Mathematical Notes, Vol. 62, No. 3, 1997

Del Pezzo Surfaces With Nonrational Singularities

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UDC 512.774.42

ABSTRACT. Normal algebraic surfaces X with the property $\mathrm{rk}(\mathrm{Div}(X)\otimes\mathbb{Q}/\equiv) = 1$, numerically ample canonical classes, and nonrational singularities are classified. It is proved, in particular, that any such surface X is a contraction of an exceptional section of a (possibly singular) relatively minimal ruled surface $\widetilde{X}$ with a nonrational base. Moreover, $\widetilde{X}$ is uniquely determined by the surface X.

KEY WORDS: numerical del Pezzo surface, relatively minimal ruled surface, numerically ample Weil divisor, normal algebraic surface.

Introduction

F. Sakai's works naturally carry over questions on the classification of algebraic surfaces to the category of normal algebraic surfaces. For a Weil divisor on such a surface, it is possible to formally define its numerical inverse image, which has good functorial properties and allows the construction of intersections of Weil $\mathbb{Q}$-divisors over $\mathbb{Q}$ (see [1]). Numerical del Pezzo surfaces and relatively minimal ruled surfaces play the same role in the Sakai classification as smooth surface with Kodaira dimension $-\infty$ in the classification of smooth algebraic surfaces.

Note that in [2] a narrower class of del Pezzo surfaces with nonrational singularities was classified.

We assume that all surfaces under consideration are normal, complex, and algebraic.

§1. Ruled surfaces

Theorem 1. *Let $\widetilde{X}$ be a smooth surface, C a smooth curve, and $\widetilde{\pi}\colon \widetilde{X} \to C$ a surjective morphism whose fibers are isomorphic to $\mathbb{P}^1$. Then*

(1) $\widetilde{X} \cong \mathbb{P}_C(\mathcal{E})$, *where $\mathcal{E}$ is a rank-2 locally free sheaf such that $H^0(\mathcal{E}) \neq 0$ and $H^0(\mathcal{E}\otimes\mathcal{F}) = 0$ for any $\mathcal{F} \in \mathrm{Pic}(\widetilde{X})$ with $\deg(\mathcal{F}) < 0$;*

(2) $e = -\deg(\mathcal{E})$ *is an invariant of the surface $\widetilde{X}$;*

(3) *there exists a section C_0 of the ruled surface $\widetilde{\pi}\colon \widetilde{X} \to C$ such that $C_0^2 = -e$;*

(4) $\mathrm{Pic}(\widetilde{X}) \cong \mathbb{Z}C_0 \oplus \widetilde{\pi}^*\,\mathrm{Pic}(C)$;

(5) $K_{\widetilde{X}} \sim -2C_0 + \widetilde{\pi}^*(K_C + \wedge^2\mathcal{E})$; *in particular,* $K_{\widetilde{X}} \equiv -2C_0 + (2g(C) - 2 - e)F$, *where F is the fiber of the morphism $\widetilde{\pi}$;*

(6) *if $e > 2g(C) - 2$, then the sheaf $\mathcal{E}$ is decomposable;*

(7) $C_\lambda^2 \geq -e$ *for any section C_λ of the ruled surface $\widetilde{\pi}\colon \widetilde{X} \to C$.*

The proof of Theorem 1 is given in [3].

Definition 1. A surface $\widehat{X}$ is *ruled* if there exists a surjective morphism $\widehat{\pi}\colon \widehat{X} \to C$ of $\widehat{X}$ onto a curve C such that the general fiber of $\widehat{\pi}$ is isomorphic to $\mathbb{P}^1$.

Remark 1. The curve C in Definition 1 is smooth, because the surface $\widehat{X}$ is normal.

Definition 2. A ruled surface $\widetilde{\pi}\colon \widetilde{X} \to C$ is *relatively minimal* if each fiber of the morphism $\widetilde{\pi}$ is irreducible (but possibly reduced).

Translated from *Matematicheskie Zametki*, Vol. 62, No. 3, pp. 451–467, September, 1997.
Original article submitted February 2, 1996.

0001–4346/97/6234–0377 $18.00

Lemma 1. *For every ruled surface $\widehat{\pi}\colon \widehat{X} \to C$, there exists a commutative diagram*

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\rho} & \widetilde{X} \\ {\scriptstyle\widehat{\pi}}\downarrow & & \downarrow{\scriptstyle\widetilde{\pi}} \\ C & \cong & C \end{array}$$

such that the morphism $\rho\colon \widehat{X} \to \widetilde{X}$ is birational and $\widetilde{\pi}\colon \widetilde{X} \to C$ is a relatively minimal ruled surface.

Proof. Let F be a reducible fiber of the morphism $\widehat{\pi}\colon \widehat{X} \to C$. Then

$$\left(\sum_{i=1}^{n} \lambda_i F_i\right)^2 \le 0 \quad \text{and} \quad \left(\sum_{i=1}^{n} \lambda_i F_i\right)^2 = 0 \iff \sum_{i=1}^{n} \lambda_i F_i = \lambda F,$$

where F_i are components of the fiber F and $\lambda_i, \lambda \in \mathbb{Q}$ (see [4]). Therefore, for any proper subset $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$, the intersection form of the divisors F_{i_j} with $j = 1, \dots, k$ is negative definite; hence all the divisors F_{i_j} are contractible (see [1]). This immediately implies the assertion of Lemma 1. □

Lemma 2. *To a relatively minimal ruled surface $\widetilde{\pi}\colon \widetilde{X} \to C$ with a section C_0, there corresponds canonically a smooth relatively minimal ruled surface $\widetilde{\pi}^s\colon \widetilde{X}^s \to C$ such that the diagram*

$$\begin{array}{ccc} \widetilde{X} & \overset{\varphi}{\dashrightarrow} & \widetilde{X}^s \\ {\scriptstyle\widetilde{\pi}}\downarrow & & \downarrow{\scriptstyle\widetilde{\pi}^s} \\ C & \cong & C \end{array},$$

where φ is a birational morphism, is commutative.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} & & \widehat{X} & & \\ & {\scriptstyle p}\swarrow & & \searrow{\scriptstyle q} & \\ \widetilde{X} & & \dashrightarrow & & \widetilde{X}^s \\ & {\scriptstyle\widetilde{\pi}}\searrow & & \swarrow{\scriptstyle\widetilde{\pi}^s} & \\ & & C & & \end{array}, \qquad (1)$$

where $\widehat{X}$ is the minimal resolution of the singularities of $\widetilde{X}$ and $\widetilde{X}^s$ is a smooth model of $\widehat{X}$ relatively minimal over C. To prove the lemma, we must show that the morphism q can be selected canonically. The fibers of the morphism p do not contain (-1)-curves, but the surface $\widetilde{\pi} \circ p\colon \widehat{X} \to C$ is not relatively minimal; therefore, each reducible fiber of the morphism $\widetilde{\pi} \circ p$ contains exactly one (-1)-curve, which is the preimage of the corresponding fiber of $\widetilde{\pi}$. Let us select q so that $q = q_1 \circ \cdots \circ q_K$ for some $K \in \mathbb{N}_{\ge 0}$ (if $K = 0$, then $\widehat{X} \cong \widetilde{X} \cong \widetilde{X}^s$), where

(1) for each $i = 1, \dots, K$, the morphism $q_i\colon \widehat{X}^i \to \widehat{X}^{i-1}$ ($\widehat{X}^K = \widehat{X}$ and $\widehat{X}^0 = \widetilde{X}^s$) is the composition of blow-ups in the fiber of the morphism $\widetilde{\pi} \circ q_1 \circ \cdots \circ q_{i-1}$ over a point $x_i \in C$, and all x_i are pairwise different;

(2) for each $i = 1, \dots, K$, $q_i^*(q_i \circ \cdots \circ q_K(p^{-1}(C_0))) \neq q_i^{-1}(q_i \circ \cdots \circ q_K(p^{-1}(C_0)))$.

