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Birationally rigid del Pezzo fibrations

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Abstract. In this paper we study del Pezzo fibrations $z : X \to \mathbb{P}^1$ of degree 1 and 2 such that X is smooth, $\operatorname{rk}\operatorname{Pic}(X) = 2$ and $K_X^2 \notin \overline{\mathbb{NE}}(X)$. These are examples of smooth birationally rigid 3-fold Mori fibre spaces. We describe all birational transformations of the 3-fold X into elliptic fibrations, fibrations of surfaces of Kodaira dimension zero, and canonical Fano 3-folds.

Let X be a smooth 3-fold[1] of Picard rank 2 and $\tau : X \to \mathbb{P}^1$ be a flat morphism whose generic fiber is a del Pezzo surface of degree 1 or 2. The following result is proved in [18].

Theorem 1. *Suppose that $K_X^2 \notin \operatorname{Int}(\overline{\mathbb{NE}}(X))$. Then X is not birational to the following:*

- *a conic bundle;*
- *a fibration[2] of rational surfaces that is not equivalent[3] to τ;*
- *a Fano 3-fold of Picard rank 1 having terminal $\mathbb{Q}$-factorial singularities.*

The main purpose of this paper is to prove the following result.

Theorem 2. *Suppose that $K_X^2 \notin \overline{\mathbb{NE}}(X)$. Then X is birational neither to a fibration whose generic fiber is a surface of Kodaira dimension zero nor to a Fano 3-fold*

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[1] All varieties are assumed to be projective, normal, and defined over the field $\mathbb{C}$.
[2] For every fibration $\pi : Y \to Z$ we assume that $\dim(Y) > \dim(Z) \neq 0$ and $\pi_*(\mathcal{O}_Y) = \mathcal{O}_Z$.
[3] Fibrations $\tau : U \to Z$ and $\bar{\tau} : \bar{U} \to \bar{Z}$ are called equivalent if there are birational maps $\alpha : U \dashrightarrow \bar{U}$ and $\beta : Z \dashrightarrow \bar{Z}$ such that the diagram

$$\begin{tikzcd} U \arrow[r, dashed, "\alpha"] \arrow[d, "\pi"'] & \bar{U} \arrow[d, "\bar{\pi}"] \\ Z \arrow[r, dashed, "\beta"'] & \bar{Z} \end{tikzcd}$$

is commutative and α induces the birational isomorphism between the generic fibers of τ and $\bar{\tau}$.

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having canonical singularities. Let $\rho : X \dashrightarrow Y$ be a birational map, where Y is a 3-fold such that there is an elliptic fibration $\psi : Y \to \mathbb{P}^2$. Then there is a rational map $\alpha : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ such that the diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, "\tau"'] & Y \arrow[d, "\psi"] \\ \mathbb{P}^1 & \mathbb{P}^2 \arrow[l, dashed, "\alpha"] \end{tikzcd}$$

is commutative.

The conditions of Theorem 2 cannot be weakened.

Example 3. Let $\lambda : X \to \mathbb{P}^1 \times \mathbb{P}^2$ be a double cover branched over a smooth divisor of bi-degree $(k, 4)$ for $k \geq 4$. Consider the projections $pr_1 : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ and $pr_2 : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$, and put $\tau = pr_1 \circ \lambda$. Then X is smooth, $\tau : X \to \mathbb{P}^1$ is a del Pezzo fibration of degree 2, and $\mathrm{rk}\ \mathrm{Pic}(X) = 2$. Moreover it is easy to see that

$$K_X^2 \notin \overline{\mathbb{NE}}(X) \iff k \geq 5$$

and $K_X^2 \in \partial\overline{\mathbb{NE}}(X)$ in the case $k = 4$. Let $\psi = pr_2 \circ \lambda$. Then $\psi : X \to \mathbb{P}^2$ is an elliptic fibration when $k = 4$ and there is no rational map $\alpha : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ that makes the diagram

$$\begin{tikzcd} X \arrow[r, "id_X"] \arrow[d, "\tau"'] & X \arrow[d, "\psi"] \\ \mathbb{P}^1 & \mathbb{P}^2 \arrow[l, dashed, "\alpha"] \end{tikzcd}$$

commutative. Let $\beta : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ be a projection from a point and $\gamma = \beta \circ \psi$. Then the generic fiber of γ is a K3 surface in the case $k = 4$ (cf. Theorem 27).

