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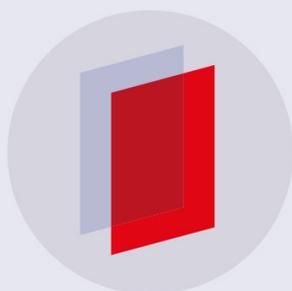
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Elliptic structures on weighted three-dimensional Fano hypersurfaces

I. A. Cheltsov

Abstract. We classify birational transformations into elliptic fibrations of a general quasi-smooth hypersurface in $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ of degree $\sum_{i=1}^{4} a_i$ that has terminal singularities.

§ 1. Introduction

Let X be a quasi-smooth hypersurface in $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ of degree $d = \sum_{i=1}^{4} a_i$ that has terminal singularities, where $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$. Then X is a Fano threefold, and there are exactly 95 possibilities for the four-tuple (a_1, a_2, a_3, a_4). We shall use the symbol n to denote the place of such a family in the list in [1].

Suppose that the hypersurface X is general. The following result is proved in [2].

Theorem 1.1. *The hypersurface X is birationally rigid* [1] *and, in particular, non-rational.*

There are finitely many involutions $\tau_1, \ldots, \tau_{k_n} \in \operatorname{Bir}(X)$ that generate the group $\operatorname{Bir}(X)$ up to biregular automorphisms (see [2]). In the case when $n \notin \{7, 20, 60\}$ and $k_n > 0$, the hypersurface X can be birationally transformed into an elliptic fibration that is invariant under the induced action of the group $\operatorname{Bir}(X)$. This fact is used in [4] to find the relations between the involutions $\tau_1, \ldots, \tau_{k_n}$.

It is natural to try to classify all birational transformations of the hypersurface X into elliptic fibrations. This is equivalent to the following problem: find all rational maps $X \dashrightarrow \mathbb{P}^2$ whose generic fibre is birational to an elliptic curve. Here are a few examples.

Example 1.2. Let $n = 1$. Then X is a quartic threefold. Let $\xi \colon X \dashrightarrow \mathbb{P}^2$ be projection from a line contained in X. Then a generic fibre of the map ξ is an elliptic curve.

Example 1.3. Let $n = 2$. Then X is a hypersurface in $\mathbb{P}(1, 1, 1, 1, 2)$ of degree 5 which has one singular point of type $\frac{1}{2}(1, 1, 1)$. There is a commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\ \gamma\ } & Z \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \eta} \\
X & \dashrightarrow[\ \psi\ ] & \mathbb{P}^3,
\end{array}
$$

---

[1] Let V be a Fano variety of Picard rank 1 having terminal $\mathbb{Q}$-factorial singularities. Then V is said to be *birationally rigid* (see [3]) if it is not birational to any other Mori fibre space, and birationally superrigid if it is birationally rigid and $\operatorname{Bir}(V) = \operatorname{Aut}(V)$.

AMS 2000 Mathematics Subject Classification. 14E05.

where ψ is the natural projection, π is a weighted blow-up of the singular point of the hypersurface X with weights $(1,1,1)$, γ is a birational morphism that contracts 15 irreducible smooth rational curves $C_1, \ldots, C_{15}$, and η is a double cover. Put $\xi = \chi \circ \psi$, where $\chi \colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ is the projection from the point $\eta \circ \gamma(C_i)$. Then a generic fibre of the map ξ is an elliptic curve.

Example 1.4. Let $n = 17$. Then X is a hypersurface in $\mathbb{P}(1,1,3,4,4)$ of degree 12 whose singularities consist of three singular points of type $\frac{1}{4}(1,1,3)$. There is a commutative diagram

$$
\begin{array}{ccc}
U & \xleftarrow{\ \ \beta\ \ } & W \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \omega} \\
X & \dashrightarrow[\ \xi\ ] & \mathbb{P}(1,1,4),
\end{array}
$$

where ξ is a projection, α is a weighted blow-up of a singular point of type $\frac{1}{4}(1,1,3)$ with weights $(1,1,3)$, β is a weighted blow-up with weights $(1,1,2)$ of the singular point that is contained in the exceptional divisor of the morphism α, and ω is an elliptic fibration.

Example 1.5. Let $n = 26$. Then X is a hypersurface in $\mathbb{P}(1,1,3,5,6)$ of degree 15 that has two singular points of type $\frac{1}{3}(1,1,2)$. There is a commutative diagram

$$
\begin{array}{ccc}
 & U & \\
{\scriptstyle \sigma}\swarrow & & \searrow{\scriptstyle \omega} \\
X \dashrightarrow[\ \xi\ ] & & \mathbb{P}(1,1,6),
\end{array}
$$

where ξ is a projection, σ is a weighted blow-up of a singular point of type $\frac{1}{3}(1,1,2)$ with weights $(1,1,2)$, and ω is the morphism given by the linear system $|-6K_X|$. Then the normalization of a generic fibre of the rational map ξ is an elliptic curve.

Example 1.6. Let $n = 31$. Then X is a hypersurface in $\mathbb{P}(1,1,4,5,6)$ of degree 16 that has a singular point of type $\frac{1}{5}(1,1,4)$. There is a commutative diagram

$$
\begin{array}{ccc}
U & \xleftarrow{\ \ \beta\ \ } & W \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \omega} \\
X & \dashrightarrow[\ \xi\ ] & \mathbb{P}(1,1,6),
\end{array}
$$

where ξ is a projection, α is a weighted blow-up of the singular point of type $\frac{1}{5}(1,1,4)$ with weights $(1,1,4)$, β is a weighted blow-up with weights $(1,1,3)$ of the singular point that is contained in the exceptional divisor of the morphism α, and ω is an elliptic fibration.

Example 1.7. Let $n \in \{7, 11, 19\}$. Then $a_2 = a_3$, and the hypersurface X has $\frac{d}{a_2}$ singular points of type $\frac{1}{a_2}(1, 1, a_2 - 1)$. Let $\xi \colon X \dashrightarrow \mathbb{P}(1, a_1, a_2)$ be the rational

map induced by a linear subsystem in the linear system $|-a_2 K_X|$ consisting of surfaces that pass through a given singular point of type $\frac{1}{a_2}(1, 1, a_2 - 1)$. Then the normalization of a generic fibre of the map ξ is an elliptic curve.

Example 1.8. Let $n \in \{7, 9, 20, 30, 36, 44, 49, 51, 64\}$. Then X can be given by the equation

$$w^2 t + wg(x, y, z, t) + f(x, y, z, t) = 0$$

or by the equation

$$tz^k + \sum_{i=0}^{k-1} g_i(x, y, t, w)z^i = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x, y, z, t, w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(t) = a_3$, $\mathrm{wt}(w) = a_4$, and g_i is a quasi-homogeneous polynomial. Let $\xi \colon X \dashrightarrow \mathbb{P}(1, a_1, a_3)$ be the rational map given by a linear system consisting of surfaces that are cut out by $f(x, y, t) = 0$, where $f(x, y, t)$ is a quasi-homogeneous polynomial of degree $a_1 a_3$. Then the normalization of a generic fibre of the map ξ is an elliptic curve.

Example 1.9. Let $n \notin \{1, 2, 3, 7, 11, 19, 60, 75, 84, 87, 93\}$ and $\xi \colon X \dashrightarrow \mathbb{P}(1, a_1, a_2)$ be the natural projection. Then the normalization of a generic fibre of the map ξ is an elliptic curve.

The purpose of this paper is to prove the following result.[2]

Theorem 1.10. *Let $\rho \colon X \dashrightarrow \mathbb{P}^2$ be a rational map whose generic fibre is birational to an elliptic curve. Then there is a commutative diagram*

$$
\begin{array}{ccc}
X & \xrightarrow{\quad \sigma \quad} & X \\
\Big\downarrow{\scriptstyle \xi} & & \Big\downarrow{\scriptstyle \rho} \\
\mathbb{P}(1, a_1, a_i) & \xrightarrow[\quad \varphi \quad]{} & \mathbb{P}^2,
\end{array}
$$

where φ is a birational map, $\sigma \in \mathrm{Bir}(X)$, and ξ is one of the dominant rational maps constructed in Examples 1.2–1.9.

Corollary 1.11. *Let $\rho \colon X \dashrightarrow \mathbb{P}^2$ be a rational map whose generic fibre is birational to an elliptic curve. Suppose that $n \notin \{1, 2, 7, 9, 11, 17, 19, 20, 26, 30, 31, 36, 44, 49, 51, 64\}$. Then there is a commutative diagram*

$$
\begin{array}{ccc}
 & X & \\
{\scriptstyle \psi}\swarrow & & \searrow{\scriptstyle \rho} \\
\mathbb{P}(1, a_1, a_2) & \xrightarrow[\quad \varphi \quad]{} & \mathbb{P}^2,
\end{array}
$$

where ψ is the natural projection and φ is a birational map.

Corollary 1.12. *The hypersurface X can be birationally transformed into an elliptic fibration if and only if $n \notin \{3, 60, 75, 84, 87, 93\}$.*

---

[2]In the case when $n \in \{1, 3, 14, 22, 28, 34, 37, 39, 52, 53, 57, 59, 60, 66, 70, 72, 73, 75, 78, 81, 84, 86, 87, 88, 89, 90, 92, 93, 94, 95\}$, the assertion of Theorem 1.10 is proved in [4]–[6].

We illustrate our technique by proving the following result.

Proposition 1.13. *The assertion of Theorem* 1.10 *holds for* $n = 14$.

Proof. Let $n = 14$. Then X is a hypersurface in $\mathbb{P}(1,1,1,4,6)$ of degree 12 with just one singular point, which is of type $\frac{1}{2}(1,1,1)$. Let $\xi\colon X \dashrightarrow \mathbb{P}^2$ be the natural projection and $\pi\colon U \to X$ a weighted blow-up with weights $(1,1,1)$ of the singular point of X. Then $\xi \circ \pi$ is a morphism.

Let $\rho\colon X \dashrightarrow \mathbb{P}^2$ be a rational map such that the normalization of its generic fibre is an irreducible elliptic curve. Consider commutative diagram

$$
\begin{array}{ccc}
 & V & \\
{\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\beta} \\
X \dashrightarrow[\rho] & & \mathbb{P}^2,
\end{array}
$$

where V is smooth, α is a birational morphism and β is a morphism. Let $\mathcal{M}$ be the proper transform of $|\beta^*(\mathcal{O}_{\mathbb{P}^2}(1))|$ on the hypersurface X. To complete the proof, we have to show that the proper transform of the linear system $\mathcal{M}$ on the variety U lies in the fibres of the fibration $\xi \circ \pi$.

There is a natural number $k > 0$ such that $\mathcal{M} \sim -kK_X$. Consider the mobile linear system $\left(X, \frac{1}{k}\mathcal{M}\right)$. Then

$$
K_V + \frac{1}{k}\mathcal{B} \sim_{\mathbb{Q}} \alpha^*\left(K_X + \frac{1}{k}\mathcal{M}\right) + \sum_{i=1}^{\delta} a_i E_i \sim_{\mathbb{Q}} \sum_{i=1}^{\delta} a_i E_i,
$$

where E_i is an α-exceptional divisor, a_i is a rational number and δ is the number of exceptional divisors of α. It follows from [2] that $a_i \geqslant 0$ for every i, but that there is a j such that $a_j \leqslant 0$ by Lemma 2.1. Put $Z_j = \alpha(E_j)$.

Suppose that Z_j is a smooth point of X. Let S_1 and S_2 be general surfaces in the linear system $\mathcal{M}$. Then $\mathrm{mult}_{Z_j}(S_1 \cdot S_2) \geqslant 4k^2$ by [7], Lemma 1.10. But the linear system $|-4K_X|$ induces a double cover $X \to \mathbb{P}(1,1,1,4)$. Thus, we have

$$
2k^2 = H \cdot S_1 \cdot S_2 \geqslant \mathrm{mult}_{Z_j}(S_1 \cdot S_2) \geqslant 4k^2,
$$

where H is a sufficiently general divisor in the linear system $|-4K_X|$ that passes through the point Z_j, a contradiction.

It follows from Corollary 2.8 that Z_j is not a curve, which implies that Z_j is the unique singular point of the hypersurface X. Let $\mathcal{D}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2. Hence, the linear system $\mathcal{D}$ lies in the fibres of the elliptic fibration $\xi \circ \pi$, which completes the proof.

Let us describe the structure of the paper. We give some auxiliary results in §2. The first steps of the proof of Theorem 1.10 are done in §3, where we prove Theorem 1.10 for $n \in \{1, 3, 5, 11, 14, 22, 28, 34, 37, 39, 52, 53, 57, 59, 60, 66, 70, 72, 73, 75, 78, 81, 84, 86, 87, 88, 89, 90, 92, 93, 94, 95\}$. Then we prove Theorem 1.10 in the remaining cases.

§ 2. Preliminaries

Let X be a threefold having terminal $\mathbb{Q}$-factorial singularities, $\mathcal{M}$ a linear system on X such that $\mathcal{M}$ does not have fixed components, and λ an arbitrary non-negative rational number. In this section we consider technical results describing properties of the mobile log pair $(X, \lambda\mathcal{M})$ which are used in the proof of Theorem 1.10. Elementary properties of mobile log pairs can be found in [3]. As usual, the set of centres of canonical singularities of $(X, \lambda\mathcal{M})$ is denoted by $\mathbb{CS}(X, \lambda\mathcal{M})$.

The main idea in the proof of Theorem 1.10 is to use iteratively the following result, which is a generalization of the classical Noether–Fano inequality.

Lemma 2.1. *Let $\rho\colon X \dashrightarrow \mathbb{P}^2$ be a rational map whose generic fibre is birational to an elliptic curve and $\pi\colon V \to X$ a resolution of the indeterminacies of ρ. Suppose that $\mathcal{M}$ is a proper transform of the linear system $|\rho\circ\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))|$, the divisor $-K_X$ is nef and big and the equivalence $K_X + \lambda\mathcal{M} \sim_{\mathbb{Q}} 0$ holds. Then the singularities of the log pair $(X, \lambda\mathcal{M})$ are not terminal.*

Proof. See [3], proof of Theorem 1.4.4.

In the course of the proof of Theorem 1.10, applications of Lemma 2.1 are usually followed by applications of the following well-known result.

Theorem 2.2. *Let O be a singular point of X of type $\frac{1}{r}(1, a, r - a)$, where a and r are coprime natural numbers such that $r > a$, and let $\mathrm{mult}_O(\mathcal{M})$ be a rational number such that*

$$\mathcal{D} \sim_{\mathbb{Q}} \pi^*(\mathcal{M}) - \mathrm{mult}_O(\mathcal{M})G,$$

where $\pi\colon U \to X$ is a weighted blow-up of O with weights $(1, a, r - a)$, G is the exceptional divisor of π and $\mathcal{D}$ is a proper transform of $\mathcal{M}$ on the variety U. Suppose that $\mathbb{CS}(X, \lambda\mathcal{M})$ contains either the point O or a curve passing through O. Then $\mathrm{mult}_O(\mathcal{M}) \geqslant 1/(r\lambda)$.

Proof. This is proved in [8].

In the course of the proof of Theorem 1.10, applications of Theorem 2.2 are usually followed by applications of the following result.

Lemma 2.3. *With the assumptions and notation of Theorem 2.2, suppose that the singularities of the log pair $(X, \lambda\mathcal{M})$ are canonical, $\mathbb{CS}(X, \lambda\mathcal{M}) = \{O\}$ and $\mathbb{CS}(U, \lambda\mathcal{D}) \neq \varnothing$. Then the following assertions hold:*

the set $\mathbb{CS}(U, \lambda\mathcal{D})$ does not contain smooth points of $G \cong \mathbb{P}(1, a, r - a)$;

if the set $\mathbb{CS}(U, \lambda\mathcal{D})$ contains a curve L, then $L \in \big|\mathcal{O}_{\mathbb{P}(1, a, r-a)}(1)\big|$, and every singular point of the surface G is contained in the set $\mathbb{CS}(U, \lambda\mathcal{D})$.

Proof. We consider only the case when $r = 5$ and $a = 2$ because the proof is similar in the general case. Thus, we have $G \cong \mathbb{P}(1, 2, 3)$.

Let P and Q be singular points of G and L the curve in $\left|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)\right|$. Then L passes through P and Q, but $\text{mult}_O(\mathcal{M}) = 1/(5\lambda)$ by Theorem 2.2, which implies that $\mathcal{D}|_G \sim_{\mathbb{Q}} \lambda L$.

Suppose that the set $\mathbb{CS}(U, \lambda\mathcal{D})$ contains a subvariety Z of the subvariety U that is different from the curve L and the points P and Q. We will show that this assumption leads to a contradiction, which is enough to complete the proof by Theorem 2.2. We have $Z \subset G$.

Suppose that Z is a point. Then Z is smooth on the variety U, which implies the inequality $\text{mult}_Z(\mathcal{D}) > 1/\lambda$. Let C be a general curve in the linear system $\left|\mathcal{O}_{\mathbb{P}(1,2,3)}(6)\right|$ that passes through Z. Then C is not contained in the base locus of the linear system $\mathcal{D}$. Hence, we have $1/\lambda = C \cdot \mathcal{D} \geqslant \text{mult}_Z(C)\,\text{mult}_Z(\mathcal{D}) > 1/\lambda$, which is a contradiction.

Therefore, the subvariety Z is a curve. Then $\text{mult}_Z(\mathcal{D}) \geqslant 1/\lambda$. Let C be a sufficiently general curve in the linear system $\left|\mathcal{O}_{\mathbb{P}(1,2,3)}(6)\right|$. Then

$$\frac{1}{\lambda} = C \cdot \mathcal{D} \geqslant \text{mult}_Z(\mathcal{D})C \cdot Z \geqslant \frac{C \cdot Z}{\lambda}\,,$$

which implies that $C \cdot Z = 1$. Hence, the curve Z is contained in $\left|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)\right|$. Thus, the curves L and Z coincide, which is a contradiction. The lemma is proved.

In the course of the proof of Theorem 1.10, applications of Lemma 2.3 are sometimes followed by applications of the following result.

Lemma 2.4. *Let C be a curve on X such that $C \in \mathbb{CS}(X, \lambda\mathcal{M})$. Suppose that the complete linear system $|-mK_X|$ is base-point-free for some natural number $m > 0$. Then $-K_X \cdot C \leqslant -K_X^3$.*

Proof. Let M_1 and M_2 be general surfaces in $\mathcal{M}$. Then

$$\text{mult}_C(M_1 \cdot M_2) \geqslant \text{mult}_C(M_1)\,\text{mult}_C(M_2) \geqslant \frac{1}{\lambda^2}\,.$$

Let H be a general surface in $|-mK_X|$. Then

$$\frac{-mK_X^3}{\lambda^2} = H \cdot M_1 \cdot M_2 \geqslant (-mK_X \cdot C)\,\text{mult}_C(M_1 \cdot M_2) \geqslant \frac{-mK_X \cdot C}{\lambda^2}\,,$$

which implies that $-K_X \cdot C \leqslant -K_X^3$.

We now consider a simple result that will substantially simplify the proof of Theorem 1.10.

Lemma 2.5. *Suppose that the linear system $\mathcal{M}$ is not composed of a pencil. Then there is no proper Zariski-closed subset $\Sigma \subsetneq X$ such that*

$$\text{Supp}(S_1) \cap \text{Supp}(S_2) \subset \Sigma \subsetneq X,$$

where S_1 and S_2 are general divisors in the linear system $\mathcal{M}$.

Proof. Suppose that there is a proper Zariski-closed subset $\Sigma \subset X$ such that the set-theoretic intersection of the sufficiently general divisors S_1 and S_2 of the linear

system $\mathcal{M}$ is contained in the set Σ. We claim that this assumption leads to a contradiction.

Let $\rho\colon X \dashrightarrow \mathbb{P}^n$ be a rational map induced by the linear system $\mathcal{M}$, where n is the dimension of $\mathcal{M}$. Then there is a commutative diagram

$$
\begin{array}{ccc}
 & W & \\
{\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\beta} \\
X & \dashrightarrow{\scriptstyle\rho} & \mathbb{P}^n,
\end{array}
$$

where W is a smooth variety, α is a birational morphism and β is a morphism. Let Y be the image of β. Then $\dim(Y) \geqslant 2$ because $\mathcal{M}$ is not composed of a pencil.

Let Λ be a Zariski-closed subset of W such that the morphism

$$\alpha|_{W\setminus\Lambda}\colon W \setminus \Lambda \longrightarrow X \setminus \alpha(\Lambda)$$

is an isomorphism, and let Δ be the union of the subset $\Lambda \subset W$ and the closure of the proper transform of the set $\Sigma \setminus \alpha(\Lambda)$ on W. Then Δ is a Zariski-closed proper subset of W.

Let B_1 and B_2 be general hyperplane sections of the variety Y and let D_1 and D_2 be proper transforms of the divisors B_1 and B_2 on the variety W, respectively. Then $\alpha(D_1)$ and $\alpha(D_2)$ are general divisors of the linear system $\mathcal{M}$. Hence, in the set-theoretic sense we have

$$\varnothing \neq \beta^{-1}\big(\mathrm{Supp}(B_1) \cap \mathrm{Supp}(B_2)\big) = \mathrm{Supp}(D_1) \cap \mathrm{Supp}(D_2) \subset \Delta \subsetneqq W \qquad (2.1)$$

because $\dim(Y) \geqslant 2$. However, the set-theoretic identity (2.1) is an absurdity.

The following result is implied by [9], Lemma 0.3.3, and Lemma 2.5.

