}$, and set $G_i^j = \psi_{j,i}^{-1}(G_i) \subset X_j$, $j \geqslant i$; $\mathcal{D}_{X_i} = \psi_{i,0}^{-1}(\mathcal{D}_X)$; $F_\omega^i = \psi_{i,0}^{-1}(F_\omega) \subset X_i$, $i = 0, \dots, N$. In this notation we have the equivalences

$$K_{X_N} + \mu\mathcal{D}_{X_N} \sim_{\mathbb{Q}} \psi_{N,0}^*(rF_\omega) + \sum_{i=1}^{N} a(X, \mu\mathcal{D}_X, G_i) G_i^N,$$

$$\psi_{N,0}^*(F_\omega) \sim F_\omega^N + \sum_{i=1}^{N} b(X, F_\omega, G_i) G_i^N;$$

here $a(X,\mu\mathcal{D}_X,G_i)\in\mathbb{Q}$, $b(X,F_\omega,G_i)\in\mathbb{N}$, and by construction $a(X,\mu\mathcal{D}_X,G_N)=a_t$, $b(X,F_\omega,G_N)=b_t$,

$$-K_X\cdot C_\omega\leqslant 2-\frac{4a(X,\mu\mathcal{D}_X,G_N)}{b(X,F_\omega,G_N)},$$

and $a(X,\mu\mathcal{D}_X,G_N)<0$. Moreover, we can assume in accordance with Remark 5.21 that $a(X,\mu\mathcal{D}_X,G_N)/b(X,F_\omega,G_N)<a(X,\mu\mathcal{D}_X,G_i)/b(X,F_\omega,G_i)$ for $i<N$.

Remark 5.24. A priori we must assume that the 3-fold X_i is either quasi-projective or singular for $i>K$, which has no influence on the subsequent arguments; however, it must be pointed out that it follows easily by the inequality $a(X,\mu\mathcal{D}_X,G_N)<0$, Remark 5.21, and Lemma 5.23 that P_K is a line in $G_K\cong\mathbb{P}^2$ and P_i is a section of the ruled surface $\psi_{i-1,i-2}\big|_{G_{i-1}}\colon G_{i-1}\to P_{i-1}\cong\mathbb{P}^1$ for $i>K$.

The main object under consideration in the rest of this section is the effective 1-dimensional cycle

$$T_0=\mu^2D_1\cdot D_2=Z_X+C_\omega+\sum_{\omega\neq\lambda\in\mathbb{P}^1}C_\lambda,$$

where no component of Z_X lies in fibres of τ, every component of C_λ lies in the fibre F_λ of τ over λ, $\lambda\neq\omega$, and $C_\omega\subset F_\omega$.

It follows from the inequality $a(X,\mu\mathcal{D}_X,G_K)<0$ that $\operatorname{mult}_O(T_0)>4$, by Theorem 3.12. On the other hand, $\operatorname{mult}_O(Z_X)\leqslant F_\omega\cdot Z_X=2$, so that $\operatorname{mult}_O(C_\omega)>2$, which is a rather strong condition by itself yet seems to be not sufficiently strong for the required contradiction.

Remark 5.25.

$$2<\operatorname{mult}_O(C_\omega)\leqslant -K_X\cdot C_\omega\leqslant 2-\frac{4a(X,\mu\mathcal{D}_X,G_N)}{b(X,F_\omega,G_N)}.$$

The following construction is well known (see [15] and [20]). We assign to each divisor G_i a vertex of an oriented graph Γ and indicate by an arrow $G_j\to G_i$ the fact that the vertex G_j is joined to G_i by an oriented edge going from G_j to G_i. We set

$$G_j\to G_i\quad\iff\quad j>i\quad\text{and}\quad P_{j-1}\subset G_i^{j-1}\subset X_{j-1},$$

so that, in particular, we always have $G_i\to G_{i-1}$ for $i=1,\dots,N$.

Remark 5.26. In the case when the discrete valuation of G_N coincides with the discrete valuation associated with an exceptional divisor of a weighted blowup of the point O with weights $(1,K,N)$ for coprime positive integers K and N in a suitable local coordinate system on X, it follows by [22] that the graph Γ is a chain, which means merely that $G_j\to G_i\iff j=i+1$.

Let $P(i)$ be the number of paths in Γ from G_N to G_i, $i\neq N$, and set $P(N)=1$. Then

$$a(X,\mu\mathcal{D}_X,G_N)=\sum_{i=1}^{K}P(i)(2-\nu_i)+\sum_{i=K+1}^{N}P(i)(1-\nu_i),$$

where $\nu_i = \mathrm{mult}_{P_{i-1}}(\mu \mathcal{D}_X^{i-1})$. We set

$$T_0^j = \psi_{j,0}^{-1}(T_0), \qquad Z_X^j = \psi_{j,0}^{-1}(Z_X), \qquad C_\omega^j = \psi_{j,0}^{-1}(C_\omega)$$

and

$$m_{T_0}^i = \mathrm{mult}_{P_{i-1}}(T_0^{i-1}), \qquad m_{Z_X}^i = \mathrm{mult}_{P_{i-1}}(Z_X^{i-1}), \qquad m_{C_\omega}^i = \mathrm{mult}_{P_{i-1}}(C_\omega^{i-1}),$$

where $i = 1, \ldots, K$. Set $D_1^i = \psi_{i,0}^{-1}(D_1)$, $D_2^i = \psi_{i,0}^{-1}(D_2)$, let $T_i \subset G_i \cong \mathbb{P}^2$ be an effective 1-dimensional cycle on the 3-fold X_i, $i = 1, \ldots, N$, defined by the relation

$$\mu^2 \mathcal{D}_{X_i}^2 \equiv \mu^2 D_1^i \cdot D_2^i = \psi_{i,i-1}^{-1}(\mu^2 D_1^{i-1} \cdot D_2^{i-1}) + T_i,$$

and set $T_i^j = \psi_{j,i}^{-1}(T_i)$ for $j \geqslant i$. Then

$$\mu^2 D_1^i \cdot D_2^i = T_0^i + T_1^i + \cdots + T_{i-1}^i + T_i$$

for $i = 1, \ldots, K$.

Set $m_{i,j} = \mathrm{mult}_{P_{j-1}}(T_i^{j-1})$ for $j > i = 1, \ldots, K$ and let d_i be the degree of T_i in $\mathbb{P}^2 \cong G_i$.

Lemma 5.27. *For $i = 1, \ldots, K$, $\sum_{j=0}^{i-1} m_{j,i} = \nu_i^2 + d_i$.*

Proof. By considering a general hyperplane section of X_{i-1} passing through P_{i-1} we reduce the situation to the well-known result on the multiplicity of an intersection on a smooth surface.

Lemma 5.28. $d_K \geqslant \sum_{i=K+1}^{N} \nu_i^2 \deg(\psi_{i-1,L}\big|_{P_{i-1}})$.

Proof. For $i > K$ we regard T_i as a subscheme of G_i. Let d_i be the intersection in G_i of T_i with a general fibre of $\psi_{i,i-1}\big|_{G_i}$. Then $d_i \geqslant \mathrm{mult}_{P_i}(\mathcal{D}_{X_i}^2) \deg\big(\psi_{i,i-1}\big|_{P_i}\big)$ for $i > K$ and $\mathrm{mult}_{P_i}(\mathcal{D}_{X_i}^2) = \nu_{i+1}^2 + d_{i+1}$ for $i \geqslant K$, so that

$$\begin{aligned} d_K \geqslant \mathrm{mult}_{P_K}(\mathcal{D}_{X_L}^2) &\geqslant \sum_{i=K+1}^{N} \nu_i^2 \prod_{j=K}^{i-1} \deg\big(\psi_{j+1,j}\big|_{P_{j+1}}\big) \\ &= \sum_{i=K+1}^{N} \nu_i^2 \deg\big(\psi_{i-1,K}\big|_{P_{i-1}}\big), \end{aligned}$$

which is the required result.

It can be proved that the curve P_K is a line in $G_K \cong \mathbb{P}^2$ and $\deg\big(\psi_{i-1,L}\big|_{P_{i-1}}\big) = 1$ for $i > K$.

Corollary 5.29. $d_K \geqslant \sum_{i=K+1}^{N} \nu_i^2$.

Consider the restriction of the function $P(i)$ to $[1, K] \subseteq [1, N]$. More precisely, we define $Q\colon [1, K] \to \mathbb{N}$ to be the function such that $Q(i)$ is the number of oriented paths in Γ going from G_N to G_i, $i = 1, \ldots, K$.