Remark 4. Suppose that the generic fiber of Z is a del Pezzo surface of degree 2. Let C be a section of the fibration τ and $\mathcal{H}_C \subset | - K_X + \tau^*(\mathcal{O}_{\mathbb{P}^1}(n))|$ be the linear system of surfaces passing through the section C for $n \gg 0$. Then the generic fiber of the rational map $\phi_{\mathcal{H}_C} : X \dashrightarrow Z_C$ is an elliptic curve, the surface Z_C is rational, a resolution of indeterminacy of the map $\phi_{\mathcal{H}_C}$ gives an elliptic fibration with a section, and there is a natural rational projection $\alpha_C : Z_C \dashrightarrow \mathbb{P}^1$ such that $\alpha_C \circ \phi_{\mathcal{H}_C} = \tau$.

The fibration τ always has a section due to the following result in [16].

Theorem 5. *Let Y be a smooth geometrically irreducible surface over a C_1-field $\mathbb{F}$ such that Y is geometrically rational. Then the surface Y has a point in $\mathbb{F}$.*

On the other hand, the following result was proved in [15].

Theorem 6. *Let Y be a projective variety and $g : Y \to R$ be a morphism with a section onto a smooth curve R. Suppose that we have a set of closed points $\{r_1, \ldots, r_k\} \in R$ such that each fiber $Y_i = g^{-1}(r_i)$ is smooth and separably rationally connected. Then for a set of closed points $y_i \in Y_i$ there is a section $C \subset Y$ of the morphism g passing through each point y_i.*

Therefore the fibration $\tau : X \to \mathbb{P}^1$ has a huge set of sections. Moreover for two general enough sections C_1 and C_2 of τ the corresponding rational maps $\phi_{\mathcal{H}_{C_1}}$ and $\phi_{\mathcal{H}_{C_2}}$ give two non-equivalent elliptic fibrations. Hence the part of Theorem 2 referring to models as elliptic fibrations cannot be improved, but it can be clarified by means of the following result (see [4]).

Proposition 7. *Let Y be a smooth del Pezzo surface of degree 1 or 2 defined over a perfect field $\mathbb{F}$ such that* $\mathrm{rk}\ \mathrm{Pic}(Y) = 1$, *let $\rho : Y \dashrightarrow V$ be a birational map and $\omega : V \to R$ be an elliptic fibration. Then there is a commutative diagram*

$$\begin{tikzcd} Y \arrow[d, dashed, "\sigma"] & W \arrow[l, "\alpha"'] \arrow[dr, "\phi_{|-nK_W|}"] & \\ Y \arrow[r, dashed, "\rho"] & V \arrow[r, "\omega"] & R \end{tikzcd}$$

for $n \gg 0$, where σ is a birational map, and α is a birational morphism such that $K_W^2 = 0$, the divisor $-K_W$ is nef, and the linear system $|-nK_W|$ is free.

Corollary 8. *Let Y be a smooth del Pezzo surface of degree 1 of Picard rank 1 defined over a perfect field $\mathbb{F}$, $\rho : Y \dashrightarrow V$ be a birational map, where V is a smooth surface equipped with a morphism $\pi : V \to S$ such that π is a relatively minimal elliptic fibration with connected fibers. Then the following holds:*

- *ρ is a blow up of the surface Y at some $\mathbb{F}$-point $P \in Y$;*
- *there is a curve $C \in |-K_Y|$ such that $P \in \hat{C}$, where $\hat{C} = C \setminus \mathrm{Sing}(C)$;*
- *the curve $\hat{C}$ is a group scheme with $\mathrm{id}_{\hat{C}} = O$, where $O = \mathrm{Bs}\,|-K_Y|$;*
- *there is $n \in \mathbb{N}$ such that $P^n = \mathrm{id}_{\hat{C}}$ in $\hat{C}$;*
- *$\pi = \phi_{|-nK_V|}$.*

Birational transformations into elliptic fibrations were used in [2], [3], and [13] in the proof of the following result.