Corollary 2.6. *Suppose that $\mathcal{M}$ is not composed of a pencil. Let D be a divisor on X that is big and nef. Then $D \cdot S_1 \cdot S_2 > 0$, where S_1 and S_2 are sufficiently general surfaces in the linear system $\mathcal{M}$.*

The proof of Lemma 2.5 implies the following result.

Lemma 2.7. *Suppose that $\mathcal{M}$ is not composed of a pencil. Let $\mathcal{D}$ be a linear system on X that does not have fixed components. Then there is no Zariski-closed subset $\Sigma \subsetneqq X$ such that*

$$\mathrm{Supp}(S) \cap \mathrm{Supp}(D) \subset \Sigma \subsetneqq X,$$

where S and D are sufficiently general divisors of the linear system $\mathcal{M}$ and $\mathcal{D}$, respectively.

Lemma 2.5 and the proof of Lemma 2.4 imply the following result.

Corollary 2.8. *With the assumptions and notation of Lemma 2.4, suppose that the linear system $\mathcal{M}$ is not composed of a pencil and the divisor $-K_X$ is nef and big. Then $-K_X \cdot C < -K_X^3$.*

Many applications of Lemma 2.7 use the following simple result.

Lemma 2.9. *Let S be a surface and D an effective divisor on S such that $D \equiv \sum_{i=1}^{r} a_i C_i$, where $a_i \in \mathbb{Q}$ and $C_1, \ldots, C_r$ are irreducible curves on S whose intersection form is negative definite. Then $D = \sum_{i=1}^{r} a_i C_i$.*

Proof. Let $D = \sum_{i=1}^{k} c_i B_i$, where B_i is an irreducible curve on S and c_i is a nonnegative rational number. Suppose that

$$\sum_{i=1}^{k} c_i B_i \neq \sum_{i=1}^{r} a_i C_i$$

and none of the B_i are among the curves $C_1, \ldots, C_r$. We may assume that not all of $c_1, \ldots, c_k$ are zero. We claim that these assumptions lead to a contradiction, which implies the desired result.

The intersection form of the curves $C_1, \ldots, C_r$ is negative definite. Thus, we have

$$0 \geqslant \left(\sum_{a_i > 0} a_i C_i \right) \left(\sum_{a_i > 0} a_i C_i \right)$$

$$= \left(\sum_{i=1}^{k} c_i B_i \right) \left(\sum_{a_i > 0} a_i C_i \right) - \left(\sum_{a_i \leqslant 0} a_i C_i \right) \left(\sum_{a_i > 0} a_i C_i \right) \geqslant 0,$$

which implies that

$$\sum_{c_i \geqslant 0} c_i B_i \equiv \sum_{a_i \leqslant 0} a_i C_i.$$

Hence, we have $c_i = 0$ and $a_i = 0$ for every i, which is a contradiction.

§3. Beginning of the classification

We shall use the notation and assumptions of §1. In this section we begin the proof of Theorem 1.10. Suppose that there is a birational map $\rho \colon X \dashrightarrow V$ and an elliptic fibration $\nu \colon V \to \mathbb{P}^2$ such that V is smooth and the fibres of ν are connected. We must show that there is a commutative diagram

$$
\begin{array}{ccccc}
X & \dashrightarrow{\sigma} & X & \dashrightarrow{\rho} & V \\
{\scriptstyle \xi} \downarrow & & & & \downarrow {\scriptstyle \nu} \\
\mathbb{P}(1, a_1, a_i) & & \dashrightarrow{\zeta} & & \mathbb{P}^2,
\end{array}
\tag{3.1}
$$

where ζ and σ are birational maps and ξ is one of the rational maps constructed in Examples 1.2–1.9.

The commutative diagram (3.1) implies the commutative diagram

$$
\begin{array}{ccc}
X & \dashrightarrow{\rho} & V \\
{\scriptstyle \xi} \downarrow & & \downarrow {\scriptstyle \nu} \\
\mathbb{P}(1, a_1, a_i) & \dashrightarrow{\zeta} & \mathbb{P}^2
\end{array}
\tag{3.2}
$$

in the case when $\xi \circ \sigma = \chi \circ \xi$ for every $\sigma \in \mathrm{Bir}(X)$, where $\chi \in \mathrm{Bir}(\mathbb{P}(1, a_1, a_i))$.

Example 3.1. Let $\psi\colon X \dashrightarrow \mathbb{P}(1, a_1, a_2)$ be a projection and σ any birational automorphism of the threefold X. Suppose that $n \notin \{1, 2, 3, 7, 11, 19, 20, 36, 60, 75,$ $84, 87, 93\}$. Then it follows from [2] that there is a birational automorphism χ of the surface $\mathbb{P}(1, a_1, a_2)$ such that $\psi \circ \sigma = \chi \circ \psi$.

Let $\mathcal{M}$ be a proper transform of the linear system $|\nu^*(\mathcal{O}_{\mathbb{P}^2}(1))|$ on the hypersurface X. Then $\mathcal{M} \sim_{\mathbb{Q}} -kK_X$ for some natural number k, but the singularities of the log pair $\left(X, \frac{1}{k}\mathcal{M}\right)$ are not terminal by Lemma 2.1.

Remark 3.2. It follows from [2] that there is a birational automorphism $\sigma \in \mathrm{Bir}(X)$ such that the singularities of the log pair $(X, \frac{1}{k'}\sigma(\mathcal{M}))$ are canonical, where $k' \in \mathbb{N}$ is such that $\mathcal{M} \sim_{\mathbb{Q}} -k'K_X$.

We may assume that the singularities of the log pair $\left(X, \frac{1}{k}\mathcal{M}\right)$ are canonical.

Theorem 3.3. *The set* $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ *does not contain smooth points of X when* $n \neq 1$ *or* 2.

Proof. This follows from the proof of Theorem 5.1.2 in [2].

The following corollary is implied by Lemma 2.4.

Corollary 3.4. *The set* $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ *of centres of canonical singularities does not contain curves that do not contain singular points of X when $n \geqslant 6$.*

In particular, the set $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ contains a singular point of X when $n \geqslant 6$ by Theorem 2.2.

Proposition 3.5. *Suppose that the set* $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ *contains a singular point O of the hypersurface X that is a singularity of type $\frac{1}{r}(1, r - a, a)$, where a and r are coprime natural numbers and $r > a$. Let $\pi\colon Y \to X$ be a weighted blow-up of the point O with weights $(1, a, r - a)$. Then $-K_Y^3 \geqslant 0$.*

Proof. Suppose that $-K_Y^3 = -K_X^3 - 1/(ra(r-a)) < 0$. Let E be the π-exceptional divisor and $\mathcal{B}$ a proper transform of $\mathcal{M}$ on the variety Y. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2.

Let $\overline{\mathbb{NE}}(Y)$ be the closure in $\mathbb{R}^2$ of the cone generated by effective one-dimensional cycles of the variety Y. Then $-E \cdot E$ generates an extremal ray of $\overline{\mathbb{NE}}(Y)$, but it follows from [2], Corollary 5.4.6, that there are integers $b > 0$ and $c \geqslant 0$ such that the cycle $-K_Y \cdot (-bK_Y + cE)$ is numerically equivalent to an effective, irreducible and reduced curve Γ on the variety Y that generates the extremal ray of the cone $\overline{\mathbb{NE}}(Y)$ different from the ray generated by $-E \cdot E$.

Let S_1 and S_2 be general surfaces in $\mathcal{B}$. Then $S_1 \cdot S_2 \in \overline{\mathbb{NE}}(Y)$, but $S_1 \cdot S_2 \equiv k^2 K_Y^2$, which implies that the cycle $S_1 \cdot S_2$ generates an extremal ray of the cone $\overline{\mathbb{NE}}(Y)$ that contains the curve Γ. Moreover, for every effective cycle $C \in \mathbb{R}^+\Gamma$ we have

$$\mathrm{Supp}(C) = \mathrm{Supp}(S_1 \cdot S_2)$$

because $S_1 \cdot \Gamma < 0$ and $S_2 \cdot \Gamma < 0$, which is impossible by Lemma 2.5 because the linear system $\mathcal{B}$ is not composed of a pencil.

The following result is implied by Proposition 3.5.

Proposition 3.6. *The assertion of Theorem* 1.10 *holds for* $n \in \{14, 22, 28, 34, 37,$
$39, 52, 53, 57, 59, 66, 70, 72, 73, 78, 81, 86, 88, 89, 90, 92, 94, 95\}.$

Proof. We must show the existence of a commutative diagram

$$
\begin{array}{ccc}
X & \dashrightarrow^{\rho} & V \\
{\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\nu} \\
\mathbb{P}(1, a_1, a_2) & \dashrightarrow_{\varphi} & \mathbb{P}^2,
\end{array}
\tag{3.3}
$$

where ψ is the natural projection and φ is a birational map.

It follows from Theorems 3.3, Lemma 2.4 and Theorem 2.2 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$
contains a singular point P of the hypersurface X of type $\frac{1}{r}(1, a, r - a)$, where a
and r are coprime natural numbers and $r > a$.

Let $\pi \colon Y \to X$ be a weighted blow-up of the point P with weights $(1, a, r - a)$
and $\mathcal{B}$ the proper transform of the linear system $\mathcal{M}$ on the variety Y. Then
the equivalence $\mathcal{B} \sim_{\mathbb{Q}} -kK_Y$ holds by Theorem 2.2. Moreover, for every value
of n, either the inequality $-K_Y^3 < 0$ holds or the inequality $-K_Y^3 = 0$ holds.

Then $-K_Y^3 = 0$ by Proposition 3.5. Then the linear system $|-rK_Y|$ does not
have base points for $r \gg 0$ and induces a morphism $\eta \colon Y \to \mathbb{P}(1, a_1, a_2)$ such that
the diagram

is commutative. Let C be a generic fibre of the morphism η and S a general surface
in the linear system $\mathcal{B}$. Then $S \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2. Hence, the equality
$S \cdot C = 0$ holds, which implies that the surface S lies in the fibres of the elliptic
fibration η. The latter implies the existence of the commutative diagram (3.3).

The following result implies Corollary 1.12.

Lemma 3.7. *Our assumptions imply that* $n \notin \{3, 60, 75, 84, 87, 93\}.$

Proof. It follows from Proposition 3.5 that $n \notin \{75, 84, 87, 93\}$.

Suppose that $n = 3$. Then the hypersurface X is smooth and the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$
contains an irreducible curve Γ such that $-K_X \cdot \Gamma = 1$ by Lemma 2.4. Let $\gamma \colon \bar{X} \to X$
be a blow-up of the curve Γ and G the exceptional divisor of γ. Then the divisor
$\gamma^*(-3K_X) - G$ is nef and big. But

$$
\left(\eta^*(-3K_X) - G\right) \cdot \bar{S}_1 \cdot \bar{S}_1 = 0,
$$

where $\bar{S}_i$ is the proper transform of a general surface in $\mathcal{M}$ on $\bar{X}$, which is impossible
by Corollary 2.6.

We have $n = 60$. Then X is a hypersurface in $\mathbb{P}(1, 4, 5, 6, 9)$ of degree 24.

It is easy to check that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain curves by Corollary 2.8,
which implies that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ consists of the singular point O of the
hypersurface X that is a quotient singularity of type $\frac{1}{9}(1, 4, 5)$ by Proposition 3.5.

Let $\pi\colon Y \to X$ be a weighted blow-up of the singular point O with weights $(1, 4, 5)$ and $\mathcal{D}$ a proper transform of the linear system $\mathcal{M}$ on the threefold Y. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2. Let P and Q be the points of Y contained in the π-exceptional divisor that are singularities of types $\frac{1}{4}(1, 1, 3)$ and $\frac{1}{5}(1, 1, 4)$, respectively. Then $\mathbb{CS}(Y, \frac{1}{k}\mathcal{D}) \subseteq \{P, Q\}$ by Lemmas 2.3 and 2.4.

Suppose that $\mathbb{CS}(Y, \frac{1}{k}\mathcal{D})$ contains the point Q. Let $\alpha\colon U \to Y$ be a weighted blow-up of Q with weights $(1, 1, 4)$ and $\mathcal{B}$ a proper transform of $\mathcal{M}$ on the variety U. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2, the linear system $|-4K_U|$ is a proper transform of the pencil $|-4K_X|$ and the base locus of the pencil $|-4K_U|$ consists of an irreducible reduced curve Z on U such that the curve $\pi \circ \alpha(Z)$ is a base curve of the pencil $|-4K_X|$. Let H be a general surface in $|-4K_U|$. Then $Z^2 = -1/30$ on H. But the equivalence $\mathcal{B}|_H \sim_{\mathbb{Q}} kZ$ holds, which is impossible by Lemmas 2.7 and 2.9.

We see that $\mathbb{CS}(Y, \frac{1}{k}\mathcal{D}) = \{P\}$. Let $\beta\colon W \to Y$ be a weighted blow-up of the point P with weights $(1, 1, 3)$ and D the proper transform of the general surface in $|-5K_X|$ on the variety W. Then D is nef and big, but the equality $D \cdot H_1 \cdot H_2 = 0$ holds, where H_1 and H_2 are general surfaces in the proper transform of $\mathcal{M}$ on W, which is impossible by Corollary 2.6.

We now consider a very simple case when there are many ways of birationally transforming the hypersurface X into elliptic fibrations.

Proposition 3.8. *Suppose that $n = 11$. Then the diagram (3.2) exists, where ξ is one of the five rational maps constructed in Example 1.7.*

Proof. The threefold X is a hypersurface in $\mathbb{P}(1, 1, 2, 2, 5)$ of degree 10 whose singularities consist of points P_1, P_2, P_3, P_4 and P_5 that are singularities of types $\frac{1}{2}(1, 1, 1)$. The hypersurface X is birationally superrigid. It follows from the construction in Example 1.7 that there is a commutative diagram

$$
\begin{array}{ccc}
 & U_i & \\
{\scriptstyle \pi_i}\swarrow & & \searrow{\scriptstyle \eta_i} \\
X \dashrightarrow[\xi_i] & & \mathbb{P}(1, 1, 2),
\end{array}
$$

where ξ_i is a projection, π_i is the weighted blow-up of P_i with weights $(1, 1, 1)$ and η_i is an elliptic fibration. It follows from Theorem 2.2, Lemma 2.4 and Theorem 3.3 that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the singular point P_i for some $i \in \{1, 2, 3, 4, 5\}$. Let $\mathcal{D}_i$ be the proper transform of $\mathcal{M}$ on U_i. Then $\mathcal{D}_i \sim_{\mathbb{Q}} -kK_{U_i}$ by Theorem 2.2, which implies the existence of the commutative diagram (3.2) for $\xi = \xi_i$.

The following result is obtained in [10].

Theorem 3.9. *Suppose that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains an irreducible curve C on the hypersurface X and $n \geqslant 3$. Then there are different surfaces S_1 and S_2 in the linear system $|-K_X|$ such that the curve C is a component of the curve $S_1 \cap S_2$.*

The following result is obtained in the thesis of D. Ryder.

Proposition 3.10. *The assertion of Theorem 1.10 holds for $n = 5$.*

Proof. Suppose that $n = 5$. Then X is a general hypersurface in $\mathbb{P}(1, 1, 1, 2, 3)$ of degree 7, the equality $-K_X^3 = 7/6$ holds, and the singularities of the hypersurface X consist of two isolated singular points P and Q of types $\frac{1}{2}(1, 1, 1)$ and $\frac{1}{3}(1, 1, 2)$, respectively.

The hypersurface X can be given by the equation

$$w^2 f_1(x, y, z) + f_4(x, y, z, t)w + f_7(x, y, z, t) = 0 \subset \mathbb{P}(1, 1, 1, 2, 3)$$
$$\cong \mathrm{Proj}\big(\mathbb{C}[x, y, z, t, w]\big),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = \mathrm{wt}(z) = 1$, $\mathrm{wt}(t) = 2$ and $\mathrm{wt}(w) = 3$, f_i is a general quasi-homogeneous polynomial of degree i, and Q is given by the equations $x = y = z = t = 0$. There is a commutative diagram
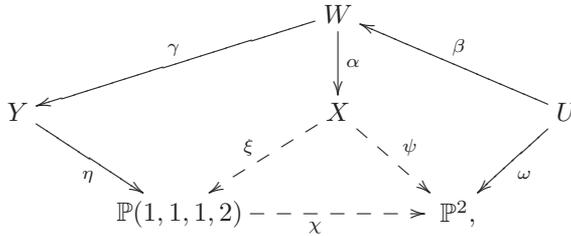
where χ and ξ are the natural projections, the morphism ω is an elliptic fibration, the morphism α is a weighted blow-up of the point Q with weights $(1, 1, 2)$, the morphism γ is the birational morphism that contracts 14 smooth irreducible rational curves $C_1, \ldots, C_{14}$ into 14 isolated ordinary double points $P_1, \ldots, P_{14}$, respectively, of the variety Y, the morphism η is a double cover branched over the surface $R \subset \mathbb{P}(1, 1, 1, 2)$ that is given by the equation

$$f_4(x, y, z, t)^2 - 4f_1(x, y, z)f_7(x, y, z, t) = 0 \subset \mathbb{P}(1, 1, 1, 2) \cong \mathrm{Proj}\big(\mathbb{C}[x, y, z, t]\big)$$

and has 14 isolated ordinary double points $\eta(P_1), \ldots, \eta(P_{14})$, and β is the composite of the weighted blow-ups with the weights $(1, 1, 1)$ of two singular points of the variety W that are singularities of types $\frac{1}{2}(1, 1, 1)$.

It follows from Theorem 3.9 that the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ does not contain curves (see the proof of Lemma 8.3).

Suppose that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{Q\}$. Let O be the singular point of the threefold W such that $\alpha(O) = Q$ and $\mathcal{D}$ the proper transform of $\mathcal{M}$ on the threefold W. Then $O \in \mathbb{CS}\big(W, \frac{1}{k}\mathcal{D}\big)$ by Theorem 2.2 and Lemmas 2.1 and 2.3. Let $\mathcal{B}$ be the proper transform of $\mathcal{D}$ on the variety Y. Then it follows from Theorem 2.2 and Lemmas 2.3 and 2.4 that $\mathbb{CS}\big(Y, \frac{1}{k}\mathcal{B}\big)$ contains a curve C such that $-K_Y \cdot C = 1/2$ and $\chi \circ \eta(C)$ is a point.

There is an irreducible curve Z on Y such that $\eta(Z) = \eta(C)$ and $Z \neq C$. Let S be a general surface in the linear system $|-K_Y|$ that contains the curve C. Then $Z^2 < 0$ on S. But $\mathcal{B}|_S \sim_{\mathbb{Q}} kC + kZ$ on S, which is impossible by Lemma 2.7 because $\mathcal{B}$ is not composed of a pencil.

It follows from Theorem 2.2 and Lemmas 2.3 and 2.4 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P, Q\}$. Hence, we have $O \in \mathbb{CS}\big(W, \frac{1}{k}\mathcal{D}\big)$ by Corollary 2.8 and Lemmas 2.1 and 2.3. Thus, the proper transform of $\mathcal{M}$ on U is contained in the fibres of the elliptic fibration ω by Theorem 2.2.

The assertion of Theorem 3.9 implies the following result.

Lemma 3.11. *Suppose that $a_2 \neq 1$. Then the set $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ does not contain curves.*

Proof. Suppose that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ contains a curve C. Then the assertion of Theorem 3.9 implies that there are surfaces S_1 and S_2 in the pencil $|-K_X|$ such that C is contained in the intersection $S_1 \cap S_2$. Now the inequality $a_2 \neq 1$ implies that the cycle $S_1 \cdot S_2$ is a reduced and irreducible curve. Hence, we have $-K_X \cdot C = -K_X^3$, which is impossible by Lemma 2.4.

We illustrate the application of Lemma 3.11 by proving the following result.

Proposition 3.12. *The assertion of Theorem 1.10 holds for $n = 18$.*

Proof. Let $n = 18$. Then X is a hypersurface in $\mathbb{P}(1, 2, 2, 3, 5)$ of degree 12 whose singularities consist of points $O_1, \ldots, O_6$ that are quotient singularities of type $\frac{1}{2}(1, 1, 1)$ and a point P that is a quotient singularity of type $\frac{1}{5}(1, 2, 3)$. The equality $-K_X^3 = 1/5$ holds and there is a commutative diagram

$$
\begin{array}{ccc}
U & \xleftarrow{\ \ \beta\ \ } & W \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\eta} \\
X & \dashrightarrow{\ \psi\ } & \mathbb{P}(1, 2, 2),
\end{array}
$$

where ψ is the natural projection, the morphism α is the weighted blow-up of the point P with weights $(1, 2, 3)$, β is the weighted blow-up with weights $(1, 1, 2)$ of the singular point of the variety U that is a quotient singularity of type $\frac{1}{3}(1, 1, 2)$, and η is an elliptic fibration.

It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) = \{P\}$.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U and let Q and O be the singular points of U contained in the exceptional divisor of the birational morphism α that are singularities of types $\frac{1}{3}(1, 1, 2)$ and $\frac{1}{2}(1, 1, 1)$, respectively. Then it follows from Theorem 2.2 and Lemmas 2.1, 2.3 and 2.4 that $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$, the set $\mathbb{CS}\left(U, \frac{1}{k}\mathcal{D}\right)$ is non-empty and the set $\mathbb{CS}\left(U, \frac{1}{k}\mathcal{D}\right)$ contains either Q or O.

