NON-EXISTENCE OF ELLIPTIC STRUCTURES ON FANO COMPLETE INTERSECTIONS OF INDEX ONE

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Abstract. We prove the absence of birational transformations into elliptic fibrations of a general enough complete intersection $\cap_{i=1}^{k} F_i \subset \mathbb{P}^M$, where F_i is a hypersurface of degree $d_i \geq 2$, relation $\sum_{i=1}^{k} d_i = M > 3k$ holds and $M \neq 4$.

Let $X = \cap_{i=1}^{k} F_i \subset \mathbb{P}^M$ be a general complete intersection[1], where $F_i \subset \mathbb{P}^M$ is a hypersurface of degree $d_i \geq 2$ and $\sum_{i=1}^{k} d_i = M > 3k$. Then $-K_X \sim \mathcal{O}_{\mathbb{P}^M}(1)|_X$, $\mathrm{Pic}(X) = \mathbb{Z}K_X$ and X is a Fano variety. We may assume $d_1 \leq d_2 \leq \cdots \leq d_k$. Moreover, it is easy to see that the inequality $M > 3k$ implies $\dim(X) \geq 3$ and $d_k \geq 4$.

In the case $k = 1$ the variety X is a hypersurface of degree $M \geq 4$ in $\mathbb{P}^M$, but the dimension of X is at least 5 when $k \geq 2$. In dimension 5 the variety X is either a complete intersection of a quadric and a quntic in $\mathbb{P}^7$ or a complete intersection of a cubic and a quartic in $\mathbb{P}^7$.

The purpose of this paper is to prove the following result.

Theorem 1. *Let $\rho : V \dashrightarrow X$ be a birational map, $\tau : V \to Z$ be a fibration whose general fiber has Kodaira dimension zero, and $M \neq 4$. Then $Z \cong \mathbb{P}^1$.*

Corollary 2. *Let $M \neq 4$. Then X is not birational to elliptic and K3 fibrations.*

Birational transformations into elliptic fibrations were used in [2] and [12] in the proof of potential density[2] of rational points on smooth Fano 3-folds, where the following result was proved.

Theorem 3. *The set of rational points is potentially dense on all smooth Fano 3-folds with a possible exception of a double cover of $\mathbb{P}^3$ ramified in a smooth sextic surface.*

The possible exception appears in Theorem 3 because a smooth sextic double solid is the only smooth Fano 3-fold that is not birational an elliptic fibration (see [3]). It should be pointed out that a double cover of $\mathbb{P}^3$ branched over a sextic with one ordinary double point is birational to a unique elliptic fibration (see [6]).

Remark 4. The condition $M \neq 4$ in Theorem 3 is crucial. Indeed, every smooth quartic 3-fold contains a line and the corresponding projection gives a birational transformation into an elliptic fibration. The generality condition in Theorem 3 is crucial as well, because a smooth quintic 4-fold can be birationally transformed into a K3 fibration if and only if it contains a plane (see [4]). A smooth complete intersection of a quadric and a quartic in $\mathbb{P}^6$ is birational to an elliptic fibration if and only if it contains a plane (see [9]).

The author is very grateful to M.Grinenko, V.Iskovskikh, Yu.Prokhorov, A.Pukhlikov and V.Shokurov for fruitful conversations. The author would like to thank the referee for many useful remarks. All varieties are assumed to be projective, normal and defined over $\mathbb{C}$.

[1]The generality of X is considered in the sense of [17].

[2]The set of rational points of a variety V defined over a number field $\mathbb{F}$ is called potentially dense if for a finite extension of fields $\mathbb{K}/\mathbb{F}$ the set of $\mathbb{K}$-rational points of the variety V is Zariski dense.

Our methods also proves the following result.

Theorem 5. *Let $\rho : X \dashrightarrow V$ be a birational map, where V is a Fano variety with canonical singularities. Then ρ is an isomorphism.*

Actually, the claims similar to Theorems 1 and 5 hold for many birationally rigid varieties (see [3]). The birational rigidity of X was proved in [13], [15], [17]. Therefore, the variety X is not birationally equivalent to a fibration whose general fiber has Kodaira dimension $-\infty$. In particular, X is not rational. However, X is always birational to a fibration whose general fiber has Kodaira dimension zero.

