Izvestiya: Mathematics

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To cite this article: I A Cheltsov 2006 *Izv. Math.* **70** 605

Izvestiya: Mathematics **70**:3 605–639 ©2006 RAS(DoM) and LMS
Izvestiya RAN: Ser. Mat. **70**:3 185–221 DOI 10.1070/IM2006v070n03ABEH002321

Local inequalities and birational superrigidity of Fano varieties

I. A. Cheltsov

Abstract. We obtain local inequalities for log canonical thresholds and multiplicities of movable log pairs. We prove the non-rationality and birational superrigidity of the following Fano varieties: a double covering of a smooth cubic hypersurface in $\mathbb{P}^n$ branched over a nodal divisor that is cut out by a hypersurface of degree $2(n-3) \geqslant 10$; a cyclic triple covering of a smooth quadric hypersurface in $\mathbb{P}^{2r+2}$ branched over a nodal divisor that is cut out by a hypersurface of degree $r \geqslant 3$; a double covering of a smooth complete intersection of two quadric hypersurfaces in $\mathbb{P}^n$ branched over a smooth divisor that is cut out by a hypersurface of degree $n-4 \geqslant 6$.

§ 1. Introduction

Let X be a variety,[1] O a smooth point of X and $\mathcal{M}$ a linear system on X that has no fixed components. Suppose that $\dim(X) \geqslant 3$ and O is a centre of canonical singularities of the log pair $\bigl(X, \frac{1}{n}\mathcal{M}\bigr)$. Namely, there is a birational morphism $f\colon \overline{X} \to X$ such that we have an equivalence

$$K_{\overline{X}} + \frac{1}{n}\,\overline{\mathcal{M}} \sim_{\mathbb{Q}} f^*\biggl(K_X + \frac{1}{n}\,\mathcal{M}\biggr) + \sum_{i=1}^{k} a(X, M_X, E_i)E_i$$

in a neighbourhood of O and the inequality $a(X, M_X, E_j) \leqslant 0$ for some j, where $\overline{\mathcal{M}}$ is the proper transform of $\mathcal{M}$ on $\overline{X}$, the E_i are exceptional divisors of the birational morphism f and the $a(X, M_X, E_i)$ are rational numbers.

The following result is proved in [59]. It is known as the $4n^2$-*inequality*.

Theorem 1. *We have* $\operatorname{mult}_O(S_1 \cdot S_2) \geqslant 4n^2$ *for general divisors* S_1, S_2 *of the linear system* $\mathcal{M}$.

The Noether–Fano–Iskovskikh inequality (see [16], Theorem 64) and Theorem 1 imply the birational superrigidity [2] and, in particular, the non-rationality of smooth quartic threefolds (see [5], [4], [59], [32]). The local inequality in Theorem 1 cannot

[1] All varieties are assumed to be projective, normal and defined over $\mathbb{C}$.

[2] Let V be a Fano variety having terminal and $\mathbb{Q}$-factorial singularities such that $\operatorname{rk}\operatorname{Pic}(V) = 1$. We say that V is *birationally superrigid* if the following conditions hold: V can not be birationally transformed into a fibration $\tau\colon Y \to Z$ such that $\dim(Y) > \dim(Z) \neq 0$ and the Kodaira dimension of a general fibre of τ is $-\infty$; V is not birational to a Fano variety of Picard rank 1 having terminal and $\mathbb{Q}$-factorial singularities that is not biregular to V; the groups $\operatorname{Bir}(V)$ and $\operatorname{Aut}(V)$ coincide.

This work is partially supported by the CRDF (grant no. RUM1-2692MO-05).

AMS 2000 *Mathematics Subject Classification.* 14E05, 14E07, 14E08, 14J40, 14J45.

be improved if $\dim(X) = 3$ (see [31], [46]). One might expect this inequality to admit considerable improvement for higher-dimensional X, but this does not seem to be the case.

There are also local inequalities for singular points. They are similar to the inequality in Theorem 1, and one can use them to prove the non-rationality of many higher-dimensional mildly singular Fano varieties. For example, the following result of [32] yields the non-rationality of any factorial quartic threefold having at most isolated ordinary double points (see [6], [32], [57]).

Theorem 2. *Let P be an isolated ordinary double point on X such that the linear system $\mathcal{M}$ consists of $\mathbb{Q}$-Cartier divisors*[3] *and P is a centre of canonical singularities of the log pair $\bigl(X, \frac{1}{n}\mathcal{M}\bigr)$. Then we have $\operatorname{mult}_P(S) \geqslant n$, where S is a general divisor of $\mathcal{M}$ and $\operatorname{mult}_P(S)$ is the positive integer such that $\widetilde{S} \sim_{\mathbb{Q}} g^*(S) - \operatorname{mult}_P(S)E$. Here $g\colon \widetilde{X} \to X$ is the blow-up of P, E is the exceptional divisor of the birational morphism g and $\widetilde{S}$ is the proper transform of S on $\widetilde{X}$. We also have the strict inequality $\operatorname{mult}_P(S) > n$ if $\dim(X) > 3$.*

As in the case of Theorem 1, the inequality in Theorem 2 cannot be improved in the case when $\dim(X) = 3$ nor, perhaps, in the case when $\dim(X) \geqslant 4$.

The purpose of this paper is to obtain the following local inequalities, which imply the birational superrigidity and, in particular, the non-rationality of many higher-dimensional Fano varieties of degree[4] 6 or 8.

Theorem 3. *Let Y be a variety of dimension $r \geqslant 4$, $\mathcal{H}$ a linear system on Y without fixed components, S_1 and S_2 general divisors of $\mathcal{H}$ and P a smooth point of Y such that P is a centre of canonical singularities of the log pair $\bigl(Y, \frac{1}{n}\mathcal{H}\bigr)$ for some $n \in \mathbb{N}$. Suppose that the singularities of the log pair $\bigl(Y, \frac{1}{n}\mathcal{H}\bigr)$ are canonical outside P. Let $\pi\colon \widehat{Y} \to Y$ be the blow-up of P, and let Π be the exceptional divisor of the morphism π. Then there is a linear subspace $\Lambda \subset \Pi \cong \mathbb{P}^{r-1}$ of codimension 2 such that*

$$\operatorname{mult}_P(S_1 \cdot S_2 \cdot \Delta) \geqslant 8n^2$$

for every effective divisor Δ on Y that satisfies the following hypotheses:

1) *Δ contains P and is smooth at P*;

2) *the divisor $\pi^{-1}(\Delta)$ contains Λ*;

3) *Δ contains no subvarieties of Y of codimension 2 that are contained in the base locus of $\mathcal{H}$.*

Theorem 4. *Let V be a variety of dimension $r \geqslant 4$, $\mathcal{H}$ a linear system on V without fixed components, S_1 and S_2 general divisors of $\mathcal{H}$ and P an isolated ordinary double point of V such that P is a centre of canonical singularities of the log pair $\bigl(V, \frac{1}{n}\mathcal{H}\bigr)$ for some $n \in \mathbb{N}$. Suppose that the singularities of the log pair $\bigl(V, \frac{1}{n}\mathcal{H}\bigr)$ are canonical outside P. Let $\pi\colon \widehat{V} \to V$ be the blow-up of P and let E be the exceptional divisor of the birational morphism π.* (*One can identify E with a smooth quadric hypersurface in $\mathbb{P}^r$.*) *Then there is a linear subspace $\Lambda \subset \mathbb{P}^r$ of codimension 3 such*

[3] A Weil divisor is called a $\mathbb{Q}$-*Cartier divisor* if some non-zero multiple of it is a Cartier divisor.
[4] *The degree of a Fano variety* V is the number $(-K_V)^n$, where $n = \dim(V)$.

that Λ is contained in the quadric E and

$$\mathrm{mult}_P(S_1 \cdot S_2 \cdot \Delta) = \mathrm{mult}_P(S_1|_\Delta \cdot S_2|_\Delta) \geqslant 6n^2$$

for every effective divisor Δ on V that satisfies the following hypotheses:

1) *Δ contains the point P*;

2) *P is an ordinary double point of Δ*;

3) *the divisor $\pi^{-1}(\Delta)$ contains Λ*;

4) *Δ contains no subvarieties of V of codimension 2 that are contained in the base locus of $\mathcal{H}$.*

One can show that the inequalities in Theorems 3 and 4 are strict for varieties of dimension 5 or more. Thus we may regard Theorem 3 (resp. Theorem 4) as a natural higher-dimensional generalization of Theorem 1 (resp. Theorem 2). Theorems 3 and 4 may be called the $8n^2$-*inequality* and $6n^2$-*inequality* respectively. We mention that all local intersections in Theorem 4 are well defined via the intersections of the corresponding cycles on the blow-up of the variety at the ordinary double point (see Definition 7).

We note that the condition on the linear subspace in Theorem 4 becomes vacuous for varieties of dimension 6 or more by Lefschetz' theorem (see [27], [21]). This yields the following corollary.

Corollary 1. *Let V be a variety of dimension $r \geqslant 6$, $\mathcal{H}$ a linear system on V without fixed components, S_1 and S_2 general divisors of $\mathcal{H}$ and P an isolated ordinary double point of V such that P is a centre of canonical singularities of the log pair $\big(V, \frac{1}{n}\mathcal{H}\big)$ for some $n \in \mathbb{N}$. Suppose that the log pair $\big(V, \frac{1}{n}\mathcal{H}\big)$ has canonical singularities outside P. Then* $\mathrm{mult}_O(S_1 \cdot S_2) > 6n^2$.

We use Theorems 3 and 4 to prove the following result.

Theorem 5. *The following Fano varieties are birationally superrigid*:

1) *a double covering of a smooth hypersurface $V \subset \mathbb{P}^n$ of degree 3 branched over an effective divisor $R \subset V$ such that R has at most isolated ordinary double points, R is cut out on V by a hypersurface in $\mathbb{P}^n$ of degree $2(n-3)$ and $n \geqslant 9$*;

2) *a cyclic triple covering of a smooth quadric hypersurface $Q \subset \mathbb{P}^{2r+2}$ branched over an effective divisor $S \subset Q$ such that S has at most isolated ordinary double points and S is cut out on Q by a hypersurface in $\mathbb{P}^{2r+2}$ of degree $3r \geqslant 12$*;

3) *a double covering of a smooth complete intersection $Y \subset \mathbb{P}^n$ of two quadric hypersurfaces branched over a smooth effective divisor $D \subset Y$ that is cut out on Y by a hypersurface in $\mathbb{P}^n$ of degree $2(n-4) \geqslant 12$.*

All varieties in Theorem 5 are complete intersections in weighted projective spaces. In particular, we can effectively apply Theorem 5 to construct examples of rationally connected [5] non-rational higher-dimensional varieties.

Example 1. Let Q be the quadric hypersurface in $\mathbb{P}^{10}$ given by the equation

$$\sum_{i=0}^{10} x_i^2 = 0 \subset \mathbb{P}^{10} \cong \mathrm{Proj}(\mathbb{C}[x_0, \dots, x_{10}])$$

[5] A variety is said to be *rationally connected* if any pair of general points of it can be joined by a rational curve (see [54]–[56], [50]). For example, every unirational variety is rationally connected.

and let $\psi\colon X \to Q$ be a cyclic triple covering branched over a divisor that is cut out on the quadric $Q \subset \mathbb{P}^{10}$ by the hypersurface

$$x_0^6x_1^6 + x_2^6x_3^6 + x_4^6x_5^6 + x_6^6x_7^6 + x_8^6x_9^6 = 0 \subset \mathbb{P}^{10} \cong \operatorname{Proj}(\mathbb{C}[x_0, \dots, x_{10}]).$$

Then X is a Fano variety, $\dim(X) = 9$, $(-K_X)^9 = 6$ and simple calculations show that X is smooth. The variety X is birationally superrigid and non-rational by Theorem 5. We may regard X as the weighted complete intersection in $\mathbb{P}(1^{11}, 4)$ given by the equations

$$\sum_{i=0}^{10} x_i^2 = z^3 - x_0^6x_1^6 + x_2^6x_3^6 + x_4^6x_5^6 + x_6^6x_7^6 + x_8^6x_9^6 = 0 \subset \mathbb{P}(1^{11}, 4)$$
$$\cong \operatorname{Proj}(\mathbb{C}[x_0, \dots, x_{10}, z]).$$

We note that Fano varieties satisfying the hypotheses of Theorem 5 are expected to be birationally superrigid because their degree and singularities are sufficiently small. Informally speaking, we expect Fano varieties to become *more rational* when their anticanonical degree gets bigger and their singularities get worse. For example, it follows from the classification of smooth Fano threefolds that a smooth Fano threefold is rational if its degree is bigger than 24 (see [44]). Thus the claim of Theorem 5 looks rather natural. Unfortunately, all existing proofs of the birational superrigidity of higher-dimensional Fano varieties use the projective geometry of their anticanonical map. It is natural to expect that some results on the birational superrigidity of Fano varieties can be proved without using properties of their anticanonical rings. For example, we expect the following to be true.

Conjecture 1. *Let X be a smooth Fano variety of dimension n such that $\operatorname{rk}\operatorname{Pic}(X) = 1$ and $(-K_X)^n \leqslant n$. Then X is birationally superrigid.*

Conjecture 1 has been proved only in dimension 3 (via the classification of smooth Fano threefolds; see [44]). On the other hand, is is supported by the many new examples of birationally superrigid higher-dimensional Fano varieties given in Theorem 5. The proof of Theorem 5 shows that the main difficulty in proving Conjecture 1 is to find an appropriate way to apply existing local inequalities to Fano varieties without using their projective geometry. It is most likely that the proof of Conjecture 1 will be very hard, but we have every hope that the following conjecture will be proved in the near future by existing methods.

Conjecture 2. *Let X be a smooth Fano variety of dimension n such that $\operatorname{rk}\operatorname{Pic}(X) = 1$ and $(-K_X)^n = 1$. Then X is birationally superrigid.*

The geometrical meaning of Theorem 5 is akin to that of the theorem of Noether which asserts that the group of birational automorphisms of $\mathbb{P}^2$ is generated by projective automorphisms and the Cremona involution (see [31]). There are many interesting problems related to the latter theorem, and one of them is that of the birational classification of plane elliptic pencils considered in [24]. The ideas in [24] are rigorously justified in [3], where it is proved that every elliptic pencil on the projective plane can be birationally transformed into a so-called Halphen pencil. One can consider a similar problem for Fano varieties that satisfy the hypotheses of Theorem 5. Our methods enable us to prove the following result.

Theorem 6. *Birationally superrigid Fano varieties that satisfy the hypotheses of Theorem 5 cannot be birationally transformed into elliptic fibrations.*

Birational transformations into elliptic fibrations are used in [25], [26], [40] to study the potential density [6] of rational points on Fano threefolds. The following result is obtained there.