Suppose that $\mathbb{CS}\left(U, \frac{1}{k}\mathcal{D}\right)$ contains O. Let $\pi \colon Y \to U$ be the weighted blow-up of O with weights $(1, 1, 1)$, F the π-exceptional divisor and $\mathcal{H}$ and $\mathcal{P}$ the proper transforms of $\mathcal{M}$ and $|-3K_U|$ on Y, respectively. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2, but

$$
\mathcal{P} \sim_{\mathbb{Q}} \pi^*(-3K_U) - \frac{1}{2}F,
$$

and the base locus of the linear system $\mathcal{P}$ consists of the irreducible curve Z such that $\alpha \circ \pi(Z)$ is the base curve of the linear system $|-3K_X|$. Moreover, for a general surface S in $\mathcal{P}$, the inequality $S \cdot Z > 0$ holds, which implies that the divisor $\pi^*(-6K_U) - F$ is nef and big. Let D_1 and D_2 be general surfaces in the linear system $\mathcal{H}$. Then

$$
\left(\pi^*(-6K_U) - F\right) \cdot D_1 \cdot D_2 = \left(\pi^*(-6K_U) - F\right) \cdot \left(\pi^*(-kK_U) - \frac{k}{2}F\right)^2 = 0,
$$

which is impossible by Corollary 2.6.

Therefore, the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ contains the point Q. Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on W. Then the equivalence $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ holds by Theorem 2.2, which easily implies the desired assertion.

We conclude this section by proving the following result, which is obtained in [5] and [6].

Proposition 3.13. *The assertion of Theorem* 1.10 *holds for* $n = 1$.

Proof. Let X be a general hypersurface in $\mathbb{P}^4$ of degree 4. Then we must show that there is a line $L \subset X$ such that there is a commutative diagram

$$
\begin{array}{ccc}
X & \dashrightarrow^{\;\rho\;} & V \\
\psi \downarrow & & \downarrow \nu \\
\mathbb{P}^2 & \dashrightarrow_{\;\sigma\;} & \mathbb{P}^2,
\end{array}
\tag{3.4}
$$

where ψ is a projection from L and σ is a birational map.

Suppose that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains a point P of the quartic X. Let H be a general hyperplane section of X passing through P. Then it follows from [7], Lemma 1.10, that

$$
4k^2 \leqslant \operatorname{mult}_P(D_1 \cdot D_2) \leqslant D_1 \cdot D_2 \cdot H = 4k^2,
$$

where D_1 and D_2 are general surfaces in $\mathcal{M}$. Therefore, the support of the effective one-dimensional cycle $D_1 \cdot D_2$ is contained in the union of a finite number of lines on the quartic X that pass through the point P. This is impossible by Lemma 2.5.

Therefore, $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains a curve C. Thus, the inequality $\operatorname{mult}_C(\mathcal{M}) \geqslant k$ holds, but it follows from Lemma 2.4 that $\deg(C) \leqslant 3$.

Suppose that C is not contained in any plane in $\mathbb{P}^4$. Then C is either a smooth curve of degree 3 or 4, or a rational curve of degree 4 that has one double point.

Suppose that C is smooth. Let $\alpha\colon U \to X$ be the blow-up of C, F the exceptional divisor of α and $\mathcal{D}$ the proper transform of $\mathcal{M}$ on the variety U. Then the base locus of the linear system $|\alpha^*(-\deg(C)K_X) - F|$ does not contain curves. We have

$$
\big(\alpha^*(-\deg(C)K_X) - F\big) \cdot D_1 \cdot D_2 < 0,
$$

where D_1 and D_2 are general surfaces in the linear system $\mathcal{D}$, which is a contradiction.

Thus, the curve C is a quartic curve with a double point P. Let $\beta\colon W \to X$ be the composite of the blow-up of P and the blow-up of the proper transform of C. Let G and E be the exceptional divisors of β such that $\beta(E) = C$ and $\beta(G) = P$. Then the base locus of the linear system $|\beta^*(-4K_X) - E - 2G|$ does not contain curves. We have

$$
\big(\beta^*(-4K_X) - E - 2G\big) \cdot D_1 \cdot D_2 < 0,
$$

where D_1 and D_2 are general surfaces in the linear system $\mathcal{D}$, which is a contradiction. Hence, we see that C is contained in some two-dimensional linear subspace of $\mathbb{P}^4$.

Suppose that $\deg(C) \neq 1$. Then we have the following possibilities:

(i) C is a smooth conic;

(ii) C is a smooth plane cubic;

(iii) C is a plane singular cubic.

Suppose that C is smooth. Let $\alpha\colon U \to X$ be a blow-up of C, F the exceptional divisor of the birational morphism α and $\mathcal{D}$ the proper transform of $\mathcal{M}$ on U. Then one can easily check that the base locus of the linear system $|\alpha^*(-\deg(C)K_X) - F|$ does not contain curves. Therefore, the divisor $\alpha^*(-\deg(C)K_X) - F$ is nef and big. On the other hand, we have

$$\big(\alpha^*\big(-\deg(C)K_X\big) - F\big) \cdot D_1 \cdot D_2 = 0,$$

where D_1 and D_2 are general surfaces in the linear system $\mathcal{D}$, which is impossible by Corollary 2.6.

Hence, C is a plane cubic with a double point P. Let $\beta\colon W \to X$ be the composite of the blow-up of P and the blow-up of the proper transform of C. Let G and E be the exceptional divisors of the morphism β such that $\beta(E) = C$ and $\beta(G) = P$. Then the base locus of the linear system $|\beta^*(-3K_X) - E - 2G|$ does not contain curves, which implies that the divisor $\beta^*(-3K_X) - E - 2G$ is nef and big. On the other hand, the inequality

$$\big(\beta^*(-3K_X) - E - 2G\big) \cdot D_1 \cdot D_2 \leqslant 0$$

holds, where D_1 and D_2 are general surfaces in $\mathcal{D}$, which is impossible by Corollary 2.6.

Thus, we see that C is a line. The equality $\mathrm{mult}_C(D) = k$ implies the existence of the commutative diagram (3.4) for $L = C$.

§4. The case $n = 2$: a hypersurface of degree 5 in $\mathbb{P}(1,1,1,1,2)$

We use the notation and assumptions of §3. Let $n = 2$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,1,1,2)$ of degree 5, the equality $-K_X^3 = 5/2$ holds and the singularities of the hypersurface X consist of a point O that is a quotient singularity of type $\frac{1}{2}(1,1,1)$.

The hypersurface X can be given by the equation

$$w^2 f_1(x,y,z,t) + f_3(x,y,z,t)w + f_5(x,y,z,t) = 0 \subset \mathbb{P}(1,1,1,1,2)$$
$$\cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = \mathrm{wt}(z) = \mathrm{wt}(t) = 1$ and $\mathrm{wt}(w) = 2$, $f_i(x,y,z,t)$ is a homogeneous polynomial of degree i, and the point O is given by the equations $x = y = z = t = 0$. Let $\psi\colon X \dashrightarrow \mathbb{P}^3$ be the natural projection. Then there is a commutative diagram
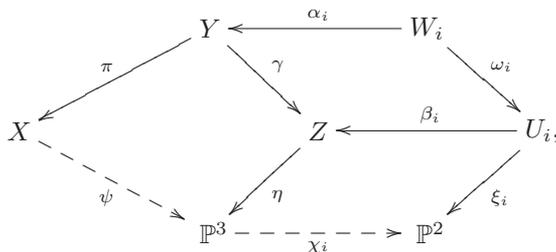
where π is the weighted blow-up of the point O with weights $(1, 1, 1)$, the morphism γ is the birational morphism that contracts 15 smooth rational curves $C_1, \ldots, C_{15}$ to 15 isolated ordinary double points $P_1, \ldots, P_{15}$, respectively, of the variety Z, η is a double cover branched over the surface $R \subset \mathbb{P}^3$ of degree 6 given by the equation

$$f_3(x, y, z, t)^2 - 4f_1(x, y, z, t)f_5(x, y, z, t) = 0 \subset \mathbb{P}^3 \cong \mathrm{Proj}\big(\mathbb{C}[x, y, z, t]\big)$$

and has 15 isolated ordinary double points $\eta(P_1), \ldots, \eta(P_{15})$, α_i is a blow-up of C_i, β_i is a blow-up of the point P_i, w_i is a birational morphism, χ_i is a projection from $\eta(P_i)$, and ξ_i is an elliptic fibration. Moreover, the points $\eta(P_1), \ldots, \eta(P_{15})$ are given by the equations $f_3 = f_1 = f_5 = 0$.

Remark 4.1. Let τ be the birational involution of X induced by the double covering η. According to [2], the group $\mathrm{Bir}(X)$ is generated by τ and biregular automorphisms of X. Moreover, it follows from [2] that generic fibres of $\chi_i \circ \psi$ are $\mathrm{Bir}(X)$-invariant.

In the rest of this section we prove the following result.

Proposition 4.2. *There is a commutative diagram*

$$
\begin{array}{ccc}
U_i & - - -\overset{\rho}{-} - - \rightarrow & V \\
{\scriptstyle \xi_i}\downarrow & & \downarrow{\scriptstyle \nu} \\
\mathbb{P}^2 & - - -\underset{\varphi}{-} - - \rightarrow & \mathbb{P}^2
\end{array}
\tag{4.1}
$$

for some $i \in \{1, \ldots, 15\}$, where φ is a birational map.

Proof. Let $\mathcal{B}_i$ be the proper transform of the linear system $\mathcal{M}$ on the variety W_i. To prove the existence of the commutative diagram (4.1), it is enough to show that $\mathcal{B}_i$ lies in the fibres of the elliptic fibration $\xi_i \circ w_i$. The latter follows easily from the equivalence $\mathcal{B}_i \sim_{\mathbb{Q}} -kK_{W_i}$.

Lemma 4.3. *Suppose that the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains a smooth point of X. Then the commutative diagram (4.1) exists for some $i \in \{1, \ldots, 15\}$.*

Proof. Suppose that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains a smooth point P of X. Let S be a sufficiently general surface in $|-K_X|$ that passes through P. In the case when $P \notin \bigcup_{i=1}^{15} \pi(C_i)$, the surface S does not contain irreducible components of the effective cycle $D_1 \cdot D_2$, where D_1 and D_2 are general surfaces in the linear system $\mathcal{M}$. Therefore, in this case we have

$$\mathrm{mult}_P(D_1 \cdot D_2) \leqslant D_1 \cdot D_2 \cdot S = -k^2 K_X^3 = \frac{5}{2}k^2,$$

which is impossible by [7], Lemma 1.10. Thus, the point P is contained in $\pi(C_i)$ for some i. Following the arguments of [2], put $C = \pi(C_i)$ and

$$\mathcal{M}|_S = \mathcal{L} + \mathrm{mult}_C(\mathcal{M})C,$$

where $\mathcal{L}$ is a linear system on the surface S without fixed components. Then the log pair

$$\left(S, \frac{1}{k}\mathcal{L} + \frac{\mathrm{mult}_C(\mathcal{M})}{k}C\right)$$

is not log terminal at P by [11], Theorem 7.5. Let L_1 and L_2 be general curves in $\mathcal{L}$. Then

$$\mathrm{mult}_P(L_1 \cdot L_2) \geqslant 4\left(1 - \frac{\mathrm{mult}_C(\mathcal{M})}{k}\right)k^2$$

by [7], Theorem 3.1. On the other hand, the equality

$$L_1 \cdot L_2 = \frac{5}{2}k^2 - \mathrm{mult}_C(\mathcal{M})k - \frac{3}{2}\mathrm{mult}_C^2(\mathcal{M})$$

holds on the surface S because $C^2 = -3/2$. Hence, we have

$$\frac{5}{2}k^2 - \mathrm{mult}_C(\mathcal{M})k - \frac{3}{2}\mathrm{mult}_C^2(\mathcal{M}) \geqslant 4\left(1 - \frac{\mathrm{mult}_C(\mathcal{M})}{k}\right)k^2,$$

which implies the equality $\mathrm{mult}_C(\mathcal{M}) = k$. Thus, the curve $\pi(C_i)$ is also contained in the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. It now follows from Theorem 2.2 that the equivalence $\mathcal{B}_i \sim_{\mathbb{Q}} -kK_{W_i}$ holds, which implies the desired assertion.

Hence, we may assume that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain smooth points of X.

Lemma 4.4. *Let C be a curve on X such that $C \cap \mathrm{Sing}(X) = \varnothing$. Then $C \notin \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$.*

Proof. Suppose that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains C. Then $\mathrm{mult}_C(\mathcal{M}) = k$. Let H be a very ample divisor on X. Then $H \sim_{\mathbb{Q}} -\lambda K_X$ holds for some natural number λ. Thus, we have

$$\frac{5\lambda k^2}{2} = -\lambda k^2 K_X^3 = H \cdot S_1 \cdot S_2 \geqslant \mathrm{mult}_C^2(\mathcal{M})H \cdot C \geqslant -\lambda k^2 K_X \cdot C,$$

where S_1 and S_2 are general surfaces in $\mathcal{M}$. Therefore, we have the following possibilities:

(i) the equality $-K_X \cdot C = 1$ holds and C is smooth and rational;
(ii) the equality $-K_X \cdot C = 2$ holds and C is smooth and rational;
(iii) the equality $-K_X \cdot C = 2$ holds and the arithmetic genus of C is 1.

Let $\sigma \colon \check{X} \to X$ be the blow-up of the ideal sheaf of C and G the exceptional divisor of the birational morphism σ. Then the variety $\check{X}$ is smooth in the neighbourhood of G whenever C is smooth. Moreover, in the case when C has an ordinary double point, the singularities of $\check{X}$ in the neighbourhood of G consist of a single isolated ordinary double point. In the case when C has a cuspidal singularity, the singularities of the variety $\check{X}$ in the neighbourhood of G consist of an isolated double point such that, in the neighbourhood of this point, $\check{X}$ is locally isomorphic to the hypersurface

$$x_1^2 + x_2^2 + x_3^2 + x_4^3 = 0 \subset \mathbb{C} \cong \mathrm{Spec}\left(\mathbb{C}[x_0, x_1, x_2, x_3]\right).$$

Let $\check{S}_1$ and $\check{S}_2$ be the proper transforms of S_1 and S_2 on $\check{X}$, respectively.

Suppose that $-K_X \cdot C = 1$. Then C is cut out in the set-theoretic sense by the surfaces in the linear system $|-2K_X|$ that pass through C. Moreover, the scheme-theoretic intersection of two general surfaces in $|-2K_X|$ passing through C is reduced at a general point of C. Thus, the divisor $\sigma^*(-2K_X) - G$ is nef and big (see [2], Lemma 5.2.5). But the equality

$$\left(\sigma^*(-2K_X) - G\right) \cdot \check{S}_1 \cdot \check{S}_2 = 0$$

holds, which is impossible by Corollary 2.6.

Suppose that $-K_X \cdot C = 2$ and C is smooth and rational. Then $\sigma^*(-2K_X) - G$ is nef because C is cut out in the set-theoretic sense by the surfaces in the linear system $|-2K_X|$ that pass through it, but the scheme-theoretic intersection of two general surfaces in $|-2K_X|$ passing through C is reduced at a general point of C. We have

$$0 > -3k^2 = \left(\sigma^*(-2K_X) - G\right) \cdot \check{S}_1 \cdot \check{S}_2 \geqslant 0,$$

which is a contradiction.

Hence, the arithmetic genus of C is 1 and $-K_X \cdot C = 2$. C is the set-theoretic intersection of the surfaces in $|-4K_X|$ that pass through C. Moreover, the scheme-theoretic intersection of two general surfaces in $|-4K_X|$ passing through C is reduced at a general point of C. Hence, the divisor $\sigma^*(-4K_X) - G$ is nef and big. On the other hand, the equality

$$\left(\sigma^*(-4K_X) - G\right) \cdot \check{S}_1 \cdot \check{S}_2 = 0$$

holds, which is impossible by Corollary 2.6.

It follows from Theorem 2.2 that $O \in \left(X, \frac{1}{k}\mathcal{M}\right)$. Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on Y. Then it follows from Theorem 2.2 that $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$. Thus, the commutative diagram (4.1) exists in the case when the set $\mathbb{CS}\left(Y, \frac{1}{k}\mathcal{D}\right)$ contains the curve C_i. Therefore, we may assume that

$$\mathbb{CS}\left(Y, \frac{1}{k}\mathcal{D}\right) \cap \{C_1, \dots, C_{15}\} = \varnothing.$$

Lemma 4.5. $\mathbb{CS}\left(Y, \frac{1}{k}\mathcal{D}\right)$ *does not contain smooth points of* Y.

Proof. $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ does not contain smooth points of X. Therefore, to complete the proof it is enough to show that $\mathbb{CS}\left(Y, \frac{1}{k}\mathcal{D}\right)$ does not contain points of the exceptional divisor of the morphism π, and this follows from Lemma 2.3.

Put $\mathcal{H} = \gamma(\mathcal{D})$. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_Z$ and the singularities of the log pair $\left(Z, \frac{1}{k}\mathcal{H}\right)$ are canonical. Thus, it follows from Lemma 2.1 that $\mathbb{CS}\left(Z, \frac{1}{k}\mathcal{H}\right)$ is non-empty.

Lemma 4.6. $\mathbb{CS}\left(Z, \frac{1}{k}\mathcal{H}\right)$ *does not contain points of* Z.

Proof. It follows from Lemma 4.5 that smooth points of Z are not contained in $\mathbb{CS}\left(Z, \frac{1}{k}\mathcal{H}\right)$. On the other hand, if $P_i \in \mathbb{CS}\left(Z, \frac{1}{k}\mathcal{H}\right)$, then $\mathbb{CS}\left(Y, \frac{1}{k}\mathcal{D}\right)$ contains either the curve C_i or one of its points, which is impossible.

Thus, there is a curve Γ on Z that is contained in $\mathbb{CS}\left(Z, \frac{1}{k}\mathcal{H}\right)$. In particular, the inequality $\mathrm{mult}_\Gamma(\mathcal{H}) = k$ holds.

Lemma 4.7. *The equality* $-K_Z \cdot \Gamma = 1$ *holds.*

Proof. Let H be a general divisor of the linear system $|-K_Z|$. Then

$$2k^2 = H \cdot D_1 \cdot D_2 \geqslant \mathrm{mult}_\Gamma(D_1 \cdot D_2) H \cdot \Gamma \geqslant -k^2 K_Z \cdot \Gamma,$$

where D_1 and D_2 are sufficiently general surfaces in the linear system $\mathcal{H}$. Therefore, the inequality $-K_Z \cdot \Gamma \leqslant 2$ holds. Moreover, if the equality $-K_Z \cdot \Gamma = 2$ holds, then the support of the effective cycle $D_1 \cdot D_2$ coincides with the curve Γ. The latter is impossible by Lemma 2.5.

The curve $\eta(\Gamma)$ is a line in $\mathbb{P}^3$ and $\eta|_\Gamma \colon \Gamma \to \eta(\Gamma)$ is an isomorphism. However, the arguments used in the proof of Lemma 4.7 easily imply that the set $\mathbb{CS}(Z, \frac{1}{k}\mathcal{H})$ does not contain subvarieties of Z besides the curve Γ, and the inequality $\mathrm{mult}_\Gamma(D_1 \cdot D_2) < 2k^2$ holds, where D_1 and D_2 are general surfaces in $\mathcal{H}$.

Lemma 4.8. *The line* $\eta(\Gamma)$ *is contained in the ramification surface* R *of the double cover* η.

Proof. Suppose that $\eta(\Gamma)$ is not contained in R. Let S be a general surface in $|-K_Z|$ that passes through Γ. Then

$$\mathcal{H}|_S = \mathrm{mult}_\Gamma(\mathcal{H})\Gamma + \mathrm{mult}_\Omega(\mathcal{H})\Omega + \mathcal{L},$$

where $\mathcal{L}$ is a linear system on the surface S that does not have fixed components and Ω is a smooth rational curve on Z such that $\eta(\Omega) = \eta(\Gamma)$ but $\Omega \neq \Gamma$. We have

$$\mathrm{Sing}(Z) \cap \Gamma = \{P_{i_1}, \ldots, P_{i_r}\} \subsetneq \{P_1, \ldots, P_{15}\} = \mathrm{Sing}(Z),$$

but P_{i_j} is an ordinary double point of S. The equalities $\Gamma^2 = \Omega^2 = -2 + r/2$ hold on S but $r \leqslant 3$. Hence, the inequality $\Omega^2 < 0$ holds on S. We have

$$(k - \mathrm{mult}_\Omega(\mathcal{H}))\Omega^2 = (\mathrm{mult}_\Gamma(\mathcal{H}) - k)\Gamma \cdot \Omega + L \cdot \Omega = L \cdot \Omega \geqslant 0,$$

where L is a general curve in $\mathcal{L}$. Therefore, the inequality $\mathrm{mult}_\Omega(\mathcal{H}) \geqslant k$ holds. Then $\Omega \in \mathbb{CS}(Z, \frac{1}{k}\mathcal{H})$, which is impossible.

Let H be a general hyperplane in $\mathbb{P}^3$ passing through the line $\eta(\Gamma)$. Then the curve

$$\Delta = H \cap R = \eta(\Gamma) \cup \Upsilon$$

is reduced and $\eta(\Gamma) \not\subset \mathrm{Supp}(\Upsilon)$, where Υ is a plane curve of degree 5. Moreover, the reducible curve Δ is singular at every singular point $\eta(P_i)$ of R that lies on $\eta(\Gamma)$, but the set $\eta(\Gamma) \cap \Upsilon$ contains at most 5 points. On the other hand, we have

$$\mathrm{Sing}(\Delta) \cap \eta(\Gamma) = \Upsilon \cap \eta(\Gamma),$$

which implies that $|\mathrm{Sing}(Z) \cap \Gamma| \leqslant 5$. Moreover, R is given by the equation

$$f_3(x, y, z, t)^2 = 4f_1(x, y, z, t)f_5(x, y, z, t) \subset \mathbb{P}^3 \cong \mathrm{Proj}\big(\mathbb{C}[x, y, z, t]\big)$$

and singular points of R are given by the equations $f_1 = f_3 = f_5 = 0$. We may assume that the equations $f_1 = f_3 = 0$ and $f_1 = f_5 = 0$ define irreducible curves in $\mathbb{P}^3$, which implies that at most 3 points of the subset $\mathrm{Sing}(R) \subset \mathbb{P}^3$ lie on a single line. Therefore, Bertini's theorem implies that the intersection $\eta(\Gamma) \cap \Upsilon$ contains at least two different points O_1 and O_2 that are not contained in the set $\mathrm{Sing}(R)$.