Remark 5.30.

$$Q(i) = \sum_{G_j \to G_i} Q(j) + \sum_{G_j \to G_i}^{j>K} P(j) \geqslant \sum_{G_j \to G_i} Q(j).$$

Lemma 5.31.
$$\sum_{i=1}^{K} Q(i)m_{0,i} \geqslant \sum_{i=1}^{K} Q(i)\nu_i^2 + Q(K)d_K.$$

Proof. For $j=1,\dots,K$ we multiply the equality
$$\sum_{i=0}^{j-1} m_{i,j} = \nu_j^2 + d_j$$
by $Q(j)$, take the sum:
$$\sum_{j=1}^{K}\sum_{i=0}^{j-1} Q(j)m_{i,j} = \sum_{j=1}^{K} Q(j)\nu_j^2 + \sum_{j=1}^{K} Q(j)d_j,$$
and change the order of summation:
$$\sum_{j=1}^{K}\sum_{i=0}^{j-1} Q(j)m_{i,j} = \sum_{i=0}^{K-1}\sum_{j=i+1}^{K} Q(j)m_{i,j},$$
where $m_{i,j} \leqslant d_i$ for $1 \leqslant i < j \leqslant K$. Moreover, $m_{i,j} > 0 \iff G_j \to G_i$. Hence
$$\begin{aligned}\sum_{i=0}^{K-1}\sum_{j=i+1}^{K} Q(j)m_{i,j} &\leqslant \sum_{j=1}^{K} Q(j)m_{0,j} + \sum_{i=1}^{K-1}\sum_{G_j\to G_i} Q(j)d_i\\ &\leqslant \sum_{j=1}^{K} Q(j)m_{0,j} + \sum_{i=1}^{K-1} Q(i)d_i,\end{aligned}$$
so that
$$\sum_{j=1}^{K} Q(j)m_{0,j} + \sum_{i=1}^{K-1} Q(i)d_i \geqslant \sum_{j=1}^{K} Q(j)\nu_j^2 + \sum_{j=1}^{K} Q(j)d_j,$$
and finally,
$$\sum_{j=1}^{K} Q(j)m_{0,j} \geqslant \sum_{j=1}^{K} Q(j)\nu_j^2 + Q(K)d_K,$$
as required.

Corollary 5.32. $\sum_{i=1}^{K} P(i)m_{0,i} \geqslant \sum_{i=1}^{N} P(i)\nu_i^2$.

Set $\Sigma_0 = \sum_{i=1}^{K} P(i)$ and $\Sigma_1 = \sum_{i=K+1}^{N} P(i)$. Then elementary calculus shows that
$$\sum_{i=1}^{K} P(i)m_{0,i} \geqslant \sum_{i=1}^{N} P(i)\nu_i^2 \geqslant \frac{(2\Sigma_0 + \Sigma_1 - a(X,\mu\mathcal{D}_X,G_N))^2}{(\Sigma_0+\Sigma_1)},$$
because $\sum_{i=1}^{N} P(i)\nu_i = 2\Sigma_0 + \Sigma_1 - a(X,\mu\mathcal{D}_X,G_N)$.

Recall that $m_{0,i} = m^i_{T_0} = m^i_{Z_X} + m^i_{C_\omega}$ for $i = 1, \dots, K$ and $m^i_{C_\omega} = 0$ for $i > M$, where the positive integer $M \leqslant K$ is the largest such that $P_{M-1} \in F^{M-1}_\omega$; moreover, $\operatorname{mult}_O(Z_X) \geqslant m^i_{Z_X}$ for $i = 1, \dots, K$, and the similar inequality $\operatorname{mult}_O(C_\omega) \geqslant m^i_{C_\omega}$ holds for $i = 1, \dots, M$. Thus, we have

$$\operatorname{mult}_O(Z_X)\Sigma_0 + \operatorname{mult}_O(C_\omega)\Sigma_0' \geqslant \frac{(2\Sigma_0 + \Sigma_1 - a(X, \mu\mathcal{D}_X, G_N))^2}{(\Sigma_0 + \Sigma_1)},$$

where $\Sigma_0' = \sum_{i=1}^M P(i)$. On the other hand, $\operatorname{mult}_O(Z_X) \leqslant F_\omega \cdot Z_X = 2$,

$$\operatorname{mult}_O(C_\omega) \leqslant -K_X \cdot C_\omega \leqslant 2 - \frac{4a(X, \mu\mathcal{D}_X, G_N)}{b(X, F_\omega, G_N)},$$

and we have $b(X, F_\omega, G_N) \geqslant \sum_{i=1}^M q_i P(i) \geqslant \Sigma_0'$, where $q_i = \operatorname{mult}_{P_{i-1}}(F^{i-1}_\omega)$ for $i = 1, \dots, M$.

Combining all these inequalities we obtain

$$2(\Sigma_0 - \Sigma_0')(\Sigma_0 + \Sigma_1) + (\Sigma_1 + a(X, \mu\mathcal{D}_X, G_N))^2 \leqslant 0,$$

which yields $a(X, \mu\mathcal{D}_X, G_N)) = -\Sigma_1$ and $\Sigma_0 = \Sigma_0'$. Moreover,

$$\sum_{i=1}^N P(i)\nu_i^2 = \frac{\bigl(2\Sigma_0 + \Sigma_1 - a(X, \mu\mathcal{D}_X, G_N)\bigr)^2}{(\Sigma_0 + \Sigma_1)},$$

which is possible only for $\nu_1 = \nu_2 = \dots = \nu_N = \nu$. Hence

$$\nu(\Sigma_0 + \Sigma_1) = \sum_{i=1}^N P(i)\nu_i = 2\Sigma_0 + \Sigma_1 - a(X, \mu\mathcal{D}_X, G_N) = 2(\Sigma_0 + \Sigma_1),$$

so that $\nu = 2$.

We see that each of the above inequalities must be an equality. In particular, the set $\mathcal{J}$ has cardinality 1, by Remark 5.20. This means that F_ω is the only fibre of τ such that the singularities of the log pair $(X, \mu\mathcal{D}_X)$ are not canonical at points in F_ω. Moreover, it follows by the equality

$$-K_X \cdot C_\omega = 2 + 4r = -K_X \cdot \sum_{\lambda \in \mathbb{P}^1} C_\lambda$$

that all 1-cycles C_λ are empty for $\lambda \neq \omega$. The equivalence

$$T_0 = Z_X + C_\omega \equiv Z + (2 + 4r)C,$$

where Z is a fibre of $f\big|_E \colon E \to \widetilde{L}$ and C is a curve in F_ω with $-K_X \cdot C = 1$, shows that the support of the cycle Z_X lies in the fibres of $f\big|_E$. In particular, $O \in F_\omega \cap Z_\delta$, where Z_δ is a fibre of $f\big|_E \colon E \to \widetilde{L}$ over some point $\delta \in \widetilde{L}$. However, since $2 = F_\omega \cdot Z_X = \operatorname{mult}_O(Z_X)$, it follows that the support of Z_X lies in Z_δ, which must therefore be a reducible fibre of $f\big|_E$. In particular, δ must be one of the points O_i defined in § 1, and $Z_\delta = Z_i = Z_i^0 \cup Z_i^1$. On the other hand $\mathcal{D}_X$ is not composed of a pencil, therefore there exists a point $P \in X$ such that $P \notin F_\omega \cup E$ and the linear subsystem $\mathcal{D}_P \subset \mathcal{D}_X$ consisting of surfaces in the linear system $\mathcal{D}_X$ passing through P has no fixed components. Hence $P \in A \cap B$ for two sufficiently general divisors A and B in $\mathcal{D}_P$. However, we can replace the divisors D_1 and D_2 in the linear system $\mathcal{D}_X$ that we used before by A and B, respectively. Hence $P \in A \cap B \subset F_\omega \cup E$ in the set-theoretic sense, which contradicts our choice of the point P. The proof of Theorem 1.3 is now complete.

§ 6. Proof of Theorem 1.4

In the notation and assumptions of § 1, assume further the existence of a birational transformation $\beta\colon V \dashrightarrow Y$ such that Y is a canonical Fano 3-fold. The main aim of this section is the proof of the following result.