Theorem 7. *The rational points are potentially dense on all smooth Fano threefolds except possibly for the double covering of $\mathbb{P}^3$ branched over a smooth sextic surface.*

The possible exception appears in Theorem 7 because no birational transformations into elliptic fibrations are known for the double covering of $\mathbb{P}^3$ branched over a smooth sextic surface. On the other hand, it follows from [29] that the double covering of $\mathbb{P}^3$ branched over a smooth sextic surface is not birational to an elliptic fibration, and it follows from the classification of smooth Fano threefolds (see [44]) that it is the only smooth Fano threefold with this property.

The well-known *weak Lang conjecture* asserts that the rational points are not potentially dense on varieties of general type. It is known only for curves and subvarieties of abelian varieties (see [35], [36]). On the other hand, the geometry of birationally superrigid varieties is reminiscent of the geometry of varieties of general type. Hence we can expect that the rational points are not potentially dense on some birationally superrigid varieties. This may be regarded as a possible way of constructing an example of a rationally connected non-unirational variety (see [52], Conjecture 4.1.6).

§ 2. Preliminaries

In this section we consider properties of movable log pairs (see [19], [16]) and some results related to the Shokurov connectedness principle (see [18]).

Definition 1. A *movable log pair* (X, M_X) is a pair consisting of a variety X and a finite formal linear combination $M_X = \sum_{i=1}^n a_i \mathcal{M}_i$ (referred to as a movable boundary), where the $\mathcal{M}_i$ are linear systems on X without fixed components and the a_i are non-negative rational numbers.

One can naturally define the image of a movable boundary under a birational map because the base loci of the components of a movable boundary contain no divisors.

Remark 1. Let (X, M_X) be a movable log pair. We can naturally regard the self-intersection M_X^2 as an effective cycle of codimension 2 on X provided that X has $\mathbb{Q}$-factorial singularities. Namely, write $M_X = \sum_{i=1}^n a_i \mathcal{M}_i$, where the $\mathcal{M}_i$ are linear systems without fixed components. For every index i, choose two sufficiently general divisors S_i and $\widehat{S}_i$ of $\mathcal{M}_i$, then put $M_X^2 = \sum_{i,j=1}^n a_i a_j \, S_i \cdot \widehat{S}_j$.

One can define the discrepancies, terminality, canonicity, log terminality and log canonicity for movable log pairs in the same way as for ordinary log pairs (see [48]). Moreover, the singularities of a movable log pair coincide with those of

[6]Rational points of a variety V defined over a number field $\mathbb{F}$ are potentially dense if there is a finite extension $\mathbb{F} \subset \mathbb{K}$ of fields such that the set of all $\mathbb{K}$-points of V is Zariski dense in V.

the underlying variety outside the union of the base loci of the components of the movable boundary. It follows that every movable log pair is birational to a movable log pair having terminal singularities (see [42]).

Definition 2. A proper irreducible subvariety $Y \subset X$ is called a *centre of canonical singularities of a movable log pair* (X, M_X) if one can find a smooth variety W, a birational morphism $f\colon W \to X$ and an exceptional divisor $E_1 \subset W$ of f such that

$$K_W + f^{-1}(M_X) \sim_{\mathbb{Q}} f^*(K_X + M_X) + \sum_{i=1}^{k} a(X, M_X, E_i)E_i,$$

where the $a(X, M_X, E_i)$ are rational numbers, the E_i are exceptional divisors of the birational morphism f, $f(E_1) = Y$ and $a(X, M_X, E_1) \leqslant 0$. The set of all centres of canonical singularities of the log pair (X, M_X) is denoted by $\mathbb{CS}(X, M_X)$.

Definition 2 implies that a movable log pair (X, M_X) has terminal singularities if and only if $\mathbb{CS}(X, M_X) = \varnothing$.

Remark 2. Let (X, M_X) be a movable log pair with terminal singularities. Then the singularities of the movable log pair $(X, \varepsilon M_X)$ are also terminal for every sufficiently small rational number $\varepsilon > 1$.

Remark 3. Let (X, M_X) be a movable log pair and let Z be a proper irreducible subvariety of X such that X is smooth at a general point of Z. Then elementary properties of blow-ups of smooth varieties at smooth subvarieties ensure that

$$Z \in \mathbb{CS}(X, M_X) \Rightarrow \mathrm{mult}_Z(M_X) \geqslant 1,$$

and $\mathrm{mult}_Z(M_X) \geqslant 1 \Rightarrow Z \in \mathbb{CS}(X, M_X)$ in the case when $\mathrm{codim}(Z \subset X) = 2$.

Remark 4. Let (X, M_X) be a movable log pair, H a general hyperplane section of X and Z a proper irreducible subvariety of X such that $\dim(Z) \geqslant 1$ and $Z \in \mathbb{CS}(X, M_X)$. Then every component of $Z \cap H$ is contained in the set $\mathbb{CS}(H, M_X|_H)$.

We mainly use movable log pairs whose boundaries consist of a single linear system without fixed components (see [32]). However, we sometimes need to consider more complicated movable log pairs. Let us illustrate this by proving the following result [7] obtained in [58].

Theorem 8. *Let $\psi\colon X \to \mathrm{Spec}(\mathcal{O})$ and $\varphi\colon V \to \mathrm{Spec}(\mathcal{O})$ be fibrations into del Pezzo surfaces of degree d, where $\mathcal{O}$ is a discrete valuation ring. Let $\rho\colon X \dashrightarrow V$ be a birational map inducing an isomorphism of general fibres of ψ and φ such that the diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, "\psi"'] & V \arrow[d, "\varphi"] \\ \mathrm{Spec}(\mathcal{O}) \arrow[r, "\mathrm{id}"] & \mathrm{Spec}(\mathcal{O}) \end{tikzcd}$$

[7] Theorem 8 also holds for fibrations into del Pezzo surfaces of degree 1, but the proof must be slightly modified (see [58]). The arguments in the proof of Theorem 8 can be applied in much more general situations (see [17]). For example, it is easy to prove a result similar to Theorem 8 for fibrations into smooth hypersurfaces in $\mathbb{P}^n$ of degree $n \geqslant 4$ using the results of [14], [17]. However, the claim of Theorem 8 holds under much weaker and more natural assumptions (see [1], [2]). It seems that our proof does not reflect the geometrical meaning of Theorem 8 (see [8]).

commutes. If $1 \neq d \leqslant 4$ *and the scheme-theoretic fibres of* ψ *and* φ *over the closed point of* $\operatorname{Spec}(\mathcal{O})$ *are smooth, then* ρ *is an isomorphism.*

Proof. Put $\Lambda = |-K_X|$, $\Gamma = |-K_V|$, $\bar{\Lambda} = \rho(\Lambda)$, $\overline{\Gamma} = \rho^{-1}(\Gamma)$ and

$$M_X = \frac{1+\varepsilon}{2}\,\Lambda + \frac{1+\varepsilon}{2}\,\overline{\Gamma}, \qquad M_V = \frac{1+\varepsilon}{2}\,\bar{\Lambda} + \frac{1+\varepsilon}{2}\,\Gamma,$$

where ε is a sufficiently small positive rational number. Then the movable log pairs (X, M_X) and (V, M_V) are birationally equivalent while the divisors $K_X + M_X$ and $K_V + M_V$ are ample. Therefore Lemma 36 of [16] shows that ρ is an isomorphism provided that the singularities of the log pairs (X, M_X) and (V, M_V) are canonical.

To complete the proof, it suffices to assume that the singularities of (X, M_X) are not canonical and show that this leads to a contradiction.

The singularities of the log pair $\bigl(X, \frac{1+\varepsilon}{2}\,\overline{\Gamma}\bigr)$ are not canonical because we can disregard the linear system Λ. Indeed, Λ has no base points because ψ is a fibration into del Pezzo surfaces of degree $d > 1$.

Let $\overline{X}$ be the scheme-theoretic fibre of ψ over the closed point of $\operatorname{Spec}(\mathcal{O})$. Then the singularities of the log pair $\bigl(\overline{X}, \frac{1+\varepsilon}{2}\,\overline{\Gamma}|_{\overline{X}}\bigr)$ are not log canonical by Theorem 17.6 in [53] or by Theorem 13 below. In particular, the singularities of the log pair $\bigl(\overline{X}, \frac{1}{2}\,\overline{\Gamma}|_{\overline{X}}\bigr)$ are not log terminal because we can choose $\varepsilon > 0$ to be as small as we wish. On the other hand, elementary properties of del Pezzo surfaces of degree $d \leqslant 4$ imply that the singularities of the log pair $\bigl(\overline{X}, \frac{1}{2}\,S\bigr)$ are always log terminal for every curve S of the linear system $|-K_{\overline{X}}|$ (see [58]).

For the rest of the section we do not impose any restrictions on the coefficients or components of the boundary. In particular, the boundaries need not be effective or movable. However, we assume that all log canonical divisors are $\mathbb{Q}$-Cartier divisors.

Definition 3. Consider an arbitrary log pair (X, B_X) and a birational morphism $f\colon V \to X$. A log pair (V, B^V) is called the *log pullback of the log pair* (X, B_X) if

$$B^V = f^{-1}(B_X) - \sum_{i=1}^{n} a(X, B_X, E_i)E_i, \qquad K_V + B^V \sim_{\mathbb{Q}} f^*(K_X + B_X),$$

where $a(X, B_X, E_i) \in \mathbb{Q}$ and the E_i are exceptional divisors of f.

Definition 4. Let (X, B_X) be an arbitrary log pair. A proper irreducible subvariety $Y \subset X$ is called a *centre of log canonical singularities of* (X, B_X) if one can find a birational morphism $f\colon W \to X$ and a divisor $E \subset W$ (not necessarily f-exceptional) such that E is contained in the support of the effective part of the divisor $\lfloor B^Y \rfloor$.

Definition 5. We denote by $\mathbb{LCS}(X, B_X)$ the set of all centres of log canonical singularities of the log pair (X, B_X) and by $\operatorname{LCS}(X, B_X)$ the union of all centres of its log canonical singularities. The set $\operatorname{LCS}(X, B_X)$ is regarded as a proper subset of X and is usually called the *locus of log canonical singularities.*

Remark 5. Let (X, B_X) be a log pair, H a sufficiently general hyperplane section of X and $Z \subset X$ a proper irreducible subvariety such that $\dim(Z) \geqslant 1$ and $Z \in \mathbb{LCS}(X, B_X)$. Then every irreducible component of $Z \cap H$ is contained in the set $\mathbb{LCS}(H, B_X|_H)$.

We consider a log pair (X, B_X), where $B_X = \sum_{i=1}^{k} a_i B_i$, $a_i \in \mathbb{Q}$, and the B_i are either effective, irreducible and reduced divisors on X, or linear systems on X without fixed components. We say that the boundary B_X is *effective* if each of the a_i is non-negative, and we say that the boundary B_X is *movable* if every B_i is a linear system without fixed components.

Example 2. Let O be a smooth point on X belonging to the set $\mathbb{LCS}(X, B_X)$, $f\colon V \to X$ the blow-up of O and E the exceptional divisor of f. Then either $E \in \mathbb{LCS}(V, B^V)$ or there is a proper subvariety $Z \subset E$ belonging to the set $\mathbb{LCS}(V, B^V)$. The exceptional divisor E belongs to $\mathbb{LCS}(V, B^V)$ if and only if the inequality $\mathrm{mult}_O(B_X) \geqslant \dim(X)$ holds.

Suppose that $f\colon Y \to X$ is a birational morphism, Y is smooth and the union of the proper transforms of all the components of B_X and all f-exceptional divisors forms a divisor with simple normal crossings. Then the birational morphism f is called a *log resolution of the log pair* (X, B_X).

Definition 6. The subscheme associated with the ideal sheaf

$$\mathcal{I}(X, B_X) = f_*(\lceil -B^Y \rceil)$$

is called the *subscheme of log canonical singularities of the log pair* (X, B_X) and is denoted by $\mathcal{L}(X, B_X)$.

We have $\mathrm{Supp}(\mathcal{L}(X, B_X)) = \mathrm{LCS}(X, B_X) \subset X$.

The following result is the Shokurov vanishing theorem (see [18], [20] and [16], Theorem 42).

Theorem 9. *Suppose that B_X is effective. Let H be any numerically effective big divisor on X such that the divisor $K_X + B_X + H$ is numerically equivalent to a Cartier divisor D. Then $H^i(X, \mathcal{I}(X, B_X) \otimes D) = 0$ for every $i > 0$.*

We now consider two simple applications of Theorem 9, which are special cases of a much more general result (see [14], [17], [38]).

Lemma 1. *Suppose that $V = \mathbb{P}^1 \times \mathbb{P}^1$ and B_V is an effective boundary on V of bidegree (a, b), where $a, b \in \mathbb{Q} \cap [0, 1)$. Then $\mathbb{LCS}(V, B_V) = \varnothing$.*

Proof. Write $B_V = \sum_{i=1}^{k} a_i B_i$, where the a_i are positive rational numbers and the B_i are irreducible reduced curves. Then $a_i \leqslant \max(a, b) < 1$ for every i because the intersections of B_V with the fibres of the two projections to $\mathbb{P}^1$ are equal to a and b respectively. In particular, the set $\mathbb{LCS}(V, B_V)$ does not contain curves.

Suppose that the set $\mathbb{LCS}(V, B_V)$ contains a point $O \in V$. Let H be an arbitrary ample $\mathbb{Q}$-divisor on V of bidegree $(1 - a, 1 - b)$. Then there is a Cartier divisor D on V such that $D \sim_{\mathbb{Q}} K_V + B_V + H$ and $H^0(\mathcal{O}_V(D)) = 0$. However, it follows from Theorem 9 that the natural map

$$H^0(\mathcal{O}_V(D)) \to H^0(\mathcal{O}_{\mathcal{L}(V, B_V)}(D)) \to 0$$

is surjective, but $H^0(\mathcal{O}_{\mathcal{L}(V, B_V)}(D)) = H^0(\mathcal{O}_{\mathcal{L}(V, B_V)}) \neq 0$. This is a contradiction.