Remark 4.9. The hyperplane H is tangent to R at the points O_1 and O_2.

Let L_j be a general line on the plane H that passes through the point O_j, $\widetilde{O}_j$ the proper transform of O_j on Z and $\tilde{L}_j$ the proper transform of L_j on Z. Then L_j is tangent to R at O_j and the curve $\tilde{L}_j$ is irreducible and singular at the point $\widetilde{O}_j$, but $-K_Z \cdot \tilde{L}_j = 2$. Let $\widetilde{H}$ be the proper transform of H on Z and D a general surface in the linear system $\mathcal{H}$. Then

$$D|_{\widetilde{H}} = \mathrm{mult}_\Gamma(\mathcal{H})\Gamma + \Psi,$$

where Ψ is an effective divisor on $\widetilde{H}$ such that $\Gamma \not\subset \mathrm{Supp}(\Psi)$. Let $\Lambda_j = (D\setminus\Gamma)\cap\tilde{L}_j$. Then

$$2k = D \cdot \tilde{L}_j \geqslant \mathrm{mult}_{\widetilde{O}_j}(\tilde{L}_j)\,\mathrm{mult}_\Gamma(D) + \sum_{P\in\Lambda_j} \mathrm{mult}_P(D)\,\mathrm{mult}_P(\tilde{L}_j)$$

$$\geqslant 2k + \sum_{P\in\Lambda_j} \mathrm{mult}_P(D)\,\mathrm{mult}_P(\tilde{L}_j),$$

which implies that $D \cap \tilde{L}_j \subset \Gamma$. On the other hand, when we vary the lines L_1 and L_2 on the plane H, the curves $\tilde{L}_1$ and $\tilde{L}_2$ span two different pencils on the surface $\widetilde{H}$, whose base loci consist of the points $\widetilde{O}_1$ and $\widetilde{O}_2$, respectively. Hence, we have $\mathrm{Supp}(D) \cap \mathrm{Supp}(\widetilde{H}) = \mathrm{Supp}(\Gamma)$, where $\widetilde{H}$ is a general divisor in $|-K_Z|$ that passes through Γ and D is a general divisor in $\mathcal{H}$. But this is impossible by Lemma 2.7. The assertion of Proposition 4.2 is proved.

§5. The case $n = 4$: a hypersurface of degree 6 in $\mathbb{P}(1,1,1,2,2)$

We use the notation and assumptions of §3. Let $n = 4$. Then X is a general hypersurface in $\mathbb{P}(1,1,1,2,2)$ of degree 6. The equality $-K_X^3 = 3/2$ holds. The singularities of X consist of points P_1, P_2 and P_3 that are quotient singularities of type $\frac{1}{2}(1,1,1)$.

Let $\psi\colon X \dashrightarrow \mathbb{P}^2$ be the natural projection. Then a generic fibre of the rational map ψ is an elliptic curve and the composite $\psi \circ \eta$ is a morphism, where $\eta\colon Y \to X$ is the composite of the weighted blow-ups of the singular points P_1, P_2 and P_3 with weights $(1,1,1)$.

Proposition 5.1. *The assertion of Theorem 1.10 holds for $n = 4$.*

Proof. To prove this, we must show the existence of a commutative diagram

$$\begin{array}{ccc} X & \overset{\rho}{\dashrightarrow} & V \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\nu} \\ \mathbb{P}^2 & \underset{\varphi}{\dashrightarrow} & \mathbb{P}^2, \end{array} \qquad (5.1)$$

where $\varphi \in \mathrm{Bir}(\mathbb{P}^2)$. Let Z be an element of $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$. We have the following possibilities:

(i) Z is a curve contained in $X \setminus \mathrm{Sing}(X)$;
(ii) Z is a curve containing a singular point of X;
(iii) Z is a singular point of X.

Suppose that Z is an irreducible curve that does not contain singular points of X. Then the equality $-K_X \cdot Z = 1$ holds by Lemma 2.4, which implies that Z is smooth. Let $\gamma \colon W \to X$ be the blow-up of Z and G the exceptional divisor of the morphism γ. Then the divisor $\gamma^*(-4K_X) - G$ is nef, which implies that

$$\big(\gamma^*(-4K_X) - G\big) \cdot \bar{S}_1 \cdot \bar{S}_2 \geqslant 0,$$

where the $\bar{S}_i$ are the proper transforms on W of sufficiently general surfaces in $\mathcal{M}$. We have

$$0 \leqslant \big(\gamma^*(-4K_X) - G\big) \cdot \bar{S}_1 \cdot \bar{S}_2 = -k^2,$$

which is a contradiction.

Suppose that Z is an irreducible curve that passes through some singular point of X. Then the inequality $-K_X \cdot Z \leqslant 1$ holds by Lemma 2.4. The curve Z is contracted by the rational map ψ to a point, and either $-K_X \cdot Z = 1/2$ or $-K_X \cdot Z = 1$.

Let F be a sufficiently general surface in $|-K_X|$ that passes through Z. Then F is smooth outside P_1, P_2 and P_3, which are isolated ordinary double points of F. Let $\widetilde{Z}$ be a fibre of ψ over the point $\psi(Z)$. Then the generality of X implies that Z is an irreducible component of $\widetilde{Z}$.

Suppose that $\widetilde{Z}$ consists of two irreducible components, Z and $\bar{Z}$. Then the inequality $\bar{Z}^2 < 0$ holds on F. But the equivalence $\mathcal{M}|_F \sim_{\mathbb{Q}} kZ + k\bar{Z}$ holds. On the other hand, we have

$$\mathcal{M}|_F = \mathrm{mult}_Z(\mathcal{M})Z + \mathrm{mult}_{\bar{Z}}(\mathcal{M})\bar{Z} + \mathcal{F},$$

where $\mathcal{F}$ is a linear system on F that does not have fixed components. We have $(k - \mathrm{mult}_{\bar{Z}}(\mathcal{M}))\bar{Z} \sim_{\mathbb{Q}} \mathcal{F}$, which implies that $\mathrm{mult}_{\bar{Z}}(\mathcal{M}) = k$, and the support of the effective cycle $S_1 \cdot S_2$ is contained in $Z \cup \bar{Z}$. But this is impossible by Lemma 2.5.

Suppose that $\widetilde{Z}$ consists of three irreducible components, Z, $\widehat{Z}$ and $\check{Z}$. Then

$$-K_X \cdot \widehat{Z} = -K_X \cdot \check{Z} = -K_X \cdot Z = \frac{1}{2} \,,$$

and the intersection form of the curves $\widehat{Z}$ and $\check{Z}$ on F is negative definite. Thus, the support of the cycle $S_1 \cdot S_2$ is contained in the union $Z \cup \widehat{Z} \cup \check{Z}$, where S_1 and S_2 are general surfaces in $\mathcal{M}$. However, this is impossible by Lemma 2.5. Hence, we see that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_1, P_2, P_3\}$.

Let $\pi \colon U \to X$ be the composite of the weighted blow-ups with weights $(1,1,1)$ of the singular points of X that are contained in the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ and $\mathcal{D}$ the proper transform of $\mathcal{M}$ on U. Then it follows from Theorem 2.2 that the equivalence $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ holds, but the divisor $-K_U$ is nef.

Suppose that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ contains a subvariety Δ of U. Then Δ is contained in an exceptional divisor, G say, of π. Then Δ is a line on the surface $G \cong \mathbb{P}^2$ by Lemma 2.3. On the other hand, the linear system $|\pi^*(-2K_X) - G|$ does not have base points and the divisor $\pi^*(-2K_X) - G$ is nef and big. It follows from [9], Lemma 0.3.3, that there is a proper Zariski-closed subset $\Lambda \subset U$ that contains all curves on U having trivial intersection with the divisor $\pi^*(-2K_X) - G$. We have

$$2k^2 = \big(\pi^*(-2K_X) - G\big) \cdot \widetilde{S}_1 \cdot \widetilde{S}_2 \geqslant \mathrm{mult}_\Delta^2(\mathcal{D})\big(\pi^*(-2K_X) - G\big) \cdot \Delta = 2k^2,$$

where $\widetilde{S}_1$ and $\widetilde{S}_2$ are the proper transforms of S_1 and S_2 on U, respectively. Thus, the support of the effective cycle $\widetilde{S}_1 \cdot \widetilde{S}_2$ is contained in the subset $\Lambda \cup \Delta$, which is impossible by Lemma 2.5. Hence, the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ is empty.

The equality $-K_U^3 = 0$ holds by Lemma 2.1. Then $\pi = \eta$ and $-K_U \sim (\psi \circ \pi)^*(\mathcal{O}_{\mathbb{P}^2}(1))$. Hence, the linear system $\mathcal{D}$ lies in the fibres of the elliptic fibration $\psi \circ \pi$. Therefore, the commutative diagram (5.1) exists.

§ 6. The case $n = 6$: a hypersurface of degree 8 in $\mathbb{P}(1, 1, 1, 2, 4)$

We use the notation and assumptions of § 3. Let $n = 6$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1, 1, 1, 2, 4)$ of degree 8, the equality $-K_X^3 = 1$ holds and the singularities of X consist of points P_1 and P_2 that are singularities of type $\frac{1}{2}(1, 1, 1)$. There is a commutative diagram

$$
\begin{array}{ccc}
 & U & \\
{\scriptstyle \pi} \swarrow & & \searrow {\scriptstyle \eta} \\
X \dashrightarrow_{\psi} & & \mathbb{P}^2,
\end{array}
\tag{6.1}
$$

where ψ is the natural projection, π is the composite of the weighted blow-ups of the singular points P_1 and P_2 with weights $(1, 1, 1)$ and η is an elliptic fibration.

Proposition 6.1. *The assertion of Theorem 1.10 holds for $n = 6$.*

Proof. It follows from Theorem 3.3 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain smooth points of X. Therefore, $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains a singular point of X by Corollary 3.4 and Theorem 2.2.

Remark 6.2. Suppose that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains both of the points P_1 and P_2. Then it follows easily from Theorem 2.2 that the assertion of the Theorem 1.10 holds for X. We may thus assume that $P_1 \in \mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \not\ni P_2$.

Lemma 6.3. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain curves.*

Proof. Suppose that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains an irreducible curve C. Then $-K_X \cdot C = 1/2$ by Lemma 2.4.

Let $\check{C}$ be the proper transform of C on U. Then $-K_U \cdot \check{C} = 0$, which implies that $\check{C}$ is a component of a fibre of η. Therefore, C is contracted by the rational map $\psi \colon X \dashrightarrow \mathbb{P}^2$ to a point. In particular, C is smooth and rational.

Let S be a general surface in $|-K_X|$ that contains C. Then S is smooth outside the points P_1 and P_2, which are isolated ordinary double points on S. Let F be a fibre of the rational map ψ over the point $\psi(C)$. Then F consists of two irreducible components and C is one of them. Let Z be the other component of F. Then $Z^2 < 0$ on S but

$$
\mathcal{M}|_S = \operatorname{mult}_C(\mathcal{M})C + \operatorname{mult}_Z(\mathcal{M})Z + \mathcal{L},
$$

where $\mathcal{L}$ is a linear system with no fixed components. We have $(k - \operatorname{mult}_Z(\mathcal{M}))Z \sim_{\mathbb{Q}} \mathcal{L}$, which implies that $\operatorname{mult}_Z(\mathcal{M}) = k$. It follows from Theorem 2.2 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains P_2, which is a contradiction.

Thus, the set $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ consists of the point P_1. Let $\zeta\colon Y \to X$ be the weighted blow-up of P_1 with weights $(1, 1, 1)$ and $\mathcal{D}$ the proper transform of $\mathcal{M}$ on Y. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2 and $\mathbb{CS}\left(Y, \frac{1}{k}\mathcal{D}\right) \neq \varnothing$ by Lemma 2.1.

Let T be a subvariety of Y contained in $\mathbb{CS}\left(Y, \frac{1}{k}\mathcal{D}\right)$. Then $\zeta(T) \in \mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$, which implies that $T \subset G$, where G is the exceptional divisor of ζ. It follows from Lemma 2.3 that T is a line in $G \cong \mathbb{P}^2$. But this is impossible by Lemma 2.4.

§ 7. The case $n = 7$: a hypersurface of degree 8 in $\mathbb{P}(1, 1, 2, 2, 3)$

We use the notation and assumptions of § 3. Let $n = 7$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1, 1, 2, 2, 3)$ of degree 8 whose singularities consist of points P_1, P_2, P_3 and P_4 that are quotient singularities of type $\frac{1}{2}(1, 1, 1)$ and a point Q that is a quotient singularity of type $\frac{1}{3}(1, 1, 2)$.

Let $\xi_i\colon X \dashrightarrow \mathbb{P}(1, 1, 2)$ be a projection such that there is a commutative diagram
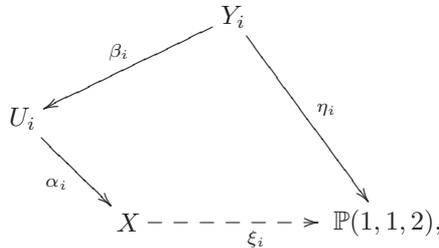
$$
\begin{array}{ccc}
 & Y_i & \\
\beta_i \nearrow & & \searrow \eta_i \\
U_i & & \\
\alpha_i \searrow & & \\
 & X \dashrightarrow[\xi_i] \mathbb{P}(1, 1, 2), &
\end{array}
$$

where α_i is the weighted blow-up of P_i with weights $(1, 1, 1)$, β_i is the weighted blow-up of the proper transform of Q on the variety U_i with weights $(1, 1, 2)$, and η_i is an elliptic fibration.

Let $\xi_0\colon X \dashrightarrow \mathbb{P}(1, 1, 2)$ be a projection such that there is a commutative diagram

$$
\begin{array}{ccc}
U_0 \xleftarrow{\;\beta_0\;} Y_0 & & \\
\alpha_0 \downarrow & \searrow \eta_0 & \\
X \dashrightarrow[\xi_0] & & \mathbb{P}(1, 1, 2),
\end{array}
$$
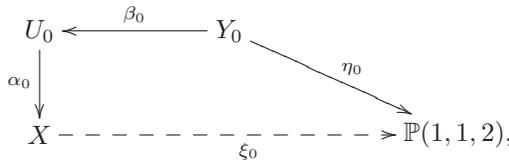
where α_0 is the weighted blow-up of Q with weights $(1, 1, 2)$, β_0 is the blow-up with weights $(1, 1, 1)$ of the singular point of U_0 that dominates Q, and η_0 is an elliptic fibration.

Proposition 7.1. *There is a commutative diagram*

$$
\begin{array}{ccccc}
X & \dashrightarrow[\sigma] & X & \dashrightarrow[\rho] & V \\
\xi_i \downarrow & & & & \downarrow \nu \\
\mathbb{P}(1, 1, 2) & \dashrightarrow & & [\varphi] & \mathbb{P}^2
\end{array}
\tag{7.1}
$$

for some $i \in \{0, 1, 2, 3, 4\}$, where σ and φ are birational maps.

Proof. The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ is non-empty and does not contain smooth points of X. It follows from Lemma 3.11 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_1, P_2, P_3, P_4, Q\}$.

Lemma 7.2. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain two points of the set $\{P_1, P_2, P_3, P_4\}$.*

Proof. Suppose that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the points P_1 and P_2. Let $\pi\colon W \to X$ be the composite of the weighted blow-ups of P_1 and P_2 with weights $(1, 1, 1)$ and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2. But the base locus of $|-K_W|$ consists of a curve C such that $C^2 = -1/3$ on a general surface in the pencil $|-K_W|$. But this is impossible by Lemma 2.7.

Let $\mathcal{D}_i$ be the proper transform of $\mathcal{M}$ on U_i.

Remark 7.3. If the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point Q, then $\mathbb{CS}(U_0, \frac{1}{k}\mathcal{D}_0)$ is non-empty by Lemma 2.1. Similarly, $\mathbb{CS}(U_i, \frac{1}{k}\mathcal{D}_i)$ is non-empty if $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains P_i.

Lemma 7.4. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not consist of the point P_i.*

Proof. Suppose that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_i\}$. Then the set $\mathbb{CS}(U_i, \frac{1}{k}\mathcal{D}_i)$ contains an irreducible subvariety $Z \subset U_i$ by Lemma 2.1. Let E_i be the exceptional divisor of α_i. Then Z is a line on the surface $E_i \cong \mathbb{P}^2$ by Lemma 2.3, which is impossible by Lemma 2.4.

Therefore, if $P_i \in \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$, then $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_i, Q\}$. This implies that the proper transform of $\mathcal{M}$ on Y_i lies in the fibres of the elliptic fibration η_i. Thus, if $P_i \in \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$, then the commutative diagram (7.1) exists. Thus, we may assume that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ consists of the point Q.

Let O be a singular point of the variety U_0 contained in the exceptional divisor of the morphism α_0. Then O is contained in $\mathbb{CS}(U_0, \frac{1}{k}\mathcal{D}_0)$ by Lemma 2.3, which implies the existence of the commutative diagram (7.1) by Theorem 2.2. The assertion is proved.

§ 8. The case $n = 8$: a hypersurface of degree 9 in $\mathbb{P}(1, 1, 1, 3, 4)$

We use the notation and assumptions of § 3. Let $n = 8$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1, 1, 1, 3, 4)$ of degree 9, the equality $-K_X^3 = 3/4$ holds and the singularities of X consist of one point O that is a quotient singularity of type $\frac{1}{4}(1, 1, 3)$.

Proposition 8.1. *The assertion of Theorem 1.10 holds for $n = 8$.*

Proof. The hypersurface X can be given by the equation

$$w^2 z + f_5(x, y, z, t)w + f_9(x, y, z, t) = 0 \subset \mathbb{P}(1, 1, 1, 3, 5)$$
$$\cong \mathrm{Proj}\big(\mathbb{C}[x, y, z, t, w]\big), \quad (8.1)$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = \mathrm{wt}(z) = 1$, $\mathrm{wt}(t) = 3$, $\mathrm{wt}(w) = 4$, and f_i is a quasi-homogeneous polynomial of degree i. The point O is given by $x = y = z = t = 0$.

There is a commutative diagram
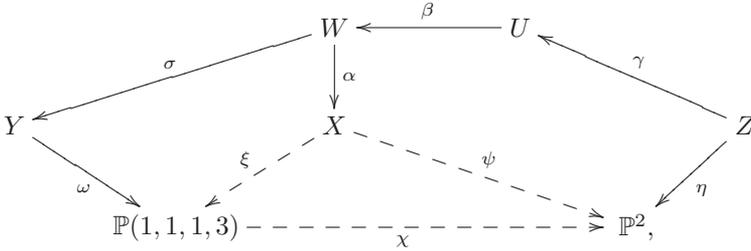
where ξ, ψ and χ are natural projections, α is the weighted blow-up of the singular point O with weights $(1, 1, 3)$, β is the weighted blow-up with weights $(1, 1, 2)$ of the singular point of W that is a quotient singularity of type $\frac{1}{3}(1, 1, 2)$, γ is the weighted blow-up with weights $(1, 1, 1)$ of the singular point of U that is a quotient singularity of type $\frac{1}{2}(1, 1, 1)$, η is an elliptic fibration, σ is a birational morphism that contracts 15 smooth rational curves to 15 isolated ordinary double points $P_1, \ldots, P_{15}$ of Y, respectively, and ω is a double cover branched over the surface $R \subset \mathbb{P}(1, 1, 1, 3)$ given by the equation

$$f_5(x, y, z, t)^2 - 4z f_9(x, y, z, t) = 0 \subset \mathbb{P}(1, 1, 1, 3) \cong \mathrm{Proj}\big(\mathbb{C}[x, y, z, t]\big)$$

and has 15 isolated ordinary double points $\omega(P_1), \ldots, \omega(P_{15})$ given by $z = f_5 = f_9 = 0$.

Let G be the exceptional divisor of the morphism α, F the exceptional divisor of the morphism β, P the singular point of W, Q the singular point of U, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on W, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on U, $\mathcal{P}$ the proper transform of $\mathcal{M}$ on Z, and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on Y.

Remark 8.2. The divisors $-K_W$ and $-K_U$ are nef and big, and $\omega \circ \sigma(G)$ is given by $z = 0$.

It follows from Theorem 3.3 that the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ does not contain smooth points of X. It follows from Corollary 3.4 and Theorem 2.2 that $O \in \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$.

Lemma 8.3. *The set* $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ *does not contain curves.*

Proof. Let L be a curve in $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$. It follows from Theorem 3.9 that there are two different surfaces D and D' in the linear system $|-K_X|$ such that the irreducible curve L is a component of the cycle $D \cdot D'$, which is reduced and contains at most two components.