Proposition 6.1. *There exists a birational automorphism σ of V such that σ is a uniquely-defined composite of Bertini involutions of the generic fibre of the del Pezzo fibration τ, and either $\beta\circ\sigma$ is biregular or $\beta\circ\sigma = \alpha\circ\rho_{i,k}$ for some biregular automorphism α of the Fano 3-fold $V_{i,k}$.*

Corollary 6.2. *Either $Y \cong V$ or $Y \cong V_{i,k}$.*

Corollary 6.3. *The sequence $1 \to \Gamma \to \mathrm{Bir}(V) \to \mathrm{Aut}(V) \to 1$ is exact, where Γ is a free product of Bertini involutions of the generic fibre of the fibration τ by del Pezzo surfaces of degree 2 regarded as a del Pezzo surface of degree 2 with Picard group $\mathbb{Z}$ over the field $\mathbb{C}(x)$.*

Set $\mathcal{D}_V = \beta^{-1}(\mathcal{D}_Y)$, where $\mathcal{D}_Y = |-tK_Y|$ for $t \gg 0$. Then $\mathcal{D}_V \subset |-nK_V|$ for some $n \in \mathbb{N}$. Let $\mathcal{D}_X$ be the proper transform of $\mathcal{D}_V$ on X, and F a fibre of the fibration $\tau\colon X \to \mathbb{P}^1$ by del Pezzo surfaces. Then

$$\mathcal{D}_X \sim f^*(-nK_V) - mE \sim -nK_X - mE \sim f^*(K_V) \sim n(F+E) - mE,$$

where $n > m = \mathrm{mult}_{\widetilde{L}}(\mathcal{D}_V) \geqslant 0$. Set $\mu = 1/(n-m)$ and $r = m/(n-m)$. Then $K_X + \mu\mathcal{D}_X \sim_{\mathbb{Q}} rF$.

Remark 6.4. The following equivalences hold:

$$m = 0 \quad\iff\quad r = 0 \quad\iff\quad \mu = \frac{1}{n} \quad\iff\quad \widetilde{L} \notin \overline{\mathbb{CS}}\biggl(V, \frac{1}{n}\mathcal{D}_V\biggr).$$

Lemma 6.5. *If $\mathbb{CS}(X, \mu\mathcal{D}_X) = \varnothing$, then β is biregular.*

Proof. Assume that $r > 0$. Then $(X, \varepsilon\mathcal{D}_X)$ – is a canonical model for some $\varepsilon > \mu$. Hence $\varkappa(Y, \varepsilon\mathcal{D}_Y) = \varkappa(X, \varepsilon\mathcal{D}_X) = 3$ and $(Y, \varepsilon\mathcal{D}_Y)$ is also a canonical model, so that the birational map $\beta\circ f$ is biregular by Theorem 2.15, which is a contradiction because $-K_X$ is not ample.

Hence $r = 0$, $\mu = \frac{1}{n}$, the log pair $(X, \frac{1}{n}\mathcal{D}_X)$ is the log pullback of the log pair $\bigl(V, \frac{1}{n}\mathcal{D}_V\bigr)$, and $\widetilde{L} \notin \overline{\mathbb{CS}}\bigl(V, \frac{1}{n}\mathcal{D}_V\bigr)$. In particular, the log pair $\bigl(V, \frac{1}{n}\mathcal{D}_V\bigr)$ is semiterminal. Then $(V, \varepsilon\mathcal{D}_V)$ is a canonical model for some $\varepsilon \in \mathbb{Q}_{>1/n}$. In particular, $\varkappa(Y, \varepsilon\mathcal{D}_Y) = \varkappa(V, \varepsilon\mathcal{D}_V) = 3$ and $(Y, \varepsilon\mathcal{D}_Y)$ is a canonical model. Hence the map β is biregular, by Theorem 2.15.

Lemma 6.6. *Let $C \subset X$ be a curve lying in the fibres of the fibration τ such that $C \in \mathbb{CS}(X, \mu\mathcal{D}_X)$. Then the log pair $(X, \mu\mathcal{D}_X)$ is canonical at the generic point of C.*

Proof. See the proof of Lemma 5.11.

Lemma 6.7. *There exists a uniquely-defined composite σ of Bertini involutions of the generic fibre of τ such that the log pair $(X, \mu_\sigma \sigma(\mathcal{D}_X))$ has canonical singularities at generic points of curves on X, where $\mu_\sigma \in \mathbb{Q}_{>0}$ is defined by the relation $K_X + \mu_\sigma \sigma(\mathcal{D}_X) \sim_{\mathbb{Q}} r_\sigma F$ for some non-negative rational number r_σ.*

Proof. See the proof of Lemma 5.12.

Remark 6.8. For the proof of Proposition 6.1 we can replace the birational map β by the composite $\beta \circ \sigma^{-1}$ and assume that the singularities of the log pair $(X, \mu\mathcal{D}_X)$ are canonical at generic points of curves on X.

Lemma 6.9. *Assume that the log pair $(X, \mu\mathcal{D}_X)$ has canonical singularities. Then $r = 0$.*

Proof. Assume that $r > 0$. Then $\varkappa(X, \mu\mathcal{D}_X) = 1$, while $\varkappa(Y, \mu\mathcal{D}_Y) \in \{-\infty, 0, 3\}$ by the construction of the linear system $\mathcal{D}_Y$, which contradicts the birational invariance of the Kodaira dimension of a movable log pair.

Let $h\colon U \to X$ be a birational morphism with smooth U and regular $\beta \circ f \circ h$, and suppose that

$$K_U + \mu\mathcal{D}_U \sim_{\mathbb{Q}} h^*(rF) + \sum_{i=1}^{k} a_i E_i,$$

where $\mathcal{D}_U = h^{-1}(\mathcal{D}_X)$, E_i is an h-exceptional divisor, and $a_i \in \mathbb{Q}$. We define the subset $\mathcal{J}$ of $\mathbb{P}^1$ as the image of the exceptional divisors E_i with $a_i < 0$. For $\lambda \in \mathcal{J}$ we set

$$h^*(F_\lambda) \sim h^{-1}(F_\lambda) + \sum_{j=1}^{k_\lambda} b_j E_j,$$

where F_λ is the fibre of τ over λ and $b_i \in \mathbb{N}$. Finally, we have $\mathcal{I} = \bigcup_{\lambda \in \mathcal{J}} \mathcal{I}_\lambda$, where for $\lambda \in \mathcal{J}$ we define $\mathcal{I}_\lambda \subset \{1, \dots, k\}$ as follows: $i \in \mathcal{I}_\lambda$ if and only if $h(E_i)$ is a point in the fibre F_λ and $a_i < 0$.

Proposition 6.10.

$$r + \sum_{\lambda \in \mathcal{J}} \min\left\{ \frac{a_i}{b_i} \;\middle|\; h(E_i) \in F_\lambda \quad \textit{and} \quad a_i < 0 \right\} \leqslant 0.$$

Proof. Assume that the claim fails. Then there exist positive rational numbers ε and c_λ such that $r = \varepsilon + \sum_{\lambda \in \mathcal{J}} c_\lambda$ and $c_\lambda + \min\{a_i/b_i \mid h(E_i) \in F_\lambda \text{ and } a_i < 0\} > 0$. In that case

$$K_U + \mu\mathcal{D}_U \sim_{\mathbb{Q}} h^*(\varepsilon F) + \sum_{\lambda \in \mathcal{J}} \left(h^*(c_\lambda F_\lambda) + \sum_{i \in \mathcal{I}_\lambda} a_i E_i \right) + \sum_{i \notin \mathcal{I}} a_i E_i,$$

and for each $\lambda \in \mathcal{J}$ the divisor $h^*(c_\lambda F_\lambda) + \sum_{i \in \mathcal{I}_\lambda} a_i E_i$ is effective by the choice of c_λ, while the divisor $\sum_{i \notin \mathcal{I}} a_i E_i$ is effective because the log pair $(X, \mu\mathcal{D}_X)$ is canonical in curves. Thus, $\varkappa(U, \mu\mathcal{D}_U) = 1$, whereas $\varkappa(Y, \mu\mathcal{D}_Y) \in \{-\infty, 0, 3\}$ by construction, which contradicts the birational invariance of the Kodaira dimension.

Lemma 6.11. $r = 0$.

Proof. Let Z be a fibre of $f\big|_E\colon E \to \widetilde{L}$, let C be a curve in fibres of τ with $-K_X\cdot C=1$, and D_1, D_2 two sufficiently general surfaces in $\mathcal{D}_X$. Then $\overline{\mathbb{NE}}(X) = \mathbb{R}_{\geqslant 0}Z \oplus \mathbb{R}_{\geqslant 0}C$ and

$$\mu^2 D_1 \cdot D_2 = Z_X + \sum_{\lambda\in\mathbb{P}^1} C_\lambda \equiv Z + (2+4r)C,$$

where no component of the effective 1-cycle Z_X lies in fibres of τ and each component of the effective cycle C_λ lies in the fibre F_λ of τ over $\lambda \in \mathbb{P}^1$. Hence

$$-K_X \cdot \sum_{\lambda\in\mathbb{P}^1} C_\lambda = 2 + 4r \leqslant 2 - 4\sum_{\lambda\in\mathcal{J}} \min\biggl\{\frac{a_i}{b_i}\biggm| h(E_i) \in F_\lambda \ \text{ and } \ a_i < 0\biggr\}$$

by Proposition 6.10. Assume that $r > 0$. Then there exists $\omega \in \mathbb{P}^1$ such that

$$-K_X \cdot C_\omega \leqslant 2 - 4\frac{a_t}{b_t}\,,$$

where $a_t/b_t = \min\{a_i/b_i \mid h(E_i) \in F_\omega \text{ and } a_i < 0\}$. We can now repeat verbatim the proof of Theorem 1.3, starting with Lemma 5.22, to obtain a contradiction.