Lemma 2. *Suppose that $V \subset \mathbb{P}^n$ is a smooth hypersurface, B_V is an effective boundary on V and $B_V \equiv rH$ for $r \in \mathbb{Q} \cap [0,1)$, where $H \in |\mathcal{O}_{\mathbb{P}^n}(1)|_V|$. If $\deg(V) < n$, then the singularities of the log pair (V, B_V) are log terminal.*

Proof. Put $k = \deg(V)$. Suppose that the set $\mathbb{LCS}(V, B_V)$ contains some subvariety $Z \subset V$, but $k < n$. Then Theorem 2 of [7] yields that $\dim(Z) = 0$. Hence the set $\mathbb{LCS}(V, B_V)$ contains at most closed points of V. It follows that the support of the scheme of log canonical singularities $\mathcal{L}(V, B_V)$ is zero-dimensional and $H^0(\mathcal{O}_{\mathcal{L}(V,\, B_V)}) \neq 0$.

We have

$$K_V + B_V + (1-r)H \equiv (k-n)H$$

and $H^0(\mathcal{O}_V((k-n)H)) = 0$ because $k < n$. However the sequence

$$H^0(\mathcal{O}_V((k-n)H)) \to H^0(\mathcal{O}_{\mathcal{L}(V,\, B_V)}((k-n)H)) \to 0$$

is exact by Theorem 9, but $H^0(\mathcal{O}_{\mathcal{L}(V,\, B_V)}((k-n)H)) = H^0(\mathcal{O}_{\mathcal{L}(V,\, B_V)}) \neq 0$. This is a contradiction.

We can apply the idea behind the proofs of Lemmas 1 and 2 to obtain much more general results. Namely, given an arbitrary Cartier divisor D on X, we consider the exact sequence of sheaves

$$0 \to \mathcal{I}(X, B_X) \otimes D \to \mathcal{O}_X(D) \to \mathcal{O}_{\mathcal{L}(X,\, B_X)}(D) \to 0$$

and the corresponding exact sequence of cohomology groups

$$H^0(\mathcal{O}_X(D)) \to H^0(\mathcal{O}_{\mathcal{L}(X,\, B_X)}(D)) \to H^1(\mathcal{I}(X, B_X) \otimes D).$$

Applying Theorem 9, we get the following connectedness theorems (see [18]).

Theorem 10. *Suppose that B_X is effective and the divisor $-(K_X + B_X)$ is numerically effective and big. Then the set $\mathrm{LCS}(X, B_X)$ is connected.*

Theorem 11. *Suppose that B_X is effective and the divisor $-(K_X + B_X)$ is numerically effective and big with respect to some morphism $g : X \to Z$ with connected fibres. Then the set $\mathrm{LCS}(X, B_X)$ is connected in the neighbourhood of every fibre of g.*

The proof of Theorem 9 can also be used to obtain the following result, which is Theorem 17.4 in [53].

Theorem 12. *Suppose that the divisor $-(K_X + B_X)$ is numerically effective and big with respect to some morphism $g\colon X \to Z$ with connected fibres, and the inequality $\mathrm{codim}(g(B_i) \subset Z) \geqslant 2$ holds whenever $b_i < 0$. Then $\mathrm{LCS}(Y, B^Y)$ is connected in the neighbourhood of every fibre of the morphism $g \circ f\colon Y \to Z$.*

We have defined a centre of canonical singularities and the set of all centres of canonical singularities for movable log pairs (see Definition 2). Similar definitions also work for ordinary log pairs.

Theorem 13. *Suppose that the boundary B_X is effective and Z is an element of $\mathbb{CS}(X, B_X)$ contained in the support of an effective irreducible Cartier divisor H on X such that H is not a component of B_X and H is smooth at a general point of the subvariety Z. Then $\mathbb{LCS}(H, B_X|_H) \neq \varnothing$.*

Proof. Consider the log pair $(X, B_X + H)$. Then

$$\{Z, H\} \subset \mathbb{LCS}(X, B_X + H).$$

Let $f\colon W \to X$ be a log resolution of the log pair $(X, B_X + H)$. Then

$$K_W + \widehat{H} \sim_{\mathbb{Q}} f^*(K_X + B_X + H) + \sum_{E \neq \widehat{H}} a(X, B_X + H, E)E,$$

where $\widehat{H} = f^{-1}(H)$. Applying Theorem 12 to the log pullback of the log pair $(X, B_X + H)$ on the variety W, we see that there is an exceptional divisor E of the morphism f such that $f(E) = Z$, $a(X, B_X, E) \leqslant -1$ and $\widehat{H} \cap E \neq \varnothing$. Hence the equivalence

$$K_{\widehat{H}} \sim (K_W + \widehat{H})|_{\widehat{H}} \sim_{\mathbb{Q}} f|_{\widehat{H}}^*(K_H + B_X|_H) + \sum_{E \neq \widehat{H}} a(X, B_X + H, E)E|_{\widehat{H}}$$

completes the proof.

The following result can be proved in the same way as Theorem 13.

Corollary 2. *Suppose that the boundary B_X is movable and effective and the singularities of the log pair (X, B_X) are log terminal in a punctured neighbourhood of a point O of X such that X has an isolated hypersurface singularity at O and $O \in \mathbb{CS}(X, M_X)$. Then O is contained in the set $\mathbb{LCS}(S, B_S)$, where $S = \bigcap_{i=1}^{k} H_i$, $B_S = B_X|_S$ and H_i is a general hyperplane section of X through O.*

The following result is Theorem 3.1 in [32], which gives the simplest known proof of the $4n^2$-inequality (see [59]). This proof uses Theorem 13.

Theorem 14. *Let H be a surface, O a smooth point of H, M_H an effective movable boundary on H and Δ_1, Δ_2 irreducible reduced curves on H that intersect each other normally at O. Suppose that*

$$O \in \mathbb{LCS}(H, (1 - a_1)\Delta_1 + (1 - a_2)\Delta_2 + M_H)$$

for some non-negative rational numbers a_1 and a_2. Then

$$\operatorname{mult}_O(M_H^2) \geqslant \begin{cases} 4a_1a_2 & \text{if } a_1 \leqslant 1 \text{ or } a_2 \leqslant 1, \\ 4(a_1 + a_2 - 1) & \text{if } a_1 > 1 \text{ and } a_2 > 1. \end{cases}$$

Theorem 14 can easily be proved by induction on the number of blow-ups that must be done in order to obtain the corresponding negative discrepancy.

Proof of Theorem 2. Let O be an isolated ordinary double point of the variety X and B_X an effective boundary on X such that B_X is a $\mathbb{Q}$-Cartier divisor and $O \in \mathbb{CS}(X, B_X)$. We must show that $\operatorname{mult}_O(B_X) \geqslant 1$ if $\dim(X) \geqslant 3$, where $\operatorname{mult}_O(B_X)$ is a rational number such that

$$B_W \sim_{\mathbb{Q}} f^*(B_X) - \operatorname{mult}_O(B_X)E.$$

Here $f\colon W \to X$ is the blow-up of O, $B_W = f^{-1}(B_X)$ and E is the exceptional divisor of the birational morphism f. By Corollary 2, we may assume that X is a threefold.

Suppose that $\mathrm{mult}_O(B_X) < 1$. Then the equivalence

$$K_W + B_W \sim_{\mathbb{Q}} f^*(K_X + B_X) + (1 - \mathrm{mult}_O(B_X))E$$

yields the existence of a subvariety $Z \subset E$ that is a centre of canonical singularities of the log pair (W, B_W). Therefore the set $\mathbb{LCS}(E, B_W|_E)$ is non-empty by Theorem 13. This contradicts Lemma 1 because $E \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 2 can be generalized in two ways.

Proposition 1. *Suppose that* $\dim(X) = 3$, B_X *is an effective boundary, the set* $\mathbb{CS}(X, B_X)$ *contains an isolated double point* O *of* X *and* X *is locally isomorphic to the hypersurface*

$$y^3 = \sum_{i=1}^{3} x_i^2 \subset \mathbb{C}^4 \cong \mathrm{Spec}(\mathbb{C}[x_1, x_2, x_3, y])$$

in a neighbourhood of O*. Then* $\mathrm{mult}_O(B_X) \geqslant \frac{1}{2}$, *where* $\mathrm{mult}_O(B_X)$ *is a rational number defined by the equivalence* $B_W \sim_{\mathbb{Q}} f^*(B_X) - \mathrm{mult}_O(B_X)E$*. Here* $f\colon W \to X$ *is the blow-up of* O, $B_W = f^{-1}(B_X)$ *and* E *is the exceptional divisor of* f*.*

Proof. The variety W is smooth and the exceptional divisor E is a cone in $\mathbb{P}^3$ over a smooth conic. Moreover, the restriction $-E|_E$ is rationally equivalent to a hyperplane section of the cone $E \subset \mathbb{P}^3$. Hence we have the equivalence

$$K_W + B_W \sim_{\mathbb{Q}} f^*(K_X + B_X) + (1 - \mathrm{mult}_O(B_X))E.$$

Suppose that $\mathrm{mult}_O(B_X) < \frac{1}{2}$. Then

$$\mathbb{CS}(W, B_W) \subset \mathbb{CS}(W, B_W + (\mathrm{mult}_O(B_X) - 1)E)$$

because $\mathrm{mult}_O(B_X) - 1 < 0$. On the other hand, the log pair

$$(W, B_W + (\mathrm{mult}_O(B_X) - 1)E)$$

is a log pullback of the log pair (X, B_X), but $O \in \mathbb{CS}(X, B_X)$. Hence there is an irreducible proper subvariety $Z \subset E$ such that $Z \in \mathbb{CS}(W, B_W)$. It follows from Theorem 13 that the set $\mathbb{LCS}(E, B_W|_E)$ is non-empty.

Let $B_E = B_W|_E$. Then the set $\mathbb{LCS}(E, B_E)$ contains no curves on E since otherwise the intersection of B_E with a ruling of the cone E would be strictly bigger than $\frac{1}{2}$, which is impossible because $\mathrm{mult}_O(B_X) < \frac{1}{2}$. Hence we see that $\dim(\mathrm{Supp}(\mathcal{L}(E, B_E))) = 0$.

Let H be a hyperplane section of the cone $E \subset \mathbb{P}^3$. Then

$$K_E + B_E + (1 - \mathrm{mult}_O(B_X))H \sim_{\mathbb{Q}} -H$$

and $H^0(\mathcal{O}_E(-H)) = 0$. The sequence of cohomology groups

$$H^0(\mathcal{O}_E(-H)) \to H^0(\mathcal{O}_{\mathcal{L}(E, B_E)}) \to H^1(E, \mathcal{I}(E, B_E) \otimes \mathcal{O}_E(-H))$$

is exact and $H^1(E, \mathcal{I}(E, B_E) \otimes \mathcal{O}_E(-H)) = 0$ by Theorem 9. Hence we have $H^0(\mathcal{O}_{\mathcal{L}(E,\, B_E)}) = 0$, which is impossible because $\mathbb{LCS}(E, B_E) \neq \varnothing$.

Using Corollary 2 and Theorem 2, we get the following result.

Proposition 2. *Suppose that B_X is an effective boundary consisting of a $\mathbb{Q}$-Cartier divisor and $\dim(X) \geqslant 4$. Let O be an isolated double point on X such that X is locally isomorphic in a neighbourhood of O to the hypersurface*

$$y^3 = \sum_{i=1}^{\dim(X)} x_i^2 \subset \mathbb{C}^{\dim(X)+1} \cong \operatorname{Spec}(\mathbb{C}[x_1, \dots, x_{\dim(X)}, y])$$

and $O \in \mathbb{CS}(X, B_X)$. Then $\operatorname{mult}_O(B_X) > 1$, where $\operatorname{mult}_O(B_X)$ is a rational number given by the equivalence $B_W \sim_{\mathbb{Q}} f^*(B_X) - \operatorname{mult}_O(B_X)E$. Here $f\colon W \to X$ is the blow-up of O, $B_W = f^{-1}(B_X)$ and E is the exceptional divisor of f.

The following generalization of Theorem 2 of [7] is Lemma 3.18 in [15].

Proposition 3. *Let V be a smooth complete intersection $\bigcap_{i=1}^k G_i \subset \mathbb{P}^m$ of hypersurfaces G_i with $m - k > 2$, D an effective divisor on V and S an irreducible subvariety of V such that $\dim(S) \geqslant k$ and $\operatorname{codim}(S \subset V) \geqslant 2$. Then we have $\operatorname{mult}_S(D) \leqslant n$, where $n \in \mathbb{N}$ is such that $D \sim \mathcal{O}_{\mathbb{P}^m}(n)|_V$.*

Corollary 3. *Suppose that (V, B_V) is a log pair, B_V is effective, V is a smooth complete intersection $\bigcap_{i=1}^k G_i \subset \mathbb{P}^m$ of hypersurfaces G_i with $m-k > 2$ and $B_V \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}^m}(1)|_V$. Then the set $\mathbb{CS}(V, B_V)$ contains no subvarieties of V of dimension greater than or equal to k.*

§ 3. The $8n^2$-inequality

In this section we prove Theorem 3. Let X be a variety with $\dim(X) \geqslant 3$ and $B_X = \sum_{i=1}^n a_i \mathcal{B}_i$ a movable boundary on X, where the $\mathcal{B}_i$ are linear systems on X without fixed components and the a_i are non-negative rational numbers. Then we have the following result, which is Corollary 3.5 in [32].

Proposition 4. *Let O be a smooth point of X, $f\colon V \to X$ the blow-up of O, E the exceptional divisor of f and $B_V = f^{-1}(B_X)$. Suppose that $\dim(X) = 3$ and $O \in \mathbb{CS}(X, B_X)$, but $\operatorname{mult}_O(B_X) < 2$. Then there is a line $L \subset E \cong \mathbb{P}^2$ such that $L \in \mathbb{LCS}(V, B_V + (\operatorname{mult}_O(B_X) - 1)E)$.*

Proof. Since the assertion is local with respect to X, we may assume that $X \cong \mathbb{C}^3$ and O is the origin in $\mathbb{C}^3$.

Let H be a sufficiently general hyperplane section of X through the point O. Put $T = f^{-1}(H)$. Then

$$K_V + B_V + (\operatorname{mult}_O(B_X) - 1)E + T \sim_{\mathbb{Q}} f^*(K_X + B_X + H)$$

and $\operatorname{mult}_O(B_X) \geqslant 1$ (see Remark 3).

Since the hyperplane section H is general, it follows from Theorem 13 that $O \in \mathbb{LCS}(H, B_X|_H)$. Moreover, we have

$$\mathbb{LCS}(T, B_V|_T + (\operatorname{mult}_O(B_X) - 1)E|_T) \neq \varnothing$$

because $\mathrm{mult}_O(B_X) < 2$ by hypothesis (see Example 2). Applying Theorem 11 to the morphism f, we see that the set

$$\mathbb{LCS}(T, B_V|_T + (\mathrm{mult}_O(B_X) - 1)E|_T)$$

consists of a single point $P \in E \cap T$. On the other hand, since H is general, P is the intersection of T and some element of $\mathbb{LCS}(V, B_V + (\mathrm{mult}_O(B_X) - 1)E)$. Hence the set

$$\mathbb{LCS}(T, B_V + (\mathrm{mult}_O(B_X) - 1)E)$$

contains a curve $L \subset E$ such that $L \cap T$ consists of the point P. Therefore the curve L is a line in $E \cong \mathbb{P}^2$. This completes the proof.