Let $\mathcal{P}$ be the pencil in $|-K_X|$ generated by D and D'. Then we may assume that D is a sufficiently general surface in $\mathcal{P}$. Applying Lemma 2.7 together with the proof of Lemma 2.4 to $\mathcal{M}$ and $\mathcal{P}$, we immediately obtain a contradiction in the case when the equality $-K_X \cdot L = 3/4$ holds. Therefore, we may assume that either $-K_X \cdot L = 1/4$ or $-K_X \cdot L = 1/2$. We consider only the case $-K_X \cdot L = 1/4$ because the case $-K_X \cdot L = 1/2$ is simpler and very similar.

Let S_W be the proper transform on W of the surface given by $z = 0$ and L_W the proper transform on W of L. Then S_W must contain L_W because

$$S_W \sim_{\mathbb{Q}} \alpha^*(-K_X) - \frac{5}{4}G,$$

but $G \cdot L_W \geqslant 1/3$. Thus, either the curve L_W is contracted by σ or the curve $\omega(L_Y)$ is a ruling of the cone $\mathbb{P}(1,1,1,3)$ contained in the surface $\omega \circ \sigma(G)$, where $L_Y = \sigma(L_W)$.

Suppose that L_W is not contracted by σ. Then $\omega(L_Y)$ is not contained in the surface R, which implies that $\omega(L_Y)$ contains at most one singular point of R different from the point $\omega \circ \sigma(P)$. Moreover, the curve $\omega(L_Y)$ must contain a singular point of R different from $\omega \circ \sigma(P)$ because otherwise $-K_X \cdot L = 3/4$. Thus, we may assume that $\omega(L_Y)$ contains the point $\omega(P_1)$.

Let D_Y and D'_Y be the proper transforms on Y of D and D', respectively. Then P_1 is an isolated ordinary double point of D_Y. Thus, we see that the proper transform of the curve L' on the threefold W is contracted to the point P_1 by σ and

$$D_Y \cdot D'_Y = L_Y + \bar{L}_Y,$$

where $\bar{L}_Y$ is a ruling of the cone $G \cong \mathbb{P}(1,1,3)$. In particular, we have $-K_X \cdot L' = 1/4$, which is impossible by the equality $-K_X \cdot L = 1/4$. Hence, L_W is contracted by σ.

Let L'_W be the proper transform on W of L' and $L'_Y = \sigma(L'_W)$. Then $\omega(L'_Y)$ is a ruling of the cone $\mathbb{P}(1,1,1,3)$ contained in the surface $\omega \circ \sigma(G)$. $\omega(L'_Y)$ is not contained in R, which implies that $\omega(L'_Y)$ contains at most one singular point of R different from $\omega \circ \sigma(P)$. $\omega(L'_Y)$ must contain a singular point of R different from $\omega \circ \sigma(P)$ because $-K_X \cdot L' \neq 3/4$. Thus, we may assume that $\omega(L'_Y)$ contains $\omega(P_1)$.

Since P_1 is an isolated ordinary double point of D_Y and $\sigma(L_W) = P_1$, we have

$$D_Y \cdot D'_Y = L'_Y + \bar{L}'_Y,$$

where $\bar{L}'_Y$ is a ruling of $G \cong \mathbb{P}(1,1,3)$. The intersection $L_W \cap L'_W$ consists of a point $O' \notin G$. Hence, the intersection $L \cap L'$ contains the point $\alpha(O')$, which is different from O.

The surface D is smooth at the point $\alpha(O')$ but $(L + L') \cdot L' = 1/2$ on D, which implies that $L' \cdot L' < 0$ on D. Therefore, we have

$$\mathcal{M}|_D = m_1 L + m_2 L' + \mathcal{L} \equiv kL + kL',$$

where $\mathcal{L}$ is a linear system on D that does not have fixed components and m_1 and m_2 are natural numbers such that $m_1 \geqslant k$. In particular, we have

$$0 \leqslant (m_1 - k)L' \cdot L + \mathcal{L} \cdot L' = (k - m_2)L' \cdot L',$$

which implies that $m_2 = m_1 = k$ and $\mathcal{M}|_D = kL + kL'$, which is impossible by Lemma 2.7.

It follows from Theorem 3.3 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{O\}$. Hence, the assertion of Theorem 2.2 implies that $\mathcal{D} \sim_{\mathbb{Q}} -kK_W$. But it follows from Corollary 2.8 and Lemmas 2.1 and 2.3 that $\mathbb{CS}(W, \frac{1}{k}\mathcal{D}) = \{P\}$.

The equivalence $\mathcal{H} \sim_{\mathbb{Q}} -kK_U$ holds by Theorem 2.2. Hence, the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{H})$ contains the point Q by Lemmas 2.1 and 2.3. We see that $\mathcal{P}$ lies in the fibres of η by Theorem 2.2.

§9. The case $n = 9$: a hypersurface of degree 9 in $\mathbb{P}(1, 1, 2, 3, 3)$

We use the notation and assumptions of §3. Let $n = 9$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1, 1, 2, 3, 3)$ of degree 9, the equality $-K_X^3 = 1/2$ holds and the singularities of X consist of points P_1, P_2 and P_3 that are quotient singularities of type $\frac{1}{3}(1, 1, 2)$ and a point O that is a quotient singularity of type $\frac{1}{2}(1, 1, 1)$.

Let $\xi \colon X \dashrightarrow \mathbb{P}(1, 1, 2)$ be a projection. There is a commutative diagram
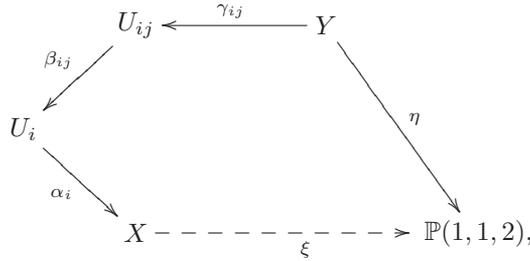
where α_i is the blow-up of P_i with weights $(1, 1, 2)$, β_{ij} is the weighted blow-up with weights $(1, 1, 2)$ of the proper transform of P_j on the variety U_i, γ_{ij} is the weighted blow-up with weights $(1, 1, 2)$ of the proper transform of P_k on U_{ij}, and η is an elliptic fibration, where $i \neq j$ and $k \notin \{i, j\}$.

Remark 9.1. The divisors $-K_{U_i}$ and $-K_{U_{ij}}$ are nef and big.

There is a commutative diagram
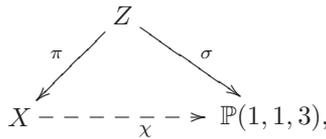
where χ is a projection, π is the blow-up of Q with weights $(1, 1, 1)$ and σ is an elliptic fibration.

Proposition 9.2. *Either there is a commutative diagram*

$$
\begin{array}{ccc}
X & \overset{\rho}{\dashrightarrow} & V \\
{\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \nu} \\
\mathbb{P}(1, 1, 2) & \underset{\varphi}{\dashrightarrow} & \mathbb{P}^2
\end{array}
\tag{9.1}
$$

or there is a commutative diagram

$$
\begin{array}{ccccc}
X & \overset{\omega}{\dashrightarrow} & X & \overset{\rho}{\dashrightarrow} & V \\
{\scriptstyle \chi}\downarrow & & & & \downarrow{\scriptstyle \nu} \\
\mathbb{P}(1, 1, 3) & & \underset{\upsilon}{\dashrightarrow} & & \mathbb{P}^2,
\end{array}
\tag{9.2}
$$

where φ, ω and υ are birational maps.

Proof. Note that this proposition implies the assertion of Theorem 1.10 for $n = 9$.

Lemma 9.3. *Suppose that* $Q \in \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. *Then there is a commutative diagram* (9.2).

Proof. Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on Z. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_X$ by Theorem 2.2, which implies the equality $S \cdot C = 0$, where S is a general surface in $\mathcal{M}$ and C is a generic fibre of the morphism σ. Thus, the commutative diagram (9.2) exists.

We may assume that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_1, P_2, P_3\}$ by Theorem 3.3 and Lemma 3.11.

Lemma 9.4. *The commutative diagram* (9.1) *exists whenever* $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_1, P_2, P_3\}$.

Proof. See the proof of Lemma 9.3.

Therefore, we may assume that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain the point P_3 but does contain the point P_1.

Lemma 9.5. *The set* $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ *contains the point* P_2.

Proof. Suppose that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ consists of the point P_1. Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U_1. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_{U_1}$ by Theorem 2.2, and Lemma 2.1 implies that the set $\mathbb{CS}(U_1, \frac{1}{k}\mathcal{B})$ is non-empty.

Let Z be an element of the set $\mathbb{CS}(U_1, \frac{1}{k}\mathcal{B})$ and G the exceptional divisor of the birational morphism α_1. Then it follows from Lemmas 2.3 and 2.4 that Z is a singular point of G which is a quotient singularity of type $\frac{1}{2}(1, 1, 1)$ on U_1.

Let $\delta \colon W \to U_1$ be the weighted blow-up of the point Z with weights $(1, 1, 1)$, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on W, and F a general surface in the linear system $|-K_W|$. Then the inequality $\Delta^2 = -1/2$ holds on F, but $\mathcal{D}|_F \sim_{\mathbb{Q}} k\Delta$ by Theorem 2.2, which is impossible by Lemmas 2.9 and 2.7.

We can now apply the arguments of the proof of Lemma 9.5 to get a contradiction.

§ 10. The case $n = 10$: a hypersurface of degree 10 in $\mathbb{P}(1, 1, 1, 3, 5)$

We use the notation and assumptions of § 3. Let $n = 10$. Then X is a hypersurface in $\mathbb{P}(1, 1, 1, 3, 5)$ of degree 10, the equality $-K_X^3 = 2/3$ holds and the singularities of X consist of a point O that is a quotient singularity of type $\frac{1}{3}(1, 1, 2)$. There is a commutative diagram
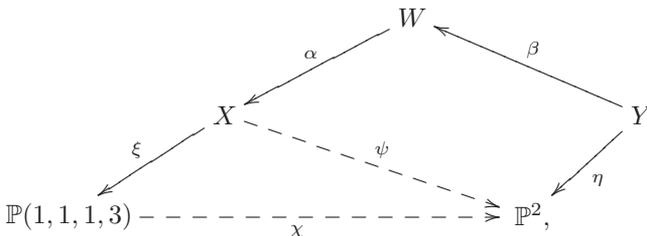
where ξ, ψ and χ are natural projections, α is the weighted blow-up of O with weights $(1,1,1)$, β is the weighted blow-up with weights $(1,1,1)$ of the singular point of W, and η is an elliptic fibration.

Proposition 10.1. *The assertion of Theorem* 1.10 *holds for* $n = 10$.

Proof. Let Q be the unique singular point of the variety W, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on W and $\mathcal{H}$ the proper transform of $\mathcal{M}$ on Y.

Lemma 10.2. *The set* $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ *does not contains curves.*

Proof. Suppose that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ contains a curve C. Then $-K_X \cdot C = 1/3$ by Lemma 2.4, which implies that C is contracted by the rational map ψ to a point.

The variety $\mathbb{P}(1,1,1,3)$ is cone whose vertex is the point $\xi(O)$. The curve $\xi(C)$ is a ruling of the cone $\mathbb{P}(1,1,1,3)$. The generality of the hypersurface X implies that $\xi(C)$ is not contained in the ramification surface of the morphism ξ. Thus, there is an irreducible smooth rational curve Z on X such that $Z \neq C$ and $\xi(Z) = \xi(C)$.

Let S be a general surface in the linear system $|-K_X|$ that contains the curves C and Z. Then $Z^2 < 0$ on S. But $\mathcal{M}|_S \sim_{\mathbb{Q}} kC + kZ$, which easily leads to a contradiction using Lemmas 2.7 and 2.9 because $\text{mult}_C(\mathcal{M}) = k$.

The set $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ consists of the point O by Theorem 3.3. The set $\mathbb{CS}\left(W, \frac{1}{k}\mathcal{D}\right)$ contains the point Q by Lemmas 2.1 and 2.3. But $\mathcal{H} \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2, which implies that $\mathcal{H}$ is contained in the fibres of η. The assertion is proved.

§ 11. The case $n = 12$: a hypersurface of degree 10 in $\mathbb{P}(1,1,2,3,4)$

We use the notation and assumptions of § 3. Let $n = 12$. Then X is a general hypersurface in $\mathbb{P}(1,1,2,3,4)$ of degree 10 whose singularities consist of points P_1 and P_2 that are singularities of type $\frac{1}{2}(1,1,1)$, a point P_3 that is a singularity of type $\frac{1}{3}(1,1,2)$ and a point P_4 that is a singularity of type $\frac{1}{4}(1,1,3)$.

Proposition 11.1. *The assertion of Theorem* 1.10 *holds for* $n = 12$.

There is a commutative diagram
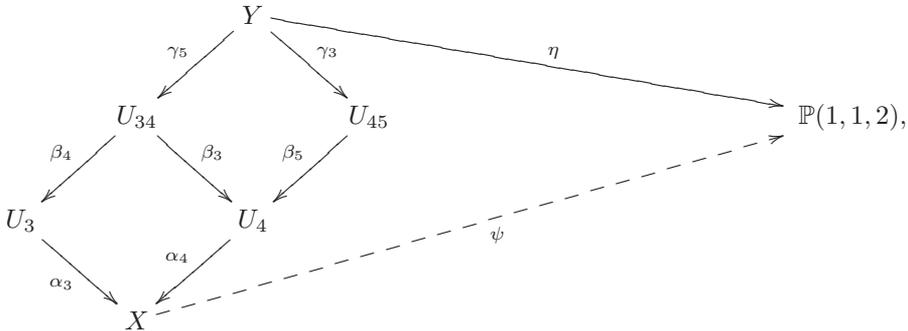
where ψ is a projection, α_3 is the weighted blow-up of P_3 with weights $(1,1,2)$, α_4 is the weighted blow-up of P_4 with weights $(1,1,3)$, β_4 is the weighted blow-up with weights $(1,1,3)$ of the proper transform of the point P_4 on U_3, β_3 is the weighted blow-up with weights $(1,1,2)$ of the proper transform of the point P_3 on U_4, β_5 is the

weighted blow-up with weights $(1,1,2)$ of the singular point of U_4 that is contained in the exceptional divisor of α_4, γ_3 is the weighted blow-up with weights $(1,1,2)$ of the proper transform of the point P_3 on the variety U_{45}, γ_5 is the weighted blow-up with weights $(1,1,2)$ of the singular point of U_{34} that is contained in the exceptional divisor of the morphism β_4, and η is an elliptic fibration.

Remark 11.2. The divisors $-K_{U_3}$, $-K_{U_4}$, $-K_{U_{34}}$ and $-K_{U_{45}}$ are nef and big.

Proof of Proposition 11.1. It follows from Theorem 3.3 and Lemma 3.11 that

$$\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) \subseteq \{P_1, P_2, P_3, P_4\}.$$

Lemma 11.3. *The set* $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ *does not contain* P_1 *or* P_2.

Proof. We assume that $P_1 \in \mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ and seek a contradiction.

Let $\pi\colon W \to X$ be the weighted blow-up of the singular point P_1 with weights $(1,1,1)$, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W, S a general surface in the pencil $|-K_W|$, and Z the base curve of $|-K_W|$. Then $Z^2 = -1/24$ on S. But $\mathcal{B}|_S \sim_{\mathbb{Q}} kZ$ by Theorem 2.2, which is impossible by Lemmas 2.9 and 2.7.

Let $\mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{D}_{34}$ and $\mathcal{D}_{45}$ be the proper transforms of $\mathcal{M}$ on U_3, U_4, U_{34} and U_{45}, respectively. Then it follows from Lemma 2.1 that $\mathbb{CS}\left(U_\mu, \frac{1}{k}\mathcal{D}_\mu\right) \neq \varnothing$ in the case when $\mathcal{D}_\mu \sim_{\mathbb{Q}} -kK_{U_\mu}$.

Lemma 11.4. *Suppose that the set* $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ *contains the point* P_4. *Let* $\bar{P}_3$ *be the proper transform of* P_3 *on* U_4 *and* $\bar{P}_5$ *the singular point of* U_4 *such that* $\alpha_4(\bar{P}_5) = P_4$. *Then* $\mathcal{D}_4 \sim_{\mathbb{Q}} -kK_{U_4}$ *and* $\mathbb{CS}\left(U_4, \frac{1}{k}\mathcal{D}_4\right) \subseteq \{\bar{P}_3, \bar{P}_5\}$.

Proof. The equivalence $\mathcal{D}_4 \sim_{\mathbb{Q}} -kK_{U_4}$ follows from Theorem 2.2. We must show that $\mathbb{CS}\left(U_4, \frac{1}{k}\mathcal{D}_4\right) \subseteq \{\bar{P}_3, \bar{P}_5\}$. Suppose that the set $\mathbb{CS}\left(U_4, \frac{1}{k}\mathcal{D}_4\right)$ is not contained in $\{\bar{P}_3, \bar{P}_5\}$. Let C be an element of $\mathbb{CS}\left(U_4, \frac{1}{k}\mathcal{D}_4\right)$ that is different from the points $\bar{P}_3$ and $\bar{P}_5$. Then $\alpha_4(C)$ is contained in the set $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$, which implies that $\alpha_4(C) = P_4$.

Let G be an exceptional divisor of the blow-up α_4. Then $G \cong \mathbb{P}(1,1,3)$, and we can identify G with a cone over a smooth rational curve in $\mathbb{P}^3$ of degree 3. It follows from Lemma 2.3 that C is a ruling of the cone G. But this is impossible by Corollary 2.8.

Lemma 11.5. *Suppose that* $P_3 \in \mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$. *Let* $\bar{P}_4$ *be the proper transform of* P_4 *on* U_3. *Then* $\mathcal{D}_3 \sim_{\mathbb{Q}} -kK_{U_3}$ *and* $\mathbb{CS}\left(U_3, \frac{1}{k}\mathcal{D}_3\right) = \{\bar{P}_4\}$.

Proof. See the proof of Lemma 9.5.

It follows from Theorem 2.2 that either $\mathcal{D}_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$ or $\mathcal{D}_{45} \sim_{\mathbb{Q}} -kK_{U_{45}}$. Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on Y.

Lemma 11.6. *Suppose that* $\mathcal{D}_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$. *Then* $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$.

Proof. Let F be the exceptional divisor of the morphism β_3, G the exceptional divisor of the morphism β_4, $\check{P}_5$ the singular point of the surface G and $\check{P}_6$ the singular point of the surface F. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2 if the set $\mathbb{CS}\left(U_{34}, \frac{1}{k}\mathcal{D}_{34}\right)$ contains the point $\check{P}_5$.

We may assume that $\check{P}_5 \notin \mathbb{CS}(U_{34}, \frac{1}{k}\mathcal{D}_{34})$. Then $\check{P}_6 \in \mathbb{CS}(U_{34}, \frac{1}{k}\mathcal{D}_{34})$ by Lemmas 2.1 and 2.3. Let $\pi\colon W \to U_{34}$ be the weighted blow-up of the point $\check{P}_6$ with weights $(1,1,1)$, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W and S a general surface in the linear system $|-K_W|$. Then the base locus of $|-K_W|$ consists of the irreducible curve Δ such that $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$. But $\Delta^2 < 0$ on S, which is impossible by Lemmas 2.9 and 2.7.

Arguing as in the proof of Lemma 11.6, we see that $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ in the case when $\mathcal{D}_{45} \sim_{\mathbb{Q}} -kK_{U_{45}}$. But it follows from the equivalence $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ that the linear system $\mathcal{D}$ lies in the fibres of the elliptic fibration η. Hence, the assertion is proved.

§ 12. The case $n = 13$: a hypersurface of degree 11 in $\mathbb{P}(1,1,2,3,5)$

We use the notation and assumptions of § 3. Let $n = 13$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,2,3,5)$ of degree 11, the singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, a point P_2 that is a singularity of type $\frac{1}{3}(1,1,2)$ and a point P_3 that is a singularity of type $\frac{1}{5}(1,2,3)$. The equality $-K_X^3 = 11/30$ holds.

There is a commutative diagram
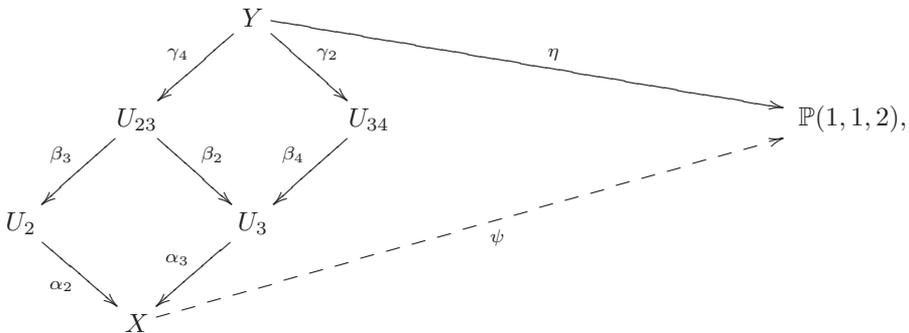
where ψ is a projection, α_2 is the weighted blow-up of P_2 with weights $(1,1,2)$, α_3 is the weighted blow-up of P_3 with weights $(1,2,3)$, β_3 is the weighted blow-up with weights $(1,2,3)$ of the proper transform of the point P_3 on U_2, β_2 is the weighted blow-up with weights $(1,1,2)$ of the proper transform of P_2 on U_3, β_4 is the weighted blow-up with weights $(1,1,2)$ of the singular point of U_3 that is the quotient singularity of type $\frac{1}{3}(1,1,2)$ contained in the exceptional divisor of α_3, γ_2 is the weighted blow-up with weights $(1,1,2)$ of the proper transform of P_2 on U_{34}, γ_4 is the weighted blow-up with weights $(1,1,2)$ of the singular point of U_{23} that is the quotient singularity of type $\frac{1}{3}(1,1,2)$ contained in the exceptional divisor of the morphism β_3, and η is an elliptic fibration.