Corollary 6.12. $\widetilde{L} \notin \overline{\mathbb{CS}}\bigl(V, \frac{1}{n}\mathcal{D}_V\bigr)$.

Lemma 6.13. *If* $\overline{\mathbb{CS}}\bigl(V, \frac{1}{n}\mathcal{D}_V\bigr) = \varnothing$, *then* β *is biregular.*

Proof. Suppose that $\overline{\mathbb{CS}}\bigl(V, \frac{1}{n}\mathcal{D}_V\bigr) = \varnothing$. Then $(V, \varepsilon\mathcal{D}_V)$ is a canonical model for some $\varepsilon \in \mathbb{Q}_{>1/n}$, and $\varkappa(Y, \varepsilon\mathcal{D}_Y) = \varkappa(V, \varepsilon\mathcal{D}_V) = 3$; hence $(Y, \varepsilon\mathcal{D}_Y)$ is a canonical model. So β is biregular by Theorem 2.15.

Remark 6.14. The linear system $\mathcal{D}_V$ does not lie in fibres of a dominant map $\chi\colon V \dashrightarrow Z$ such that Z is a curve or a surface.

Lemma 6.15. $\overline{\mathbb{CS}}\bigl(V, \frac{1}{n}\mathcal{D}_V\bigr)$ *does not contain curves or smooth points of* V.

Proof. See the proofs of Lemmas 5.3–5.6.

Lemma 6.16. $\overline{\mathbb{CS}}\bigl(V, \frac{1}{n}\mathcal{D}_V\bigr)$ *does not contain points other than the* O_i.

Proof. Assume that $\overline{\mathbb{CS}}\bigl(V, \frac{1}{n}\mathcal{D}_V\bigr)$ contains a point O on the 3-fold V such that $O \neq O_i$. Then O lies in a curve $\widetilde{L}$ by Lemma 6.15, but $\widetilde{L} \notin \overline{\mathbb{CS}}\bigl(V, \frac{1}{n}\mathcal{D}_V\bigr)$ because $r = 0$. Hence $\mathbb{CS}\bigl(X, \frac{1}{n}\mathcal{D}_X\bigr)$ contains either an irreducible fibre Z of the morphism $f\big|_E\colon E \to \widetilde{L}$ over O or a point $P \in Z$.

Assume that $\mathbb{CS}\bigl(X, \frac{1}{n}\mathcal{D}_X\bigr)$ contains a curve Z. Let $g\colon W \to X$ be a blowup of Z, $\mathcal{D}_W$ the proper transform of the linear system $\mathcal{D}_X$ on the 3-fold W, and G an exceptional divisor of g. Then the linear system $|-K_W|$ is free, the morphism $\varphi_{|-K_W|}$ is an elliptic fibration, and $\mathcal{D}_W \cdot C = 2n - \operatorname{mult}_Z(\mathcal{D}_X)$ for a sufficiently general fibre C of $\varphi_{|-K_W|}$. Thus, $\mathcal{D}_W$ lies in the fibres of $\varphi_{|-K_W|}$, which is impossible by construction.

Assume now that $\mathbb{CS}\bigl(X, \frac{1}{n}\mathcal{D}_X\bigr)$ contains a point $P \in Z$. Then

$$\mathcal{D}_X^2 = \operatorname{mult}_Z(\mathcal{D}_X^2)Z + C_X,$$

where the support of the 1-dimensional cycle C_X does not contain the curve Z, and

$$\mathrm{mult}_P(\mathcal{D}_X^2) = \mathrm{mult}_Z(\mathcal{D}_X^2) + \mathrm{mult}_P(C_X) \geqslant 4n^2$$

by Theorem 3.12. On the other hand $Z \cdot F = 2$ and $\mathcal{D}_X^2 \cdot F = 2n^2$, where F is a fibre of the fibration τ by del Pezzo surfaces of degree 2. In particular, $\mathrm{mult}_Z(\mathcal{D}_X^2) \leqslant n^2$ and $\mathrm{mult}_P(C_X) \geqslant 3n^2$. Let H be a sufficiently general divisor in $|-K_X|$ passing through the curve Z. Then H contains no irreducible component of C_X and

$$n^2 = H \cdot C_X \geqslant \mathrm{mult}_P(C_X) \geqslant 3n^2,$$

which proves the required result.

Lemma 6.17. *Suppose that $O_i \in \mathbb{CS}\big(V, \frac{1}{n}\mathcal{D}_V\big)$. Then $\beta = \alpha \circ \rho_{i,0}$ or $\beta = \alpha \circ \rho_{i,1}$ for some biregular automorphism α of $V_{i,0}$ or $V_{i,1}$, respectively.*

Proof. We can assume that $i = 1$. We set by definition $Z_1 = Z_1^0 \cup Z_1^1 = f^{-1}(O_1)$. Then $\mathbb{CS}\big(X, \frac{1}{n}\mathcal{D}_X\big)$ contains either a point $P \in Z_1$ or an irreducible component of the fibre Z_1. Moreover, we can repeat verbatim part of the proof of Lemma 5.13 to demonstrate that the first case is impossible and that $\mathbb{CS}\big(X, \frac{1}{n}\mathcal{D}_X\big)$ consists of a single irreducible component of Z_1. We can assume that $\mathbb{CS}\big(X, \frac{1}{n}\mathcal{D}_X\big) = \{Z_1^0\}$. Let $g\colon W \to X$ be a blowup of the smooth curve Z_1^0 and $\mathcal{D}_W$ the proper transform of $\mathcal{D}_X$ on W. We now repeat another part of the proof of Lemma 5.13 to show that the log pair $\big(W, \frac{1}{n}\mathcal{D}_W\big)$ is terminal. We claim that $\beta = \alpha \circ \rho_{1,0}$ for some biregular automorphism α of the singular Fano 3-fold $V_{1,0}$.

In the notation of § 4, let $\check{h} \circ \check{p} \circ \widehat{p}^{-1} \circ h^{-1}\colon W \dashrightarrow \check{W}$ be an antiflip in Z_1^1 and $\mathcal{D}_{\check{W}}$ the proper transform of the linear system $\mathcal{D}_W$ on $\check{W}$. Then the singularities of the log pair $\big(\check{W}, \frac{1}{n}\mathcal{D}_{\check{W}}\big)$ are terminal because $\big(K_W + \frac{1}{n}\mathcal{D}_W\big) \cdot Z_1^1 = 0$ and $\check{h} \circ \check{p} \circ \widehat{p}^{-1} \circ h^{-1}$ is a log flop for $\big(W, \frac{1}{n}\mathcal{D}_W\big)$; for some $\varepsilon \in \mathbb{Q}_{>1/n}$ the log par $(\check{W}, \varepsilon\mathcal{D}_{\check{W}})$ is also terminal, and

$$K_{\check{W}} + \varepsilon\mathcal{D}_{\check{W}} \sim_{\mathbb{Q}} \left(\frac{1}{n} - \varepsilon\right) K_{\check{W}},$$

where $-K_{\check{W}}$ is nef and big by Lemma 4.6, and for $n \gg 0$ the linear system $|-nK_{\check{W}}|$ defines a birational morphism $\varphi_{|-nK_{\check{W}}|}\colon \check{W} \to V_{1,0}$ contracting curves having trivial intersection with $-K_{\check{W}}$. It follows, in particular, that the log pair $(V_{1,0}, \varepsilon\mathcal{D}_{V_{1,0}})$ is a canonical model and $\varkappa(V_{1,0}, \varepsilon\mathcal{D}_{V_{1,0}}) = \varkappa(Y, \varepsilon\mathcal{D}_Y) = 3$. Hence $(Y, \varepsilon\mathcal{D}_Y)$ is also a canonical model. On the other hand $\mathcal{D}_{V_{1,0}} = \rho_{1,0}(\mathcal{D}_V)$. Thus, the birational map $\rho_{1,0} \circ \beta^{-1}$ is biregular by Theorem 2.15.

The proof of Theorem 1.4 is now complete.

§ 7. Proof of Theorem 1.15

Under the assumptions and notation of § 1, let $E \subset X$ be a smooth surface, L the unique line on the sextic S passing through one of the points $\gamma(O_i) \in \mathbb{P}^3$, and assume that there exist a birational map $\beta\colon V \dashrightarrow Y$ and a fibration $\pi\colon Y \to \mathbb{P}^1$ such that the general fibre of π is a connected smooth surface with numerically trivial canonical divisor. In this section we shall find a birational automorphism σ of the 3-fold V and a pencil $\mathcal{P} \subset |-K_V|$ such that $\rho \circ \sigma = \varphi_{\mathcal{P}}$, where $\rho = \pi \circ \beta$.