Suppose that $\dim(X) = 4$ and there is a smooth point O of X such that $O \in \mathbb{CS}(X, B_X)$, but $\mathrm{mult}_O(B_X) < 3$. Suppose that the log pair (X, B_X) has canonical singularities outside O. Let $f\colon V \to X$ be the blow-up of O, E the exceptional divisor of f and $B_V = f^{-1}(B_X)$. Then we have the following result, which may be regarded as a generalization of Proposition 4.

Proposition 5. *One of the following claims holds*:
1) *there is a surface $S \subset E$ such that*

$$S \in \mathbb{LCS}(V, B_V + (\mathrm{mult}_O(B_X) - 2)E);$$

2) *there is line $L \subset E \cong \mathbb{P}^3$ such that*

$$L \in \mathbb{LCS}(V, B_V + (\mathrm{mult}_O(B_X) - 2)E).$$

Proof. This follows from Theorem 11 and the proof of Proposition 4.

Suppose that the set $\mathbb{LCS}(V, B_V + (\mathrm{mult}_O(B_X) - 2)E)$ does not contain surfaces that are contained in the divisor E, but does contain a line $L \subset E \cong \mathbb{P}^3$. Let $g\colon W \to V$ be the blow-up of L. Put

$$F = g^{-1}(L), \qquad \overline{E} = g^{-1}(E), \qquad B_W = g^{-1}(B_V).$$

Then it follows from Definition 3 that

$$B^W = B_W + (\mathrm{mult}_O(B_X) - 3)\,\overline{E} + (\mathrm{mult}_O(B_X) + \mathrm{mult}_L(B_V) - 5)F.$$

Proposition 6. *One of the following claims holds*:
1) *the divisor F belongs to the set $\mathbb{LCS}(W, B^W + \overline{E} + 2F)$*;
2) *there is a surface $Z \subset F$ such that $g(Z) = L$ and*

$$F \in \mathbb{LCS}(W, B^W + \overline{E} + 2F).$$

We now show that Theorem 3 in dimension 4 follows from Proposition 6.

Theorem 15. *Let Y be a fourfold, $\mathcal{M}$ a linear system on Y without fixed components, S_1 and S_2 general divisors of $\mathcal{M}$ and P a smooth point of Y. Suppose that $P \in \mathbb{CS}(Y, \frac{1}{n}\mathcal{M})$ for some $n \in \mathbb{N}$, but the singularities of the log pair $(Y, \frac{1}{n}\mathcal{M})$*

are canonical outside P. Let $\pi\colon \widehat{Y} \to Y$ be the blow-up of P and Π *the exceptional divisor of π. Then there is a line $C \subset \Pi \cong \mathbb{P}^3$ such that*

$$\mathrm{mult}_P(S_1 \cdot S_2 \cdot \Delta) \geqslant 8n^2$$

for every effective divisor Δ on the variety Y such that

1) *Δ contains the point P and is smooth at P;*
2) *the line $C \subset \Pi \cong \mathbb{P}^3$ is contained in the divisor $\pi^{-1}(\Delta)$;*
3) *Δ contains no subvarieties of Y of dimension* 2 *that are contained in the base locus of $\mathcal{M}$.*

Proof. Let Δ be an effective divisor on Y such that $P \in \Delta$, Δ is smooth at P and Δ contains no surfaces that are contained in the base locus of $\mathcal{M}$. Then the linear system $\mathcal{M}|_\Delta$ has no fixed components and $S_1 \cdot S_2 \cdot \Delta$ is an effective 1-cycle.

Put $\overline{S}_1 = S_1|_\Delta$ and $\overline{S}_2 = S_2|_\Delta$. We must show that the inequality

$$\mathrm{mult}_P(\overline{S}_1 \cdot \overline{S}_2) \geqslant 8n^2 \tag{1}$$

holds, possibly under some additional hypotheses on Δ.

Put $\overline{\mathcal{M}} = \mathcal{M}|_\Delta$. Then $P \in \mathbb{LCS}\big(\Delta, \frac{1}{n}\overline{\mathcal{M}}\big)$ by Theorem 13.

Let $\bar{\pi}\colon \hat{\Delta} \to \Delta$ be a blow-up of the point P and $\overline{\Pi} = \bar{\pi}^{-1}(P)$. Then the diagram

$$\begin{tikzcd} \hat{\Delta} \arrow[r, hook] \arrow[d, "\bar{\pi}"'] & \widehat{Y} \arrow[d, "\pi"] \\ \Delta \arrow[r, hook] & Y \end{tikzcd} \tag{2}$$

commutes, and we can identify $\hat{\Delta}$ with $\pi^{-1}(\Delta)$, which implies that $\overline{\Pi} = \Pi \cap \hat{\Delta}$.

The inequality (1) is trivial in the case when $\mathrm{mult}_P(\overline{\mathcal{M}}) \geqslant 3n$. Hence we assume that $\mathrm{mult}_P(\overline{\mathcal{M}}) < 3n$.

Put $\widehat{\mathcal{M}} = \bar{\pi}^{-1}(\overline{\mathcal{M}})$. Then

$$\overline{\Pi} \notin \mathbb{LCS}\bigg(\hat{\Delta}, \frac{1}{n}\widehat{\mathcal{M}} + \bigg(\frac{1}{n}\mathrm{mult}_P(\overline{\mathcal{M}}) - 2\bigg)\overline{\Pi}\bigg).$$

Hence there is a proper subvariety $\Xi \subset \overline{\Pi} \cong \mathbb{P}^2$ such that

$$\Xi \in \mathbb{LCS}\bigg(\hat{\Delta}, \frac{1}{n}\widehat{\mathcal{M}} + \bigg(\frac{1}{n}\mathrm{mult}_P(\overline{\mathcal{M}}) - 2\bigg)\overline{\Pi}\bigg)$$

(see Example 2).

Suppose that Ξ is curve. Put $\widehat{S}_i = \bar{\pi}^{-1}(S_i)$. Then it follows from simple properties of multiplicities (see [59], Lemma 6.5) that

$$\mathrm{mult}_P(\overline{S}_1 \cdot \overline{S}_2) \geqslant \mathrm{mult}_P(\overline{\mathcal{M}})^2 + \mathrm{mult}_\Xi(\widehat{S}_1 \cdot \widehat{S}_2).$$

Applying Theorem 14 to the log pair $\big(\hat{\Delta}, \frac{1}{n}\widehat{\mathcal{M}} + \big(\frac{1}{n}\mathrm{mult}_P(\overline{\mathcal{M}}) - 2\big)\overline{\Pi}\big)$ at a general point of Ξ, we get

$$\mathrm{mult}_\Xi(\widehat{S}_1 \cdot \widehat{S}_2) \geqslant 4\big(3n^2 - n\,\mathrm{mult}_P(\overline{\mathcal{M}})\big).$$

It follows that

$$\mathrm{mult}_P(\overline{S}_1\cdot\overline{S}_2)\geqslant \mathrm{mult}_P(\overline{\mathcal{M}})^2+4\big(3n^2-n\,\mathrm{mult}_P(\overline{\mathcal{M}})\big)\geqslant 8n^2,$$

which completes the proof in this case.

Now suppose that Ξ is a point. Then Proposition 5 yields a line $C\subset\Pi\cong\mathbb{P}^3$ such that $\Xi=C\cap\hat{\Delta}$ and

$$C\in\mathbb{LCS}\bigg(\widehat{Y},\frac{1}{n}\pi^{-1}(\mathcal{M})+\bigg(\frac{1}{n}\mathrm{mult}_P(\mathcal{M})-2\bigg)\Pi\bigg).$$

The line C does not depend on the choice of Δ. Proposition 5 implies that the line $C\subset\Pi$ depends only on properties of the log pair $\big(Y,\frac{1}{n}\mathcal{M}\big)$.

Suppose that the divisor Δ satisfies the additional assumption $C\subset\pi^{-1}(\Delta)$. Then we can repeat all previous steps of the proof under the new hypotheses. This also enables us to clarify the geometrical meaning of Proposition 6. Namely, by Proposition 6, the condition $C\subset\hat{\Delta}=\pi^{-1}(\Delta)$ implies that

$$C\in\mathbb{LCS}\bigg(\hat{\Delta},\frac{1}{n}\widehat{\mathcal{M}}+\bigg(\frac{1}{n}\mathrm{mult}_P(\overline{\mathcal{M}})-2\bigg)\overline{\Pi}\bigg)$$

provided that the set $\mathbb{LCS}\big(\hat{\Delta},\frac{1}{n}\widehat{\mathcal{M}}+\big(\frac{1}{n}\mathrm{mult}_P(\overline{\mathcal{M}})-2\big)\overline{\Pi}\big)$ contains no other curves that are contained in the divisor $\overline{\Pi}$. To see this, it suffices to blow-up the divisor $\hat{\Delta}$ at the curve C in the commutative diagram (2) and use Remark 5. Now we can repeat the previous arguments for a divisor Δ with $C\subset\hat{\Delta}$ and obtain (1).

We now prove Proposition 6. Since the assertion is local with respect to X, we may assume that $X\cong\mathbb{C}^4$. Let H be a general hyperplane section of X such that H contains the point O and $L\subset f^{-1}(H)$. Put $T=f^{-1}(H)$, $S=g^{-1}(T)$. Then

$$K_W+B^W+\overline{E}+2F+S\sim_{\mathbb{Q}}(f\circ g)^*(K_X+B_X+H),$$
$$B^W+\overline{E}+2F=B_W+(\mathrm{mult}_O(B_X)-2)\overline{E}+(\mathrm{mult}_O(B_X)+\mathrm{mult}_L(B_V)-3)F,$$

which implies that

$$F\in\mathbb{LCS}(W,B^W+\overline{E}+2F)\iff \mathrm{mult}_O(B_X)+\mathrm{mult}_L(B_V)\geqslant 4$$

by Definition 4. In particular, we may assume that $\mathrm{mult}_O(B_X)+\mathrm{mult}_L(B_V)<4$. Hence we must prove the existence of a surface $Z\subset F$ such that $g(Z)=L$ and

$$F\in\mathbb{LCS}(W,B^W+\overline{E}+2F).$$

Let $\overline{H}$ be a general hyperplane section of X such that $\overline{H}$ contains O and $L\not\subset f^{-1}(\overline{H})$. Put $\overline{T}=f^{-1}(\overline{H})$, $\overline{S}=g^{-1}(\overline{T})$. Then

$$O\in\mathbb{LCS}\big(\overline{H},B_X|_{\overline{H}}\big)$$

by Theorem 13. We have

$$K_W+B^W+\overline{E}+F+\overline{S}\sim_{\mathbb{Q}}(f\circ g)^*(K_X+B_X+H),$$

which implies that $\mathbb{LCS}(\overline{S},(B^W+\overline{E}+F)|_{\overline{S}})\neq\varnothing$.

Applying Theorem 12 to the morphism $f\circ g\colon \overline{S}\to\overline{H}$, we see that one of the following claims holds:

1) the set $\mathbb{LCS}(\overline{S},(B^W+\overline{E}+F)|_{\overline{S}})$ consists of a single point, which is contained in the fibre of the morphism $g\colon F\to L$ over the point $\overline{T}\cap L$;

2) the set $\mathbb{LCS}(\overline{S},(B^W+\overline{E}+F)|_{\overline{S}})$ contains a curve that is contained in the fibre of the morphism $g\colon F\to L$ over the point $\overline{T}\cap L$.

Remark 6. Since the divisor $\overline{H}$ is general, it follows that every element of the set $\mathbb{LCS}(\overline{S},(B^W+\overline{E}+F)|_{\overline{S}})$ contained in the fibre of the $\mathbb{P}^2$-fibration g over the point $\overline{T}\cap L$ is the intersection of $\overline{S}$ and an element of the set $\mathbb{LCS}(W,B^W+\overline{E}+F)$.

The generality of the choice of $\overline{H}$ thus shows that either $\mathbb{LCS}(W,B^W+\overline{E}+F)$ contains a surface that is contained in F and dominates the curve L, or the only centre of log canonical singularities of the log pair $(W,B^W+\overline{E}+F)$ that is contained in the exceptional divisor F and dominates the curve L is a section of the $\mathbb{P}^2$-fibration $g\colon F\to L$. On the other hand, it is clear that

$$\mathbb{LCS}(W,B^W+\overline{E}+F)\subseteq\mathbb{LCS}(W,B^W+\overline{E}+2F).$$

Hence, to complete the proof of Proposition 6, we may assume that the divisor F contains a curve C such that the following conditions hold:

1) C is a section of the $\mathbb{P}^2$-fibration $g\colon F\to L$;

2) C is the unique element of the set $\mathbb{LCS}(W,B^W+\overline{E}+2F)$ that is contained in F and dominates the curve L;

3) C is the unique element of the set $\mathbb{LCS}(W,B^W+\overline{E}+F)$ that is contained in F and dominates the curve L.

We now return to the sufficiently general hyperplane section H of X such that H passes through O and $L\subset T$, where $T=f^{-1}(H)$.

The point O is a centre of log canonical singularities of the log pair $(H,M_X|_H)$ by Theorem 13, and our assumptions imply that

$$\mathbb{LCS}\bigl(S,(B^W+\overline{E}+2F)|_S\bigr)\neq\varnothing,$$

where $S=g^{-1}(T)$. Applying Theorem 12 to the log pair

$$\bigl(S,(B^W+\overline{E}+2F)|_S\bigr)$$

and the morphism $f\circ g\colon S\to H$, we see that one of the following claims holds:

1) the set $\mathbb{LCS}(S,(B^W+\overline{E}+2F)|_S)$ consists of a single point;

2) the set $\mathbb{LCS}(S,(B^W+\overline{E}+2F)|_S)$ contains the curve C.

Corollary 4. *Either $C\subset S$ or the intersection $S\cap C$ consists of a single point.*

We have $L\cong C\cong\mathbb{P}^1$ and

$$F\cong\operatorname{Proj}(\mathcal{O}_L(-1)\oplus\mathcal{O}_L(1)\oplus\mathcal{O}_L(1)).$$

It follows that $S|_F\sim B+D$, where B is the tautological line bundle on F and D is a fibre of the natural projection $g|_F\colon F\to L\cong\mathbb{P}^1$.