Theorem 9. *The set of rational points is potentially dense*[4] *on all smooth Fano 3-folds defined over a number field $\mathbb{F}$ with a possible exception of a double cover of $\mathbb{P}^3$ ramified in a smooth sextic surface.*

Remark 10. The possible exception appears in Theorem 9 because a smooth sextic double solid is the only smooth Fano 3-fold that is not birationally equivalent to an elliptic fibration (see [4]).

The results of [17], [11], and [12] together with Theorem 1 imply the following result.

Theorem 11. *Suppose that $K_X^2 \notin \mathrm{Int}(\overline{\mathbb{NE}}(X))$. Then the group of birational automorphism of the 3-fold X is generated by Bertini involutions of the generic fiber of τ and biregular automorphisms of X.*

[4] The set of rational points of a variety X defined over a number field $\mathbb{F}$ is potentially dense if for some finite extension $\mathbb{K}$ of the field $\mathbb{F}$ the set of $\mathbb{K}$-rational points of X is Zariski dense in X.

Corollary 12. *In the conditions of Theorem 11 suppose that τ is a del Pezzo fibration of degree* 1. *Then* $\mathrm{Bir}(X) = \mathrm{Aut}(X)$.

In the rest of the paper we prove Theorem 2. We use methods of the paper [18] in the form of Theorems 3.12 and 5.1 of the paper [8]. Suppose that $K_X^2 \not\in \overline{\mathbb{NE}}(X)$ and there is a birational map $\rho : X \dashrightarrow Y$ such that either

- Y is a Fano 3-fold with canonical singularities (*the Fano 3-fold case*) or
- $\pi : Y \to S$ is a fibration whose general fiber has a numerically trivial canonical divisor (*the fibration case*).

Let

$$\mathcal{D} = \begin{cases} |-tK_Y| \text{ for } t \gg 0 \text{ in the Fano 3-fold case} \\ |\pi^*(D)| \text{ for some very ample divisor } D \text{ on } S \text{ in the fibration case} \end{cases}$$

and put $\mathcal{H} = \rho^{-1}(\mathcal{D})$. Then the linear system $\mathcal{H}$ has no fixed components and

$$K_X + \frac{1}{n}\mathcal{H} \sim_{\mathbb{Q}} rF$$

for some $n \in \mathbb{N}$ and $r \in \mathbb{Q}$, where F is a fiber of τ.

Lemma 13. *The rational number r is positive.*

Proof. Suppose that $r \leq 0$. Then

$$K_X^2 \equiv \frac{1}{n^2}\mathcal{H}^2 - 2rK_X \cdot F \in \overline{\mathbb{NE}}(X),$$

which contradicts our initial assumption. □

Consider the movable log pair[5] $(X, \frac{1}{n}\mathcal{H})$.

Proposition 14. *The singularities of the log pair $(X, \frac{1}{n}\mathcal{H})$ are not terminal.*

Proof. Suppose that the singularities of the log pair $(X, \frac{1}{n}\mathcal{H})$ are terminal. Then the log pair $(X, \epsilon\mathcal{H})$ is a canonical model for some $\epsilon \in \mathbb{Q}_{>\frac{1}{n}}$. Thus $\kappa(X, \epsilon\mathcal{H}) = 3$. However

$$\kappa(X, \epsilon\mathcal{H}) = \kappa(Y, \epsilon\mathcal{D}) \leq \dim(S)$$

in the fibration case. In the Fano 3-fold case the log pair $(Y, \epsilon\mathcal{D})$ is a canonical model and the uniqueness of canonical model (see Theorem 2.9 in [5]) implies $X \cong Y$, which is impossible because $K_X^2 \not\in \overline{\mathbb{NE}}(X)$. □

[5] For basic properties of movable log pairs see [1], [8], and [5].