Remark 7.1. Unfortunately, some arguments used efficiently in §§ 5, 6 fail under the assumptions of this section.

Set $\mathcal{D}_V = \beta^{-1}\big(|\tau^*(\mathcal{O}_{\mathbb{P}^1}(1))|\big)$. Then $\mathcal{D}_V \subset |-nK_V|$ for some $n \in \mathbb{N}$.

Lemma 7.2. $\overline{\mathbb{CS}}\big(V, \frac{1}{n}\mathcal{D}_V\big) \neq \varnothing$.

Proof. Assume that $\overline{\mathbb{CS}}\big(V, \frac{1}{n}\mathcal{D}_V\big) = \varnothing$. Then $\varkappa(V, \varepsilon\mathcal{D}_V) = 3$, $\varepsilon \in \mathbb{Q}_{>1/n}$, whereas by the construction of the linear system $\mathcal{D}_V$ we have $\varkappa(V, \varepsilon\mathcal{D}_V) \leqslant 1$.

Lemma 7.3. $\mathbb{CS}\big(V, \frac{1}{n}\mathcal{D}_V\big)$ *contains no smooth points of* V.

Proof. Let $O \in \mathbb{CS}\big(V, \frac{1}{n}\mathcal{D}_V\big)$ be a point such that $O \notin \widetilde{L}$, and let $H_O \in |-K_V|$ be a general surface passing through O. Then $2n^2 = H_O \cdot \mathcal{D}_V^2 \geqslant 4n^2$ by Theorem 3.12.

Lemma 7.4. *Assume that* $\mathbb{CS}\big(V, \frac{1}{n}\mathcal{D}_V\big)$ *contains a curve* $C \neq \widetilde{L}$. *Then* $\gamma(C)$ *is a line.*

Proof. The inequality $\mathrm{mult}_C(\mathcal{D}_V) \geqslant n$ is equivalent to the fact that C lies in $\mathbb{CS}\big(V, \frac{1}{n}\mathcal{D}_V\big)$. Hence for a sufficiently general divisor H in the linear system $|-K_V|$ it follows by the inequalities

$$2n^2 = H \cdot \mathcal{D}_V^2 \geqslant \mathrm{mult}_C(\mathcal{D}_V^2) H \cdot C \geqslant \mathrm{mult}_C^2(\mathcal{D}_V) H \cdot C \geqslant n^2 H \cdot C$$

that $-K_V \cdot C \leqslant 2$. If $-K_V \cdot C = 1$, then the curve $\gamma(C)$ is a line in $\mathbb{P}^3$. We can thus assume that $-K_V \cdot C = 2$, $\mathrm{mult}_C(\mathcal{D}_V) = n$ and $\mathrm{mult}_C(\mathcal{D}_V^2) = n^2$, the curve $\gamma(C)$ is a conic in $\mathbb{P}^3$, the map $\gamma\big|_C \colon C \to \gamma(C)$ is an isomorphism, and the support of the effective 1-cycle $\mathcal{D}_V^2$ consists of C.

Assume first that $C \cap \widetilde{L} = \varnothing$. In this case let $g \colon W \to V$ be a blowup of C, let G be an exceptional divisor of g and $\mathcal{D}_W$ the proper transform of $\mathcal{D}_V$ on the 3-fold W. We claim that the effective divisor $g^*(-3K_V) - G$ is numerically effective. Assume that $\gamma(C) \not\subset S$ and consider the curve $\widetilde{C} \subset W$ such that $\gamma \circ g(\widetilde{C}) = \gamma(C)$ and $g(\widetilde{C}) \neq C$. Then $\widetilde{C}$ is the unique curve in the base locus of the linear system $|g^*(-2K_V) - G|$, and $(g^*(-2K_V) - G) \cdot \widetilde{C} = -2$, which proves that $g^*(-3K_V) - E$ is numerically effective. Now suppose that $\gamma(C) \subset S$. In this case the base locus of the linear system $|g^*(-2K_V) - G|$ lies in G. Let s_∞ be an exceptional section of the ruled surface $g\big|_G \colon G \to C$. It is easy to see that in this case the numerical effectiveness of $g^*(-3K_V) - G$ follows from the inequality $(g^*(-3K_V) - G)\big|_G \cdot s_\infty \geqslant 0$; however, elementary calculations demonstrate the equalities

$$(g^*(-3K_V) - G)\big|_G \cdot s_\infty = 6 + \frac{s_\infty^2}{2}$$

and $G^3 = 0$. We must show that $s_\infty^2 \geqslant -12$. Suppose that $\mathcal{N}_{C/V} \cong \mathcal{O}_C(m) \oplus \mathcal{O}_C(n)$ for $m \geqslant n$. Then

$$m + n = \deg(\mathcal{N}_{C/V}) = -K_V \cdot C + 2g(C) - 2 = 0,$$

and the exact sequence

$$0 \to \mathcal{N}_{C/\widetilde{S}} \to \mathcal{N}_{C/V} \to \mathcal{N}_{\widetilde{S}/V}\big|_C \to 0$$

yields $n \geqslant \deg(\mathcal{N}_{C/\widetilde{S}}) = -6$, where $\widetilde{S} = \gamma^{-1}(S)$. Hence $s_\infty^2 = n - m = 2n \geqslant -12$ and the divisor $g^*(-3K_V) - G$ is numerically effective. In particular, we have the inequality

$$6n^2 - \operatorname{mult}_C(\mathcal{D}_V)(6\operatorname{mult}_C(\mathcal{D}_V) + 4n) = (g^*(-3K_V) - G)\cdot\mathcal{D}_W^2 \geqslant 0,$$

which leads to a contradiction. Hence $C \cap \widetilde{L} \neq \varnothing$.

Set $\widehat{C} = f^{-1}(C)$ and let $h\colon U \to X$ be a blowup of the curve C; let H be an exceptional divisor of h, $\mathcal{D}_U$ the proper transform of $\mathcal{D}_V$ on U, D_1 and D_2 two sufficiently general divisors in $\mathcal{D}_U$, and set $t = E\cdot\widehat{C} = 2 - F\cdot\widehat{C}$, where F is a fibre of τ. Then

$$T_U = D_1\cdot D_2 = \sum_{O_i\neq\delta\in\widetilde{L}\cap C} p_\delta Z_\delta + \sum_{O_i\in\widetilde{L}\cap C}(p_{i,0}\overline{Z}_i^0 + p_{i,1}\overline{Z}_i^1) + C_H,$$

where Z_δ is a proper transform on U of an irreducible fibre of f over $\delta \in \widetilde{L}$, $\overline{Z}_i^k$ is the proper transform on U of the component Z_i^k of the fibre $Z_i = Z_i^0 \cup Z_i^1$ of f over O_i, the integers p_δ and $p_{i,k}$ are non-negative, and the support of the cycle C_H lies in H. Moreover,

$$(2-t)n^2 = h^*(F)\cdot\mathcal{D}_U^2 = \sum 2p_\delta + \sum p_{i,0} + \sum p_{i,1},$$

because the 1-cycle C_H is either empty or is contracted by the morphism h in view of the equality $\operatorname{mult}_C^2(\mathcal{D}_V) = \operatorname{mult}_C(\mathcal{D}_V^2)$. It is easy to see that the divisor $h^*(-3K_X) - H$ is not numerically effective. Nevertheless, we always have

$$(h^*(-3K_X) - H)\cdot T_U \geqslant -H\cdot C_H - \sum p_\delta - \sum p_{i,0} - \sum p_{i,1} \geqslant (t-2)n^2 \geqslant -n^2$$

and therefore

$$6n^2 - \operatorname{mult}_C(\mathcal{D}_V)(6\operatorname{mult}_C(\mathcal{D}_V) + 4n) = (h^*(-3K_X) - H)\cdot T_U \geqslant -n^2,$$

which leads to a contradiction that proves the required result.

Lemma 7.5. *Assume that $\mathbb{CS}\big(V, \frac{1}{n}\mathcal{D}_V\big)$ contains a curve C with $-K_V\cdot C = 2$. Then there exists a pencil $\mathcal{P}$ of* K3 *surfaces in the linear system $|-K_V|$ such that $\rho = \varphi_{\mathcal{P}}$.*

Proof. The image of C on $\mathbb{P}^3$ is a line, by Lemma 7.4. Thus, there exists a pencil $\mathcal{P} \subset |-K_V|$ of surfaces passing through the C. Let $g\colon W \to V$ be a birational morphism resolving indeterminacy of the rational map $\varphi_{\mathcal{P}}$ such that W is smooth and there exists a unique g-exceptional divisor G on W dominating C. Let $\mathcal{D}_W$ be the proper transform of $\mathcal{D}_V$ on W and D_W a sufficiently general fibre of the morphism $\varphi_{\mathcal{P}}\circ g\colon W \to \mathbb{P}^1$. Then the equality $\operatorname{mult}_C(\mathcal{D}_V) = n$ yields

$$\mathcal{D}_W\big|_{D_W} \sim \sum_{i=1}^k a_i G_i,$$

where the G_i are g-exceptional divisors whose images are points in V, and the a_i are integers. On the other hand the linear system $\mathcal{D}_W$ has no fixed components, and therefore $\varphi_{\mathcal{D}_W} = \varphi_{\mathcal{P}}\circ g$ and $\rho = \varphi_{\mathcal{P}}$.