Lemma 3. *The group $H^1(\mathcal{O}_W(S-F))$ vanishes.*

Proof. The divisor $-g^*(E) - F$ intersects non-negatively every curve that is contained in the divisor $\overline{E}$. On the other hand, we have

$$(-g^*(E) - F)|_F \sim B + D,$$

which implies that the divisor $-4g^*(E) - 4F$ is numerically effective and big with respect to the morphism $h = f \circ g$. However, we know that $X \cong \mathbb{C}^4$ and

$$K_W - 4g^*(E) - 4F = S - F,$$

which gives $H^1(\mathcal{O}_W(S - F)) = 0$ by the Kawamata–Viehweg vanishing theorem (see [48]).

Therefore, the natural restriction map

$$H^0(\mathcal{O}_W(S)) \to H^0(\mathcal{O}_F(S|_F))$$

is surjective, but the linear system $|S|_F|$ has no base points (see [61], § 2.8).

Corollary 5. *The curve C is not contained in the divisor S.*

Put $\tau = g|_F$. Let $\mathcal{I}_C$ be the ideal sheaf of the curve C on the scroll F. Then $R^1\,\tau_*(B \otimes \mathcal{I}_C) = 0$ and the map

$$\pi\colon \mathcal{O}_L(-1) \oplus \mathcal{O}_L(1) \oplus \mathcal{O}_L(1) \to \mathcal{O}_L(k)$$

is surjective, where $k = B \cdot C$. On the other hand, the map π is given by an element of the group

$$H^0(\mathcal{O}_L(k+1)) \oplus H^0(\mathcal{O}_L(k-1)) \oplus H^0(\mathcal{O}_L(k-1)),$$

which implies that $k \geqslant -1$.

Lemma 4. *The equality $k = 0$ is impossible.*

Proof. Suppose that $k = 0$. Then π is given by a matrix

$$(ax + by, 0, 0),$$

where a and b are complex numbers and $(x : y)$ are the homogeneous coordinates on $L \cong \mathbb{P}^1$. Hence the map π is not surjective over the point of L at which the linear form $ax + by$ vanishes.

Therefore, the divisor B has non-trivial intersection with the curve C. Hence the intersection of S and C either is trivial, or consists of more than one point. But we have already proved that $S \cap C$ consists of a single point. This contradiction proves Proposition 6.

Proof of Theorem 3. We consider only the case $\dim(Y) = 5$ because the proof for $\dim(Y) > 5$ is similar.

Let H_1, H_2, H_3 be sufficiently general hyperplane sections of Y through the point P. Put $\overline{Y} = \bigcap_{i=1}^{3} H_i$ and $\overline{\mathcal{M}} = \mathcal{M}|_{\overline{Y}}$. Then $\overline{Y}$ is a surface, $\overline{Y}$ is smooth at P and $P \in \mathbb{LCS}\big(\overline{Y}, \frac{1}{n}\overline{\mathcal{M}}\big)$ by Theorem 13.

Let $\pi\colon \widehat{Y} \to Y$ be the blow-up of P, Π the exceptional divisor of π and $\widehat{\mathcal{M}} = \pi^{-1}(\mathcal{M})$. Since $P \in \mathbb{LCS}\big(\overline{Y}, \frac{1}{n}\,\overline{\mathcal{M}}\big)$, it follows that the set

$$\mathbb{LCS}\bigg(\widehat{Y}, \frac{1}{n}\,\widehat{\mathcal{M}} + \bigg(\frac{1}{n}\,\mathrm{mult}_P(\mathcal{M}) - 2\bigg)\Pi\bigg)$$

contains a subvariety $Z \subset \Pi$ with $\dim(Z) \geqslant 2$.

The inequality $\dim(Y) \geqslant 5$ implies that

$$Z \in \mathbb{LCS}\bigg(\widehat{Y}, \frac{\varepsilon}{n}\,\widehat{\mathcal{M}} + \bigg(\frac{\varepsilon}{n}\,\mathrm{mult}_P(\mathcal{M}) - 2\bigg)\Pi\bigg)$$

for some positive rational number $\varepsilon < 1$, a standard observation in such problems. Indeed, we have use Theorem 13 twice during the proof of Theorem 15, but now we use Theorem 13 three times. Hence the first iterative usage of Theorem 13 shows that P is a centre of log canonical singularities of the log pair $\big(H_1, \frac{1}{n}\,\mathcal{M}|_{H_1}\big)$, and it follows that P is a centre of canonical singularities of the log pair $\big(H_1, \frac{\varepsilon}{n}\,\mathcal{M}|_{H_1}\big)$ for some positive rational number $\varepsilon < 1$. The second iterative usage of Theorem 13 yields that

$$P \in \mathbb{LCS}\bigg(H_1 \cap H_2, \frac{\varepsilon}{n}\,\mathcal{M}\bigg|_{H_1 \cap H_2}\bigg),$$

whence $Z \in \mathbb{LCS}\big(\widehat{Y}, \frac{\varepsilon}{n}\,\widehat{\mathcal{M}} + \big(\frac{\varepsilon}{n}\,\mathrm{mult}_P(\mathcal{M}) - 2\big)\Pi\big)$.

If $\dim(Z) = 4$, then the theorem becomes trivial because $\dim(Z) = 4$ means that $Z = \Pi$ and $\mathrm{mult}_P(\mathcal{M}) \geqslant 3n$.

If $\dim(Z) = 3$, then we can repeat all the arguments at the beginning of the proof of Theorem 15. This yields the inequality $\mathrm{mult}_P(S_1 \cdot S_2 \cdot \Delta) > 8n^2$ for every effective Cartier divisor Δ on Y such that Δ contains P, is smooth at P and contains no three-dimensional subvarieties of Y that are contained in the base locus of $\mathcal{M}$.

We note that if $\dim(Z) \geqslant 3$, then there is no need to find linear subspaces $\Lambda \subset \Pi$ of codimension 2 with $\Lambda \subset \pi^{-1}(\Delta)$ since this condition is vacuous for $\dim(Z) \geqslant 3$.

Now suppose that $\dim(Z) = 2$. Then Theorem 12 yields that Z is a linear subspace in $\Pi \cong \mathbb{P}^4$ of codimension 2. Moreover, Z depends only on properties of the movable log pair $\big(Y, \frac{1}{n}\,\mathcal{M}\big)$ and does not depend on choice of the divisors H_1, H_2, H_3.

Put $\Lambda = Z$. Let H be a general hyperplane section of Y through the point P and Δ an effective divisor on Y such that $P \in \Delta$, the divisor Δ is smooth at P, the divisor $\pi^{-1}(\Delta)$ contains Λ and Δ contains no three-dimensional subvarieties of Y that are contained in the base locus of $\mathcal{M}$. Then

$$\mathrm{mult}_P(S_1 \cdot S_2 \cdot \Delta) > 8n^2 \iff \mathrm{mult}_P(S_1|_H \cdot S_2|_H \cdot \Delta|_H) > 8n^2$$

by the generality in the choice of H. On the other hand, it follows from Theorem 15 and the inequality $\varepsilon < 1$ that $\mathrm{mult}_P(S_1|_H \cdot S_2|_H \cdot \Delta|_H) > 8n^2$.

§ 4. The $6n^2$-inequality

In this section we prove Theorem 4. Let X be a variety, O an isolated ordinary double point of X and B_X an effective movable boundary on X. For simplicity we assume that $B_X = \frac{1}{n}\mathcal{M}$, where $\mathcal{M}$ is a linear system on X without fixed components and $n \in \mathbb{N}$. Suppose that all divisors of $\mathcal{M}$ are $\mathbb{Q}$-Cartier divisors.

Let $\pi\colon V \to X$ be the blow-up of O, E the exceptional divisor of the birational morphism π and $B_V = f^{-1}(B_X)$. Then

$$B_V \sim_{\mathbb{Q}} \pi^*(B_X) - \operatorname{mult}_O(B_X)$$

for some positive rational number $\operatorname{mult}_O(B_X)$, which can naturally be regarded as a multiplicity of the movable boundary B_X at the ordinary double point O. We identify E with a smooth quadric in $\mathbb{P}^{\dim(X)}$.

Let H_i $(i = 1, \dots, \dim(X) - 2)$ be general hyperplane sections of X through O and let S_1, S_2 be sufficiently general divisors of the linear system $\mathcal{M}$. Then we can define the numbers $\operatorname{mult}_O(H_i)$ and $\operatorname{mult}_O(S_i)$ similarly to $\operatorname{mult}_O(B_X)$. We have

$$\operatorname{mult}_O(H_i) = 1, \qquad \operatorname{mult}_O(S_1) = \operatorname{mult}_O(S_2) = n \operatorname{mult}_O(B_X).$$

Put $\widehat{H}_i = \pi^{-1}(H_i)$ and $\widehat{S}_i = \pi^{-1}(S_i)$.

Definition 7. The multiplicity of the cycle $S_1 \cdot S_2$ at the point O is the rational number

$$\operatorname{mult}_O(S_1 \cdot S_2) = 2\operatorname{mult}^2_O(S_1) + \sum_{P \in E} \operatorname{mult}_P(\widehat{S}_1 \cdot \widehat{S}_2) \prod_{i=1}^{\dim(X)-2} \operatorname{mult}_P(\widehat{H}_i),$$

where the sum is taken over the finitely many points of the intersection

$$E \cap \widehat{S}_1 \cap \widehat{S}_2 \bigcap_{i=1}^{\dim(X)-2} \widehat{H}_i,$$

and $\widehat{S}_1 \cdot \widehat{S}_2$ is the scheme-theoretic intersection. Put $\operatorname{mult}_O(B_X^2) = \frac{1}{n^2}\operatorname{mult}_O(S_1 \cdot S_2)$.

We have $\operatorname{mult}_O(B_X^2) \geqslant 2\operatorname{mult}^2_O(B_X)$.

Remark 7. The following inequality holds for every subvariety $Z \subset E$ that has codimension 2 in V:

$$\operatorname{mult}_O(B_X^2) \geqslant 2\operatorname{mult}^2_O(B_X) + \operatorname{mult}_Z(B_V^2).$$

This may be regarded as the inequality

$$\operatorname{mult}_O(S_1 \cdot S_2) \geqslant 2\operatorname{mult}^2_O(S_1) + \operatorname{mult}_Z(\widehat{S}_1 \cdot \widehat{S}_2)$$

divided by n^2, where the intersection $\widehat{S}_1 \cdot \widehat{S}_2$ is scheme-theoretic.

The multiplicity $\operatorname{mult}_O(B_X^2)$ has nice geometric properties.

Example 3. Let X be a hypersurface in $\mathbb{P}^6$ of degree 6 such that the singularities of X are isolated ordinary double points and O is one of these points. Then the group $\operatorname{Cl}(X)$ is generated by the class of a hyperplane section H of X (see Lemma 5). Hence we have $S_i \sim kH$, where $k \in \mathbb{N}$. Suppose that $n = k$. Then

$$n^2 \operatorname{mult}_O(B_X^2) = \operatorname{mult}_O(S_1 \cdot S_2) \leqslant 6n^2.$$

Suppose that $\dim(X) \geqslant 4$ and $O \in \mathbb{CS}(X, B_X)$, but the singularities of the movable log pair (X, B_X) are canonical outside O. To prove Theorem 4, we must find a linear subspace $\Lambda \subset E$ of codimension 3 in $\mathbb{P}^r$ (where $r = \dim(X)$ and the divisor E is identified with a smooth quadric in $\mathbb{P}^r$) such that

$$\mathrm{mult}_O(S_1 \cdot S_2 \cdot \Delta) = \mathrm{mult}_O(S_1|_\Delta \cdot S_2|_\Delta) \geqslant 6n^2 \tag{3}$$

for every effective divisor Δ on X that satisfies the following conditions:

1) Δ contains the point O;

2) O is an isolated ordinary double point of Δ;

3) the divisor $\pi^{-1}(\Delta)$ contains Λ;

4) Δ contains no subvarieties of X of codimension 2 that are contained in the base locus of $\mathcal{M}$.

We must also prove that the inequality (3) is strict if $\dim(X) \geqslant 5$ and, in particular, that

$$\mathrm{mult}_O(S_1 \cdot S_2) > 6n^2$$

for $\dim(X) \geqslant 6$ because in this case the quadric E cannot contain the linear subspace Λ by Lefschetz' theorem (see [27], [21]).

Proof of Corollary 1. We prove the inequality $\mathrm{mult}_O(S_1 \cdot S_2) > 6n^2$ only for $\dim(X) = 6$ because the general proof is similar.

Suppose that $\dim(X) = 6$. Then

$$K_V + B_V \sim_{\mathbb{Q}} \pi^*(K_X + B_X) + (4 - \mathrm{mult}_O(B_X))E.$$

Let H_1, H_2, H_3 be general hyperplane sections of X through O. Put $\check{X} = \bigcap_{i=1}^3 H_i$. Then $\check{X}$ is a three-dimensional variety and O is an isolated ordinary double point of $\check{X}$.

Put $\check{\mathcal{M}} = \mathcal{M}|_{\check{X}}$. Then O is a centre of log canonical singularities of the log pair $\bigl(\check{X}, \frac{1}{n}\check{\mathcal{M}}\bigr)$ by Corollary 2. Moreover, O is a centre of log canonical singularities of the log pair $\bigl(\check{X}, \frac{\varepsilon}{n}\check{\mathcal{M}}\bigr)$ for some rational number $\varepsilon < 1$. Indeed, we have applied Corollary 2 three times during the reduction of X to $\check{X}$, and canonical singularities become log canonical during the first application (see the proof of Theorem 3).

Let $\check{\pi}\colon \check{V} \to \check{X}$ be the blow-up of O and let $\check{E} = \check{\pi}^{-1}(O)$. Then the diagram

$$\begin{tikzcd}
\check{V} \arrow[r, hook] \arrow[d, "\check{\pi}"'] & V \arrow[d, "\pi"] \\
\check{X} \arrow[r, hook] & X
\end{tikzcd}$$

commutes and the threefold $\check{V}$ can be identified with the subvariety $\pi^{-1}(\check{X}) \subset V$. In particular, we have $\check{E} = E \cap \check{V}$.

Since the divisors H_1, H_2, H_3 are general, we have

$$\mathrm{mult}_O(\check{\mathcal{M}}) = \mathrm{mult}_O(\mathcal{M}).$$

We can assume that $\mathrm{mult}_O(\mathcal{M}) < 2n$ since otherwise the inequality $\mathrm{mult}_O(S_1 \cdot S_2) > 6n^2$ is obvious (see Remark 7).

Let $\mathcal{B} = \pi^{-1}(\mathcal{M})$ and $\check{\mathcal{B}} = \check{\pi}^{-1}(\check{\mathcal{M}})$. Then $\check{\mathcal{B}} = \mathcal{B}|_{\check{V}}$.