Lemma 15. *Suppose that the singularities of the log pair* $(X, \frac{1}{n}\mathcal{H})$ *are canonical. Then the Fano 3-fold case is impossible, in the fibration case the generic fiber of* π *is an elliptic curve, the surface* S *is rational and the following diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, "\tau"'] & Y \arrow[d, "\pi"] \\ \mathbb{P}^1 & S \arrow[l, dashed, "\alpha"] \end{tikzcd}$$

commutates, where α *is a dominant rational map.*

Proof. In the Fano 3-fold case

$$1 = \kappa(X, \frac{1}{n}\mathcal{H}) = \kappa(Y, \frac{1}{n}\mathcal{D}) \in \{-\infty, 0, 3\}$$

by construction of the linear system $\mathcal{D}$, which is a contradiction.

In the fibration case consider the commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "h"'] \arrow[dr, "g"] & \\ X \arrow[rr, dashed, "\rho"'] & & Y \end{tikzcd}$$

with U is a smooth 3-fold, and h and g birational morphisms. Then

$$K_U + \frac{1}{n}\mathcal{R} \sim_{\mathbb{Q}} h^*(rF) + \sum_{i=1}^{k} a_i E_i,$$

where E_i is a h-exceptional divisor, $\mathcal{R}$ is the proper transform of $\mathcal{D}$ on U, and each a_i is a non-negative rational number. Let C be the proper transform on U of a sufficiently general curve lying in a fiber of π. Then $K_U \cdot C = 0$ and $\mathcal{R} \cdot C = 0$. Thus, we have

$$(K_U + \frac{1}{n}\mathcal{R}) \cdot C = h^*(\epsilon F) \cdot C + \sum_{i=1}^{k} a_i E_i \cdot C \geq rF \cdot h(C),$$

which implies $F \cdot C = 0$. Therefore the curve $h(C)$ lies in a fiber of τ. Hence there is a rational map $\alpha : S \dashrightarrow \mathbb{P}^1$ such that $\alpha \circ \pi \circ \rho = \tau$ and π is an elliptic fibration. $\square$

The following result is due to [18] (see Step 1 in the proof of Theorem 5.1 in [8]).

Lemma 16. *Let* C *be a curve contained in a fiber of* τ. *Then* $\mathrm{mult}_C(\mathcal{H}) \leq n$.

Lemma 17. *The singularities of the log pair* $(X, \frac{1}{n}\mathcal{H})$ *are canonical in codimension one in the case when the generic fiber of the fibration* τ *is a del Pezzo surface of degree* 1.

Proof. Suppose that $\mathrm{mult}_C(\mathcal{H}) > n$ for some curve $C \subset X$. Let F be a general enough fiber of the morphism τ. Then

$$n^2 = \mathcal{H}^2 \cdot F \geq \mathrm{mult}_C(\mathcal{H}^2)F \cdot C > n^2,$$

because $F \cdot C \neq 0$ by Lemma 16, which is a contradiction. □

The following result is classical and due to [17] (see also [7] and [18]).

Proposition 18. *Suppose that τ is a del Pezzo fibration of degree 2. Then there is a composition σ of Bertini involutions of the generic fiber of τ such that the singularities of the movable log pair $(X, \frac{1}{\bar{n}}\sigma(\mathcal{H}))$ are canonical in codimension one, where $\bar{n} \in \mathbb{N}$ such that $K_X + \frac{1}{\bar{n}}\sigma(\mathcal{H}) \sim_{\mathbb{Q}} \bar{r}F$ for $\bar{r} \in \mathbb{Q}_{>0}$.*

Remark 19. In the following we may assume that the singularities of the log pair $(X, \frac{1}{n}\mathcal{H})$ are canonical in codimension one. Indeed in the case when τ is a del Pezzo fibration of degree 2 we may substitute ρ by $\rho \circ \sigma^{-1}$ and use the fact that $\tau \circ \sigma = \tau$, where σ is the composition σ of Bertini involutions of the generic fiber of τ from Proposition 18.