It is easy to see that conditions (1)–(2) determine the morphism q uniquely. □

Remark 2. The proof of Lemma 2 yields an easy algorithm for constructing all relatively minimal ruled surfaces. It is sufficient to take a smooth relatively minimal ruled surface and then reconstruct some of its fibers as follows:

(1) blow up a point on the fiber;
(2) blow up the intersection point of the blown up curve and the preimage of the fiber (two (-1)-curves);
(3) successively perform blow-ups of a point on the current (-1)-curve in such a way that the fiber will contain only one (-1)-curve;
(4) contract all curves in the fiber except the unique (-1)-curve.

Note that nonuniqueness in the reverse passage from a singular surface to a smooth one consists in the appearance of two (-1)-curves in the fiber of the nonsingular ruled surface when the first blow-up is performed.

Theorem 2. *If $\widetilde{\pi}\colon \widetilde{X} \to C$ is a relatively minimal ruled surface, then*

(1) *$\widetilde{X}$ is a projective surface;*
(2) *$\widetilde{X}$ has no singularities worse than rational;*
(3) $R^1\widetilde{\pi}_*(\mathcal{O}_{\widetilde{X}}) = 0$;
(4) *all fibers with reduced structures are smooth and isomorphic to $\mathbb{P}^1$;*
(5) $\operatorname{rk}(\operatorname{Div}(\widetilde{X}) \otimes \mathbb{Q}/\equiv) = 2$.

Proof. (1) See [5].

(2) Consider the commutative diagram (1), where p is the minimal resolution of the singularities of $\widetilde{X}$ and q is a birational morphism onto the relatively minimal smooth ruled surface $\widetilde{\pi}^s\colon \widetilde{X}^s \to C$. It is well known that

$$R^1\widetilde{\pi}^s_*(\mathcal{O}_{\widetilde{X}^s}) = 0, \qquad R^0\widetilde{\pi}^s_*(\mathcal{O}_{\widetilde{X}^s}) = \mathcal{O}_C \qquad \text{and} \quad R^1q_*(\mathcal{O}_{\widehat{X}}) = 0, \qquad R^0q_*(\mathcal{O}_{\widehat{X}}) = \mathcal{O}_{\widetilde{X}^s}.$$

The Leray spectral sequence implies that

$$R^1(\widetilde{\pi} \circ p)_*(\mathcal{O}_{\widehat{X}}) = 0 \qquad \text{and} \qquad R^0(\widetilde{\pi} \circ p)_*(\mathcal{O}_{\widehat{X}}) = \mathcal{O}_C.$$

Suppose that $F = \sum_{i=1}^n a_i F_i$, where the F_i are the irreducible components of the fiber F and $a_i \in \mathbb{N}$. Then $R^1(\widetilde{\pi} \circ p)_*(\mathcal{O}_{\widehat{X}}) = 0$ implies that $H^1(\mathcal{O}_F) = 0$. Indeed, let $\mathcal{I}_F$ be the sheaf of the ideals of the scheme F; then the exact sequence

$$0 \to \mathcal{I}_F \to \mathcal{O}_{\widehat{X}} \to \mathcal{O}_F \to 0$$

implies the exact sequence

$$\begin{array}{ccccc} 0 & & H^1(\mathcal{O}_F) & & \\ \| & & \| & & \\ R^1(\widetilde{\pi} \circ p)_*(\mathcal{O}_{\widehat{X}}) & \longrightarrow & R^1(\widetilde{\pi} \circ p)_*(\mathcal{O}_F) & \longrightarrow & R^2(\widetilde{\pi} \circ p)_*(\mathcal{I}_F) \end{array} \quad ;$$

on the other hand, $R^2(\widetilde{\pi} \circ p)_*(\mathcal{I}_F) = 0$ from dimension considerations. Therefore, all singularities of $\widetilde{X}$ are rational (see [6]), as well as those of any surface obtained from $\widehat{X}$ by contracting components of the fibers of $\widetilde{\pi} \circ p$.

(3) As proved above, all singularities of $\widetilde{X}$ are rational, i.e., in the notation introduced in (2), we have

$$R^1p_*(\mathcal{O}_{\widehat{X}}) = 0 \qquad \text{and} \qquad R^0p_*(\mathcal{O}_{\widehat{X}}) = \mathcal{O}_{\widetilde{X}}.$$

The Leray spectral sequence implies that

$$R^1\widetilde{\pi}_*(\mathcal{O}_{\widetilde{X}}) = 0 \qquad \text{and} \qquad R^0\widetilde{\pi}_*(\mathcal{O}_{\widetilde{X}}) = \mathcal{O}_C.$$

(4) The argument from (2) and (3) shows that if F is a reduced fiber of $\widetilde{\pi}$, then $H^1(\mathcal{O}_F) = 0$ and $F \cong \mathbb{P}^1$.

Note that Remark 2 allows us to find fundamental cycles (see [6]) of singularities of the surface $\widetilde{X}$. The intersection of the only (-1)-curve in a given reducible fiber of $\widetilde{\pi} \circ p$ with the corresponding fundamental cycles equals one, which implies (4) (see [6]).

(5) See [1]. □

Remark 3. The proof of Theorem 2 implies that all singularities of a ruled surface are rational.

§2. Numerical del Pezzo surfaces

Definition 3. A Weil divisor D on a surface X is called *numerically ample* if for each curve $C \in X$, the inequalities $DC > 0$ and $D^2 > 0$ hold.

Definition 4. A surface X is said to be a *numerical del Pezzo surface* if $-K_X$ is a numerically ample Weil divisor.

Lemma 3. *Let X be a numerical del Pezzo surface. Then*

(1) $H^i(\mathcal{O}_X) = 0$ *for* $i = 1, 2$;
(2) X *is a projective surface.*

For the proof of (1), see [1], and for that of (2), see [5].

Lemma 4. *Let X be a numerical del Pezzo surface and $f \colon \widehat{X} \to X$ a resolution of singularities of X. Then*

(1) $H^1(\mathcal{O}_{\widehat{X}}) \cong H^0(R^1 f_*(\mathcal{O}_{\widehat{X}}))$ *and* $H^2(\mathcal{O}_{\widehat{X}}) = 0$;
(2) $\operatorname{kod}(\widehat{X}) = -\infty$.

Proof. (1) Lemma 3, the normality of X, and the Leray spectral sequence imply the exact sequence

$$\begin{array}{ccccccccc} & & H^1(\mathcal{O}_X) = 0 & & & & H^2(\mathcal{O}_X) = 0 & & \\ & & \| & & & & \| & & \\ 0 & \longrightarrow & H^1(R^0 f_*(\mathcal{O}_{\widehat{X}})) & \longrightarrow & H^1(\mathcal{O}_{\widehat{X}}) & \longrightarrow & H^0(R^1 f_*(\mathcal{O}_{\widehat{X}})) & \longrightarrow & H^2(R^0 f_*(\mathcal{O}_{\widehat{X}})) \\ & & & & & & \longrightarrow & H^2(\mathcal{O}_{\widehat{X}}) & \longrightarrow 0 \end{array},$$

which proves the required assertion.