We have the equivalence

$$K_V + \frac{1}{n}\mathcal{B} + \Big(\frac{1}{n}\operatorname{mult}_O(\mathcal{M}) - 1\Big)E + \widetilde{H}_1 + \widetilde{H}_2 + \widetilde{H}_3 \sim_{\mathbb{Q}} \pi^*\Big(K_X + \frac{1}{n}\mathcal{M} + H_1 + H_2 + H_3\Big),$$

where $\widetilde{H}_i = \pi^{-1}(H_i)$. Hence,

$$K_{\check{V}} + \frac{1}{n}\check{\mathcal{B}} + \Big(\frac{1}{n}\operatorname{mult}_O(\mathcal{M}) - 1\Big)\check{E} \sim_{\mathbb{Q}} \check{\pi}^*\Big(K_{\check{X}} + \frac{1}{n}\check{\mathcal{M}}\Big),$$

but the inequality $\operatorname{mult}_O(\mathcal{M}) < 2n$ implies that

$$E \notin \mathbb{LCS}\Big(V, \frac{1}{n}\mathcal{B} + \Big(\frac{1}{n}\operatorname{mult}_O(\mathcal{M}) - 1\Big)E\Big),$$
$$\check{E} \notin \mathbb{LCS}\Big(\check{V}, \frac{1}{n}\check{\mathcal{B}} + \Big(\frac{1}{n}\operatorname{mult}_O(\mathcal{M}) - 1\Big)\check{E}\Big).$$

Since H_1, H_2, H_3 are general, it follows that there are proper irreducible subvarieties $\Omega \subsetneq E$ and $\check{\Omega} \subsetneq \check{E}$ such that

$$\Omega \in \mathbb{LCS}\Big(V, \frac{1}{n}\mathcal{B} + \Big(\frac{1}{n}\operatorname{mult}_O(\mathcal{M}) - 1\Big)E\Big),$$
$$\check{\Omega} \in \mathbb{LCS}\Big(\check{V}, \frac{1}{n}\check{\mathcal{B}} + \Big(\frac{1}{n}\operatorname{mult}_O(\mathcal{M}) - 1\Big)\check{E}\Big)$$

and $\check{\Omega} \subseteq \Omega \cap \check{V}$.

We have $\check{\Omega} = \Omega \cap \check{V}$ for $\dim(\check{\Omega}) > 0$.

We may assume that Ω and $\check{\Omega}$ have the biggest dimension among all subvarieties with similar properties. We apply the general connectedness principle (see Theorem 12) to the log pair $\big(\check{V}, \frac{1}{n}\check{\mathcal{B}} + \big(\frac{1}{n}\operatorname{mult}_O(\mathcal{M}) - 1\big)\check{E}\big)$ and the morphism $\check{\pi}$. If $\dim(\check{\Omega}) = 0$, then it follows immediately that the point $\check{\Omega}$ is the unique subvariety of $\check{E}$ which is a centre of log canonical singularities of the log pair

$$\Big(\check{V}, \frac{1}{n}\check{\mathcal{B}} + \Big(\frac{1}{n}\operatorname{mult}_O(\mathcal{M}) - 1\Big)\check{E}\Big).$$

In particular, we always have the set-theoretic equation

$$\check{\Omega} = \Omega \cap \check{V}.$$

Suppose that $\dim(\check{\Omega}) = 0$. Since $\check{\Omega} = \Omega \cap \check{V}$, we see that Ω is a three-dimensional linear subspace of $\mathbb{P}^6$ and is contained in the smooth five-dimensional quadric $E \subset \mathbb{P}^6$, which is impossible by Lefschetz' theorem. Hence we have $\dim(\check{\Omega}) \geqslant 1$ and, therefore, $\dim(\Omega) = 4$.

We have the log pair $\big(V, \frac{1}{n}\mathcal{B} + \big(\frac{1}{n}\operatorname{mult}_O(\mathcal{M}) - 1\big)E\big)$ and its centre of log canonical singularities $\Omega \subset E$ of dimension 4. In particular, we can apply Theorem 14 to the log pair $\big(V, \frac{1}{n}\mathcal{B} + \big(\frac{1}{n}\operatorname{mult}_O(\mathcal{M}) - 1\big)E\big)$ at a general point of Ω. This yields the inequality

$$\operatorname{mult}_\Omega\big(\widetilde{S}_1 \cdot \widetilde{S}_2\big) \geqslant 4\big(2n^2 - n\operatorname{mult}_O(\mathcal{M})\big),$$

where $\widetilde{S}_i = \pi^1(S_i)$. Hence we see that

$$\mathrm{mult}_O(S_1 \cdot S_2) \geqslant 2\,\mathrm{mult}_O(\mathcal{M})^2 + \mathrm{mult}_\Omega\big(\widetilde{S}_1 \cdot \widetilde{S}_2\big) \geqslant 4\big(2n^2 - n\,\mathrm{mult}_O(\mathcal{M})\big) \geqslant 6n^2.$$

As already mentioned, the point O is a centre of log canonical singularities of the movable log pair $(\check{X}, \frac{\varepsilon}{n}\check{\mathcal{M}})$ for some positive rational number $\varepsilon < 1$. Therefore the subvariety $\Omega \subset E$ is a centre of log canonical singularities of the log pair

$$\left(V, \frac{\varepsilon}{n}\mathcal{B} + \left(\frac{\varepsilon}{n}\,\mathrm{mult}_O(\mathcal{M}) - 1\right)E\right).$$

It follows that

$$\mathrm{mult}_O(S_1 \cdot S_2) \geqslant 2\,\mathrm{mult}_O(\mathcal{M})^2 + \mathrm{mult}_\Omega\big(\widetilde{S}_1 \cdot \widetilde{S}_2\big) > 4\big(2n^2 - n\,\mathrm{mult}_O(\mathcal{M})\big) \geqslant 6n^2.$$

This completes the proof.

Proof of Theorem 4. We may assume that $\dim(X) \leqslant 5$ because Corollary 1 has already been proved. We may also assume that $\dim(X) = 4$ because the proof for $\dim(X) = 5$ is similar. Then we have

$$K_V + B_V \sim_{\mathbb{Q}} \pi^*(K_X + B_X) + (2 - \mathrm{mult}_O(B_X))E,$$

where $\mathrm{mult}_O(B_X) > 1$ by Theorem 2.

Let Δ be an effective divisor on X such that $O \in \Delta$, the point O is an ordinary double point of the threefold Δ and Δ contains no surfaces that are contained in the base locus of $\mathcal{M}$. Then $\mathcal{M}|_\Delta$ has no fixed components and $S_1 \cdot S_2 \cdot \Delta$ is an effective 1-cycle.

Put $\overline{S}_1 = S_1|_\Delta$, $\overline{S}_2 = S_2|_\Delta$ and $\overline{\mathcal{M}} = \mathcal{M}|_\Delta$. Then the point O is a centre of log canonical singularities of the log pair $\big(\Delta, \frac{1}{n}\overline{\mathcal{M}}\big)$ by Corollary 2.

We must show that the inequality

$$\mathrm{mult}_O(\overline{S}_1 \cdot \overline{S}_2) \geqslant 6n^2$$

holds, possibly under some additional hypotheses on Δ.

Let $\bar{\pi}\colon \hat{\Delta} \to \Delta$ be the blow-up of O and let $\overline{E} = \bar{\pi}^{-1}(O)$. Then the diagram

$$\begin{tikzcd}
 & \hat{\Delta} \arrow[r, hook] \arrow[dl, "\bar{\pi}"'] & V \arrow[dr, "\pi"] & \\
\Delta \arrow[rrr, hook] & & & X
\end{tikzcd}$$

commutes and $\hat{\Delta}$ can be identified with the divisor $\pi^{-1}(\Delta) \subset V$. Hence we have $\overline{E} = E \cap \hat{\Delta}$.

If $\mathrm{mult}_O(\overline{\mathcal{M}}) \geqslant 2n$, then Remark 7 shows that

$$\mathrm{mult}_O(\overline{S}_1 \cdot \overline{S}_2) \geqslant 8n^2$$

and the theorem follows. Hence we may assume that $\mathrm{mult}_O(\overline{\mathcal{M}}) < 2n$.

Put $\widehat{\mathcal{M}} = \bar{\pi}^{-1}(\overline{\mathcal{M}})$. Since $\operatorname{mult}_O(\overline{\mathcal{M}}) < 2n$, it follows that

$$\overline{E} \notin \mathbb{LCS}\Big(\hat{\Delta}, \frac{1}{n}\widehat{\mathcal{M}} + \Big(\frac{1}{n}\operatorname{mult}_O(\overline{\mathcal{M}}) - 1\Big)\overline{E}\Big).$$

Hence there is a subvariety $\Xi \subsetneq \overline{E}$ such that

$$\Xi \in \mathbb{LCS}\Big(\hat{\Delta}, \frac{1}{n}\widehat{\mathcal{M}} + \Big(\frac{1}{n}\operatorname{mult}_O(\overline{\mathcal{M}}) - 1\Big)\overline{E}\Big).$$

The divisor $\overline{E}$ can be identified with a smooth quadric in $\mathbb{P}^3$.

Suppose that Ξ is a curve. Put $\widehat{S}_i = \bar{\pi}^{-1}(S_i)$. Then we have

$$\operatorname{mult}_O(\overline{S}_1 \cdot \overline{S}_2) \geqslant 2\operatorname{mult}_O(\overline{\mathcal{M}})^2 + \operatorname{mult}_\Xi(\widehat{S}_1 \cdot \widehat{S}_2)$$

(see Remark 7), and an application of Theorem 14 to the log pair $\big(\hat{\Delta}, \frac{1}{n}\widehat{\mathcal{M}} + \big(\frac{1}{n}\operatorname{mult}_P(\overline{\mathcal{M}}) - 1\big)\overline{E}\big)$ at a general point of Ξ yields that

$$\operatorname{mult}_\Xi(\widehat{S}_1 \cdot \widehat{S}_2) \geqslant 4\big(2n^2 - n\operatorname{mult}_O(\overline{\mathcal{M}})\big).$$

It follows that

$$\operatorname{mult}_O(\overline{S}_1 \cdot \overline{S}_2) \geqslant 2\operatorname{mult}_O(\overline{\mathcal{M}})^2 + 4\big(2n^2 - n\operatorname{mult}_O(\overline{\mathcal{M}})\big) \geqslant 6n^2,$$

which completes the proof.

Hence we may assume that Ξ is a point. Then there is a line $\Lambda \subset E \subset \mathbb{P}^4$ such that

$$\Lambda \in \mathbb{LCS}\Big(V, \frac{1}{n}\mathcal{B} + \Big(\frac{1}{n}\operatorname{mult}_O(\mathcal{M}) - 1\Big)E\Big),$$

and $\Xi = \Lambda \cap \hat{\Delta}$. Indeed, assuming that Δ is sufficiently general and applying Theorem 12 to the log pair $\big(\hat{\Delta}, \frac{1}{n}\widehat{\mathcal{M}} + \big(\frac{1}{n}\operatorname{mult}_O(\overline{\mathcal{M}}) - 1\big)\overline{E}\big)$ and the birational morphism $\hat{\pi}$, we see (as in the proof of Proposition 4) that the set

$$\mathbb{LCS}\Big(V, \frac{1}{n}\mathcal{B} + \Big(\frac{1}{n}\operatorname{mult}_O(\mathcal{M}) - 1\Big)E\Big)$$

contains either a surface that is contained in the divisor E or a line $\Lambda \subset E \subset \mathbb{P}^4$ such that $\Xi = \Lambda \cap \hat{\Delta}$. In the former case, the required inequality follows because the set

$$\mathbb{LCS}\Big(\hat{\Delta}, \frac{1}{n}\widehat{\mathcal{M}} + \Big(\frac{1}{n}\operatorname{mult}_O(\overline{\mathcal{M}}) - 1\Big)\overline{E}\Big)$$

must contain a curve contained in $\overline{E}$ (for the original choice of Δ). Hence, to complete the proof of Theorem 4, we can assume that Ξ is a point and there is a line $\Lambda \subset E \subset \mathbb{P}^4$ such that $\Xi = \Lambda \cap \hat{\Delta}$ and $\Lambda \in \mathbb{LCS}\big(V, \frac{1}{n}\mathcal{B} + \big(\frac{1}{n}\operatorname{mult}_O(\mathcal{M}) - 1\big)E\big)$.

Previous arguments imply that Λ does not depend on the choice of Δ but only on properties of the log pair (X, B_X). Hence we may assume that Δ is chosen in such a way that $\Lambda \subset \pi^{-1}(\Delta)$.

We can now repeat all the previous arguments under the new hypotheses. Moreover, the condition $\Lambda \subset \hat{\Delta} = \pi^{-1}(\Delta)$ implies that

$$\Lambda \in \mathbb{LCS}\Big(\hat{\Delta}, \frac{1}{n}\,\widehat{\mathcal{M}} + \Big(\frac{1}{n}\operatorname{mult}_O(\overline{\mathcal{M}}) - 1\Big)\overline{E}\Big)$$

because the boundary $\frac{1}{n}\mathcal{B} + \big(\frac{1}{n}\operatorname{mult}_O(\mathcal{M}) - 1\big)E$ is effective by the inequality $\operatorname{mult}_O(\mathcal{M}) > n$ (see Theorem 2). Hence we can apply Theorem 13 or Theorem 12 to a general point of the line Λ on the variety V.

Thus the set

$$\mathbb{LCS}\Big(\hat{\Delta}, \frac{1}{n}\,\widehat{\mathcal{M}} + \Big(\frac{1}{n}\operatorname{mult}_O(\overline{\mathcal{M}}) - 1\Big)\overline{E}\Big)$$

contains a curve that is contained in $\overline{E}$. It follows from Theorem 14 that

$$\operatorname{mult}_O(\overline{S}_1 \cdot \overline{S}_2) \geqslant 2\operatorname{mult}_O(\overline{\mathcal{M}})^2 + 4\big(2n^2 - n\operatorname{mult}_O(\overline{\mathcal{M}})\big) \geqslant 6n^2.$$

This completes the proof of Theorem 4.

We note that the proof of Theorem 4 is similar to that of Theorem 15. However, since the boundary

$$\frac{1}{n}\,\mathcal{B} + \Big(\frac{1}{n}\operatorname{mult}_O(\mathcal{M}) - 1\Big)E$$

is effective, the last step in the proof of Theorem 4 is much simpler than the corresponding step of the proof of Theorem 15, where the boundary need not be effective and we are unable to use the connectedness principle. This explains why the proof of Theorem 15 uses Theorem 12 together with simple but important local calculations (see Lemmas 3 and 4).