Consider a commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "h"'] \arrow[dr, "g"] & \\ X \arrow[rr, dashed, "\rho"'] & & Y \end{tikzcd}$$

with U is smooth, and h and g birational morphisms. Let $\mathcal{R}$ be the proper transform of the linear system $\mathcal{H}$ on U. Then

$$K_U + \frac{1}{n}\mathcal{R} \sim_{\mathbb{Q}} h^*(rF) + \sum_{i=1}^{k} a_i E_i,$$

where E_i is an h-exceptional divisor and $a_i \in \mathbb{Q}$. Now let us consider a (finite or empty) subset $\mathcal{J} \subset \mathbb{P}^1$ that is defined as follows:

$$\mathcal{J} = \{\lambda \in \mathbb{P}^1 \mid h(E_i) \text{ is a point on } \tau^{-1}(\lambda) \text{ for some } E_i \text{ with } a_i < 0\}.$$

For every point λ in $\mathcal{J}$ we have

$$h^*(F_\lambda) \sim h^{-1}(F_\lambda) + \sum_{j=1}^{k_\lambda} b_j E_j,$$

where $b_i \in \mathbb{N}$ and F_λ is a fiber of τ over λ. For every $\lambda \in \mathcal{J}$ let $\mathcal{I}_\lambda \subset \{1, \ldots, k\}$ be a finite subset such that $i \in \mathcal{I}_\lambda$ if and only if $a_i < 0$ and $h(E_i) \in F_\lambda$ is a point. Put $\mathcal{I} = \cup_{\lambda \in \mathcal{J}} \mathcal{I}_\lambda$.

In the following we assume that either Y is a Fano 3-fold with canonical singularities or there is no rational map $\alpha : S \dashrightarrow \mathbb{P}^1$ that makes the diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, "\tau"'] & Y \arrow[d, "\pi"] \\ \mathbb{P}^1 & S \arrow[l, dashed, "\alpha"] \end{tikzcd}$$

commutative. Let us show that this assumption leads to a contradiction.

Corollary 20. *The set $\mathcal{J}$ is not empty.*

The following result is known as the existence of *the super-maximal singularity* in the notations of the paper [19].

Proposition 21. *The inequality*

$$r + \sum_{\lambda \in \mathcal{J}} \min\{\frac{a_i}{b_i} \mid h(E_i) \in F_\lambda \text{ and } a_i < 0\} \leq 0$$

holds.

Proof. Suppose the claim is false. Then there are positive rationals ϵ and c_λ such that

$$c_\lambda + \min\{\frac{a_i}{b_i} \mid h(E_i) \in F_\lambda \text{ and } a_i < 0\} > 0$$

and $r = \epsilon + \sum_{\lambda \in \mathcal{J}} c_\lambda$. Then

$$K_U + \frac{1}{n}\mathcal{R} \sim_{\mathbb{Q}} h^*(\epsilon F) + \sum_{\lambda \in \mathcal{J}} \Big(h^*(c_\lambda F_\lambda) + \sum_{i \in \mathcal{I}_\lambda} a_i E_i \Big) + \sum_{i \notin \mathcal{I}} a_i E_i$$

and the divisor

$$h^*(c_\lambda F_\lambda) + \sum_{i \in \mathcal{I}_\lambda} a_i E_i$$

is effective for $\lambda \in \mathcal{J}$ by the choice of c_λ. The divisor $\sum_{i \notin \mathcal{I}} a_i E_i$ is effective because the singularities of $(X, \frac{1}{n}\mathcal{H})$ are canonical in codimension one. Thus $\kappa(U, \frac{1}{n}\mathcal{R}) = 1$.