(2) If there exists an effective divisor $D \in |K_{\widehat{X}}|$, then $K_X = f_*(D)$, which is impossible, because $-D$ is a numerically ample divisor on a projective surface (see Lemma 3). □

Corollary. *A numerical del Pezzo surface is rational if and only if its singularities are rational.*

§3. Numerical del Pezzo surfaces with nonrational singularities

Theorem 3. *Let X be a numerical del Pezzo surface with nonrational singularities, and let $f \colon \widehat{X} \to X$ be its minimal resolution of singularities. Then*

(1) *there exists a morphism π such that $\widehat{\pi} \colon \widehat{X} \to C$ is a ruled surface and $g(C) = H^1(\mathcal{O}_{\widehat{X}}) \neq 0$;*
(2) *the morphism f contracts one smooth curve E not lying in the fibers of the morphism $\widehat{\pi}$; moreover, E is a section of the morphism $\widehat{\pi}$;*
(3) *if $\widetilde{\pi}^s \colon \widetilde{X}^s \to C$ is a model of the ruled surface $\widehat{X}$ and $\widetilde{\pi}^s$ is relatively minimal over C, then*

$$\widehat{\pi} = \widetilde{\pi}^s \circ \rho, \qquad \widetilde{X}^s \cong \mathbb{P}_C(\mathcal{E}), \qquad e > 2g(C) - 2 \qquad \text{and} \qquad \rho(E)^2 = -e,$$

where $\mathcal{E}$ is a decomposable locally free sheaf of rank 2 and e an invariant of $\mathbb{P}_C(\mathcal{E})$.

Proof. (1) The assertion of the theorem immediately follows from Lemma 4 and the corollary.

(2) Note that the morphism f contracts at least one curve not lying in the fibers of $\widehat{\pi}$, because otherwise, all singularities of X would be rational by Remark 3. Let E_j, where $j = 1, \dots, k$, be the irreducible reduced curves not lying in the fibers of $\widehat{\pi}$ and contracted by f. Then

$$K_{\widehat{X}} \equiv f^*(K_X) - \sum_{i=1}^{n} a_i F_i - \sum_{j=1}^{k} b_j E_j,$$

where F_i are exceptional curves of f lying in the fibers of $\widehat{\pi}$ and $a_i, b_j \in \mathbb{Q}_{\geq 0}$ for $i = 1, \dots, n$ and $j = 1, \dots, k$. The adjunction formula gives $(K_{\widehat{X}} + E_r)E_r \geq 2g(\widetilde{E}_r) - 2$, where $r \in \{1, \dots, k\}$ and $\widetilde{E}_r$ is a normalization of the curve $\widetilde{E}_r$. By the Hurwitz formula, $2g(\widetilde{E}_r) - 2 \geq 2g(C) - 2 \geq 0$; therefore,

$$(1 - b_r)E_r^2 \geq \Big(-\sum_{i=1}^{n} a_i F_i - \sum_{j=1, j\neq r}^{k} b_j E_j - (b_r - 1)E_r\Big)E_r \geq 0.$$

Thus, all the b_j are greater than or equal to one. If L is a fiber of $\widehat{\pi}$, then

$$-2 = K_{\widehat{X}}L = \Big(f^*(K_X) - \sum_{i=1}^{n} a_i F_i - \sum_{j=1}^{k} b_j E_j\Big)L < \Big(-\sum_{j=1}^{k} b_j E_j\Big)L;$$

therefore, $k = 1$, $b = b_1 < 2$, and $E = E_1 \cong \widetilde{E}_1$ is a section of the ruled surface $\widehat{\pi}\colon \widehat{X} \to C$.

(3) Let C_0 be a section of the ruled surface $\widetilde{\pi}^s\colon \widetilde{X}^s \to C$ such that $C_0^2 = -e$. Then

$$\rho(E) \equiv C_0 + dF,$$

where F is a fiber of the morphism $\widetilde{\pi}^s$ and $d \in \mathbb{N}$ by Theorem 1. In the notation introduced in (2), we have

$$\rho\Big(\sum_{i=1}^{n} a_i F_i\Big) \equiv aF, \qquad K_{\widetilde{X}^s} + \rho\Big(\sum_{i=1}^{n} a_i F_i + bE\Big) \equiv (b-2)C_0 + (2g(C) - 2 - e + a + db)F,$$

where $a \in \mathbb{Q}_{\geq 0}$. If $C_0 \neq \rho(E)$, then $\rho(E)C_0 = d - e \geq 0$ and

$$\begin{aligned} bd - be + 2g(C) - 2 + e + a &= \Big(K_{\widetilde{X}^s} + \rho\Big(\sum_{i=1}^{n} a_i F_i + bE\Big)\Big)C_0 \\ &= \Big(K_{\widehat{X}} + \sum_{i=1}^{n} a_i F_i + bE\Big)\rho^*(C_0) = f^*(K_X)\rho^*(C_0) < 0. \end{aligned}$$

But if $e \geq 0$, then

$$bd - be + 2g(C) - 2 + e + a > b(d - e) \geq 0,$$

and if $e < 0$, then

$$bd - be + 2g(C) - 2 + e + a > e(1 - b) \geq 0.$$

Therefore, $C_0 = \rho(E)$. Similarly,

$$\begin{aligned} be + 2g(C) - 2 + e + a &= \Big(K_{\widetilde{X}^s} + \rho\Big(\sum_{i=1}^{n} a_i F_i + bE\Big)\Big)C_0 \\ &= \Big(K_{\widehat{X}} + \sum_{i=1}^{n} a_i F_i + bE\Big)\rho^*(C_0) = f^*(K_X)\rho^*(C_0). \end{aligned}$$

Note that if $\rho^{-1}(C_0) \neq \rho^*(C_0)$, then $f^*(K_X)\rho^*(C_0) < 0$, because in this case, $\rho^*(C_0)$ contains a (-1)-curve that cannot be contracted by the morphism f. Suppose that $C_0^2 = -e \geq 0$; then $\rho^{-1}(C_0) \neq \rho^*(C_0)$ and

$$0 > f^*(K_X)\rho^*(C_0) = (1 - b)e + 2g(C) - 2 + a \geq 0.$$

Therefore, $e > 0$ and

$$0 > f^*(K_X)\rho^*(C_0) = (1 - b)e + 2g(C) - 2 + a \geq -e + 2g(C) - 2.$$

By Theorem 1, this implies that the sheaf $\mathcal{E}$ is decomposable. □

Theorem 4. *Let the conditions of Theorem 3 be fulfilled, and let* $\mathrm{rk}(\mathrm{Div}(X)\otimes\mathbb{Q}/\equiv)=1$. *Then* X *is a contraction of a section of a relatively minimal ruled surface* $\widetilde{\pi}\colon \widetilde{X}\to C$, *and* $h^1(\mathcal{O}_{\widetilde{X}})=g(C)>0$. *Moreover, the surface* $\widetilde{X}$ *is uniquely determined by* X.

Proof. Let $f\colon \widehat{X}\to X$ be the minimal resolution of the singularities of X. By Theorem 3, $\widehat{X}$ is then a ruled surface $\widehat{\pi}\colon \widehat{X}\to C$ such that $g(C)>0$ and f contracts one section and the components of reducible fibers of the morphism $\widehat{\pi}$. Let

$$F^{\lambda}=\sum_{i=1}^{j_\lambda} a_i F_i^{\lambda}, \qquad \text{where} \quad \lambda=1,\dots,N \quad \text{and} \quad a_i\in\mathbb{N},$$

be the reducible fibers of $\widehat{\pi}$. Then

$$\mathrm{rk}(\mathrm{Div}(\widehat{X})\otimes\mathbb{Q}/\equiv)=2+\sum_{\lambda=1}^{N}(j_\lambda-1).$$

On the other hand,

$$\mathrm{rk}(\mathrm{Div}(\widehat{X})\otimes\mathbb{Q}/\equiv)=1+\text{the number of curves contracted by } f.$$

Therefore, f cannot contract only one component in each reducible fiber, and we have the commutative diagram

$$\begin{array}{ccccc} \widehat{X} & \xrightarrow{p} & \widetilde{X} & \xrightarrow{g} & X \\ {\scriptstyle\widehat{\pi}}\downarrow & & \downarrow{\scriptstyle\widetilde{\pi}} & & \\ C & \cong & C & & \end{array},$$

where $f=g\circ p$, $\widetilde{\pi}\colon \widetilde{X}\to C$ is a relatively minimal ruled surface, and g is a morphism contracting a section of $\widetilde{\pi}$.