The arguments used in the proof of Theorem 4 and Proposition 2 yield the following result.

Theorem 16. *Let Y be a variety of dimension $r \geqslant 4$, P an isolated double point on Y such that Y is locally isomorphic to the hypersurface*

$$y^3 = \sum_{i=1}^{r} x_i^2 \subset \mathbb{C}^{r+1} \cong \operatorname{Spec}(\mathbb{C}[x_1, \dots, x_r, y])$$

in a neighbourhood of P, $\mathcal{R}$ a linear system on Y without fixed components and $n \in \mathbb{N}$. Suppose that $P \in \mathbb{CS}\big(Y, \frac{1}{n}\,\mathcal{R}\big)$ and the singularities of the log pair $\big(X, \frac{1}{n}\,\mathcal{R}\big)$ are canonical outside P. Take two general divisors M_1 and M_2 of the linear system $\mathcal{R}$ and define the number $\operatorname{mult}_P(M_1 \cdot M_2)$ *as in Definition 7. Let $\zeta\colon U \to Y$ be the blow-up of P and G the exceptional divisor of ζ. We identify G with a quadric cone in $\mathbb{P}^r$. Then there is a linear subspace $\Upsilon \subset G$ of codimension 3 in $\mathbb{P}^r$ such that*

$$\operatorname{mult}_P(M_1 \cdot M_2 \cdot \Theta) = \operatorname{mult}_P(M_1|_\Theta \cdot M_2|_\Theta) \geqslant 6n^2 \qquad (4)$$

for every effective divisor Θ on Y satisfying the following conditions:

1) *Θ contains the point P*;

2) *either P is an ordinary double point of Θ or Θ is locally isomorphic to the hypersurface*

$$y^3 = \sum_{i=1}^{r-1} x_i^2 \subset \mathbb{C}^r \cong \operatorname{Spec}(\mathbb{C}[x_1, \dots, x_{r-1}, y])$$

in a neighbourhood of P;

3) *the divisor $\zeta^{-1}(\Theta)$ contains Υ;*

4) *Θ contains no subvarieties of Y of codimension 2 that are contained in the base locus of $\mathcal{R}$.*

The inequality (4) is strict for $r \geqslant 5$.

Corollary 6. *In the notation and under the hypotheses of Theorem 16, suppose that $r \geqslant 7$. Then*

$$\operatorname{mult}_P(M_1 \cdot M_2) > 6n^2.$$

§ 5. Double cubics

Let $\psi\colon X \to V \subset \mathbb{P}^n$ be a double covering branched over an irreducible reduced divisor $R \subset V$, where V is a smooth cubic hypersurface and $n \geqslant 4$. Then $\operatorname{rk}\operatorname{Pic}(V) = 1$ by Lefschetz' theorem, and

$$-K_X \sim \psi^*(\mathcal{O}_{\mathbb{P}^n}(r-2-n)|_V),$$

where $r \in \mathbb{N}$ is such that $R \sim \mathcal{O}_{\mathbb{P}^n}(2r)|_V$.

Suppose that the singularities of R are isolated ordinary double points. If $r \geqslant n-2$, then the variety X is not uniruled. On the other hand, if $r \leqslant n-3$, then X is rationally connected because X is a Fano variety with terminal singularities (see [50], [64]). Moreover, we always have

$$\operatorname{rk}\operatorname{Pic}(X) = \operatorname{rk}\operatorname{Cl}(X) = 1$$

(see [34] and Lemma 5).

In this section we prove the following result.

Theorem 17. *Suppose that $n = r+3 \geqslant 9$. Then X is birationally superrigid.*

In particular, X is non-rational if $n = r+3 \geqslant 9$.

The birational superrigidity of X is proved in [9] provided that X is sufficiently general and $n = r+3 \geqslant 5$. If $n = r+3 = 4$, then X is a complete intersection of a quadric and a cubic in $\mathbb{P}^5$. This variety is not birationally superrigid, but is birationally rigid [8] if it is sufficiently general (see [45]).

In the case when $n > r+3$, the only known way of proving the non-rationality of X is the method in [49]. This method easily yields the following result (we omit the proof).

[8] Let X be a Fano variety with terminal and $\mathbb{Q}$-factorial singularities and such that $\operatorname{rk}\operatorname{Pic}(X) = 1$. We say that X is *birationally rigid* if the following conditions hold: X cannot be birationally transformed into a fibration $\tau\colon Y \to Z$ such that $\dim(Y) > \dim(Z) \neq 0$ but the Kodaira dimension of a general fibre of τ is $-\infty$ and X is not birational to a Fano variety of Picard rank 1 having terminal and $\mathbb{Q}$-factorial singularities that is not biregular to X.

Proposition 7. *Suppose that $r \geqslant \frac{n+5}{2} > 4$ and X is very general. Then X is non-ruled and, in particular, non-rational.*

First we must show that X is $\mathbb{Q}$-factorial. We deduce this from the following result, although it is also expected to be a corollary of Lefschetz' theorem (see [27], [21], [39]).

Lemma 5. *The groups* $\mathrm{Cl}(X)$ *and* $\mathrm{Pic}(X)$ *are generated by the divisor* $\psi^*(\mathcal{O}_{\mathbb{P}^n}(1))$.

Proof. Let D be a Weil divisor on X. We must show that $D \sim \psi^*(\mathcal{O}_{\mathbb{P}^n}(r))$ for some $r \in \mathbb{Z}$.

Let H be a general divisor of the linear system $|\psi^*(\mathcal{O}_{\mathbb{P}^n}(k))|$ for $k \gg 0$. Then H is a smooth weighted complete intersection (see [43]) and $\dim(X) \geqslant 3$.

The group $\mathrm{Pic}(H)$ is generated by the divisor $\psi^*(\mathcal{O}_{\mathbb{P}^n}(1))|_H$ by [39], Chapter XI, Theorem 3.13 (see also [34], Lemma 3.2.2, [33], Lemma 3.5 or [28]). Hence there is $r \in \mathbb{Z}$ such that $D|_H \sim \psi^*(\mathcal{O}_{\mathbb{P}^n}(r))|_H$.

Put $\Delta = D - \psi^*(\mathcal{O}_{\mathbb{P}^n}(r))$. The sequence of sheaves

$$0 \to \mathcal{O}_X(\Delta) \otimes \mathcal{O}_X(-H) \to \mathcal{O}_X(\Delta) \to \mathcal{O}_H \to 0$$

is exact because the sheaf $\mathcal{O}_X(\Delta)$ is locally free in a neighbourhood of H. Therefore the sequence

$$0 \to H^0(\mathcal{O}_X(\Delta)) \to H^0(\mathcal{O}_H) \to H^1(\mathcal{O}_X(\Delta) \otimes \mathcal{O}_X(-H))$$

is exact. On the other hand, the sheaf $\mathcal{O}_X(\Delta)$ is reflexive (see [41]). Hence there is an exact sequence of sheaves

$$0 \to \mathcal{O}_X(\Delta) \to \mathcal{E} \to \mathcal{F} \to 0,$$

where $\mathcal{E}$ is locally free and $\mathcal{F}$ has no torsion. Hence the sequence

$$H^0(\mathcal{F} \otimes \mathcal{O}_X(-H)) \to H^1(\mathcal{O}_X(\Delta - H)) \to H^1(\mathcal{E} \otimes \mathcal{O}_X(-H))$$

is exact. However, the group $H^0(\mathcal{F} \otimes \mathcal{O}_X(-H))$ is trivial because $\mathcal{F}$ has no torsion. The group $H^1(\mathcal{E} \otimes \mathcal{O}_X(-H))$ is trivial by the Enriques–Severi–Zariski lemma (see [63]) since X is normal. Hence we have

$$H^1(\mathcal{O}_X(\Delta) \otimes \mathcal{O}_X(-H)) = 0, \qquad H^0(\mathcal{O}_X(\Delta)) = \mathbb{C}.$$

We similarly see that $H^0(\mathcal{O}_X(-\Delta)) = \mathbb{C}$. Therefore the divisor Δ is rationally equivalent to zero. It follows that $D \sim \psi^*(\mathcal{O}_{\mathbb{P}^n}(r))$.

Proof of Theorem 17. Suppose that X is not birationally superrigid but $n = r+3 \geqslant 9$. Let us show that these assumptions lead to a contradiction.

It is well known (see Theorem 64 in [16]) that there is a linear system $\mathcal{M}$ on X such that $\mathcal{M}$ has no fixed components but the singularities of the log pair $\bigl(X, \frac{1}{m}\mathcal{M}\bigr)$ are not canonical, where m is a positive integer such that $\mathcal{M} \sim -mK_X$.

In particular, the set $\mathbb{CS}\bigl(X, \frac{1}{m}\mathcal{M}\bigr)$ contains an irreducible reduced subvariety $Z \subset X$ such that $Z \in \mathbb{CS}\bigl(X, \frac{\mu}{m}\mathcal{M}\bigr)$ for some positive rational number $\mu < 1$. We may assume without loss of generality that Z is a subvariety of maximal dimension with these properties.

Corollary 7. *Let S be a divisor in $\mathcal{M}$. Then* $\mathrm{mult}_Z(S) > m$.

We have $\dim(Z) \leqslant \dim(X) - 2 = n - 3$.

Lemma 6. *We have* $\dim(Z) \neq 0$.

Proof. Suppose that Z is a point. Let us show that this assumption leads to a contradiction.

Suppose that Z is a smooth point of X. Let S_1 and S_2 be general divisors of the linear system $\mathcal{M}$, $f\colon U \to X$ the blow-up of the point Z and E the exceptional divisor of f. Then Theorem 3 yields a linear subspace $\Pi \subset E \cong \mathbb{P}^{n-2}$ of dimension $n - 4$ such that

$$\mathrm{mult}_Z(S_1 \cdot S_2 \cdot D) > 8m^2$$

for every divisor $D \in |-K_X|$ satisfying the following conditions:

1) D contains the point Z and is smooth at Z;

2) the divisor $f^{-1}(D)$ contains the subvariety $\Pi \subset U$;

3) D contains no subvarieties of X of codimension 2 that are contained in the base locus of $\mathcal{M}$.

We consider a linear subsystem $\mathcal{H} \subset |\mathcal{O}_{\mathbb{P}^n}(1)|_V|$ such that

$$H \in \mathcal{H} \iff \Pi \subset (\psi \circ f)^{-1}(H).$$

Then there is a linear subspace $\Sigma \subset \mathbb{P}^n$ of dimension at most $n - 3$ such that all divisors of the linear system $\mathcal{H}$ are cut out on the hypersurface V by hyperplanes in $\mathbb{P}^n$ containing Σ. In particular, the base locus of $\mathcal{H}$ consists of the intersection $\Sigma \cap V$. On the other hand, we have $\Sigma \not\subset V$ by Lefschetz' theorem, whence $\dim(\Sigma \cap V) = n - 4$.

Let H be a general divisor in $\mathcal{H}$ and $D = \psi^{-1}(H) \in |-K_X|$. Then D satisfies the following conditions:

1) D contains the point Z and is smooth at Z;

2) the divisor $f^{-1}(D)$ contains the subvariety $\Pi \subset U$.

Suppose that D contains a subvariety $\Gamma \subset X$ of codimension 2 that is contained in the base locus of $\mathcal{M}$. Then

$$\dim(\psi(\Gamma)) = n - 3,$$

but $\psi(\Gamma) \subset \Sigma \cap V$ and $\dim(\Sigma \cap V) = n - 4$, which is a contradiction.

Hence we see that D contains no subvarieties of X of codimension 2 that are contained in the base locus of $\mathcal{M}$.

Let $H_1, H_2, \dots, H_k$ be general divisors of the linear system $|-K_X|$ that pass through the point Z, where $k = \dim(X) - 3$. Then

$$6m^2 = H_1 \cdot \ldots \cdot H_k \cdot S_1 \cdot S_2 \cdot D \geqslant \mathrm{mult}_Z(S_1 \cdot S_2 \cdot D) > 8m^2,$$

which is a contradiction.

Therefore Z is an isolated ordinary double point on X. We can now use the previous arguments together with Corollary 1 to obtain a contradiction.

Lemma 7. *We have* $\dim(Z) \geqslant \dim(X) - 4$.

Proof. Suppose that $\dim(Z) \leqslant \dim(X) - 5$. Let $H_1, H_2, \ldots, H_k$ be general hyperplane sections of the hypersurface V, where $k = \dim(Z) > 0$. Put

$$\overline{V} = \bigcap_{i=1}^{k} H_i, \qquad \overline{X} = \psi^{-1}(\overline{V}), \qquad \bar{\psi} = \psi|_{\overline{X}} \colon \overline{X} \to \overline{V}$$

and $\overline{\mathcal{M}} = \mathcal{M}|_{\overline{X}}$. Then $\overline{V} \subset \mathbb{P}^{n-k}$ is a smooth cubic hypersurface, the induced morphism $\bar{\psi}$ is a double covering branched over a smooth divisor $R \cap \overline{V}$, and the linear system $\mathcal{M}$ has no fixed components.

By Lefschetz' theorem, the hypersurface $\overline{V}$ contains no linear subspaces of $\mathbb{P}^{n-k}$ of dimension $n-k-3$ (because $n-k \geqslant 6$).

Let P be an arbitrary point of the intersection $Z \cap \overline{X}$. Then $P \in \mathbb{CS}\big(\overline{X}, \frac{1}{m}\overline{\mathcal{M}}\big)$ and the proof of Lemma 6 immediately leads to a contradiction.

Lemma 8. *We have* $\dim(Z) \neq \dim(X) - 2$.

Proof. Suppose that $\dim(Z) = \dim(X) - 2$. Let S_1 and S_2 be general divisors of the linear system $\mathcal{M}$ and let $H_1, H_2, \ldots, H_{n-3}$ be general divisors of the linear system $|-K_X|$. Then

$$6m^2 = H_1 \cdot \ldots \cdot H_{n-3} \cdot S_1 \cdot S_2 > m^2(-K_X)^{n-3} \cdot Z$$

because $\mathrm{mult}_Z(\mathcal{M}) > m$. Therefore we have $(-K_X)^{n-3} \cdot Z < 6$.

We note that

$$(-K_X)^{n-3} \cdot Z = \begin{cases} \deg(\psi(Z) \subset \mathbb{P}^n) & \text{if } \psi|_Z \text{ is birational,} \\ 2\deg(\psi(Z) \subset \mathbb{P}^n) & \text{if } \psi|_Z \text{ is not birational.} \end{cases}$$

By Lefschetz' theorem, $\deg(\psi(Z))$ must be divisible by 3. It follows that the morphism $\psi|_Z$ is birational and $\deg(\psi(Z)) = 3$. Therefore, either the scheme-theoretic intersection $\psi(Z) \cap R$ is singular at every point or $\psi(Z) \subset R$. Applying Lefschetz' theorem to a hyperplane section of the complete intersection $R \subset \mathbb{P}^n$, we immediately obtain a contradiction.