In the Fano 3-fold case we have

$$\kappa(Y, \frac{1}{n}\mathcal{D}) \in \{-\infty, 0, 3\},$$

which is a contradiction. Therefore there is a fibration $\pi : Y \to S$ whose sufficiently general fiber has a numerically trivial canonical divisor. Let $C \subset U$ be the proper transform of a sufficiently general curve lying in a fiber of the fibration π. Then

$$K_U \cdot C = \mathcal{R} \cdot C = 0$$

and we have

$$(K_U + \frac{1}{n}\mathcal{R}) \cdot C = h^*(\epsilon F) \cdot C + \sum_{\lambda \in \mathcal{J}} \Big(h^*(c_\lambda F_\lambda) + \sum_{i \in \mathcal{I}_\lambda} a_i E_i \Big) \cdot C$$
$$+ \sum_{i \notin \mathcal{I}} a_i E_i \cdot C \geq \epsilon F \cdot h(C),$$

which implies that the curve $h(C)$ lies in a fiber of the fibration τ. Therefore π is an elliptic fibration and there is a rational map $\alpha : S \dashrightarrow \mathbb{P}^1$ such that the diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, "\tau"'] & Y \arrow[d, "\pi"] \\ \mathbb{P}^1 & S \arrow[l, dashed, "\alpha"] \end{tikzcd}$$

is commutative, which is impossible by our assumption. □

Corollary 22. *There are positive rational numbers c_λ such that $\sum_{\lambda \in \mathcal{J}} c_\lambda = r$ and*

$$\mathbb{CS}(X, \frac{1}{n}\mathcal{H} - c_\lambda F_\lambda) \cap F_\lambda \neq \varnothing,$$

where $\mathbb{CS}$ stands for a set of centers of canonical singularities of a log pair (see [5]).

For every $\lambda \in \mathcal{J}$ the set $\mathbb{CS}(X, \frac{1}{n}\mathcal{H} - c_\lambda F_\lambda) \cap F_\lambda$ consists of finite number of points due to inclusion $\mathbb{CS}(X, \frac{1}{n}\mathcal{H}) \subset \mathbb{CS}(X, \frac{1}{n}\mathcal{H} - c_\lambda F_\lambda)$ and Remark 19.

Lemma 23. *For every $\lambda \in \mathcal{J}$ the points in $\mathbb{CS}(X, \frac{1}{n}\mathcal{H} - c_\lambda F_\lambda) \cap F_\lambda$ are smooth on F_λ.*

Proof. Let O be a point in $\mathbb{CS}(X, \frac{1}{n}\mathcal{H} - c_\lambda F_\lambda) \cap F_\lambda$ that is singular on F_λ. Then there is a pencil $\mathcal{L} \subset | -\frac{2}{d} K_{F_\lambda}|$ of curves singular at the point O. Therefore we have

$$2n = \mathcal{H} \cdot \mathcal{L} \geq \operatorname{mult}_O(\mathcal{H}) \operatorname{mult}_O(\mathcal{L}) \geq 2 \operatorname{mult}_O(\mathcal{H}),$$

but the inequality $c_\lambda > 0$ implies $\operatorname{mult}_O(\mathcal{H}) > n$. This contradiction implies the claim. □

Consider two sufficiently general divisors D_1 and D_2 in $\mathcal{H}$. Put $d = K_X^2 \cdot F$ and

$$\frac{1}{n^2} D_1 \cdot D_2 = Z + \sum_{\lambda \in \mathbb{P}^1} C_\lambda,$$

where Z is an effective cycle whose components do not lie in fibers of τ and C_λ is an effective cycle contained in the fiber F_λ of τ over the point $\lambda \in \mathbb{P}^1$. Let C be a curve in a fiber of τ with $-K_X \cdot C = 1$. Then $C_\lambda \equiv \beta_\lambda C$ for some $\beta_\lambda \in \mathbb{Q}_{\geq 0}$. Put $\beta = \sum_{\lambda \in \mathbb{P}^1} \beta_\lambda$.