Example 6. Let $\mathcal{H} \subset |-K_X|$ be a pencil and $\rho : V \to X$ be a resolution of the indeterminacy of a rational map $\phi_{\mathcal{H}} : X \dashrightarrow \mathbb{P}^1$. Then the general fiber of $\phi_{\mathcal{H}} \circ \rho$ has Kodaira dimension zero.

The following claim was conjectured in [17].

Conjecture 7. *Let $\rho : V \dashrightarrow X$ be a birational map, $\tau : V \to Z$ be a fibration whose general fiber has Kodaira dimension zero, and $M \neq 4$. Then there is a pencil $\mathcal{H} \subset |-K_X|$ such that $\tau = \phi_{\mathcal{H}} \circ \rho$.*

Note, that in the case $k = 1$ both Theorems 1 and 5 and Conjecture 7 were proved in the paper [5]. The similar claims for $M = 4$ were proved in [3] and [8].

Remark 8. The hardest part of the proof of Theorem 1 is implicitly contained in the proof of the birational rigidity of X in [17]. Moreover, there is a very short sketch of the proof of Theorem 1 in [17]. The given proof of Theorem 1 uses the main technical result of [17], but it is more explicit and somehow simpler due to Lemma 13, whose proof is based on constructions in [14], [15], [18].

In the rest of the paper we will prove Theorem 1 and 5.

Remark 9. We may assume $k \geq 2$.

Let $\rho : V \dashrightarrow X$ be a non-biregular birational map such that one of the following holds: there is a fibration $\tau : V \to Z \not\cong \mathbb{P}^1$ whose general fiber has Kodaira dimension zero; the variety V is a canonical Fano variety. In the former case put $M_X = \lambda\rho(|\tau^*(H)|)$ for a very ample divisor H on Z, in the latter case put $M_X = \frac{\lambda}{n}\rho(|-nK_V|)$ for some $n \in \mathbb{Z}_{\gg 0}$, where $\lambda \in \mathbb{Q}_{>0}$. Now choose such λ that $K_X + M_X \sim_{\mathbb{Q}} 0$.

Remark 10. The movable boundary M_X is not contained in the fibers of any dominant rational map $\gamma : X \dashrightarrow \mathbb{P}^1$.

The proof of the birational rigidity of X in [17] implies the canonicity of the singularities of the log pair (X, M_X) (see [10]).

Claim 11. *The singularities of (X, M_X) are non-terminal.*

Proof. Suppose (X, M_X) is terminal. Then $(X, \epsilon M_X)$ is canonical model for some $\epsilon \in \mathbb{Q}_{>1}$ and $\kappa(X, \epsilon M_X) = \dim(X)$. In the fibration case we have $\kappa(X, \epsilon M_X) \leq \dim(Z)$, and in the Fano case we have $\lambda = 1$. Hence, both log pairs $(X, \epsilon M_X)$ and $(V, \epsilon M_V)$ are canonical models. Thus, ρ must be an isomorphism, which contradicts the initial assumption. □

Therefore, the set of centers of canonical singularities $\mathbb{CS}(X, M_X)$ is not empty. The following result is a corollary of the main claim of [17] and the inequality of [13] in the higher-dimensional form (see [15], [16] and [11]).

Claim 12. *The set of centers of canonical singularities* $\mathbb{CS}(X, M_X)$ *does not contain subvarieties of* X *of codimension greater than* 3.

Proof. Suppose $\mathbb{CS}(X, M_X)$ contains a subvariety $Z \subset X$ whose codimension is greater than 3. Let $dim(Z) = d$. Put $Y = X \cap_{i=1}^{d} H_i$ and $M_Y = M_X|_Y$ for sufficiently general hyperplanes $H_i \subset \mathbb{P}^M$. Let O be a point in the intersection $Z \cap_{i=1}^{d} H_i$. Then O is an element of the set $\mathbb{CS}(Y, M_Y)$ and $\dim(Y) \geq 4$.

Take a general hyperplane section H of Y passing through O. Then the connectedness theorem of V.V.Shokurov (see [11]) implies $O \in \mathbb{LCS}(H, M_Y|_H)$. The dimension of H is at most 3 and we can iterate the above construction. We get a movable log pair $(S, M_Y|_S)$ such that S is smooth surface containing the point O and the log-pair $(S, M_Y|_S)$ is not log-canonical in the point $O \in S$.