Since $h^1(\mathcal{O}_{\widehat{X}})=g(C)$ and all singularities of $\widetilde{X}$ are rational by Theorem 2, the Leray spectral sequence implies that $h^1(\mathcal{O}_{\widetilde{X}})=g(C)>0$.

The uniqueness of the surface $\widetilde{X}$ follows from its construction. □

Theorem 5. *Let the conditions of Theorem 4 be fulfilled. Then to the surface* X *there corresponds canonically a smooth relatively minimal ruled surface* $\widetilde{\pi}^s\colon \widetilde{X}^s\to C$ *such that* $\widetilde{X}^s\cong\mathbb{P}_C(\mathcal{E})$, *where* $\mathcal{E}$ *is a rank-2 locally free sheaf,* $e>2g(C)-2$ *(*e *is an invariant of the ruled surface* $\mathbb{P}_C(\mathcal{E})$*), the sheaf* $\mathcal{E}$ *is decomposable, and* $q\big(p^{-1}(E)\big)^2=-e$.

The proof of Theorem 5 follows from Theorems 3 and 4 and Lemma 2.

§4. The construction

Consider a pair $(\widehat{\pi}\colon \widehat{X}\to C,\ C_0)$, where $\widehat{\pi}\colon \widehat{X}\to C$ is a smooth ruled surface and C_0 its section. We say that a pair $(\widehat{\pi}'\colon \widehat{X}'\to C,\ C_0')$ is *obtained by an elementary transformation* φ *associated to a point* $x\in C$ from the pair $(\widehat{\pi}\colon \widehat{X}\to C,\ C_0)$ if there exists a commutative diagram

$$\begin{array}{ccc} \widehat{X}' & \xrightarrow{\varphi} & \widehat{X} \\ {\scriptstyle\widehat{\pi}'}\downarrow & & \downarrow{\scriptstyle\widehat{\pi}} \\ C & \cong & C \end{array}$$

such that

(1) $\widehat{\pi}'\colon \widehat{X}'\to C$ is a smooth ruled surface;
(2) φ is a birational morphism and a composition of blow-ups in the fiber of the morphism $\widehat{\pi}$ over the point $x\in C$;
(3) the fiber of the morphism $\widehat{\pi}$ over the point $x\in C$ is irreducible;
(4) the fiber of $\widehat{\pi}'$ over x contains exactly one (-1)-curve;
(5) $C_0'=\varphi^{-1}(C_0)$ and $\varphi^*(C_0)\neq C_0'$.

Definition 5. A sequence of pairs of integers (α_i^1, α_i^2) with $i \in \mathbb{N}_{\geq 3}$ *has property* $(*)$ if

(1) $(\alpha_3^1, \alpha_3^2) = (1, 1)$;

(2) for $i \geq 4$,

$$(\alpha_i^1, \alpha_i^2) = \begin{cases} (\alpha_{i-1}^1, \alpha_{i-1}^1 + \alpha_{i-1}^2), & \text{or} \\ (\alpha_{i-1}^1 + \alpha_{i-1}^2, \alpha_{i-1}^2), & \text{or} \\ (0, \alpha_{i-1}^1 + \alpha_{i-1}^2). \end{cases}$$

Consider a pair $(\widehat{\pi}'\colon \widehat{X}' \to C,\ C_0')$ obtained by an elementary transformation φ associated to a point $x \in C$ from a pair $(\widehat{\pi}\colon \widehat{X} \to C,\ C_0)$. Let us introduce the following notation:

(1) $\widehat{X}_0 = \widehat{X}$, $\widehat{\pi}_0 = \widehat{\pi}$, and F_1 is the fiber of the morphism $\widehat{\pi}_0$ over the point x;

(2) $\chi_{1,0}\colon \widehat{X}_1 \to \widehat{X}_0$ is a blow-up of the point $F_1 \cap C_0$, $\widehat{\pi}_1 = \widehat{\pi}_0 \circ \chi_{1,0}$, and F_2 is an exceptional curve of the morphism $\chi_{1,0}$;

(3) $\chi_{2,1}\colon \widehat{X}_2 \to \widehat{X}_1$ is a blow-up of the point $F_1 \cap F_2$, $\chi_{2,0} = \chi_{1,0} \circ \chi_{2,1}$, $\widehat{\pi}_2 = \widehat{\pi}_1 \circ \chi_{2,1}$, and F_3 is an exceptional curve of the morphism $\chi_{2,1}$;

(4) $\chi_{i+1,i}\colon \widehat{X}_{i+1} \to \widehat{X}_i$ is a blow-up of a point on F_{i+1}, $\chi_{i+1,j} = \chi_{j+1,j} \circ \cdots \circ \chi_{i+1,i}$ for $j \leq i$, $\widehat{\pi}_{i+1} = \widehat{\pi}_i \circ \chi_{i+1,i}$, and F_{i+2} is an exceptional curve of the morphism $\chi_{i+1,i}$;

(5) F^r is a (possibly nonreduced) fiber of the morphism $\widehat{\pi}_r$ over the point x;

(6) $\widehat{X}_N = \widehat{X}'$, $\widehat{\pi}_N = \widehat{\pi}'$, and F_{N+1} is a unique (-1)-curve in F^N;

(7) $C_0' = \chi_{N,0}^{-1}(C_0)$ and $\bar{F}_i = \chi_{N,i-1}^{-1}(F_i)$ for $i = 1, \ldots, N-1$.

Let us denote the number of all irreducible components in the fiber of $\widehat{\pi}'$ over the point $x \in C$ by $N+1$ and put the surface $\widehat{X}'$ in correspondence with the sequence

$$(\bar{\alpha}_i^1(\widehat{X}'), \bar{\alpha}_i^2(\widehat{X}')), \qquad i = 3, \ldots, N+1, \tag{2}$$

of pairs of integers. Take $i \in \{1, \ldots, N-1\}$ and consider the surface $\widehat{X}_{i+1}$. We have

$$F^{i+1} \equiv a_{i+2}F_{i+2} + \sum_{j=1}^{i+1} a_j \chi_{i+1,j-1}^{-1}(F_j);$$

F_{i+2} is the unique (-1)-curve in F^{i+1}, and it intersects no more than two irreducible components of F^{i+1}. If F_{i+2} intersects $\chi_{i+1,k-1}^{-1}(F_k)$ and $\chi_{i+1,l-1}^{-1}(F_l)$ so that $\chi_{i+1,l-1}^{-1}(F_l)$ lies in a connected component of $F^{i+1} \setminus F_{i+2}$ meeting $\chi_{i+1,0}^{-1}(C_0)$, where $l \neq k$ and $k, l \in \{1, \ldots, i+1\}$, then we put

$$(\bar{\alpha}_{i+2}^1(\widehat{X}'), \bar{\alpha}_{i+2}^2(\widehat{X}')) = (a_k, a_l).$$

Suppose that F_{i+2} intersects only $\chi_{i+1,k-1}^{-1}(F_k)$ among all components of F^{i+1} ($k \in \{1, \ldots, i+1\}$); then $k = i+1$. In this case we put

$$(\bar{\alpha}_{i+2}^1(\widehat{X}'), \bar{\alpha}_{i+2}^2(\widehat{X}')) = (0, a_{i+1}).$$

Lemma 5. *The sequence* (2) *of pairs of integers has property* $(*)$.