Lemma 7.6. *Suppose that $\mathbb{CS}\big(V,\frac{1}{n}\mathcal{D}_V\big)$ contains a curve C such that $C\cap\widetilde{L}=\varnothing$. Then there exists a pencil $\mathcal{P}$ of* K3 *surfaces in the linear system $|-K_V|$ such that $\rho=\varphi_{\mathcal{P}}$.*

Proof. See the proof of Lemma 5.5.

Lemma 7.7. *Suppose that $\overline{\mathbb{CS}}\big(V,\frac{1}{n}\mathcal{D}_V\big)$ does not contain the curve $\widetilde{L}$, but contains some curve C on V. Then there exists a pencil $\mathcal{P}\subset|-K_V|$ such that $\rho=\varphi_{\mathcal{P}}$.*

Proof. See the proof of Lemma 5.6.

Lemma 7.8. *The set $\overline{\mathbb{CS}}\big(V,\frac{1}{n}\mathcal{D}_V\big)$ contains either a curve on V or some point O_i.*

Proof. Assume that the claim fails. Then it follows by Lemmas 7.3–7.7 that there exists a point $O\neq O_i$ such that $O\in\widetilde{L}$ and $O\in\overline{\mathbb{CS}}\big(V,\frac{1}{n}\mathcal{D}_V\big)$. The proper transform $\mathcal{D}_X$ of the linear system $\mathcal{D}_V$ on X is a linear subsystem of the linear system $|-nK_X|$, and $\mathbb{CS}\big(X,\frac{1}{n}\mathcal{D}_X\big)$ contains either a smooth irreducible fibre Z of the birational morphism f over O or a point $P\in Z$.

Assume that $\mathbb{CS}\big(X,\frac{1}{n}\mathcal{D}_X\big)$ contains a point P from Z. Then we have $\mathcal{D}_X^2=\operatorname{mult}_Z(\mathcal{D}_X^2)Z+C_X$, where the support of the one-dimensional effective cycle C_X does not contain Z and

$$\operatorname{mult}_P(\mathcal{D}_X^2)=\operatorname{mult}_Z(\mathcal{D}_X^2)+\operatorname{mult}_P(C_X)\geqslant 4n^2,$$

by Theorem 3.12. On the other hand $Z\cdot F=2$ and $\mathcal{D}_X^2\cdot F=2n^2$, where F is a fibre of the del Pezzo fibration τ. In particular, $\operatorname{mult}_Z(\mathcal{D}_X^2)\leqslant n^2$, $\operatorname{mult}_P(C_X)\geqslant 3n^2$, and

$$2n^2=H\cdot C_X\geqslant\operatorname{mult}_P(C_X)\geqslant 3n^2,$$

where H is a general surface in $|-K_X|$ passing through Z.

Suppose that $Z\in\mathbb{CS}\big(X,\frac{1}{n}\mathcal{D}_X\big)$, let $g\colon W\to X$ be a blowup of Z, $\mathcal{D}_W$ the proper transform of $\mathcal{D}_X$ on W, and G a g-exceptional divisor. Then the linear system $|-K_W|$ is free and $\varphi_{|-K_W|}$ is an elliptic fibration. Let C be a general fibre of $\varphi_{|-K_W|}$. Then $\mathcal{D}_W\cdot C=2n-\operatorname{mult}_Z(\mathcal{D}_X)$, which shows that $\mathcal{D}_W$ lies in the fibres of $\varphi_{|-K_W|}$. Moreover, $\mathbb{CS}\big(W,\frac{1}{n}\mathcal{D}_W\big)$ contains no curves not contracted by $\tau\circ g$, because otherwise

$$2n^2=\mathcal{D}_X^2\cdot F\geqslant 3n^2,$$

where F is a fibre of τ. We have already proved that $\mathbb{CS}\big(X,\frac{1}{n}\mathcal{D}_X\big)$ cannot contain points in irreducible fibres of f. Assume that $\mathbb{CS}\big(X,\frac{1}{n}\mathcal{D}_X\big)$ contains a point P_i in the reducible fibre $Z_i=Z_i^0\cup Z_i^1$ of f. Let H_{Z_i} be a general element of $|-K_X|$ passing through Z_i and suppose that

$$n^2(Z_i+2C)\equiv D_1\cdot D_2=\operatorname{mult}_{Z_i^0}(\mathcal{D}_X^2)Z_i^0+\operatorname{mult}_{Z_i^1}(\mathcal{D}_X^2)Z_i^1+C_X+R_X,$$

where D_1 and D_2 are general divisors in $\mathcal{D}_X$, C_X is an effective 1-cycle on X with components lying in the fibres of τ, R_X is an effective 1-cycle on X with components not lying in the fibres of τ, and C is a curve in the fibres of τ with $-K_X\cdot C=1$. Then

$$2n^2=H_{Z_i}\cdot(C_X+R_X)\geqslant\operatorname{mult}_P(C_X)+\operatorname{mult}_P(R_X)>2n^2$$

because $\mathrm{mult}_{P_i}(D_1 \cdot D_2) \geqslant 4n^2$ by Theorem 3.12, the equality

$$\mathrm{mult}_{Z_i^0}(\mathcal{D}_X^2) + \mathrm{mult}_{Z_i^1}(\mathcal{D}_X^2) + R_X \cdot F = 2n^2$$

holds, and $R_X \neq \varnothing$. Thus, $\mathbb{CS}\big(W, \frac{1}{n}\mathcal{D}_W\big) = \varnothing$ and $\varkappa(W, \varepsilon\mathcal{D}_W) = 2$ for some rational ε greater than $\frac{1}{n}$, whereas $\varkappa(W, \varepsilon\mathcal{D}_W) \leqslant 1$ by the construction of $\mathcal{D}_V$.

Let $\mathcal{D}_X$ be the proper transform of $\mathcal{D}_V$ on X and F a fibre of τ. Then

$$\mathcal{D}_X \sim f^*(-nK_V) - mE \sim -nK_X - mE \sim f^*(K_V) \sim n(F+E) - mE,$$

where $n > m = \mathrm{mult}_{\widetilde{L}}(\mathcal{D}_V)$. Set $\mu = 1/(n-m)$ and $r = m/(n-m)$. Then $K_X + \mu\mathcal{D}_X \sim_{\mathbb{Q}} rF$.

Remark 7.9. The following equivalences hold:

$$r > 0 \quad \Longleftrightarrow \quad m > 0 \quad \Longleftrightarrow \quad \widetilde{L} \in \overline{\mathbb{CS}}\left(V, \frac{1}{n}\mathcal{D}_V\right).$$

Lemma 7.10. $\mathbb{CS}(X, \mu\mathcal{D}_X) \neq \varnothing$.

Proof. Assume that $\mathbb{CS}(X, \mu\mathcal{D}_X) = \varnothing$. If $r > 0$, then the log pair $(X, \varepsilon\mathcal{D}_X)$ is a canonical model for some $\varepsilon \in \mathbb{Q}_{>\mu}$, whereas $\varkappa(X, \varepsilon\mathcal{D}_X) \leqslant 1$ by the construction of $\mathcal{D}_V$. If $r = 0$, then the log pair $\big(V, \frac{1}{n}\mathcal{D}_V\big)$ is semi-terminal, which contradicts Lemma 7.1.

Lemma 7.11. *Let C be a curve in $\mathbb{CS}(X, \mu\mathcal{D}_X)$ lying in fibres of τ. Then the log pair $(X, \mu\mathcal{D}_X)$ is canonical at the generic point of C.*

Proof. See the proof of Lemma 5.11.

Lemma 7.12. *There exists a composite σ of Bertini involutions of the generic fibre of τ such that the log pair $(X, \mu_\sigma\sigma(\mathcal{D}_X))$ has canonical singularities at generic points of curves in X, where $\mu_\sigma \in \mathbb{Q}_{>0}$ is defined by the relation $K_X + \mu_\sigma\sigma(\mathcal{D}_X) \sim_{\mathbb{Q}} r_\sigma F$ for some non-negative rational number r_σ.*

Proof. See the proof of Lemma 5.12.

Replacing β by $\beta \circ \sigma^{-1}$, we can assume that the log pair $(X, \mu\mathcal{D}_X)$ has canonical singularities at the generic points of curves on the 3-fold X.