Now Theorem 3.1 in [11] implies

$$mult_O(M_X^2) = mult_O((M_Y|_S)^2) > 4,$$

but Proposition 2 in [17] implies $mult_P(Y) \leq 4\frac{d_Y}{d_X}$ for any subvariety $Y \subset X$ of codimension 2 and a point $P \in Y$, where d_Y and d_X are degrees of Y and X in the given embeddings respectively. Putting the components of M_X^2 in the latter inequality we get a contradiction. □

The following result is a generalization of Theorem 2 in [14].

Lemma 13. *Let* $V = \cap_{i=1}^{k} G_i \subset \mathbb{P}^M$ *be a smooth complete intersection, where* G_i *is a hypersurface and* $M - k > 2$*. Then* $mult_S(D) \leq n$ *for any effective divisor* D *on* V *and any irreducible subvariety* $S \subset V$ *such that* $\dim(S) \geq k$ *and* $\operatorname{codim}(S) \geq 2$*, where* $n \in \mathbb{N}$ *such that* $D \equiv \mathcal{O}_{\mathbb{P}^M}(n)|_V$.

Proof. We may assume $\dim(S) = k < (M-1)/2$. Take a general enough point $P \in \mathbb{P}^M$ and a cone $C_S \in \mathbb{P}^M$ over S with a vertex P. Then $C_S \cap V = S \cup R_S$, where R_S is a curve on V. The latter holds in a scheme-theoretic sense due to the generality of P.

Let $\pi : V \to \mathbb{P}^{M-1}$ be a projection from P and $D_\pi \subset V$ be a ramification locus of the morphism π. We claim that in a set-theoretic sense $R_S \cap S = D_\pi \cap S$. Indeed, put $C_S \cap G_i = S \cup R_S^i$. Then we have $R_S^i \cap S = D_\pi^i \cap S$ for a ramification divisor $D_\pi^i \subset G_i$ of the projection $\pi^i : G_i \to \mathbb{P}^{M-1}$ from P by Lemma 3 in [18]. On the other hand, $R_S = \cap_{i=1}^{k} R_S^i$ and $D_\pi = \cap_{i=1}^{k} D_\pi^i$, which implies $R_S \cap S = D_\pi \cap S$.

Let $(z_0 : \ldots : z_M)$ be homogeneous coordinates on $\mathbb{P}^M$ such that the equation of the hypersurface G_j is $F_j = 0$ and $P = (p_0 : \ldots : p_M)$. Then the subvariety D_π is given by k equations $\sum_{i=0}^{M} \frac{\partial F_j}{\partial z_i} p_i = 0$ and linear systems $|\sum_{i=0}^{M} \lambda_i \frac{\partial F_j}{\partial z_i} = 0|$ are free on V due to the smoothness of V. Hence, the intersection $D_\pi \cap S$ consists of $d_S \prod_{i=1}^{k}(d_i - 1)$ different points, where $d_S = \deg(S \subset \mathbb{P}^M)$. On the other hand, we have $\deg(D|_{R_S}) = n d_S \prod_{i=1}^{k}(d_i - 1)$, which implies the claim. □

Claim 14. *The set of centers of canonical singularities* $\mathbb{CS}(X, M_X)$ *does not contain subvarieties of* X *of codimension* 3.

Proof. Suppose that the set of centers of canonical singularities $\mathbb{CS}(X, M_X)$ contains a subvariety $Z \subset X$ of codimension 3. Then $mult_Z(M_X) > 1$ and $M_X \equiv \mathcal{O}_{\mathbb{P}^M}(1)|_X$, but the inequality $dim(Z) < k$ holds by Lemma 13. Thus, $M < 2k + 3$. Therefore, $k = 1$ due to $M > 3k$, which is impossible due to Remark 9. □

Theorem 15. *The set $\mathbb{CS}(X, M_X)$ consists of a subvariety $S \subset X$ of codimension 2 such that $\deg(S) = \deg(X)$, $mult_S(M_X) = 1$ and $M_X^2 = S$.*

Proof. The set $\mathbb{CS}(X, M_X)$ is not empty, but $\mathbb{CS}(X, M_X)$ does not contain subvarieties of codimension greater than 2 by Claims 12 and 14. Thus, there is a subvariety $S \subset X$ of codimension 2 such that $S \in \mathbb{CS}(X, M_X)$. In particular, we have $mult_S(M_X) \geq 1$. On the other hand, the dimension of X is at least 5 due to Remark 9. Therefore, the Lefschetz theorem and $M_X \sim_{\mathbb{Q}} -K_X$ imply that the subvariety $S \subset X$ is numerically equivalent to the intersection of two hyperplane sections of X, $mult_S(M_X) = 1$ and M_X^2 consists just of the subvariety S. □