Proof. We shall use the notation

$$(\bar{\alpha}_i^1(\widehat{X}'), \bar{\alpha}_i^2(\widehat{X}')) = (\bar{\alpha}_i^1, \bar{\alpha}_i^2) \qquad \text{for} \quad i = 3, \ldots, N+1. \tag{3}$$

On the surface $\widehat{X}_2$, the relation

$$F^2 \equiv 2F_3 + \chi_{2,1}^{-1}(F_2) + \chi_{2,0}^{-1}(F_1)$$

holds. It can be verified directly that $(\bar{\alpha}_3^1, \bar{\alpha}_3^2) = (1, 1)$.

Suppose that the sequence of pairs $(\bar{\alpha}_i^1, \bar{\alpha}_i^2)$ has property $(*)$ with $i = 3, \dots, r$. Let us prove that this sequence has property $(*)$ with $i = 3, \dots, r+1$.

On the surfaces $\widehat{X}_{r-1}$ and $\widehat{X}_r$ we have the relations

$$F^{r-1} \equiv a_r F_r + \sum_{j=1}^{r-1} a_j \chi_{r-1,j-1}^{-1}(F_j) \qquad \text{and} \qquad F^r \equiv a_{r+1} F_{r+1} + \sum_{j=1}^{r} a_j \chi_{r,j-1}^{-1}(F_j).$$

Suppose that F_r intersects $\chi_{r-1,k-1}^{-1}(F_k)$ and $\chi_{r-1,l-1}^{-1}(F_l)$, and $\chi_{r-1,l-1}^{-1}(F_l)$ lies in a connected component of $F^{r-1} \setminus F_r$ intersecting $\chi_{r-1,0}^{-1}(C_0)$, where $l \neq k$ and $k, l \in \{1, \dots, r-1\}$. By assumption, $(\bar{\alpha}_r^1, \bar{\alpha}_r^2) = (a_k, a_l)$. Consider three cases.

(1) Let $\chi_{r,r-1} \colon \widehat{X}_r \to \widehat{X}_{r-1}$ be a blow-up of $F_r \cap \chi_{r-1,k-1}^{-1}(F_k)$. Then F_{r+1} intersects $\chi_{r,r-1}^{-1}(F_r)$ and $\chi_{r,k-1}^{-1}(F_k)$, and $\chi_{r,r-1}^{-1}(F_r)$ lies in a connected component of $F^r \setminus F_{r+1}$ meeting $\chi_{r,0}^{-1}(C_0)$. By definition, $(\bar{\alpha}_{r+1}^1, \bar{\alpha}_{r+1}^2) = (a_k, a_r)$, where $a_r = a_k + a_l$.

(2) Let $\chi_{r,r-1} \colon \widehat{X}_r \to \widehat{X}_{r-1}$ be a blow-up of $F_r \cap \chi_{r-1,l-1}^{-1}(F_l)$. Then F_{r+1} intersects $\chi_{r,r-1}^{-1}(F_r)$ and $\chi_{r,k-1}^{-1}(F_k)$, and $\chi_{r,k-1}^{-1}(F_k)$ lies in a connected component of $F^r \setminus F_{r+1}$ meeting $\chi_{r,0}^{-1}(C_0)$. By definition, $(\bar{\alpha}_{r+1}^1, \bar{\alpha}_{r+1}^2) = (a_r, a_l)$, where $a_r = a_k + a_l$.

(3) Finally, let $\chi_{r,r-1} \colon \widehat{X}_r \to \widehat{X}_{r-1}$ be a blow-up of a point on F_r not belonging to the union $\chi_{r-1,k-1}^{-1}(F_k) \cup \chi_{r-1,l-1}^{-1}(F_l)$. Then F_{r+1} intersects only $\chi_{r,r-1}^{-1}(F_r)$ among all components of F^r. By definition, $(\bar{\alpha}_{r+1}^1, \bar{\alpha}_{r+1}^2) = (0, a_r)$, where $a_r = a_k + a_l$.

Suppose that F_r intersects only $\chi_{r-1,r-2}^{-1}(F_{r-1})$ among all components of F^{r-1}. By assumption, $(\bar{\alpha}_r^1, \bar{\alpha}_r^2) = (0, a_{r-1})$. Note that $\chi_{r,r-1} \colon \widehat{X}_r \to \widehat{X}_{r-1}$ is a blow-up of either $F_r \cap \chi_{r-1,r-2}^{-1}(F_{r-1})$ or a point on F_r not belonging to $\chi_{r-1,r-2}^{-1}(F_{r-1})$. Consider two cases.

(1) Let $\chi_{r,r-1} \colon \widehat{X}_r \to \widehat{X}_{r-1}$ be a blow-up of $F_r \cap \chi_{r-1,r-2}^{-1}(F_{r-1})$. Then F_{r+1} intersects $\chi_{r,r-1}^{-1}(F_r)$ and $\chi_{r,r-2}^{-1}(F_{r-1})$, and $\chi_{r,r-2}^{-1}(F_{r-1})$ lies in a connected component of $F^r \setminus F_{r+1}$ meeting $\chi_{r,0}^{-1}(C_0)$. By definition, $(\bar{\alpha}_{r+1}^1, \bar{\alpha}_{r+1}^2) = (a_r, a_{r-1})$, where $a_r = a_{r-1}$.

(2) Now, let $\chi_{r,r-1} \colon \widehat{X}_r \to \widehat{X}_{r-1}$ be a blow-up of a point on F_r not belonging to $\chi_{r-1,r-2}^{-1}(F_{r-1})$. Then F_{r+1} intersects only $\chi_{r,r-1}^{-1}(F_r)$ among all components of F^r. By definition, $(\bar{\alpha}_{r+1}^1, \bar{\alpha}_{r+1}^2) = (0, a_r)$, where $a_r = a_{r-1}$.

In all the cases, the sequence of pairs $(\bar{\alpha}_i^1, \bar{\alpha}_i^2)$ has property $(*)$ with $i = 3, \dots, r+1$. The lemma is proved. □

Lemma 6. *Let a sequence of pairs (α_i^1, α_i^2) of integers have property $(*)$. Then there exists a unique pair $(\widehat{\pi}' \colon \widehat{X}' \to C,\ C_0')$ that is obtained by an elementary transformation φ associated to a point $x \in C$ from the pair $(\widehat{\pi} \colon \widehat{X} \to C,\ C_0)$ and satisfies relations* (3).

Proof. Let us find all such surfaces $\widehat{X}_r$ by induction. Suppose that we have already found the surface $\widehat{X}_r$ for some $r \in \{2, \dots, N\}$. Let us find $\widehat{X}_{r+1}$.

Suppose that F_{r+1} intersects $\chi_{r,k-1}^{-1}(F_k)$ and $\chi_{r,l-1}^{-1}(F_l)$, and $\chi_{r,l-1}^{-1}(F_l)$ lies in a connected component of $F^r \setminus F_{r+1}$ intersecting $\chi_{r,0}^{-1}(C_0)$, where $l \neq k$ and $k, l \in \{1, \dots, r\}$. Consider three cases.

(1) If $(\alpha_{r+1}^1, \alpha_{r+1}^2) = (0, \alpha_r^1 + \alpha_r^2)$, then $\chi_{r+1,r} \colon \widehat{X}_{r+1} \to \widehat{X}_r$ is a blow-up of a point on F_{r+1} not belonging to $\chi_{r,l-1}^{-1}(F_l) \cup \chi_{r,l-1}^{-1}(F_k)$.

(2) If $(\alpha_{r+1}^1, \alpha_{r+1}^2) = (\alpha_r^1, \alpha_r^2 + \alpha_r^1)$, then $\chi_{r+1,r} \colon \widehat{X}_{r+1} \to \widehat{X}_r$ is a blow-up of a point belonging to $F_{r+1} \cap \chi_{r,k-1}^{-1}(F_k)$.