Lemma 7.13. $\overline{\mathbb{CS}}\big(V, \frac{1}{n}\mathcal{D}_V\big)$ *contains some curve on V.*

Proof. Assume that $\overline{\mathbb{CS}}\big(V, \frac{1}{n}\mathcal{D}_V\big)$ contains no curves on V. In particular, it does not contain $\widetilde{L}$. By Lemma 7.8, $\overline{\mathbb{CS}}\big(V, \frac{1}{n}\mathcal{D}_V\big)$ contains some point O_i, and we can assume that $i = 1$. It follows by Lemma 7.10 that $\mathbb{CS}\big(X, \frac{1}{n}\mathcal{D}_X\big)$ contains a point $P \in Z_1$ or a component of $Z_1 = Z_1^0 \cup Z_1^1$, where Z_1 is the reducible fibre of the birational morphism f over O_1. Let D_1 and D_2 be general surfaces in $\mathcal{D}_X$. Then

$$n^2(Z_1 + 2C) \equiv n^2K_X^2 \equiv D_1 \cdot D_2 = \mathrm{mult}_{Z_1^0}(\mathcal{D}_X^2)Z_1^0 + \mathrm{mult}_{Z_1^1}(\mathcal{D}_X^2)Z_1^1 + C_X + R_X,$$

where C is a curve in the fibres of τ with $-K_X \cdot C = 1$, C_X an effective 1-cycle on X with components lying in fibres of τ, and R_X an effective 1-cycle on X with components not lying in the fibres of τ.

Assume that $\mathbb{CS}\big(X, \frac{1}{n}\mathcal{D}_X\big)$ contains a point $P \in Z_1$. Then $\mathrm{mult}_P(D_1 \cdot D_2) \geqslant 4n^2$, by Theorem 3.12. On the other hand, $Z_1^0 \cdot F = Z_1^1 \cdot F = 2$ and

$$2n^2 = D_1 \cdot D_2 \cdot F = \mathrm{mult}_{Z_1^0}(\mathcal{D}_X^2) + \mathrm{mult}_{Z_1^1}(\mathcal{D}_X^2) + R_X \cdot F,$$

where F is a fibre of τ. In particular, $\mathrm{mult}_{Z_1^0}(\mathcal{D}_X^2) + \mathrm{mult}_{Z_1^1}(\mathcal{D}_X^2) \leqslant 2n^2$ and equality holds for $R_X = \varnothing$. Let H_{Z_1} be a general surface in the linear system $|-K_X|$ passing through Z_1. Then H_{Z_1} contains no irreducible components of the 1-cycles C_X and R_X. Hence

$$\begin{aligned} H_{Z_1} \cdot (C_X + R_X) &\geqslant \mathrm{mult}_P(C_X) + \mathrm{mult}_P(R_X) \\ &\geqslant 4n^2 - \mathrm{mult}_{Z_i^0}(\mathcal{D}_X^2)\delta_P^0 + \mathrm{mult}_{Z_1^1}(\mathcal{D}_X^2)\delta_P^1, \end{aligned}$$

where $\delta_P^i = \mathrm{mult}_P(Z_1^i)$. However, $H_{Z_1} \cdot (C_X + R_X) = 2n^2$. Hence the cycle R_X is empty, the equality $\mathrm{mult}_P(C_X) = H_{Z_1} \cdot C_X = 2n^2$ holds, and either $P = Z_1^0 \cap Z_1^1$ and

$$\mathrm{mult}_{Z_1^0}(\mathcal{D}_X^2) + \mathrm{mult}_{Z_1^1}(\mathcal{D}_X^2) = 2n^2,$$

or else $P \in Z_1^k$, $\mathrm{mult}_{Z_1^k}(\mathcal{D}_X^2) = 2n^2$, and $\mathrm{mult}_{Z_1^{1-k}}(\mathcal{D}_X^2) = 0$. On the other hand, it follows in this last case that $\mathrm{mult}_{Z_1^{1-k}}(\mathcal{D}_X^2) > 0$, since $D_1 \cdot Z_{1-k}^0 = 0$. Thus, $P = Z_1^0 \cap Z_1^1$ and

$$\mathrm{mult}_{Z_1^0}(\mathcal{D}_X^2) + \mathrm{mult}_{Z_1^1}(\mathcal{D}_X^2) = 2n^2,$$

whereas the equality $\mathrm{mult}_P(C_X) = H_{Z_1} \cdot C_X$ shows that all irreducible components of the 1-cycle C_X lie in the fibre F_P of τ passing through P. Moreover, $\mathrm{mult}_P(\mathcal{D}_X) = 2n$ by Theorem 3.12. We now regard V as a hypersurface

$$u^2 = x^2 \sum_{i=0}^{4} \overline{p}_i(x,y,z)t^{4-i} + y^2 \sum_{i=0}^{4} \overline{q}_i(x,y,z)t^{4-i}$$

in the weighted projective space $\mathbb{P}(1,1,1,3)$, where $\overline{p}_i$ and $\overline{q}_i$ are homogeneous polynomials of degree i, x, y, z, and t are homogeneous coordinates of weight 1, and u is a homogeneous coordinate of weight 3. We can assume that the curve $\widetilde{L}$ is defined by the equations $x = y = 0$ and the point O_1 by $x = y = z = 0$. In that case either $\overline{q}_0 = 0$, or $\overline{p}_0 = 0$ by the definition of O_1 and our assumption that there exist precisely 8 distinct points O_i. We can assume without loss of generality that $\overline{q}_0 = 0$. Then the linear form $\overline{q}_1(x,y,z)$ does not vanish identically, because of the smoothness of the 3-fold X. Moreover, even $\overline{q}_1(0,0,z)$ does not vanish, in view of our initial assumption about the smoothness of the exceptional divisor E. Thus, the pencil $\mathcal{P}$ in the linear system $|-K_V|$ defined by the equation $Ax + B\overline{q}_1(x,y,z) = 0$, where A and B are complex coefficients, has no base components. Let $\mathcal{P}_X$ be the proper transform of $\mathcal{P}$ on the 3-fold X and D_X a general element of $\mathcal{P}_X$. Then

$$\mathrm{mult}_P(D_X) = 2, \qquad D_X \sim -K_X, \qquad \mathrm{mult}_P(D_1 \cdot D_X) = 4n,$$

and we can apply the above calculations to the cycle $D_1 \cdot D_X$ in place of the effective 1-cycle $D_1 \cdot D_2$. In particular, $D_1 \cdot D_X \subset E \cup F_P$. Let O, $O \notin E \cup F_P$ be a sufficiently general point on the surface D_1, and D_O a surface in the pencil $\mathcal{P}_X$ passing through O. Then $D_1 = D_O$ by our previous arguments, because both D_1 and D_O are irreducible. Hence $\mathcal{D}_V = \mathcal{P}$, whereas $\overline{\mathbb{CS}}(V, \mathcal{P})$ contains curves on V, which contradicts our assumptions.

Hence $\mathbb{CS}\big(X, \frac{1}{n}\mathcal{D}_X\big)$ contains no points in X, but contains one of the curves Z_1^0 and Z_1^1, or both. In each case we can repeat a suitable part of the proof of Lemma 7.8 or 5.13 to derive a contradiction.

Remark 7.14. We can assume that $r > 0$.

Let $h\colon U \to X$ be a birational morphism with smooth U and regular $\beta \circ f \circ h$, and suppose that

$$K_U + \mu\mathcal{D}_U \sim_{\mathbb{Q}} h^*(rF) + \sum_{i=1}^{k} a_i E_i,$$

where $\mathcal{D}_U = h^{-1}(\mathcal{D}_X)$, E_i is an h-exceptional divisor, and $a_i \in \mathbb{Q}$. We consider the subset $\mathcal{J}$ of $\mathbb{P}^1$ that is the image of the exceptional divisors E_i with $a_i < 0$. For $\lambda \in \mathcal{J}$ we set

$$h^*(F_\lambda) \sim h^{-1}(F_\lambda) + \sum_{j=1}^{k_\lambda} b_j E_j,$$

where F_λ is the fibre of τ over λ and $b_i \in \mathbb{N}$. Finally, we have $\mathcal{I} = \bigcup_{\lambda\in\mathcal{J}} \mathcal{I}_\lambda$, where for $\lambda \in \mathcal{J}$, $\mathcal{I}_\lambda$ is the subset of $\{1, \dots, k\}$ defined as follows: $i \in \mathcal{I}_\lambda$ if and only if $h(E_i)$ is a point in the fibre F_λ and $a_i < 0$.