Proposition 12.1. *The assertion of Theorem 1.10 holds for $n = 13$.*

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_2, P_3\}$.

Lemma 12.2. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point P_3.*

Proof. Suppose that $P_3 \notin \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. Then $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_2\}$. Let $\mathcal{D}_2$ be the proper transform of $\mathcal{M}$ on U_2. Then it follows from Theorem 2.2 that $\mathcal{D}_2 \sim_{\mathbb{Q}} -kK_{U_2}$, but the set $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$ is non-empty by Lemma 2.1.

Let E be the exceptional divisor of the morphism α_2. Then E can be identified with a cone over the smooth rational curve in $\mathbb{P}^3$ of degree 3. Let Z be a subvariety of U_2 that is contained in the set $\mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$. Then it follows from Lemmas 2.3 and 2.4 that Z is the vertex of the cone E.

The point Z is a quotient singularity of type $\frac{1}{2}(1,1,1)$ of U_2. Let $\pi\colon W \to U_2$ be the blow-up of Z with weights $(1,1,1)$ and S a sufficiently general surface in the pencil $|-K_W|$. Then the base locus of $|-K_W|$ consists of an irreducible curve Δ such that $\Delta^2 = -3/10$ on the surface S, but $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$, where $\mathcal{B}$ is the proper transform of $\mathcal{M}$ on W. We have $\mathcal{B}|_S = k\Delta$, which is impossible by Lemma 2.7.

Lemma 12.3. *The set* $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ *contains the point* P_2.

Proof. Suppose that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain P_2. Then $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ consists of the point P_3. Let $\mathcal{D}_3$ be the proper transform of $\mathcal{M}$ on U_3. Then $\mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3) \neq \varnothing$ by Lemma 2.1.

Let G be the exceptional divisor of the morphism α_3, P_4 the singular point of G that is a quotient singularity of type $\frac{1}{3}(1,1,2)$ on U_3, and P_5 the singular point of the surface G that is a quotient singularity of type $\frac{1}{2}(1,1,1)$ on U_3. Then $G \cong \mathbb{P}(1,2,3)$, and it follows from Lemma 2.3 that either $\mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3) = \{P_4\}$, or $P_5 \in \mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3)$.

Suppose that the set $\mathbb{CS}(U_3, \frac{1}{k}\mathcal{D}_3)$ contains P_5. Let $\pi\colon W \to U_3$ be the weighted blow-up of P_5 with weights $(1,1,1)$, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W, L the curve on the surface G that is contained in $|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)|$, $\bar{L}$ the proper transform of L on W and S a general surface in $|-K_W|$. Then $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta + k\bar{L}$ by Theorem 2.2 and the base locus of $|-K_W|$ consists of $\bar{L}$ and the irreducible curve Δ such that $\alpha \circ \pi(\Delta)$ is the base curve of the pencil $|-K_X|$. The equalities

$$\Delta^2 = -\frac{5}{6}, \qquad \bar{L}^2 = -\frac{4}{3}, \qquad \Delta \cdot \bar{L} = 1,$$

hold on the surface S, and this implies that the intersection form of Δ and $\bar{L}$ on S is negative definite. But this is impossible by Lemmas 2.7 and 2.9.

Hence, the set $\mathbb{CS}((U_3, \lambda\mathcal{D}_3)$ consists of the point P_4. Let D_{34} be the proper transform of $\mathcal{M}$ on U_{34}. Then $\mathbb{CS}(U_{34}, \frac{1}{k}\mathcal{D}_{34}) \neq \varnothing$ by Lemma 2.1 because the equivalence $D_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$ holds by Theorem 2.2.

Let E be the exceptional divisor of the morphism β_4 and P_6 the singular point of the surface E. Then $E \cong \mathbb{P}(1,1,3)$, P_6 is a quotient singularity of type $\frac{1}{2}(1,1,1)$ on U_{34}, and $P_6 \in \mathbb{CS}(U_{34}, \frac{1}{k}\mathcal{D}_{34})$ by Lemma 2.3.

Let $\zeta\colon Z \to U_{34}$ be the weighted blow-up of P_6 with weights $(1,1,1)$, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on Z and F a general surface in the pencil $|-K_Z|$. Then the base locus of $|-K_Z|$ consists of irreducible curves $\check{L}$ and $\check{\Delta}$ such that the equalities

$$\check{\Delta}^2 = -\frac{5}{6}, \qquad \check{L}^2 = -\frac{3}{2}, \qquad \check{\Delta} \cdot \check{L} = 1$$

hold on F. Thus, the intersection form of $\check{\Delta}$ and $\check{L}$ on F is negative definite, but $\mathcal{H}|_F \sim_{\mathbb{Q}} k\check{\Delta} + k\check{L}$ by Theorem 2.2, which is impossible by Lemmas 2.7 and 2.9.

We see that $\mathbb{CS}\bigl(X, \frac{1}{k}\mathcal{M}\bigr) = \{P_2, P_3\}$. Let $\mathcal{D}_{23}$ be the proper transform of $\mathcal{M}$ on U_{23}. Then $\mathcal{D}_{23} \sim_{\mathbb{Q}} -kK_{U_{23}}$ by Theorem 2.2, and $\mathbb{CS}\bigl(U_{23}, \frac{1}{k}\mathcal{D}_{23}\bigr) \neq \varnothing$ by Lemma 2.1.

Remark 12.4. The proof of Lemma 12.2 implies that the set $\mathbb{CS}\bigl(U_{23}, \frac{1}{k}\mathcal{D}_{23}\bigr)$ does not contain the singular point of the variety U_{23} that is contained in the exceptional divisor of β_2. But the proof of Lemma 12.3 implies that $\mathbb{CS}\bigl(U_{23}, \frac{1}{k}\mathcal{D}_{23}\bigr)$ does not contain the singular point of U_{23} that is a singularity of type $\frac{1}{2}(1,1,1)$ contained in the exceptional divisor of β_3.

Therefore, the set $\mathbb{CS}\bigl(U_{23}, \frac{1}{k}\mathcal{D}_{23}\bigr)$ contains the singular point of U_{23} that is a singularity of type $\frac{1}{3}(1,1,2)$ contained in the β_3-exceptional divisor. Let $\mathcal{P}$ be the proper transform of $\mathcal{M}$ on Y. Then $\mathcal{P} \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2, which implies the assertion of the proposition.

§13. The case $n = 15$: a hypersurface of degree 12 in $\mathbb{P}(1,1,2,3,6)$

We use the notation and assumptions of §3. Let $n = 15$. Then X is a general hypersurface in $\mathbb{P}(1,1,2,3,6)$ of degree 12, the equality $-K_X^3 = 1/3$ holds and the singularities of X consist of points P_1 and P_2 that are quotient singularities of type $\frac{1}{2}(1,1,1)$ and points P_3 and P_4 that are quotient singularities of type $\frac{1}{3}(1,1,2)$.

Proposition 13.1. *The assertion of Theorem 1.10 holds for $n = 15$.*

Proof. There is a commutative diagram
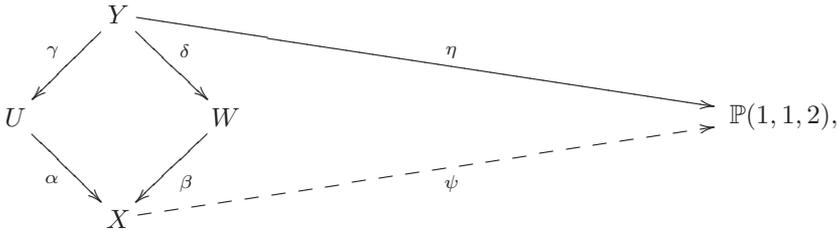
where ψ is the natural projection, α is the weighted blow-up of P_3 with weights $(1,1,2)$, β is the weighted blow-up of P_4 with weights $(1,1,2)$, γ is the weighted blow-up with weights $(1,1,2)$ of the proper transform of P_4 on U, δ is the weighted blow-up with weights $(1,1,2)$ of the proper transform of P_3 on W, and η is an elliptic fibration.

We have $\mathbb{CS}\bigl(X, \frac{1}{k}\mathcal{M}\bigr) \subseteq \{P_3, P_4\}$ by Theorem 3.3, Proposition 3.5 and Lemma 3.11.

Let $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on Y. To complete the proof we must show that $\mathcal{H}$ lies in the fibres of the morphism η, which follows easily from the condition $\mathbb{CS}\bigl(X, \frac{1}{k}\mathcal{M}\bigr) = \{P_3, P_4\}$ by Theorem 2.2.

We may assume that $P_4 \notin \mathbb{CS}\bigl(X, \frac{1}{k}\mathcal{M}\bigr) \ni P_3$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U and Q the singular point of U that is contained in the exceptional divisor of the morphism α. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2, and it follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\bigl(U, \frac{1}{k}\mathcal{B}\bigr)$ contains Q.

Let $\zeta \colon Z \to U$ be the weighted blow-up of Q with weights $(1,1,1)$, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on the threefold Z and S a general surface in $|-K_Z|$. Then the base locus of the pencil $|-K_Z|$ consists of an irreducible curve Δ such that $\Delta^2 < 0$ on S, which is impossible by Lemmas 2.9 and 2.7 because $\mathcal{D}|_S \sim_{\mathbb{Q}} -k\Delta$ by Theorem 2.2.

§ 14. The case $n = 16$: a hypersurface of degree 12 in $\mathbb{P}(1,1,2,4,5)$

We use the notation and assumptions of § 3. Let $n = 16$. Then X is a general hypersurface in $\mathbb{P}(1,1,2,4,5)$ of degree 12, the equality $-K_X^3 = 3/10$ holds and the singularities of X consist of points P_1, P_2 and P_3 that are quotient singularities of type $\frac{1}{2}(1,1,1)$ and a point P_4 that is a quotient singularity of type $\frac{1}{5}(1,1,4)$.

There is a commutative diagram

$$
\begin{array}{ccccc}
U & \xleftarrow{\ \ \beta\ \ } & W & \xleftarrow{\ \ \gamma\ \ } & Y \\
{\scriptstyle\alpha}\downarrow & & & & \downarrow{\scriptstyle\eta} \\
X & \dashrightarrow[\psi]{} & & & \mathbb{P}(1,1,2),
\end{array}
$$

where ψ is the natural projection, α is the weighted blow-up of P_4 with weights $(1,1,4)$, β is the weighted blow-up with weights $(1,1,3)$ of the singular point of U that is contained in the exceptional divisor of α, γ is the weighted blow-up with weights $(1,1,2)$ of the singular point of W that is contained in the exceptional divisor of β, and η is an elliptic fibration.

Proposition 14.1. *The assertion of Theorem 1.10 holds for $n = 16$.*

Proof. The divisors $-K_U$ and $-K_W$ are nef and big. The morphism η coincides with the map given by the linear system $|-2K_Y|$.

It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ consists of the point P_4. Let G be the exceptional divisor of the morphism α, $\bar{P}_5$ the singular point of the surface G and $\mathcal{D}$ the proper transform of $\mathcal{M}$ on U. Then G is a cone over a smooth rational curve of degree 4 and $\bar{P}_5$ is a singularity of type $\frac{1}{4}(1,1,3)$ on U.

We see that $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2 and $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big) \ne \varnothing$ by Lemma 2.1.

Lemma 14.2. *The set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ consists of the point $\bar{P}_5$.*

Proof. Suppose that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ contains a subvariety C of U that is different from the point $\bar{P}_5$. Then it follows from Lemma 2.3 that C is a ruling of the cone G, which is impossible by Corollary 2.8.

Let $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on W. Then it follows from Theorem 2.2 and Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{H}\big)$ contains the singular point of W that is contained in the exceptional divisor of β. Therefore, the proper transform of $\mathcal{M}$ on Y lies in the fibres of the morphism η by Theorem 2.2, which completes the proof.

§ 15. The case $n = 17$: a hypersurface of degree 12 in $\mathbb{P}(1, 1, 3, 4, 4)$

We use the notation and assumptions of § 3. Let $n = 17$. Then X is a general hypersurface in $\mathbb{P}(1, 1, 3, 4, 4)$ of degree 12 whose singularities consist of points P_1, P_2 and P_3 that are singularities of type $\frac{1}{4}(1, 1, 3)$. There is a commutative diagram

$$
\begin{array}{ccc}
 & Z & \\
{}^{\pi}\swarrow & & \searrow{}^{\omega} \\
X & \dashrightarrow{\psi} & \mathbb{P}(1, 1, 3),
\end{array}
$$

where ψ is a projection, π is the composite of the weighted blow-ups of P_1, P_2 and P_3 with weights $(1, 1, 3)$, and ω is an elliptic fibration.

It follows from Example 1.4 that there is a commutative diagram

$$
\begin{array}{ccc}
 & Y_i & \\
{}^{\beta_i}\swarrow & & \searrow{}^{\eta_i} \\
U_i & & \\
{}^{\alpha_i}\searrow & & \\
 & X \dashrightarrow{\xi_i} & \mathbb{P}(1, 1, 4),
\end{array}
$$

where ξ_i is a projection, α_i is the blow-up of P_i with weights $(1, 1, 3)$, β_i is the weighted blow-up with weights $(1, 1, 2)$ of the singular point of U_i that is contained in the exceptional divisor of the morphism α_i, and η_i is an elliptic fibration.

The assertion of Theorem 1.10 for $n = 17$ follows from the next result.

Proposition 15.1. *Either there is a commutative diagram*

$$
\begin{array}{ccc}
X & \dashrightarrow{\rho} & V \\
{}^{\psi}\downarrow & & \downarrow{}^{\nu} \\
\mathbb{P}(1, 1, 3) & \dashrightarrow{\varphi} & \mathbb{P}^2
\end{array}
\tag{15.1}
$$

or there is a commutative diagram

$$
\begin{array}{ccccc}
X & \dashrightarrow{\sigma} & X & \dashrightarrow{\rho} & V \\
{}^{\xi_i}\downarrow & & & & \downarrow{}^{\nu} \\
\mathbb{P}(1, 1, 4) & \dashrightarrow{\upsilon} & & & \mathbb{P}^2
\end{array}
\tag{15.2}
$$

for some $i \in \{1, 2, 3\}$, where φ, σ and υ are birational maps.

Proof. It follows from Theorem 3.3 and Lemma 3.11 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_1, P_2, P_3\}$.

Remark 15.2. It follows from Theorem 2.2 that the commutative diagram (15.1) exists in the case when the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the points P_1, P_2 and P_3.

Hence, we may assume that $P_1 \in \mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) \not\ni P_3$.

Remark 15.3. The divisor $-K_{U_1}$ is nef and big.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U_1, $\bar{P}_2$ the proper transform of P_2 on U_1 and $\bar{P}_4$ the singular point of U_1 that is contained in the exceptional divisor of α_1. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_{U_1}$ by Theorem 2.2, and it follows from Lemma 2.1 that the set $\mathbb{CS}\left(U_1, \frac{1}{k}\mathcal{D}\right)$ is non-empty.

Remark 15.4. It follows easily from Theorem 2.2 that the commutative diagram (15.2) exists in the case when $\mathbb{CS}\left(U_1, \frac{1}{k}\mathcal{D}\right)$ contains $\bar{P}_4$.

Therefore, we may assume that $\mathbb{CS}\left(U_1, \frac{1}{k}\mathcal{D}\right)$ does not contain $\bar{P}_4$. Hence, it follows from the proof of Lemma 9.5 that $\mathbb{CS}\left(U_1, \frac{1}{k}\mathcal{D}\right)$ does not contain a subvariety of U_1 that is contained in the exceptional divisor of α_1. Thus, the set $\mathbb{CS}\left(U_1, \frac{1}{k}\mathcal{D}\right)$ contains $\bar{P}_2$.

Let $\gamma\colon W \to U_1$ be the weighted blow-up of $\bar{P}_2$ with weights $(1, 1, 3)$, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W, S a general surface in $|-K_W|$, and C the base curve of the pencil $|-K_W|$. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2, the curve C is irreducible, the inequality $C^2 = -1/24$ holds on the normal surface S, and the equivalence $\mathcal{B}|_S \sim_{\mathbb{Q}} kC$ holds, which is impossible by Lemmas 2.9 and 2.7. The assertion is proved.

§16. The case $n = 19$: a hypersurface of degree 12 in $\mathbb{P}(1, 2, 3, 3, 4)$

We use the notation and assumptions of §3. Let $n = 19$. Then X is a general hypersurface in $\mathbb{P}(1, 2, 3, 3, 4)$ of degree 12, the singularities of X consist of points O_1, O_2, O_3 that are quotient singularities of type $\frac{1}{2}(1, 1, 1)$, and points P_1, P_2 P_3, P_4 that are quotient singularities of type $\frac{1}{3}(1, 1, 2)$. The equality $-K_X^3 = 1/6$ holds.

It follows from Example 1.7 that for every $i \in \{1, 2, 3, 4\}$ there is a commutative diagram

$$
\begin{array}{ccc}
 & U_i & \\
{\scriptstyle \pi_i}\swarrow & & \searrow{\scriptstyle \eta_i} \\
X \dashrightarrow[\xi_i] & & \mathbb{P}(1, 2, 3),
\end{array}
$$

where ξ_i is a projection, π_i is the blow-up of P_i with weights $(1, 1, 2)$, and η_i is an elliptic fibration.

Proposition 16.1. *There is a commutative diagram*

$$
\begin{array}{ccc}
X & \overset{\rho}{\dashrightarrow} & V \\
{\scriptstyle \xi_i}\downarrow & & \downarrow{\scriptstyle \nu} \\
\mathbb{P}(1, 2, 3) & \underset{\sigma}{\dashrightarrow} & \mathbb{P}^2
\end{array}
\tag{16.1}
$$

*for some $i \in \{1, 2, 3, 4\}$, where σ is a birational map.*

Proof. It follows from Theorem 3.3, Corollary 3.4, Theorem 2.2 and Proposition 3.5 that the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point P_i for some $i \in \{1, 2, 3, 4\}$.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U_i. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_{U_i}$ by Theorem 2.2, which implies the existence of the commutative diagram (16.1).

The assertion of Proposition 16.1 implies the assertion of Theorem 1.10 for $n = 19$.

§ 17. The case $n = 20$: a hypersurface of degree 13 in $\mathbb{P}(1, 1, 3, 4, 5)$

We use the notation and assumptions of § 3. Let $n = 20$. Then X is a general hypersurface in $\mathbb{P}(1, 1, 3, 4, 5)$ of degree 13, the equality $-K_X^3 = 13/60$ holds, and the singularities of X consist of a point P_1 that is a singularity of type $\frac{1}{3}(1, 1, 2)$, a point P_2 that is a singularity of type $\frac{1}{4}(1, 1, 3)$, and a point P_3 that is a singularity of type $\frac{1}{5}(1, 1, 4)$. There is a commutative diagram

where ψ is the natural projection, α_2 is the weighted blow-up of P_2 with weights $(1, 1, 3)$, α_3 is the weighted blow-up of P_3 with weights $(1, 1, 4)$, β_3 is the weighted blow-up with weights $(1, 1, 4)$ of the proper transform of P_3 on U_2, β_2 is the weighted blow-up with weights $(1, 1, 3)$ of the proper transform of P_2 on U_3, β_4 is the weighted blow-up with weights $(1, 1, 3)$ of the singular point of U_3 that is contained in the exceptional divisor of α_3, γ_2 is the weighted blow-up with weights $(1, 1, 3)$ of the proper transform of P_2 on U_{34}, γ_4 is the weighted blow-up with weights $(1, 1, 3)$ of the singular point of U_{23} that is contained in the exceptional divisor of the morphism β_3, and η is an elliptic fibration.

It follows from Example 1.7 that there is a commutative diagram

where ξ is a projection, α_1 is the blow-up of P_1 with weights $(1, 1, 2)$, α_3 is the weighted blow-up of P_3 with weights $(1, 1, 4)$, γ_3 is the weighted blow-up with weights $(1, 1, 4)$ of the proper transform of P_3 on U_1, γ_1 is the weighted blow-up with weights $(1, 1, 2)$ of the proper transform of P_1 on U_3, and ω is an elliptic fibration.

Proposition 17.1. *Either there is a commutative diagram*

$$
\begin{array}{ccccc}
X & \dashrightarrow^{\ \sigma\ } & X & \dashrightarrow^{\ \rho\ } & V \\
\ \downarrow{\scriptstyle\psi} & & & & \downarrow{\scriptstyle\nu} \\
\mathbb{P}(1,1,3) & \dashrightarrow[\ \upsilon\ ] & & & \mathbb{P}^2
\end{array}
\tag{17.1}
$$

or there is a commutative diagram

$$
\begin{array}{ccccc}
X & \dashrightarrow^{\ \sigma\ } & X & \dashrightarrow^{\ \rho\ } & V \\
\ \downarrow{\scriptstyle\xi} & & & & \downarrow{\scriptstyle\nu} \\
\mathbb{P}(1,1,4) & \dashrightarrow[\ \zeta\ ] & & & \mathbb{P}^2
\end{array}
\tag{17.2}
$$

where σ, υ and ζ are birational maps.

Proof. It follows from Theorem 3.3 and Lemma 3.11 that $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)\subseteq\{P_1,P_2,P_3\}$.

Lemma 17.2. *Suppose that $\{P_1,P_3\}\subseteq\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$. Then the commutative diagram* (17.2) *exists.*

Proof. Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on Y and S a general surface in $\mathcal{B}$. Then $S\sim_{\mathbb{Q}}-kK_Y$ by Theorem 2.2. We see that $S\cdot C=0$, where C is a generic fibre of ω, which implies the existence of the commutative diagram (17.2).