3) Finally, if $(\alpha_{r+1}^1, \alpha_{r+1}^2) = (\alpha_r^1 + \alpha_r^2, \alpha_r^2)$, then $\chi_{r+1,r} \colon \widehat{X}_{r+1} \to \widehat{X}_r$ is a blow-up of a point on $F_{r+1} \cap \chi_{r,l-1}^{-1}(F_l)$.

Suppose that F_{r+1} intersects only $\chi_{i+1,r-1}^{-1}(F_r)$ among all irreducible components of F^r. Consider two cases.

(1) If $(\alpha^1_{r+1}, \alpha^2_{r+1}) = (0, \alpha^2_r)$, then $\chi_{r+1,r}\colon \widehat{X}_{r+1} \to \widehat{X}_r$ is a blow-up of a point on F_{r+1} not belonging to $\chi^{-1}_{r,r-1}(F_r)$.

(2) If $(\alpha^1_{r+1}, \alpha^2_{r+1}) = (\alpha^2_r, \alpha^2_r)$, then $\chi_{r+1,r}\colon \widehat{X}_{r+1} \to \widehat{X}_r$ is a blow-up of a point on $F_{r+1} \cap \chi^{-1}_{r,r-1}(F_r)$.

It is easy to see that the surface $\widehat{X}_N = \widehat{X}'$ thus obtained is unique and satisfies relations (3). □

Definition 6. A sequence (β^1_i, β^2_i), $i \in \mathbb{N}_{\geq 3}$, of pairs of integers is *dual* to a sequence (α^1_i, α^2_i), $i \in \mathbb{N}_{\geq 3}$, of pairs of integers with property $(*)$ if

(1) $(\beta^1_3, \beta^2_3) = (0, -1)$;

(2) for $i \geq 4$,

$$(\beta^1_i, \beta^2_i) = \begin{cases} (\beta^1_{i-1}, \beta^1_{i-1} + \beta^2_{i-1} + 1) & \text{if } (\alpha^1_i, \alpha^2_i) = (\alpha^1_{i-1}, \alpha^1_{i-1} + \alpha^2_{i-1}); \\ (\beta^1_{i-1} + \beta^2_{i-1} + 1, \beta^2_{i-1}) & \text{if } (\alpha^1_i, \alpha^2_i) = (\alpha^1_{i-1} + \alpha^2_{i-1}, \alpha^2_{i-1}); \\ (0, \beta^1_{i-1} + \beta^2_{i-1} + 1) & \text{if } (\alpha^1_i, \alpha^2_i) = (0, \alpha^1_{i-1} + \alpha^2_{i-1}). \end{cases}$$

§5. Classification

Suppose we are given:

(1) a smooth relatively minimal ruled surface $\pi^0\colon X^0 \to C$ with an invariant e for which we have $X^0 \cong \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L})$, where $\mathcal{L} \in \operatorname{Pic}(C)$, C_0 is a unique section of this ruled surface, $C_0^2 = -e$, and $e = -\deg(\mathcal{L}) > 2g(C) - 2$;

(2) a set of pairwise different points $\{x_1, \dots, x_K\} \subset C$, possibly empty (for $K = 0$);

(3) smooth ruled surfaces $\pi^d\colon X^d \to C$ with sections $C_d \subset X^d$, where $d = 1, \dots, K$, such that for each $d = 1, \dots, K$, the pair $(\pi^d\colon X^d \to C,\ C_d)$ is obtained by an elementary transformation φ_d associated to a point $x_d \in C$ from the pair $(\pi^{d-1}\colon X^{d-1} \to C,\ C_{d-1})$.

If $K \geq 1$, then we apply the construction from the preceding section to put each of the surfaces X^d $(d = 1, \dots, K)$ in correspondence with the sequence of pairs of integers

$$(\bar{\alpha}^1_i(X^d), \bar{\alpha}^2_i(X^d)), \qquad i = 3, \dots, N(d) + 1,$$

and the dual sequence

$$(\bar{\beta}^1_i(X^d), \bar{\beta}^2_i(X^d)), \qquad i = 3, \dots, N(d) + 1,$$

where $N(d)+1$ is the number of irreducible components in the fiber of the morphism π^d over the point x_d. To the surface X_K we assign $2K$ sequences of pairs of integers: these are

$$(\bar{\alpha}^1_i(X^K, d), \bar{\alpha}^2_i(X^K, d)) = (\bar{\alpha}^1_i(X^d), \bar{\alpha}^2_i(X^d)) \qquad (\bar{\beta}^1_i(X^K, d), \bar{\beta}^2_i(X^K, d)) = (\bar{\beta}^1_i(X^d), \bar{\beta}^2_i(X^d))$$

with $d = 1, \dots, K$ and $i = 3, \dots, N(d) + 1$, where $N(d) + 1$ is the number of irreducible components of the fiber of π^K over x_d.

Lemma 7. *In the notation introduced in this section,*

(1) *the sequence* $(\bar{\alpha}^1_i(X^K, d), \bar{\alpha}^2_i(X^K, d))$ *with* $i = 3, \dots, N(d) + 1$ *has property* $(*)$ *for each* $d = 1, \dots, K$;

(2) *for any* K *sequences of pairs of integers*

$$(\alpha^1_i(d), \alpha^2_i(d)) \qquad \text{with} \quad d = 1, \dots, K \quad \text{and} \quad i = 3, \dots, R(d) + 1, \quad \text{where} \quad R(d) \in \mathbb{N}_{\geq 2},$$

having property $(*)$, *there exists a unique smooth ruled surface* $\pi^K\colon X^K \to C$ *with section* C_K *such that the pair* $(\pi^K\colon X^K \to C,\ C_K)$ *is obtained from the pair* $(\pi^0\colon X^0 \to C,\ C_0)$ *with the help of a sequence of elementary transformations associated to the points* $\{x_1, \dots, x_K\} \subset C$ *and*

$$(\bar{\alpha}^1_i(X^K, d), \bar{\alpha}^2_i(X^K, d)) = (\alpha^1_i(d), \alpha^2_i(d)) \qquad \text{for} \quad d = 1, \dots, K \quad \text{and} \quad i = 3, \dots, R(d) + 1,$$

where $R(d) + 1$ *is the number of irreducible components in the fiber of the morphism* π^K *over the point* x_d.

This lemma follows from Lemmas 5 and 6.

Lemma 8. *In the notation introduced in this section, let $F(d)$ be the (possibly nonreduced) fiber of the morphism π^K over the point x_d, where $d = 1\dots, K$ and $K \geq 0$. Suppose that on the surface X^K,*

$$F(d) \equiv \sum_{j=1}^{N(d)+1} a_j(d)\bar{F}_j(d) \qquad \text{and} \qquad K_{X^K} \equiv -2C_K + \sum_{d=1}^{K}\sum_{j=1}^{N(d)+1} b_j(d)\bar{F}_j(d) + (2g(C) - 2 - e)F,$$

where the sets $\bar{F}_j(d)$ are the irreducible components of the fiber of π^K over the point denoted by x_d in §4 and F, the general fiber of π^K. Then

$$\bar{\alpha}_i^1(X^K, d) + \bar{\alpha}_i^2(X^K, d) = a_i(d) \qquad \text{and} \qquad \bar{\beta}_i^1(X^K, d) + \bar{\beta}_i^2(X^K, d) + 1 = b_i(d)$$
$$\text{for} \quad d = 1, \dots, K \quad \text{and} \quad i = 3, \dots, N(d) + 1.$$

Proof. This lemma follows from elementary properties of blow-ups and the definition of the sequences $\bar{\alpha}_i^1(X^K, d)$ and $\bar{\beta}_i^1(X^K, d)$ with $d = 1, \dots, K$ and $i = 3, \dots, N(d) + 1$. □