Proposition 7.15.

$$r + \sum_{\lambda\in\mathcal{J}} \min\left\{\frac{a_i}{b_i} \;\middle|\; h(E_i) \in F_\lambda \quad \textit{and} \quad a_i < 0\right\} \leqslant 0.$$

Proof. Assume that the claim fails. Then there exist ε and c_λ in $\mathbb{Q}_{>0}$ such that $r = \varepsilon + \sum_{\lambda\in\mathcal{J}} c_\lambda$ and $c_\lambda + \min\{a_i/b_i \mid h(E_i) \in F_\lambda \text{ and } a_i < 0\} > 0$. In particular,

$$K_U + \mu\mathcal{D}_U \sim_{\mathbb{Q}} h^*(\varepsilon F) + \sum_{\lambda\in\mathcal{J}}\left(h^*(c_\lambda F_\lambda) + \sum_{i\in\mathcal{I}_\lambda} a_i E_i\right) + \sum_{i\notin\mathcal{I}} a_i E_i,$$

where for each $\lambda \in \mathcal{J}$ the divisor $h^*(c_\lambda F_\lambda) + \sum_{i\in\mathcal{I}_\lambda} a_i E_i$ is effective by our choice of the positive rational number c_λ, and the divisor $\sum_{i\notin\mathcal{I}} a_i E_i$ is effective because the singularities of the log pair $(X, \mu\mathcal{D}_X)$ are by assumption canonical at the generic points of curves on the 3-fold X. Let O be a sufficiently general point in a sufficiently general fibre D_O of the morphism $\rho \circ f \circ h$, and let C be the proper transform on U of a sufficiently general irreducible curve lying in a sufficiently general fibre of π such that C contains O. Then $K_U \cdot C = 0$ and $\mathcal{D}_U \cdot C = 0$. Thus,

$$(K_U + \mu\mathcal{D}_U) \cdot C = h^*(\varepsilon F) \cdot C + \sum_{\lambda\in\mathcal{J}}\left(h^*(c_\lambda F_\lambda) + \sum_{i\in\mathcal{I}_\lambda} a_i E_i\right) \cdot C + \sum_{i\notin\mathcal{I}} a_i E_i \cdot C = 0$$

and, in particular, $h^*(\varepsilon F)\cdot C=0$ in view of the generality of C. Let F_C be the fibre of $\tau\circ h$ passing through O. Then since $h^*(F)\cdot C=0$, it follows that $C\subset F_C$. On the other hand D_O and F_C are irreducible because of the generality of our choice of O. The generality of the choice of C means that $F_C=D_O$, which is impossible.

Let Z be a fibre of $f\big|_E\colon E\to\widetilde{L}$ and C a curve in the fibres of τ with $-K_X\cdot C=1$; we consider two sufficiently general surfaces D_1 and D_2 in $\mathcal{D}_X$. Then we have $\overline{\mathbb{NE}}(X)=\mathbb{R}_{\geqslant 0}Z\oplus\mathbb{R}_{\geqslant 0}C$ and

$$T_0=\mu^2D_1\cdot D_2=Z_X+\sum_{\lambda\in\mathbb{P}^1}C_\lambda\equiv Z+(2+4r)C,$$

where no irreducible component of the effective 1-cycle Z_X lies in fibres of τ, and every irreducible component of the effective 1-cycle C_λ lies in the fibre F_λ of τ over $\lambda\in\mathbb{P}^1$. Hence

$$-K_X\cdot\sum_{\lambda\in\mathbb{P}^1}C_\lambda=2+4r\leqslant 2-4\sum_{\lambda\in\mathcal{J}}\min\biggl\{\frac{a_i}{b_i}\biggm| h(E_i)\in F_\lambda\ \text{ and }\ a_i<0\biggr\},$$

and there exist $\omega\in\mathbb{P}^1$ and an h-exceptional divisor E_t such that $h(E_t)$ is a point O in the fibre F_ω of τ over ω, and we have

$$-K_X\cdot C_\omega\leqslant 2-4\frac{a_t}{b_t}\,,$$

where $a_t/b_t=\min\{a_i/b_i\mid h(E_i)\in F_\omega \text{ and } a_i<0\}$. In particular, we can repeat all logical implications in § 5, starting from Lemma 5.22, except the last of them because $\mathcal{D}_X$ is now a pencil. Hence, in the notation of § 5,

$$T_0=Z_X+C_\omega,\qquad a(X,\mu\mathcal{D}_X,G_N)=-\Sigma_1,\qquad \Sigma_0=\Sigma_0',$$
$$b(X,F_\omega,G_N)=\Sigma_0,\qquad r=\frac{\Sigma_1}{4\Sigma_0}\,,\qquad \nu_j=2,$$

the support of the cycle Z_X lies in some reducible fibre $Z_i=Z_i^0\cup Z_i^1$ of f, and

$$\operatorname{mult}_O(Z_X)=2n^2,\qquad \operatorname{mult}_O(C_\omega)=2+4r,$$
$$-K_X\cdot C_\omega=\operatorname{mult}_O(C_\omega),\qquad m_{T_0}^j=m_{T_0}^0.$$

Remark 7.16. The fibre F_ω is a double cover of $\gamma\circ f(F_\omega)\cong\mathbb{P}^2$ ramified in a (possibly singular) quartic curve. It follows by the smoothness of F_ω and O and the equality $-K_X\cdot C_\omega=\operatorname{mult}_O(C_\omega)$ that the support of the cycle $\gamma\circ f(C_\omega)$ is a single line in $\mathbb{P}^3$ passing through the point $\gamma(O_i)\in L$.

Lemma 7.17. $O=Z_i^0\cap Z_i^1$.

Proof. Assume that $O\neq Z_i^0\cap Z_i^1$ and $O\in Z_i^k$. Then the support of C_ω consists of a smooth curve $C\subset X$ such that $-K_X\cdot C=1$ and $O\in C$. However, we have

$$\mathcal{D}_X\big|_{F_\omega}\sim(m-n)K_{F_\omega}\sim(m-n)K_X\big|_{F_\omega}$$

and $\operatorname{mult}_O\bigl(\mathcal{D}_X\big|_{F_\omega}\bigr)\geqslant 2(n-m)$. We regard F_ω as a double cover $\gamma_{F_\omega}\colon F_\omega\to\mathbb{P}^2$ ramified in a smooth quartic $S_{F_\omega}\subset\mathbb{P}^2$. Then

$$\mathcal{D}_X\big|_{F_\omega}=2(n-m)C,$$

which shows that $\gamma_{F_\omega}(C)\subset S_{F_\omega}$. Hence $\gamma_{F_\omega}(O)\in S_{F_\omega}$ and $O=Z_i^0\cap Z_i^1$.

Lemma 7.18. $K = 1$.

Proof. Assume that $K \neq 1$, so that P_1 is a point. Since ${T_0}^0 = m^1_{T_0}$ it follows that P_1 lies in the proper transform of Z^k_i on X_1 and the support of Z_X consists of Z^k_i. On the other hand, Z^k_i intersects transversally each curve in the support of the cycle C_ω, because E is smooth at O and

$$E \cdot C_\omega = -K_X \cdot C_\omega = \mathrm{mult}_O(C_\omega),$$

which shows that the proper transform of C_ω on X_1 cannot pass through P_1. Thus, $m^0_{T_0} > m^1_{T_0}$.

Remark 7.19. It follows from the equalities $\nu_j = 2$ that the graph Γ is a chain, $\Sigma_0 = 1$, and $\Sigma_1 = N - 1$.

The restriction of $\mathcal{D}_{X_1}$ to $G_1 \cong \mathbb{P}^2$ is rationally equivalent to $\mathcal{O}_{\mathbb{P}^2}(2(n-m))$; on the other hand, $\mathrm{mult}_{P_1}(\mathcal{D}_{X_1}) = 2(n-m)$. Hence $\mathcal{D}_{X_1}\big|_{G_1} = 2(n-m)P_1$ and P_1 is a line on $G_1 \cong \mathbb{P}^2$. The equality $b(X, F_\omega, G_N) = \Sigma_0$ shows that P_1 does not lie on F^1_ω. Hence the multiplicity at a point $P_1 \cap F^1_\omega$ of the general surface in the restriction of $\mathcal{D}_{X_1}$ to F^1_ω is at least $2(n-m)$, which means that $\mathcal{D}_X\big|_{F_\omega} = 2(n-m)C_0$ for a smooth rational curve $C_0 \subset F_\omega$ passing through O such that $-K_X \cdot C_0 = 1$. In particular, $\gamma \circ f(C_0)$ is a line on S passing through $\gamma(O_i)$. Hence $\gamma \circ f(C_0) = L$ and $C_0 = F_\omega \cap E$ by our assumption that L is the unique line on the sextic S passing through one of the points $\gamma(O_i) \in \mathbb{P}^3$. However, the equality $C_0 = F_\omega \cap E$ contradicts the smoothness of E. We have thus completed the proof of Theorem 1.15.

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