Actually, we never used the assumption $Z \ncong \mathbb{P}^1$ except for Remark 10. Therefore, all the claims proved till now hold in general case as well. Moreover, to prove Conjecture 7 modulo Theorem 15 one just need to show that S is the intersection of two hyperplane section of X without using the assumption $Z \ncong \mathbb{P}^1$. It seems to us that even the single condition that S is numerically equivalent to the intersection of two hyperplane section of X must imply that S the intersection of two hyperplane section of X (see [1]) perhaps with some restrictions on $\dim(X)$.

Consider M_X as $\frac{1}{n}\mathcal{M}$, where $\mathcal{M} \subset |-nK_X|$ is a linear system without fixed components and $n \in \mathbb{N}$. Then $mult_S(\mathcal{M}) = n$ by Theorem 15, $\mathrm{Supp}(\mathcal{M}^2) = S$ and the base set of $\mathcal{M}$ consists just of S.

Claim 16. *The linear system $\mathcal{M}$ is composed from a pencil.*

Proof. Suppose $\dim(\phi_{\mathcal{M}}(X)) \neq 1$. Let $P \in X \backslash S$ be a general enough point and $\mathcal{M}_P \subset \mathcal{M}$ be a linear subsystem of divisors passing through P. Then the linear system $\mathcal{M}_P$ has no fixed components. Thus, in a set-theoretic sense $P \in \mathcal{M}_P^2 \subset \mathcal{M}^2 = S$. □

It is clear that Claim 16 contradicts Remark 10. Thus, Theorems 1 and 5 are proved.

References

[1] E.Amerik, *On codimension-two subvarieties of hypersurfaces*, J. Reine Angew. Math. **483** (1997), 61–73

[2] F.Bogomolov, Yu.Tschinkel, *On the density of rational points on elliptic fibrations*, J. Reine Angew. Math. **511** (1999), 87–93

[3] I.A.Cheltsov, *Log pairs on birationally rigid varieties*, J. Math. Sciences **102** (2000), 3843–3875

[4] ______, *On smooth quintic 4-fold*, Mat. Sbornik **191:9** (2000), 139–162

[5] ______, *Log pairs on hypersurfaces of degree N in $\mathbb{P}^N$*, Mat. Zametki **68:1** (2000), 131–138

[6] ______, *A Fano 3-fold with unique elliptic structure*, Mat. Sbornik **192:5** (2001), 145–156

[7] ______, *Log canonical thresholds on hypersurfaces*, Mat. Sbornik **192:8** (2001), 155–172

[8] ______, *Anticanonical models of Fano 3-folds of degree four*, Mat. Sbornik **194:4** (2003), 143–172

[9] ______, *Non-rationality of a four-dimensional smooth complete intersection of a quadric and a quartic not containing a plane*, Mat. Sbornik **194:11** (2003), 95–116

[10] A.Corti, *Factorizing birational maps of threefolds after Sarkisov*, J. Alg. Geometry **4** (1995), 223–254

[11] ______, *Singularities of linear systems and 3-fold birational geometry*, L.M.S. Lecture Note Series **281** (2000), 259–312

[12] J.Harris, Yu.Tschinkel, *Rational points on quartics*, Duke Math. J. **104** (2000), 477–500

[13] V.A.Iskovskikh, Yu.I.Manin, *Three-dimensional quartics and counterexamples to the Lüroth problem*, Mat. Sbornik **86:1** (1971), 140–166

[14] A.V.Pukhlikov, *Notes on theorem of V.A.Iskovskikh and Yu.I.Manin about 3-fold quartic*, Proceedings of Steklov Institute **208** (1995), 278–289

[15] ______, *Birational automorphisms of Fano hypersurfaces*, Invent. Math. **134** (1998), 401–426

[16] ———, *Essentials of the method of maximal singularities*, L.M.S. Lecture Note Series **281** (2000), 73–100
[17] ———, *Birationally rigid Fano complete intersections*, J. Reine Ang. Math. **541** (2001), 55–79
[18] ———, *Birationally rigid Fano hypersurfaces*, Izv. RAN **66:3** (2002), 159–186

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