Lemma 9. *In the notation introduced in this section, let*

$$\sum_{d=1}^{K} \frac{b_{N(d)+1}(d)}{a_{N(d)+1}(d)} < 2 - 2g(C) + e.$$

Then

(1) *there exist positive rationals $\lambda_j(d)$ and γ such that*

$$K_{X^K} \equiv -2C_K - \sum_{d=1}^{K}\sum_{j=1}^{N(d)+1} \lambda_j(d)\bar{F}_j(d) - \gamma F, \tag{4}$$

where $\bar{F}_j(d)$ is an irreducible reduced component of the fiber of the morphism π^K over the point x_d and F is the general fiber of π^K;

(2) *for $i = 3, \dots, N(d)$, we have*

$$\frac{b_i(d)}{a_i(d)} \leq \frac{b_{N(d)+1}(d)}{a_{N(d)+1}(d)};$$

(3) *the intersection form of the curves C_K and $\bar{F}_r(k)$, where $k = 1, \dots, K$ and $r = 1, \dots, N(k)$, is negative definite.*

Proof. (1) Suppose that

$$2 - 2g(C) + e = \sum_{d=0}^{K} \varepsilon_d,$$

where $\varepsilon_d \in \mathbb{Q}_{>0}$ and $\varepsilon_d > b_{N(d)+1}(d)/a_{N(d)+1}(d)$ for $d = 1, \dots, K$. Then we have relation (4), where $\bar{F}_j(d)$ is an irreducible component of the fiber of π^K over x_d, F is the general fiber of π^K, $\gamma = \varepsilon_0 > 0$, and

$$\lambda_{N(d)+1}(d) = a_{N(d)+1}(d)\varepsilon_d - b_{N(d)+1}(d) > 0 \qquad \text{for} \quad d = 1, \dots, K.$$

Let us prove that $\lambda_j(d) > 0$ for $d = 1, \dots, K$ and $j = 1, \dots, N(d)$. If this were not so, then there would exist $k \in \{1, \dots, K\}$ and $\mathcal{J} \subset \{1, \dots, N(k)\}$ such that $\bigcup_{j\subset\mathcal{J}} \bar{F}_j(k)$ would be connected and $\lambda_j(k) \leq 0$ for all $j \subset \mathcal{J}$. There is no (-1)-curve among $\bar{F}_j(k)$ with $j \subset \mathcal{J}$, and the intersection form of the curves $\bar{F}_j(k)$ is negative definite (see [4]). By the adjunction formula,

$$K_{X_K}\bar{F}_j(k) + \bar{F}_j(k)^2 \geq -2 \qquad \text{for} \quad j \subset \mathcal{J}.$$

Therefore, for all $j \subset \mathcal{J}$, we have $K_{X^K}\bar{F}_j(k) \geq 0$ and

$$0 \geq K_{X^K}\Big(\sum_{j\subset\mathcal{J}} \lambda_j(k)\bar{F}_j(k)\Big) = \Big(-2C_K - \sum_{j=1}^{N(k)+1} \lambda_j(k)\bar{F}_j(k)\Big)\Big(\sum_{j\subset\mathcal{J}} \lambda_j(k)\bar{F}_j(k)\Big)$$
$$= -\Big(\sum_{j\subset\mathcal{J}} \lambda_j(k)\bar{F}_j(k)\Big)^2 - 2C_K\Big(\sum_{j\subset\mathcal{J}} \lambda_j(k)\bar{F}_j(k)\Big) - \Big(\sum_{j=1, j\not\subset\mathcal{J}}^{N(k)+1} \lambda_j(k)\bar{F}_j(k)\Big)\Big(\sum_{j\subset\mathcal{J}} \lambda_j(k)\bar{F}_j(k)\Big).$$

On the other hand,

$$-\Big(\sum_{j\subset\mathcal{J}} \lambda_j(k)\bar{F}_j(k)\Big)^2 \geq 0;$$

the equality holds if $\lambda_j(k) = 0$ for all $j \subset \mathcal{J}$. Clearly,

$$-2C_K\Big(\sum_{j\subset\mathcal{J}} \lambda_j(k)\bar{F}_j(k)\Big) \geq 0 \qquad \text{and} \qquad -\Big(\sum_{j=1, j\not\subset\mathcal{J}}^{N(k)+1} \lambda_j(k)\bar{F}_j(k)\Big)\Big(\sum_{j\subset\mathcal{J}} \lambda_j(k)\bar{F}_j(k)\Big) \geq 0.$$

Therefore, $\lambda_j(k) = 0$ and

$$0 \leq \Big(\sum_{j\subset\mathcal{J}} \bar{F}_j(k)\Big)K_{X^K} = \Big(\sum_{j\subset\mathcal{J}} \bar{F}_j(k)\Big)\Big(-2C_K - \sum_{j=1}^{N(k)+1} \lambda_j(k)\bar{F}_j(k)\Big) < 0$$

for all $j \subset \mathcal{J}$. Hence $\lambda_j(d) > 0$ for $d = 1, \ldots, K$ and $j = 1, \ldots, N(d)$.

(2) We proved in (1) that if

$$2 - 2g(C) + e = \sum_{d=0}^{K} \varepsilon_d, \qquad \varepsilon_d \in \mathbb{Q}_{>0}, \quad \varepsilon_d > b_{N(d)+1}(d)/a_{N(d)+1}(d) \quad \text{for } d = 1, \ldots, K,$$

then we have relation (4), where $\bar{F}_j(d)$ is an irreducible component of the fiber of π^K over x_d, F is the general fiber of π^K, $\gamma = \varepsilon_0$, and $\lambda_j(d) = a_j(d)\varepsilon_d - b_j(d) > 0$ for $d = 1, \ldots, K$ and $j = 1, \ldots, N(d)$. The required expression is obtained by considering $\varepsilon_d \to b_{N(d)+1}(d)/a_{N(d)+1}(d)$.

(3) It follows from (1) and (2) that there exist positive rationals $\lambda_j(d)$ and γ satisfying relation (4). There is no (-1)-curve among C_K and $\bar{F}_r(k)$ with $k = 1, \ldots, K$ and $r = 1, \ldots, N(k)$, the intersection form of the curves $\bar{F}_r(k)$ is negative definite (see [4]), and

$$C_K^2 \leq C_0^2 = -e < 2 - 2g(C) < 0.$$

By the adjunction formula,

$$K_{X_K}C_K + C_K^2 \geq 0 \qquad \text{and} \qquad K_{X_K}\bar{F}_r(k) + \bar{F}_r(k)^2 \geq -2.$$

Therefore,

$$K_{X^K}\bar{F}_r(k) \geq 0 \quad \text{for} \quad k = 1, \ldots, K, \quad r = 1, \ldots, N(k)), \qquad \text{and} \qquad K_{X^K}C_K > 0.$$

Hence we have

$$0 \leq \bar{F}_r(k)\Big(-2C_K - \sum_{d=1}^{K}\sum_{j=1}^{N(d)+1} \lambda_j(d)\bar{F}_j(d) - \gamma F\Big), \qquad 0 < C_K\Big(-2C_K - \sum_{d=1}^{K}\sum_{j=1}^{N(d)+1} \lambda_j(d)\bar{F}_j(d) - \gamma F\Big)$$

for $k=1,\dots,K$ and $r=1,\dots,N(k)$. This implies the inequalities

$$0 \le \bar{F}_r(k)\Big(-2C_K - \sum_{d=1}^{K}\sum_{j=1}^{N(d)} \lambda_j(d)\bar{F}_j(d)\Big) \quad \text{and} \quad 0 < C_K\Big(-2C_K - \sum_{d=1}^{K}\sum_{j=1}^{N(d)} \lambda_j(d)\bar{F}_j(d)\Big)$$

for $k=1,\dots,K$ and $r=1,\dots,N(k)$, and if $\bar{F}_r(k)\cap \bar{F}_{N(k)+1}(k) \ne \varnothing$, then

$$0 < \bar{F}_r(k)\Big(-2C_K - \sum_{d=1}^{K}\sum_{j=1}^{N(d)} \lambda_j(d)\bar{F}_j(d)\Big).$$