Lemma 17.3. *The set $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$ does not contain the set $\{P_1,P_2\}$.*

Proof. Suppose that $\{P_1,P_3\}\subseteq\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$. Let $\pi\colon W\to U_1$ be the weighted blow-up with weights $(1,1,3)$ of the proper transform of P_2 on U_1 and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W. Then $\mathcal{B}\sim_{\mathbb{Q}}-kK_W$ by Theorem 2.2.

The linear system $|-K_W|$ is a pencil whose base locus is the irreducible curve Δ such that $\alpha_1\circ\pi(C)$ is the curve cut out on X by the equations $z=y=0$.

Let S be a sufficiently general surface in the linear system $|-K_W|$, $\bar{P}_3$ the proper transform of the singular point P_3 on W and $\bar{P}_5$ and $\bar{P}_6$ other singular points of W such that $\alpha_1\circ\pi(\bar{P}_5)=P_1$ and $\alpha_1\circ\pi(\bar{P}_6)=P_2$. Then $\bar{P}_5$ is a quotient singularity of type $\frac{1}{2}(1,1,1)$ on W, $\bar{P}_6$ is a quotient singularity of type $\frac{1}{3}(1,1,2)$ on W, the surface S is smooth outside the points $\bar{P}_3$, $\bar{P}_5$ and $\bar{P}_6$ and the singularities of S at the points $\bar{P}_3$, $\bar{P}_5$ and $\bar{P}_6$ are Du Val singularities of types $\mathbb{A}_4$, $\mathbb{A}_1$ and $\mathbb{A}_2$, respectively.

The equality $\Delta^2=-1/30$ holds on S. But the equivalence $\mathcal{B}|_S\sim_{\mathbb{Q}}k\Delta$ holds, which implies that $\mathcal{B}|_S=k\Delta$. We can now easily obtain a contradiction using Lemmas 2.7 and 2.9.

Lemma 17.4. *The set $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$ does not consist only of the point P_i.*

Proof. Suppose that $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)=\{P_i\}$. Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U_i. Then the equivalence $\mathcal{D}\sim_{\mathbb{Q}}-kK_{U_i}$ holds by Theorem 2.2. Moreover, it follows from Lemma 2.1 and the proof of Lemma 9.5 that the set $\mathbb{CS}\big(U_i,\frac{1}{k}\mathcal{D}\big)$ contains the singular point of U_i that is the singular point of the exceptional divisor of the birational morphism α_i.

Let $\pi\colon W \to U_i$ be the weighted blow-up with weights $(1,1,i)$ of the singular point of the variety U_i that is contained in the exceptional divisor of the morphism α_i, and let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on W. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2.

Let S be a sufficiently general surface in the pencil $|-K_W|$ and Δ the unique base curve of $|-K_W|$. Then S is normal but the curve Δ is irreducible, rational and smooth. Moreover, simple computations imply that

$$\Delta^2 = \begin{cases} -9/20 & \text{if } i = 1, \\ -1/30 & \text{if } i = 2, \\ 0 & \text{if } i = 3, \end{cases}$$

on S. However, we have the equivalence $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$, which implies (see the proof of Lemma 17.3) that the curve $\alpha_i \circ \pi(\Delta)$ is contained in the set $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ if $i \neq 3$. It follows that $i = 3$.

Let G be the exceptional divisor of α_3 and $\bar{P}_4$ the singular point of U_3 that is contained in G. Then $\bar{P}_4$ is a quotient singularity of type $\frac{1}{4}(1,1,3)$ on U_3.

It follows from Lemmas 2.3 and 2.4 that the set $\mathbb{CS}\left(U_3, \frac{1}{k}\mathcal{D}\right)$ consists of the singular point $\bar{P}_4$.

The variety W is the variety U_{34} and the birational morphism π is the morphism β_4. Thus, the divisor $-K_W$ is nef and big. Therefore, it follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\left(W, \frac{1}{k}\mathcal{B}\right)$ contains the singular point of W that is contained in the exceptional divisor of π.

Let $\mu\colon Z \to W$ be the weighted blow-up with weights $(1,1,2)$ of the singular point of W that is contained in the exceptional divisor of π and let $\mathcal{P}$ be the proper transform of $\mathcal{M}$ on Z. Then the equivalence $\mathcal{P} \sim_{\mathbb{Q}} -kK_Z$ holds by Theorem 2.2. Let F be a general surface and Γ the unique base curve of the pencil $|-K_Z|$. Then F is irreducible and normal. But Γ is irreducible, rational and smooth. The equality $\Gamma^2 = -1/24$ holds on F. But $\mathcal{P}|_S \sim_{\mathbb{Q}} k\Gamma$, which easily leads to a contradiction using Lemmas 2.7 and 2.9.

It follows from Theorem 3.3 and Lemma 3.11 that we may assume that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) = \{P_2, P_3\}$.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U_{23}. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_{U_{23}}$ by Theorem 2.2. It now follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\left(U_{23}, \frac{1}{k}\mathcal{D}\right)$ contains either the singular point of U_{23} that is contained in the exceptional divisor of the morphism β_3 or the singular point of U_{23} that is contained in the exceptional divisor of β_2.

Lemma 17.5. *The set $\mathbb{CS}\left(U_{23}, \frac{1}{k}\mathcal{D}\right)$ does not contain the singular point of U_{23} that is contained in the exceptional divisor of β_2.*

Proof. Let E be the exceptional divisor of β_2 and $\bar{P}_6$ the singular point of the surface E. Then $\bar{P}_6$ is a quotient singularity of type $\frac{1}{3}(1,1,2)$ of U_{23}. Suppose that the set $\mathbb{CS}\left(U_{23}, \frac{1}{k}\mathcal{D}\right)$ contains $\bar{P}_6$.

Let $\pi\colon W \to U_{23}$ be the weighted blow-up of $\bar{P}_6$ with weights $(1,1,2)$, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W, S a general surface in $|-K_W|$ and Δ the base curve of $|-K_W|$. Then S is normal, Δ is irreducible, and $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$ by Theorem 2.2. But the equality $\Delta^2 = -1/24$ holds on S, which is impossible by Lemmas 2.9 and 2.7.

Hence, the set $\mathbb{CS}\left(U_{23}, \frac{1}{k}\mathcal{D}\right)$ contains the singular point of U_{23} that is contained in the exceptional divisor of β_3. Thus, the existence of the commutative diagram (17.2) follows easily from Theorem 2.2. The proposition is proved.

§ 18. The case $n = 23$: a hypersurface of degree 14 in $\mathbb{P}(1,2,3,4,5)$

We use the notation and assumptions of § 3. Let $n = 23$. Then X is a general hypersurface in $\mathbb{P}(1,2,3,4,5)$ of degree 14, the singularities of X consist of points P_1, P_2 and P_3 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, a point P_4 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, a point P_5 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$ and a point P_6 that is quotient singularity of type $\frac{1}{5}(1,2,3)$. The equality $-K_X^3 = 7/60$ holds.

Proposition 18.1. *The assertion of Theorem* 1.10 *holds for* $n = 23$.

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) \subseteq \{P_5, P_6\}$. There is a commutative diagram
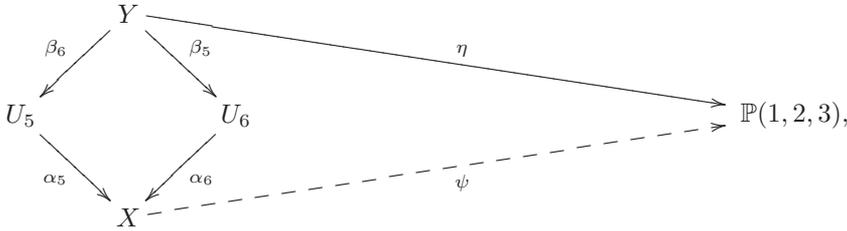
where ψ is the natural projection, α_5 is the weighted blow-up of the singular point P_5 with weights $(1,1,3)$, α_6 is the weighted blow-up of P_6 with weights $(1,2,3)$, β_5 is the weighted blow-up with weights $(1,1,3)$ of the proper transform of P_5 on U_6, β_6 is the weighted blow-up with weights $(1,2,3)$ of the proper transform of P_6 on U_5, and η is an elliptic fibration.

Lemma 18.2. *The set* $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ *contains the point* P_6.

Proof. Suppose that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ does not contain P_6. Then the set $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ consists of the point P_5. Let $\mathcal{D}_5$ be the proper transform of $\mathcal{M}$ on U_5. Then $\mathcal{D}_5 \sim_{\mathbb{Q}} -kK_{U_5}$ by Theorem 2.2, but the set $\mathbb{CS}\left(U_5, \frac{1}{k}\mathcal{D}_5\right)$ is non-empty by Lemma 2.1.

Let $\bar{P}_7$ be the singular point of U_5 that is contained in the exceptional divisor of the morphism α_5. Then $\bar{P}_7$ is a quotient singularity of type $\frac{1}{3}(1,1,2)$ on U_5, and it follows from Lemma 2.3 that $\mathbb{CS}\left(U_5, \frac{1}{k}\mathcal{D}_5\right)$ contains $\bar{P}_7$. Let $\pi\colon U \to U_5$ be the weighted blow-up of $\bar{P}_7$ with weights $(1,1,3)$. Then the linear system $|-2K_U|$ is a proper transform of the pencil $|-2K_X|$, and the base locus of $|-2K_U|$ consists of a single irreducible curve Z such that $\alpha_5 \circ \pi(Z)$ is the unique base curve of $|-2K_X|$.

Let S be a sufficiently general surface in $|-2K_U|$. Then S is normal and contains the curve Z, and the inequality $Z^2 < 0$ holds on S because $-K_U^3 < 0$. However, the equivalence $\mathcal{B}|_S \sim_{\mathbb{Q}} kZ$ holds by Theorem 2.2, where $\mathcal{B}$ is the proper transform of $\mathcal{M}$ on W. It follows from Lemma 2.9 that

$$\mathrm{Supp}(S) \cap \mathrm{Supp}(D) = \mathrm{Supp}(Z),$$

where D is a sufficiently general surface in the linear system $\mathcal{B}$. But this is impossible by Lemma 2.7 because $\mathcal{B}$ is not composed of a pencil.

It follows easily from Theorem 2.2 that the assertion of Theorem 1.10 holds for X whenever $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains P_5 and P_6. Thus, we may assume that $P_5 \notin \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. We claim that this assumption leads to a contradiction.

Let $\mathcal{D}_6$ be the proper transform of $\mathcal{M}$ on U_6. Then $\mathcal{D}_6 \sim_{\mathbb{Q}} -kK_{U_6}$ by Theorem 2.2, which implies that the set $\mathbb{CS}(U_6, \frac{1}{k}\mathcal{D}_6)$ is non-empty by Lemma 2.1. Let $\bar{P}_7$ and $\bar{P}_8$ be the singular points of U_6 that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{2}(1,1,1)$, respectively, contained in the exceptional divisor of α_6.

Lemma 18.3. *The set $\mathbb{CS}(U_6, \frac{1}{k}\mathcal{D}_6)$ does not contain the point $\bar{P}_7$.*

Proof. Suppose that $\bar{P}_7 \in \mathbb{CS}(U_6, \frac{1}{k}\mathcal{D}_6)$. Let $\gamma: W \to U_6$ be the weighted blow-up of $\bar{P}_7$ with weights $(1,1,2)$ and let S be a sufficiently general surface in $|-2K_W|$. Then S is irreducible and normal, the linear system $|-2K_W|$ is the proper transform of the pencil $|-2K_X|$, and the base locus of $|-2K_W|$ consists of the irreducible curve Δ such that the equality

$$\Delta^2 = -2K_W^3 = -\frac{1}{6}$$

holds on S. Moreover, the equivalence $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$ holds, where $\mathcal{B}$ is the proper transform of $\mathcal{M}$ on W. But this is impossible by Lemmas 2.9 and 2.7.

Therefore, the set $\mathbb{CS}(U_6, \frac{1}{k}\mathcal{D}_6)$ contains the point $\bar{P}_8$ by Lemma 2.3. $\square$

Remark 18.4. The linear system $|-3K_{U_6}|$ is the proper transform of the linear system $|-3K_X|$ and the base locus of $|-3K_{U_6}|$ consists of the irreducible fibre of $\psi \circ \alpha_6$ that passes through the singular point $\bar{P}_8$.

Let $\pi: U \to U_6$ be the weighted blow-up of the point $\bar{P}_8$ with weights $(1,1,1)$, F the exceptional divisor of the morphism π, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on U and $\mathcal{H}$ the proper transform of $|-3K_{U_6}|$ on U. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2. Simple computations imply that

$$\mathcal{H} \sim_{\mathbb{Q}} \pi^*(-3K_{U_6}) - \frac{1}{2}F,$$

and the base locus of $\mathcal{H}$ consists of the irreducible curve Z such that $\alpha_6 \circ \pi(Z)$ is the base curve of $|-3K_X|$. Moreover, the equality $S \cdot Z = 1/12$ holds, where S is a general surface in $\mathcal{H}$.

Let D_1 and D_2 be general surfaces in the linear system $\mathcal{D}$. Then $S \cdot D_1 \cdot D_2 = -k^2/4$, which is a contradiction. The assertion is proved.

§ 19. The case $n = 25$: a hypersurface of degree 15 in $\mathbb{P}(1,1,3,4,7)$

We use the notation and assumptions of § 3. Let $n = 25$. Then X is a general hypersurface in $\mathbb{P}(1,1,3,4,7)$ of degree 15, the equality $-K_X^3 = 5/28$ holds, and the singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$ and a point P_2 that is a quotient singularity of type $\frac{1}{7}(1,3,4)$.

Proposition 19.1. *The assertion of Theorem 1.10 holds for $n = 25$.*

There is a commutative diagram

where ψ is the natural projection, α_1 is the weighted blow-up of P_1 with weights $(1, 1, 3)$, α_2 is the weighted blow-up of P_2 with weights $(1, 3, 4)$, β_2 is the weighted blow-up with weights $(1, 3, 4)$ of the proper transform of P_2 on U_1, β_1 is the weighted blow-up with weights $(1, 1, 3)$ of the proper transform of P_1 on U_2, β_3 is the weighted blow-up with weights $(1, 1, 3)$ of the singular point of U_2 that is a quotient singularity of type $\frac{1}{4}(1, 1, 3)$ contained in the exceptional divisor of the morphism α_2, γ_1 is the weighted blow-up with weights $(1, 1, 3)$ of the proper transform of P_1 on U_{23}, γ_3 is the weighted blow-up with weights $(1, 1, 3)$ of the point of U_{12} that is a quotient singularity of type $\frac{1}{4}(1, 1, 3)$ contained in the exceptional divisor of β_2, and η is an elliptic fibration.

Remark 19.2. The divisors $-K_{U_1}$, $-K_{U_2}$, $-K_{U_{12}}$ and $-K_{U_{23}}$ are nef and big.

Proof of Proposition 19.1. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) \subseteq \{P_1, P_2\}$.

Lemma 19.3. *The set* $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ *contains the point* P_2.

Proof. Suppose that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ does not contain P_2. Let $\mathcal{D}_1$ be the proper transform of $\mathcal{M}$ on U_1. Then $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) = \{P_1\}$, and $\mathbb{CS}\left(U_1, \frac{1}{k}\mathcal{D}_1\right) \neq \varnothing$ by Lemma 2.1 because the equivalence $\mathcal{D}_1 \sim_{\mathbb{Q}} -kK_{U_1}$ holds by Theorem 2.2.

Let P_5 be the singular point of U_1 that is contained in the exceptional divisor of α_1. Then P_5 is a quotient singularity of type $\frac{1}{3}(1, 1, 2)$ on U_1, and it follows from Lemma 2.3 that $\mathbb{CS}\left(U_1, \frac{1}{k}\mathcal{D}_1\right)$ contains it.

Let $\pi\colon W \to U_1$ be the blow-up of P_5 with weights $(1, 1, 2)$ and let S be a sufficiently general surface in the pencil $|-K_W|$. Then S is irreducible and normal and the base locus of the pencil $|-K_W|$ consists of the irreducible curve Δ such that $\Delta^2 = -1/14$ on S. But $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$, where $\mathcal{B}$ is the proper transform of $\mathcal{M}$ on W. Therefore, we have $\mathcal{B}|_S = k\Delta$, which implies that

$$\mathrm{Supp}(S) \cap \mathrm{Supp}(D) = \mathrm{Supp}(\Delta),$$

where D is a general surface in $\mathcal{B}$. But this contradicts Lemma 2.7.

Let G be the exceptional divisor of α_2, $\mathcal{D}_2$ the proper transform of $\mathcal{M}$ on U_2, $\bar{P}_1$ the proper transform of P_1 on U_2, and $\bar{P}_3$ and $\bar{P}_4$ the singular points of U_2 that are quotient singularities of types $\frac{1}{4}(1, 1, 3)$ and $\frac{1}{3}(1, 1, 2)$, respectively, contained in

the exceptional divisor G. Then $G \cong \mathbb{P}(1,3,4)$, $\bar{P}_3$ and $\bar{P}_4$ are singular points of G, and $\mathcal{D}_2 \sim_{\mathbb{Q}} -kK_{U_2}$ by Theorem 2.2. Hence, the set $\mathbb{CS}\left(U_2, \frac{1}{k}\mathcal{D}_2\right)$ is non-empty by Lemma 2.1. Moreover, the proof of Lemma 19.3 implies that $\mathbb{CS}\left(U_2, \frac{1}{k}\mathcal{D}_2\right) \neq \{\bar{P}_1\}$.

Lemma 19.4. *The set $\mathbb{CS}\left(U_2, \frac{1}{k}\mathcal{D}_2\right)$ does not contain both $\bar{P}_3$ and $\bar{P}_4$.*

Proof. Suppose that $\{\bar{P}_3, \bar{P}_4\} \subseteq \mathbb{CS}\left(U_2, \frac{1}{k}\mathcal{D}_2\right)$. Let $\pi\colon W \to U_2$ be the composite of the weighted blow-ups of $\bar{P}_3$ and $\bar{P}_4$ with weights $(1,1,3)$ and $(1,1,2)$, respectively, and let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on W. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2.

Let S be a general surface in the pencil $|-K_W|$. Then S is irreducible and normal, but the base locus of $|-K_W|$ consists of the irreducible curves C and L such that the curve $\alpha_2 \circ \pi(C)$ is the unique base curve of $|-K_X|$, the curve $\pi(L)$ is contained in G, and $\pi(L)$ is the unique curve in $\left|\mathcal{O}_{\mathbb{P}(1,3,4)}(1)\right|$. We have

$$\mathcal{B}|_S \sim_{\mathbb{Q}} -kK_W|_S \sim_{\mathbb{Q}} kS|_S = kC + kL,$$

but the intersection form of L and C on S is negative definite. Now applying Lemma 2.9, we obtain a contradiction to Lemma 2.7.

It follows from Lemma 2.3 that $\mathbb{CS}\left(U_2, \frac{1}{k}\mathcal{D}_2\right) \subsetneq \{\bar{P}_1, \bar{P}_3, \bar{P}_4\}$.

Lemma 19.5. *The set $\mathbb{CS}\left(U_2, \frac{1}{k}\mathcal{D}_2\right)$ contains either $\bar{P}_1$ or $\bar{P}_4$.*

Proof. Suppose that the set $\mathbb{CS}\left(U_2, \frac{1}{k}\mathcal{D}_2\right)$ does not contain either of the singular points $\bar{P}_1$, $\bar{P}_4$. Then $\mathbb{CS}\left(U_2, \frac{1}{k}\mathcal{D}_2\right)$ consists of the point $\bar{P}_3$.

The linear system $|-K_{U_2}|$ is the proper transform of $|-K_X|$ and the base locus of $|-K_{U_2}|$ consists of the irreducible curves L and Δ such that $\alpha_2(\Delta)$ is the unique base curve of $|-K_X|$, L is contained in the divisor G and L is the unique curve of the linear system $\left|\mathcal{O}_{\mathbb{P}(1,3,4)}(1)\right|$.

Let $\widetilde{P}_6$ be the singular point of the variety U_{23} that is contained in the exceptional divisor of β_3 and let $\mathcal{D}_{23}$ be the proper transform of $\mathcal{M}$ on U_{23}. Then $\mathcal{D}_{23} \sim_{\mathbb{Q}} -kK_{U_{23}}$ by Theorem 2.2, and it follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\left(U_{23}, \frac{1}{k}\mathcal{D}_{23}\right)$ contains the point $\widetilde{P}_6$, which is a quotient singularity of type $\frac{1}{3}(1,1,2)$ on U_{23}.

Let $\pi\colon W \to U_{23}$ be the weighted blow-up of $\widetilde{P}_6$ with weights $(1,1,2)$, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on W, and $\bar{L}$ and $\bar{\Delta}$ be the proper transforms of L and Δ on W, respectively. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2, $|-K_W|$ is the proper transform of $|-K_{U_2}|$, and the base locus of $|-K_W|$ consists of the curves $\bar{L}$ and $\bar{\Delta}$.

Let S be a general surface in $|-K_W|$. Then S is irreducible and normal and the equivalence $\mathcal{D}|_S \sim_{\mathbb{Q}} k\bar{\Delta} + k\bar{L}$ holds. But the equalities $\bar{\Delta}^2 = -7/12$, $\bar{L}^2 = -5/6$ and $\bar{\Delta} \cdot \bar{L} = 2/3$ hold on S. Therefore, the intersection form of the curves $\bar{\Delta}$ and $\bar{L}$ on S is negative definite, which is impossible by Lemmas 2.9 and 2.7.