Lemma 24. *The inequality $\beta \leq 2rd$ holds.*

Proof. Suppose that $\beta > 2rd$. Then the equivalence

$$Z + \beta C \equiv Z + \sum_{\lambda \in \mathbb{P}^1} C_\lambda \equiv K_X^2 + 2rF \cdot K_X \equiv K_X^2 + 2rdC$$

implies $K_X^2 \equiv Z + (\beta - 2rd)C \in \overline{\mathbb{NE}}(X)$, which is a contradiction. $\square$

For every $\lambda \in \mathcal{J}$ let O_λ be a point in $\mathbb{CS}(X, \frac{1}{n}\mathcal{H} - c_\lambda F_\lambda) \cap F_\lambda$. Then

$$\mathrm{mult}_{O_\lambda}(Z) + \mathrm{mult}_{O_\lambda}(C_\lambda) \geq 4$$

due to [14] (see Corollary 7.3 in [19] or Theorem 3.1 in [8]), but Theorem 3.12 in [8] implies the existence of a rational number $t_\lambda \in [0, 1]$ such that the inequality

$$\mathrm{mult}_{O_\lambda}(Z) + t_\lambda \,\mathrm{mult}_{O_\lambda}(C_\lambda) \geq 4(1 + c_\lambda t_\lambda)$$

holds. However $\mathrm{mult}_{O_\lambda}(Z) \leq Z \cdot F_\lambda = d \leq 2$. In particular, $t_\lambda \neq 0$ and the inequalities

$$\mathrm{mult}_{O_\lambda}(C_\lambda) \geq \frac{2 + 4c_\lambda t_\lambda}{t_\lambda} > 4c_\lambda$$

hold. On the other hand, $\mathrm{mult}_{O_\lambda}(C_\lambda) \leq \frac{2}{d}\beta_\lambda$. Therefore, we have

$$2rd \geq \beta = \sum_{\lambda \in \mathbb{P}^1} \beta_\lambda \geq \sum_{\lambda \in \mathcal{J}} \beta_\lambda > 2d \sum_{\lambda \in \mathcal{J}} c_\lambda = 2dr,$$

which is a contradiction. Hence, Theorem 2 is proved.

Remark 25. There is a fibration $\tau : X \to \mathbb{P}^1$ such that the generic fiber of τ is a del Pezzo surface of degree 1 or 2, the 3-fold X is smooth, rk Pic$(X) = 2$, and $K_X^2 \in \overline{\mathbb{NE}}(X)$, but all claims of Theorem 2 hold for X.

Theorem 26. *Let $\lambda : X \to \mathrm{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ be a double cover ramified in a smooth divisor $R \sim 4M + 2L$ and $\tau : X \to \mathbb{P}^1$ be the natural projection, where M is the tautological line bundle and L is a fiber of the projection to $\mathbb{P}^1$. Then the following holds:*

- *the 3-fold X is smooth;*
- *the generic fiber of τ is a del Pezzo surface of degree* 2*,*
- rk Pic$(X) = 2$ *and* $K_X^2 \in \overline{\mathbb{NE}}(X)$*,*
- *the 3-fold X is not birational to a Fano 3-fold with canonical singularities;*
- *the 3-fold X is not birational to a fibration on surfaces of Kodaira dimension zero;*
- *any dominant rational map $\gamma : X \dashrightarrow \mathbb{P}^2$ whose generic fiber is an elliptic curve is induced by the corresponding rational map of the generic fiber of τ.*

Proof. Every step of the proof of Theorem 2 is valid in this case except for Lemma 13, but the proof of Lemma 13 gives $r \geq 0$. However in the case $r = 0$ every divisor in $\mathcal{H}$ has a negative intersection with curves whose images on V are contracted by $\phi_{|M|}$. The latter implies that $\mathcal{H}$ has a fixed component in the case $r = 0$. In the case when $r > 0$ we can proceed as in the proof of Theorem 2. □

Our technique can be applied to del Pezzo fibrations studied in [9] and [10].