Therefore, the intersection form of the curves C_K and $\bar{F}_r(k)$ with $k=1,\dots,K$ and $r=1,\dots,N(k)$ is negative definite (see [6]). □

Lemma 10. *In the notation introduced in this section,*

$$\sum_{d=1}^{K} \frac{b_{N(d)+1}(d)}{a_{N(d)+1}(d)} < 2 - 2g(C) + e \tag{5}$$

if and only if there exists a morphism $f\colon X_K \to X$ such that

(1) *X is a numerical del Pezzo surface;*

(2) $\mathrm{rk}(\mathrm{Div}(X)\otimes\mathbb{Q}/\equiv) = 1$;

(3) *f contracts the curves C_K and $\bar{F}_j(d)$ with $d=1,\dots,K$ and $j=1,\dots,N(d)$.*

Proof. *Necessity.* Let (5) be fulfilled; then Lemma 9 implies that the intersection form of the curves C_K and $\bar{F}_r(k)$ with $d=1,\dots,K$ and $j=1,\dots,N(d)$ is negative definite on the surface X_K. There exists a morphism $f\colon X_K \to X$ contracting the curves C_K and $\bar{F}_j(d)$ with $d=1,\dots,K$ and $j=1,\dots,N(d)$ (see [1]), and we have

$$\mathrm{rk}(\mathrm{Div}(X_K)\otimes\mathbb{Q}/\equiv) = 2 + \sum_{d=1}^{K} N(d);$$

therefore, $\mathrm{rk}(\mathrm{Div}(X)\otimes\mathbb{Q}/\equiv) = 1$. It is easy to see that on the surface X, the relation

$$f_*(K_{X_K}) = K_X \equiv \Big(\sum_{d=1}^{K} \frac{b_{N(d)+1}(d)}{a_{N(d)+1}(d)} - 2 - 2g(C) + e\Big) f_*(F)$$

holds. The relations $\mathrm{rk}(\mathrm{Div}(X)\otimes\mathbb{Q}/\equiv) = 1$ and

$$\sum_{d=1}^{K} \frac{b_{N(d)+1}(d)}{a_{N(d)+1}(d)} - 2 - 2g(C) + e < 0$$

imply that X is a numerical del Pezzo surface.

Sufficiency. Suppose that there exists a morphism $f\colon X_K \to X$ such that X is a numerical del Pezzo surface, $\mathrm{rk}(\mathrm{Div}(X)\otimes\mathbb{Q}/\equiv) = 1$, and f contracts the curves C_K and $\bar{F}_j(d)$ with $d=1,\dots,K$ and $j=1,\dots,N(d)$. It is easy to see that on the surface X,

$$f_*(K_{X_K}) = K_X \equiv \Big(\sum_{d=1}^{K} \frac{b_{N(d)+1}(d)}{a_{N(d)+1}(d)} - 2 - 2g(C) + e\Big) f_*(F).$$

By assumption, $\mathrm{rk}(\mathrm{Div}(X)\otimes\mathbb{Q}/\equiv) = 1$ and X is a numerical del Pezzo surface; therefore,

$$\sum_{d=1}^{K} \frac{b_{N(d)+1}(d)}{a_{N(d)+1}(d)} - 2 - 2g(C) + e < 0. \quad \square$$

Theorem 6. *There exists a one-to-one correspondence between all numerical del Pezzo surfaces X with nonrational singularities and with the property* $\operatorname{rk}(\operatorname{Div}(X) \otimes \mathbb{Q}/\equiv) = 1$ *and all triples each comprising*

(1) *a ruled surface* $\pi\colon \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L}) \to C$ *with an invariant* e *such that* $\mathcal{L} \in \operatorname{Pic}(C)$, $g(C) \geq 1$, *and* $e = -\deg(\mathcal{L}) > 2g(C) - 2$;
(2) *a set of pairwise different points* $\{x_1, \dots, x_K\} \subset C$, *possibly empty (for* $K = 0$*)*;
(3) K *sequences of pairs of integers*

$$\left(\alpha_i^1(d), \alpha_i^2(d)\right) \quad \text{with} \quad d = 1, \dots, K \quad \text{and} \quad i = 3, \dots, R(d) + 1, \quad \text{where} \quad R(d) \in \mathbb{N}_{\geq 2},$$

that have property $(*)$ *and satisfy the relation*

$$\sum_{d=1}^{K} \frac{\beta^1_{R(d)+1}(d) + \beta^2_{R(d)+1}(d) + 1}{\alpha^1_{R(d)+1}(d) + \alpha^2_{R(d)+1}(d)} < 2 - 2g(C) + e,$$

where $\left(\beta_i^1(d), \beta_i^2(d)\right)$ *with* $i = 3, \dots, R(d) + 1$ *is the sequence of pairs of integers dual to* $\left(\alpha_i^1(d), \alpha_i^2(d)\right)$ *for each* $d = 1, \dots, K$.

Theorem 6 is implied by Theorem 5 and Lemmas 7–10.

Remark 4. Theorem 6 not only classifies all numerical del Pezzo surfaces with nonrational singularities and the property $\operatorname{rk}(\operatorname{Div}(X) \otimes \mathbb{Q}/\equiv) = 1$; it also gives an effective algorithm for constructing such surfaces. The algorithm is as follows:

(1) take a smooth relatively minimal ruled surface $\mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L})$ with an invariant e and section C_0 such that $e - 2g(C) + 2 > 0$, $\mathcal{L} \in \operatorname{Pic}(C)$, and $C_0^2 = -e = \deg(\mathcal{L}) > 2g(C) - 2$;
(2) select a (possibly empty) set $\{x_1, \dots, x_K\} \subset C$ of pairwise different points;
(3) perform an elementary transformation $\varphi\colon X_K \to \mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{L})$ in the fibers of the morphism π over the points $x_1, \dots, x_K$ so that the K sequences of pairs of integers

$$\left(\alpha_i^1(X_K, d), \alpha_i^2(X_K, d)\right) \qquad \text{with} \quad d = 1, \dots, K \quad \text{and} \quad i = 3, \dots, R(d) + 1$$

satisfy the inequality

$$\sum_{d=1}^{K} \frac{\beta^1_{R(d)+1}(d)(X_K) + \beta^2_{R(d)+1}(d)(X_K) + 1}{\alpha^1_{R(d)+1}(d)(X_K) + \alpha^2_{R(d)+1}(d)(X_K)} < 2 - 2g(C) + e,$$

where $\left(\beta_i^1(d), \beta_i^2(d)\right)$ with $i = 3, \dots, R(d) + 1$ is the sequence of pairs of integers dual to $\left(\alpha_i^1(d), \alpha_i^2(d)\right)$ for each $d = 1, \dots, K$ and $R(d) + 1$ is the number of all irreducible components in the fiber of the morphism $\pi \circ \varphi$ over the point $x_d \in C$;
(4) contract the preimage of C_0 and all irreducible components of the fibers of $\pi \circ \varphi$ over the points $x_1, \dots, x_K$ except (-1)-curves.

The author is very grateful to V. A. Iskovskikh and Yu. G. Prokhorov for fruitful and interesting discussions.

This research was supported by the International Science Foundation under grant No. 90300.

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Translated by O. V. Sipacheva