Lemma 9. *We have* $\dim(Z) \leqslant \dim(X) - 5$.

Proof. Suppose that $\dim(Z) \geqslant \dim(X) - 4 \geqslant 4$. Let S be a sufficiently general divisor of the linear system $\mathcal{M}$. Put $\widehat{S} = \psi(S \cap R)$ and $\widehat{Z} = \psi(Z \cap R)$. Then $\widehat{S}$ is a divisor on a smooth complete intersection $R \subset \mathbb{P}^n$ such that

$$\mathrm{mult}_{\widehat{Z}}(\widehat{S}) > m, \qquad \widehat{S} \sim \mathcal{O}_{\mathbb{P}^n}(m)|_R.$$

This is impossible by Proposition 3 because $\dim(\widehat{Z}) \geqslant 3$.

This proves Theorem 17. We now prove the following result.

Theorem 18. *Suppose that $n = r + 3 \geqslant 9$. Then the variety X cannot be birationally transformed into an elliptic fibration.*

Proof. Suppose that X is birational to an elliptic fibration. Then Theorem 66 of [16] yields a linear system $\mathcal{M}$ on X such that $\mathcal{M}$ has no fixed components, $\mathcal{M}$ is not composed of a pencil, and the set $\mathbb{CS}\big(X, \frac{1}{m}\mathcal{M}\big)$ is non-empty, where m is a positive integer such that $\mathcal{M} \sim -mK_X$.

Let Z be an irreducible reduced subvariety of maximal dimension in X such that $Z \in \mathbb{CS}\big(X, \frac{1}{m}\mathcal{M}\big)$. The proof of Theorem 17 shows that $\dim(Z) = \dim(X) - 2$ and $(-K_X)^{n-3} \cdot Z = 6$. Hence we have

$$S_1 \cap S_2 = Z$$

in the set-theoretic sense, where S_1 and S_2 are general divisors of $\mathcal{M}$.

Let P be a general point in $X \setminus Z$ and $\mathcal{D}$ a linear subsystem of $\mathcal{M}$ that consists of divisors passing through P. Since $\mathcal{M}$ is not composed of a pencil, we see that $\mathcal{D}$ has no fixed components. Applying the previous arguments to the linear system $\mathcal{D}$ instead of $\mathcal{M}$, we see that $D_1 \cap D_2 = Z$ in the set-theoretic sense, where D_1 and D_2 are general divisors of $\mathcal{D}$. On the other hand, we have $P \in D_1 \cap D_2$ and $P \notin Z$, which is a contradiction.

We note that the proofs of Theorems 17 and 18 remain valid for $n = r + 3 \geqslant 8$ if the variety X is smooth.

§ 6. Triple quadrics

Let $\psi\colon X \to Q \subset \mathbb{P}^{2r+2}$ be a cyclic triple covering branched over an irreducible reduced divisor $R \subset Q$, where Q is smooth hypersurface of degree 2 and r is a positive integer. Then $\operatorname{rk}\operatorname{Pic}(Q) = 1$ by Lefschetz' theorem and

$$-K_X \sim \psi^*\big(\mathcal{O}_{\mathbb{P}^{2r+2}}(2k - 2r - 1)|_Q\big),$$

where $k \in \mathbb{N}$ is such that $R \sim \mathcal{O}_{\mathbb{P}^{2r+2}}(3k)|_Q$.

Suppose that the singularities of R consist of at most isolated ordinary double points. Then $\operatorname{rk}\operatorname{Pic}(X) = \operatorname{rk}\operatorname{Cl}(X) = 1$ (see Lemma 5).

Moreover, X is a Fano variety with terminal singularities if $k \leqslant r$, but X is not birationally rigid if $k < r$.

In this section we prove the following result.

Theorem 19. *Suppose that $k = r \geqslant 4$. Then X is birationally superrigid.*

In particular, X is non-rational for $k = r \geqslant 4$.

The birational superrigidity of X is proved in [60] provided that R is a sufficiently general divisor on the quadric Q and $k = r \geqslant 2$.

Proof of Theorem 19. Suppose that X is not birationally superrigid, but $k = r \geqslant 4$. Let us show that this assumption leads to a contradiction. By Theorem 64 of [16], there is a linear system $\mathcal{M}$ on X such that $\mathcal{M}$ has no fixed components, but the singularities of the log pair $(X, \frac{1}{m}\mathcal{M})$ are not canonical, where m is a positive integer such that $\mathcal{M} \sim -mK_X$. In particular, the set $\mathbb{CS}\big(X, \frac{\mu}{m}\mathcal{M}\big)$ contains a subvariety $Z \subset X$ for some rational number $\mu < 1$. We may assume that Z has maximal dimension among all subvarieties of X with this property. Then $\dim(Z) \leqslant \dim(X) - 2 = 2r - 1$.

Lemma 10. *We have* $\dim(Z) \neq 0$.

Proof. This inequality follows from the arguments in the proof of Lemma 6 if we use Corollary 6 instead of Corollary 1.

Lemma 11. *We have* $\dim(Z) \geqslant \dim(X) - 4$.

Proof. Suppose that $\dim(Z) \leqslant \dim(X) - 5$ and let $H_1, H_2, \dots, H_k$ be general hyperplane sections of the hypersurface Q, where $k = \dim(Z) > 0$. Put

$$\overline{Q} = \bigcap_{i=1}^{k} H_i, \qquad \overline{X} = \psi^{-1}(\overline{Q}), \qquad \bar{\psi} = \psi|_{\overline{X}} \colon \overline{X} \to \overline{Q}, \qquad \overline{\mathcal{M}} = \mathcal{M}|_{\overline{X}}.$$

Then $\overline{Q} \subset \mathbb{P}^{2r+2-k}$ is a smooth quadric hypersurface, the morphism $\bar{\psi}$ is a cyclic triple covering branched over a smooth divisor $R \cap \overline{Q}$, the linear system $\mathcal{M}$ has no fixed components and the quadric $\overline{Q}$ contains no linear subspaces of $\mathbb{P}^{2r+2-k}$ of dimension $2r-k-1$ by Lefschetz' theorem. Let P be any point of the intersection $Z \cap \overline{X}$. Then $P \in \mathbb{CS}\big(\overline{X}, \frac{1}{m}\overline{\mathcal{M}}\big)$, and the proof of Lemma 10 yields a contradiction.

Therefore we see that $\dim(Z) \geqslant 4$.

Let S be a general divisor of the linear system $\mathcal{M}$. Put $\widehat{S} = \psi(S \cap R)$, $\widehat{Z} = \psi(Z \cap R)$. Then

$$\mathrm{mult}_{\widehat{Z}}(\widehat{S}) > m, \qquad \widehat{S} \sim \mathcal{O}_{\mathbb{P}^{2r+2}}(m)|_R.$$

This is impossible by Proposition 3 because $\dim(\widehat{Z}) \geqslant 3$.

This proves Theorem 19. Now the proof of Theorem 18 yields the following result.

Theorem 20. *Suppose that* $k = r \geqslant 4$. *Then* X *cannot be birationally transformed into an elliptic fibration.*

The proofs of Theorems 19 and 20 remain valid for $k = r \geqslant 3$ if X is smooth.

§ 7. Double intersections of quadrics

Let $\psi \colon X \to V \subset \mathbb{P}^n$ be a double covering branched over a smooth divisor $R \subset V$, where V is a smooth complete intersection of two hypersurfaces of degree 2 and $n \geqslant 5$. Then $\mathrm{rk}\,\mathrm{Pic}(V) = 1$ by Lefschetz' theorem, and

$$-K_X \sim \psi^*(\mathcal{O}_{\mathbb{P}^n}(k - n + 3)|_V),$$

where $k \in \mathbb{N}$ is such that $R \sim \mathcal{O}_{\mathbb{P}^n}(2k)|_V$.

Suppose that $k \leqslant n - 4$. Then X is a smooth Fano variety with $\mathrm{rk}\,\mathrm{Pic}(X) = 1$, but X is not birationally rigid for $k \leqslant n - 5$.

In this section we prove the following result.

Theorem 21. *Suppose that* $k = n - 4 \geqslant 6$. *Then* X *is birationally superrigid.*

In particular, the variety X is non-rational if $k = n - 4 \geqslant 6$.

If $k = n - 4$ and $n = 5$, then X is a complete intersection of three quadrics in $\mathbb{P}^6$, which is not birationally rigid but is still non-rational (see [11], [23], [12]).

Proof of Theorem 21. Suppose that X is not birationally superrigid, but $k = n-4 \geqslant 6$. Let us show that these assumptions lead to a contradiction.

By Theorem 64 in [16], there is a linear system $\mathcal{M}$ on X such that $\mathcal{M}$ has no fixed components and the singularities of the log pair $\big(X, \frac{1}{m}\mathcal{M}\big)$ are not canonical, where m is a positive integer such that $\mathcal{M} \sim -mK_X$.

In particular, the set $\mathbb{CS}\big(X, \frac{1}{m}\mathcal{M}\big)$ contains an irreducible subvariety $Z \subset X$ such that $Z \in \mathbb{CS}\big(X, \frac{\mu}{m}\mathcal{M}\big)$ for some positive rational number $\mu < 1$. We may assume without loss of generality that Z is a subvariety of maximal dimension with this property. Then $\dim(Z) \leqslant \dim(X) - 2 = n-4$.

Lemma 12. *We have* $\dim(Z) \neq 0$.

Proof. Suppose that Z is a point. Let S_1 and S_2 be general divisors of the linear system $\mathcal{M}$, $f\colon U \to X$ the blow-up of the point Z and E the exceptional divisor of the birational morphism f. Then Theorem 3 yields a linear subspace $\Pi \subset E \cong \mathbb{P}^{n-3}$ of dimension $n-5$ such that

$$\mathrm{mult}_Z(S_1 \cdot S_2 \cdot D) > 8m^2$$

for every divisor $D \in |-K_X|$ that satisfies the following conditions:

1) D contains the point Z and is smooth at Z;

2) the divisor $f^{-1}(D)$ contains the subvariety $\Pi \subset U$;

3) D contains no subvarieties of X of codimension 2 that are contained in the base locus of $\mathcal{M}$.

Consider a linear subsystem $\mathcal{H} \subset |\mathcal{O}_{\mathbb{P}^n}(1)|_V|$ such that

$$H \in \mathcal{H} \iff \Pi \subset (\psi \circ f)^{-1}(H).$$

Then there is a linear subspace $\Sigma \subset \mathbb{P}^n$ of dimension at most $n-4$ such that the divisors of $\mathcal{H}$ are cut out on V by hyperplanes in $\mathbb{P}^n$ passing through Σ. In particular, we see that

$$\mathrm{Bs}(\mathcal{H}) = \Sigma \cap V,$$

which implies that $\dim(\Sigma \cap V) = n-5$ by Lefschetz' theorem.

Let H be a general divisor in $\mathcal{H}$. Put $D = \psi^{-1}(H) \in |-K_X|$. Then

1) D contains the point Z and is smooth at Z;

2) $f^{-1}(D)$ contains the subvariety $\Pi \subset U$.

Suppose that the divisor D contains a subvariety $\Gamma \subset X$ of codimension 2 such that Γ is contained in the base locus of $\mathcal{M}$. Then

$$\dim(\psi(\Gamma)) = n-4,$$

but $\psi(\Gamma) \subset \Sigma \cap V$ and $\dim(\Sigma \cap V) = n-5$. This is a contradiction.

Therefore D contains no subvarieties of X of codimension 2 that are contained in the base locus of $\mathcal{M}$. Let $H_1, H_2, \ldots, H_k$ be sufficiently general divisors of the linear system $|-K_X|$ that pass through the point Z, where $k = \dim(Z) - 3$. Then

$$8m^2 = H_1 \cdot \ldots \cdot H_k \cdot S_1 \cdot S_2 \cdot D \geqslant \mathrm{mult}_Z(S_1 \cdot S_2 \cdot D) > 8m^2,$$

which is a contradiction.

Lemma 13. *We have* $\dim(Z) \geqslant \dim(X) - 4$.

Proof. Suppose that $\dim(Z) \leqslant \dim(X) - 5$. Let $H_1, H_2, \dots, H_k$ be general hyperplane sections of the variety V, where $k = \dim(Z) > 0$. Put

$$\overline{V} = \bigcap_{i=1}^{k} H_i, \qquad \overline{X} = \psi^{-1}(\overline{V}), \qquad \bar{\psi} = \psi|_{\overline{X}} \colon \overline{X} \to \overline{V}, \qquad \overline{\mathcal{M}} = \mathcal{M}|_{\overline{X}}.$$

Then $\overline{V} \subset \mathbb{P}^{n-k}$ is a smooth complete intersection of two quadrics, the morphism $\bar{\psi}$ is a double covering branched over a smooth divisor $R \cap \overline{V}$ and the linear system $\mathcal{M}$ has no fixed components. By Lefschetz' theorem, the variety $\overline{V}$ contains no linear subspaces of $\mathbb{P}^{n-k}$ of dimension $n-k-4$ (because $n-k \geqslant 7$). Let P be a point of the intersection $Z \cap \overline{X}$. Then $P \in \mathbb{CS}\bigl(\overline{X}, \frac{1}{m}\overline{\mathcal{M}}\bigr)$. Hence the proof of Lemma 12 gives a contradiction.

This proves the inequality $\dim(Z) \geqslant 4$.

Let S be a general divisor of $\mathcal{M}$. We put $\widehat{S} = \psi(S \cap R)$, $\widehat{Z} = \psi(Z \cap R)$. Then

$$\operatorname{mult}_{\widehat{Z}}(\widehat{S}) > m, \qquad \widehat{S} \sim \mathcal{O}_{\mathbb{P}^n}(m)|_R.$$

This is impossible by Proposition 3 because $\dim(\widehat{Z}) \geqslant 3$.

This proves Theorem 21 and thus completes the proof of Theorem 5. The proof of Theorem 18 yields the following result, which completes the proof of Theorem 6.

Theorem 22. *Suppose that* $k = n - 4 \geqslant 6$. *Then* X *is not birational to an elliptic fibration.*

The author would like to thank M. Grinenko, V. Iskovskikh, S. Kudryavtsev, J. Park, Yu. Prokhorov, A. Pukhlikov and V. Shokurov for useful discussions.

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Received 25/JAN/05
Translated by THE AUTHOR