Theorem 27. *In the conditions of Example* 3*, let* $k = 4$*. Then every claim of Theorem* 2 *holds for the del Pezzo fibration* $\tau : X \to \mathbb{P}^1$ *with the only exceptions of the unique elliptic fibration and fibrations of K3 surfaces described in Example* 3 *up to the action of* $\mathrm{Bir}(X)$.

Proof. Every step of the proof of Theorem 2 is valid in this case except for Lemma 13, but the proof of Lemma 13 gives $r \geq 0$. Moreover

$$r = 0 \iff \mathcal{H} \subset |-nK_X|,$$

but the linear system $|-K_X|$ is free and the morphism $\phi_{|-K_X|}$ is the elliptic fibration described in Example 3. Thus in the case $r = 0$ the linear system $\mathcal{H}$ lies in fibers of the fibration $\phi_{|-K_X|}$, which implies the claim. In the case when $r > 0$ we can proceed as in the proof of Theorem 2. □

Theorem 28. *Let* $\lambda : X \to \mathrm{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ *be a double cover ramified in a smooth divisor* $R \sim 4M + 2L$ *and* $\tau : X \to \mathbb{P}^1$ *be the natural projection, where* M *is the tautological line bundle and* L *is a fiber of the projection to* $\mathbb{P}^1$*. Then the following holds:*

- *the 3-fold* X *is smooth;*
- *the generic fiber of* τ *is a del Pezzo surface;*
- $\mathrm{rk}\ \mathrm{Pic}(X) = 2$ *and* $K_X^2 \in \overline{\mathbb{NE}}(X)$*,*
- *the 3-fold* X *is not birational to a Fano 3-fold with canonical singularities;*
- *the 3-fold* X *is not birational to a fibration on surfaces of Kodaira dimension zero with the unique (up to the action of the group* $\mathrm{Bir}(X)$*) exception of the fibration of surfaces of Kodaira dimension zero given by the map* $\phi_{|-K_X|}$*;*
- *any dominant rational map* $\gamma : X \dashrightarrow \mathbb{P}^2$ *whose generic fiber is an elliptic curve is induced by the corresponding rational map of the generic fiber of* τ*.*

Proof. Every step of the proof of Theorem 2 is valid in this case except for Lemma 13, but the proof of Lemma 13 gives $r \geq 0$. Moreover $r = 0$ if and only if $\mathcal{H} \sim -nK_X$.

Let T be the reduced curve on X such that $\lambda(T)$ is contracted by $\phi_{|M|}$ and C be a curve lying in a fiber of τ such that $-K_X \cdot C = 1$. Then $K_X^2 = T$ and

$$\overline{\mathbb{NE}}(X) = \mathbb{R}_{\geq 0}T \oplus \mathbb{R}_{\geq 0}C,$$

which implies $\mathrm{Supp}(\mathcal{H}^2) = T$ in the case $r = 0$.

Suppose that $r = 0$ and $S \not\cong \mathbb{P}^1$. Then the linear system $\mathcal{H}$ is not composed from a pencil. There is a point $P \in X \setminus T$ such that the subsystem $\mathcal{H}_P \subset \mathcal{H}$ of surfaces passing through the point P has no fixed components. For two general surfaces A and B in the linear system $\mathcal{H}_P$ we have

$$P \in A \cap B \subset T$$

in the set-theoretical sense, which is a contradiction.

In the case when $r = 0$ and $S \cong \mathbb{P}^1$ the previous arguments give $\pi \circ \rho = \phi_{|-K_X|}$. In the case of $r > 0$ we can proceed as in the proof of Theorem 2 to get a contradiction. □

In the proof of Theorem 2 we use the method of [8]. Nevertheless we can use the method of [18] as well. The latter way is longer than the former one. However there are cases when the method of [8] fails, but the method of [18] succeeds (see [6]).

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