The hypersurface X can be given by the equation

$$w^2 y + wt^2 + wtf_4(x,y,z) + wf_8(x,y,z)$$
$$+ tf_{11}(x,y,z) + f_{15}(x,y,z) = 0 \subset \mathrm{Proj}\left(\mathbb{C}[x,y,z,t,w]\right),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 3$, $\mathrm{wt}(t) = 4$, $\mathrm{wt}(w) = 7$ and $f_i(x,y,t)$ is a sufficiently general quasi-homogeneous polynomial of degree i.

Remark 19.6. Suppose that the set $\mathbb{CS}\left(U_2, \frac{1}{k}\mathcal{D}_2\right)$ contains both $\bar{P}_1$ and $\bar{P}_3$. Then the assertion of Theorem 2.2 easily implies the existence of a commutative diagram

$$
\begin{array}{ccc}
X & \dashrightarrow{\rho} & V \\
\psi \downarrow & & \downarrow \nu \\
\mathbb{P}(1,1,3) & \dashrightarrow{\zeta} & \mathbb{P}^2,
\end{array}
$$

where ζ is a birational map.

Therefore, we may assume that $\mathbb{CS}\left(U_2, \frac{1}{k}\mathcal{D}_2\right)$ does not contain both $\bar{P}_1$ and $\bar{P}_3$.

Lemma 19.7. *The set* $\mathbb{CS}\left(U_2, \frac{1}{k}\mathcal{D}_2\right)$ *contains the point* $\bar{P}_1$.

Proof. Suppose that $\bar{P}_1 \notin \mathbb{CS}\left(U_2, \frac{1}{k}\mathcal{D}_2\right)$. Then $\mathbb{CS}\left(U_2, \frac{1}{k}\mathcal{D}_2\right) = \{\bar{P}_4\}$.

Let $\pi\colon W \to U_2$ be the weighted blow-up of $\bar{P}_4$ with weights $(1,1,2)$, E the exceptional divisor of π, and $\bar{G}$ and $\mathcal{B}$ the proper transforms of G and $\mathcal{M}$ on W, respectively. Then it follows from Theorem 2.2 that the equivalence $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ holds. But the proof of Lemma 19.5 implies that the set $\mathbb{CS}\left(W, \frac{1}{k}\mathcal{B}\right)$ does not contain the singular point of the variety W that is contained in the exceptional divisor of π. Therefore, the singularities of the log pair $\left(W, \frac{1}{k}\mathcal{B}\right)$ are terminal by Lemma 2.3.

Let S_x, S_y, S_z, S_t and S_w be the proper transforms on W of the surfaces that are cut out on X by the equations $x = 0$, $y = 0$, $z = 0$, $t = 0$ and $w = 0$, respectively. Then

$$
S_x \sim_{\mathbb{Q}} (\alpha_2 \circ \pi)^*(-K_X) - \frac{3}{7}E - \frac{1}{7}\bar{G},
$$

$$
S_y \sim_{\mathbb{Q}} (\alpha_2 \circ \pi)^*(-K_X) - \frac{10}{7}E - \frac{8}{7}\bar{G},
$$

$$
S_z \sim_{\mathbb{Q}} (\alpha_2 \circ \pi)^*(-3K_X) - \frac{2}{7}E - \frac{3}{7}\bar{G}, \tag{19.1}
$$

$$
S_t \sim_{\mathbb{Q}} (\alpha_2 \circ \pi)^*(-4K_X) - \frac{5}{7}E - \frac{4}{7}\bar{G},
$$

$$
S_w \sim_{\mathbb{Q}} (\alpha_2 \circ \pi)^*(-7K_X).
$$

The base locus of the pencil $|-K_W|$ consists of the irreducible curves C and L such that the curve $\alpha_2 \circ \pi(C)$ is cut out by the equations $x = y = 0$ on the hypersurface X, the curve $\pi(L)$ is contained in G, and the curve $\pi(L)$ is contained in the linear system $\left|\mathcal{O}_{\mathbb{P}(1,3,4)}(1)\right|$.

The equivalences (19.1) imply that the rational functions y/x, zy/x^4, ty/x^5 and wy^3/x^{10} are contained in the linear system $|aS_x|$, where $a = 1, 4, 5$ and 10, respectively. Therefore, the linear system $|-20K_W|$ induces a birational map $\chi\colon W \dashrightarrow X'$, where X' is a hypersurface with canonical singularities in $\mathbb{P}(1,1,4,5,10)$ of degree 20. In particular, the divisor $-K_W$ is big.

It follows from [12] that there is a birational map $\zeta\colon W \dashrightarrow Z$ such that ζ is regular outside C and L, the map ζ is an isomorphism in codimension 1, and the divisor $-K_Z$ is nef and big. Let $\mathcal{P}$ be the proper transform of $\mathcal{M}$ on Z. Then

the singularities of the log pair $\left(Z, \frac{1}{k}\mathcal{P}\right)$ are terminal because ζ is a log-flop with respect to the log pair $\left(W, \frac{1}{k}\mathcal{B}\right)$, which has terminal singularities. But it follows from Lemma 2.1 that the singularities of $\left(Z, \frac{1}{k}\mathcal{P}\right)$ are not terminal.

Hence, the set $\mathbb{CS}\left(U_2, \frac{1}{k}\mathcal{D}_2\right)$ consists of the points $\bar{P}_1$ and $\bar{P}_4$. Let $\pi\colon W \to U_2$ be the composite of the weighted blow-ups of the points $\bar{P}_1$ and $\bar{P}_4$ with weights $(1,1,3)$ and $(1,1,2)$, respectively, let $\bar{G}$ and $\mathcal{B}$ be the proper transforms of G and $\mathcal{M}$ on W respectively, and let F and E be exceptional divisors of π that dominate the points $\bar{P}_1$ and $\bar{P}_4$, respectively. Then the equivalence $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ holds by Theorem 2.2. Arguing as in the proof of Lemma 19.5, we see that the singularities of the log pair $\left(W, \frac{1}{k}\mathcal{B}\right)$ are terminal.

Let S_x, S_y, S_z, S_t and S_w be the proper transforms on W of the surfaces that are cut out on X by the equations $x = 0$, $y = 0$, $z = 0$, $t = 0$ and $w = 0$, respectively. Then

$$S_x \sim_{\mathbb{Q}} (\alpha_2 \circ \pi)^*(-K_X) - \frac{3}{7}E - \frac{1}{7}\bar{G} - \frac{1}{4}F,$$

$$S_y \sim_{\mathbb{Q}} (\alpha_2 \circ \pi)^*(-K_X) - \frac{10}{7}E - \frac{8}{7}\bar{G} - \frac{1}{4}F,$$

$$S_z \sim_{\mathbb{Q}} (\alpha_2 \circ \pi)^*(-3K_X) - \frac{2}{7}E - \frac{3}{7}\bar{G} - \frac{3}{4}F,$$

$$S_t \sim_{\mathbb{Q}} (\alpha_2 \circ \pi)^*(-4K_X) - \frac{5}{7}E - \frac{4}{7}\bar{G},$$

$$S_w \sim_{\mathbb{Q}} (\alpha_2 \circ \pi)^*(-7K_X) - \frac{11}{4}F,$$

and it follows that the rational functions y/x, zy/x^4, twy^4/x^{15} and wy^3/x^{10} are contained in the linear system $|aS_x|$, where $a = 1$, 4, 15 and 10, respectively. The linear system $|-60K_W|$ induces a birational map $\chi\colon W \dashrightarrow X'$ such that the variety X' is a hypersurface in $\mathbb{P}(1,1,4,10,15)$ of degree 30. In particular, the divisor $-K_W$ is big. We can now obtain a contradiction in the same way as in the proof of Lemma 19.7. The proposition is proved.

§20. The case $n = 26$: a hypersurface of degree 15 in $\mathbb{P}(1,1,3,5,6)$

We use the notation and assumptions of §3. Let $n = 26$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,3,5,6)$ of degree 15, the equality $-K_X^3 = 1/6$ holds, and the singularities of X consist of points P_1 and P_2 that are quotient singularities of type $\frac{1}{3}(1,1,2)$ and a point P_3 that is a quotient singularity of type $\frac{1}{6}(1,1,5)$. There is a commutative diagram

$$
\begin{array}{ccccc}
U & \xleftarrow{\ \beta\ } & W & \xleftarrow{\ \gamma\ } & Y \\
{\scriptstyle\alpha}\downarrow & & & & \downarrow{\scriptstyle\eta} \\
X & \dashrightarrow[\ \psi\ ] & \multicolumn{3}{c}{} \mathbb{P}(1,1,3),
\end{array}
$$

where ψ is the natural projection, α is the weighted blow-up of P_3 with weights $(1,1,5)$, β is the weighted blow-up with weights $(1,1,4)$ of the singular point of U

that is contained in the exceptional divisor of α, γ is the weighted blow-up with weights $(1, 1, 3)$ of the singular point of W that is contained in the exceptional divisor of β, and η is an elliptic fibration.

It follows from Example 1.5 that there is a commutative diagram

$$
\begin{array}{ccc}
 & U_i & \\
 \sigma_i \swarrow & & \searrow \omega_i \\
 X \; - - - - \underset{\xi_i}{- - \succ} & & \mathbb{P}(1, 1, 6),
\end{array}
$$

where ξ_i is a projection, σ_i is the weighted blow-up of P_i with weights $(1, 1, 2)$ and ω_i is an elliptic fibration induced by the linear system $|-6K_{U_i}|$.

It follows from [2] that the group $\mathrm{Bir}(X)$ is generated by biregular automorphisms of X and a birational involution $\tau \in \mathrm{Bir}(X)$ such that $\psi \circ \tau = \psi$ and $\xi_1 \circ \tau = \xi_2$.

The rest of the section is devoted to proving the following result.

Proposition 20.1. *Either there is a commutative diagram*

$$
\begin{array}{ccc}
 X & - - - - \overset{\rho}{- - -} \succ & V \\
 \psi \downarrow & & \downarrow \nu \\
 \mathbb{P}(1, 1, 3) & - - \underset{\varphi}{- - -} \succ & \mathbb{P}^2
\end{array}
\tag{20.1}
$$

or there is a commutative diagram

$$
\begin{array}{ccc}
 X & - - - - \overset{\rho}{- - -} \succ & V \\
 \xi_i \downarrow & & \downarrow \nu \\
 \mathbb{P}(1, 1, 6) & - - \underset{\sigma}{- - -} \succ & \mathbb{P}^2,
\end{array}
\tag{20.2}
$$

where φ and σ are birational maps and $i = 1$ or 2.

Proof. It follows from Theorem 3.3 and Lemma 3.11 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_1, P_2, P_3\}$.

Lemma 20.2. *Suppose that $P_i \in \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$, where $i \in \{1, 2\}$. Then the commutative diagram* (20.2) *exists.*

Proof. Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U_i. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_{U_i}$ by Theorem 2.2, which implies the existence of the commutative diagram (20.2).

Therefore, we may assume that the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ consists of the point P_3.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U and $\bar{P}_4$ the singular point of U that is contained in the exceptional divisor of α. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2.

Lemma 20.3. *The set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ consists of the point $\bar{P}_4$.*

Proof. Let G be an exceptional divisor of the blow-up α. Then G is a cone over a smooth rational curve of degree 5 and $\bar{P}_4$ is the vertex of G. Suppose that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ contains a subvariety C of U that is different from the point $\bar{P}_4$. Then C is a ruling of G by Lemma 2.3, which is impossible by Corollary 2.8.

Let $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on W. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2. But it follows from Lemmas 2.1 and 2.3 that $\mathbb{CS}\left(W, \frac{1}{k}\mathcal{H}\right)$ contains the singular point of W that is contained in the exceptional divisor of β. The existence of the diagram (20.1) now follows from Theorem 2.2.

§ 21. The case $n = 27$: a hypersurface of degree 15 in $\mathbb{P}(1, 2, 3, 5, 5)$

We use the notation and assumptions of § 3. Let $n = 27$. Then X is a general hypersurface in $\mathbb{P}(1, 2, 3, 5, 5)$ of degree 15, whose singularities consist of a point O that is a singularity of type $\frac{1}{2}(1, 1, 1)$, and points P_1, P_2 and P_3 that are singularities of type $\frac{1}{5}(1, 2, 3)$.

Let $\psi \colon X \dashrightarrow \mathbb{P}(1, 2, 3)$ be the natural projection. Then ψ is undefined only at the points P_1, P_2 and P_3. The generic fibre of ψ is a smooth elliptic curve. There is a commutative diagram

$$
\begin{array}{ccc}
U_{ij} & \xleftarrow{\;\gamma_{ij}\;} & Y \\
{\scriptstyle \beta_{ij}}\swarrow & & \searrow{\scriptstyle \eta} \\
U_i & & \\
{\scriptstyle \alpha_i}\searrow & & \searrow \\
& X \dashrightarrow[\psi] & \mathbb{P}(1, 2, 3),
\end{array}
$$
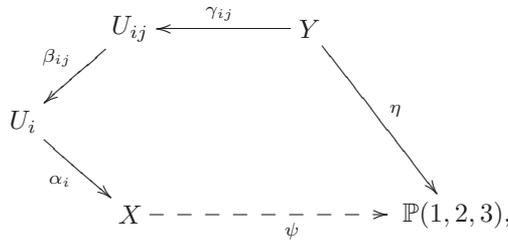
where α_i is the blow-up of P_i with weights $(1, 2, 3)$, β_{ij} is the weighted blow-up with weights $(1, 2, 3)$ of the proper transform of P_j on U_i, γ_{ij} is the weighted blow-up with weights $(1, 2, 3)$ of the proper transform of P_k on U_{ij}, and η is an elliptic fibration, where $i \neq j$ and $k \notin \{i, j\}$.

Proposition 21.1. *The assertion of Theorem* 1.10 *holds for* $n = 27$.

Proof. It follows from Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) \subseteq \{P_1, P_2, P_3\}$.

Remark 21.2. The assertion of Theorem 1.10 holds for the hypersurface X by Theorem 2.2 in the case when $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) = \{P_1, P_2, P_3\}$.

We may assume that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ contains P_1 but not P_3. We shall show that this assumption leads to a contradiction.

Lemma 21.3. *The set* $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ *contains the point* P_2.

Proof. Suppose that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ does not contain P_2. Then $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ consists of the point P_1. Let $\mathcal{D}_1$ be the proper transform of $\mathcal{M}$ on U_1. Then Theorem 2.2 implies that the equivalence $\mathcal{D}_1 \sim_{\mathbb{Q}} -kK_{U_1}$ holds. The set $\mathbb{CS}\left(U_1, \frac{1}{k}\mathcal{D}_1\right)$ is non-empty by Lemma 2.1.

Let G be the exceptional divisor of α_1 and O, Q the singular points of G that are quotient singularities of types $\frac{1}{3}(1, 1, 2)$, $\frac{1}{2}(1, 1, 1)$ on U_1, respectively. Then it follows from the assertion of Lemma 2.3 that $\mathbb{CS}\left(U_1, \frac{1}{k}\mathcal{D}_1\right)$ contains either O or Q.

The linear system $|-2K_{U_1}|$ is a pencil and its base locus consists of the irreducible curve C which passes through O and is contracted by the rational map $\psi \circ \alpha_1$ to a singular point of the surface $\mathbb{P}(1, 2, 3)$.

Suppose that the set $\mathbb{CS}\left(U_1, \frac{1}{k}\mathcal{D}_1\right)$ contains the point O. Let $\pi\colon W \to U_1$ be the weighted blow-up of O with weights $(1,1,2)$, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W and $\overline{C}$ the proper transform of C on W. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2, the linear system $|-2K_W|$ is the proper transform of the pencil $|-2K_{U_1}|$, and the base locus of $|-2K_W|$ consists of the curve $\overline{C}$. Let S be a general surface in $|-2K_W|$. Then $\mathcal{B}|_S \sim_{\mathbb{Q}} k\overline{C}$. But $\overline{C}^2 < 0$ on S. But this is impossible by Lemmas 2.7 and 2.9.

Hence, $\mathbb{CS}\left(U_1, \frac{1}{k}\mathcal{D}_1\right)$ contains Q. Let $\zeta\colon U \to U_1$ be the weighted blow-up of Q with weights $(1,1,1)$, F the exceptional divisor of ζ, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on U and $\mathcal{P}$ the proper transform of $|-3K_{U_1}|$ on U. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2. But

$$\mathcal{P} \sim_{\mathbb{Q}} \zeta^*(-3K_{U_1}) - \frac{1}{2}\,F,$$

and the base locus of $\mathcal{P}$ consists of the irreducible curve Z such that the curve $\alpha_1 \circ \zeta(Z)$ is the unique base curve of $|-3K_X|$. Therefore, we have

$$\left(\zeta^*(-3K_{U_1}) - \frac{1}{2}\,F\right) \cdot Z = \frac{1}{10}\,,$$

which implies that

$$
\begin{aligned}
-\frac{3k^2}{10} &= \left(\zeta^*(-3K_{U_1}) - \frac{1}{2}\,F\right) \cdot \left(\zeta^*(-kK_{U_1}) - \frac{k}{2}\,F\right)^2 \\
&= \left(\zeta^*(-3K_{U_1}) - \frac{1}{2}\,F\right) \cdot H_1 \cdot H_2 \geqslant 0,
\end{aligned}
$$

where H_1 and H_2 are general surfaces in $\mathcal{H}$. The resulting contradiction completes the proof of the lemma.

We have $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) = \{P_1, P_2\}$. We can now apply the proof of Lemma 21.3 to the proper transform of $\mathcal{M}$ on U_{12} to get a contradiction. The proposition is proved.

§ 22. The case $n = 29$: a hypersurface of degree 16 in $\mathbb{P}(1,1,2,5,8)$

We use the notation and assumptions of § 3. Let $n = 29$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,2,5,8)$ of degree 16, the equality $-K_X^3 = 1/5$ holds, and the singularities of X consist of points O_1, O_2 that are quotient singularities of type $\frac{1}{2}(1,1,1)$ and a point P that is a quotient singularity of type $\frac{1}{5}(1,2,3)$. There is a commutative diagram

$$
\begin{CD}
U @<\beta<< W \\
@V\alpha VV @VV\eta V \\
X @>>\psi> \mathbb{P}(1,1,2),
\end{CD}
$$

where ψ is the natural projection, α is the weighted blow-up of P with weights $(1,2,3)$, β is the weighted blow-up with weights $(1,1,2)$ of the singular point of U that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, and η is an elliptic fibration.

Proposition 22.1. *The assertion of Theorem* 1.10 *holds for* $n = 29$.

Proof. The hypersurface X is birationally superrigid. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P\}$.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2. Let G be the α-exceptional divisor and let Q, O be the singular points of G that are singularities of types $\frac{1}{3}(1,1,2)$, $\frac{1}{2}(1,1,1)$, respectively. Then it follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ either consists of the point O or contains Q.

Suppose that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ contains O. Let $\pi\colon Y \to U$ be the weighted blow-up of O with weights $(1,1,1)$, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on Y, L the curve on G that is contained in $\big|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)\big|$, $\bar{L}$ the proper transform of L on Y, and S a general surface in $|-K_Y|$. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2, and the base locus of the pencil $|-K_Y|$ consists of the curve $\bar{L}$ and the irreducible curve Δ such that $\alpha \circ \pi(\Delta)$ is the base locus of $|-K_X|$. The equalities $\Delta^2 = -1$, $\bar{L}^2 = -4/3$ and $\Delta \cdot \bar{L} = 1$ hold on S. Thus, the intersection form of the curves Δ and $\bar{L}$ on S is negative definite. We have $\mathcal{H}|_S \sim_{\mathbb{Q}} k\Delta + k\bar{L}$, which is impossible by Lemmas 2.7 and 2.9.

Therefore, $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ contains the point Q. Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on W. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2, which implies that $\mathcal{B}$ is contained in the fibres of η. The proposition is proved.

### § 23. The case $n = 30$: a hypersurface of degree 16 in $\mathbb{P}(1,1,3,4,8)$

We use the notation and assumptions of § 3. Let $n = 30$. Then X is a general hypersurface in $\mathbb{P}(1,1,3,4,8)$ of degree 16, the equality $-K_X^3 = 1/6$ holds, and the singularities of X consist of a point O that is a quotient singularity of type $\frac{1}{3}(1,1,2)$ and points P_1, P_2 that are singularities of type $\frac{1}{4}(1,1,3)$.
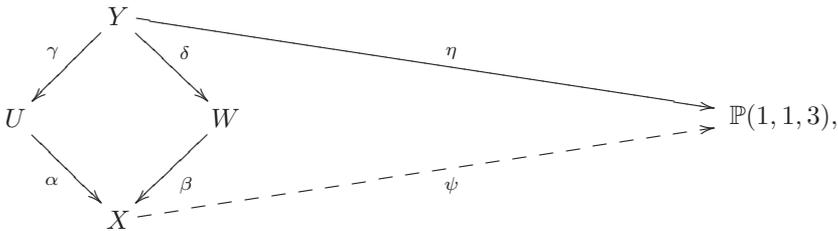
There is a commutative diagram

where ψ is the natural projection, α is the weighted blow-up of P_1 with weights $(1,1,3)$, β is the weighted blow-up of P_2 with weights $(1,1,3)$, γ is the weighted blow-up with weights $(1,1,3)$ of the proper transform of P_2 on U, δ is the weighted blow-up with weights $(1,1,3)$ of the proper transform of P_1 on W, and η is and elliptic fibration.

There is a commutative diagram
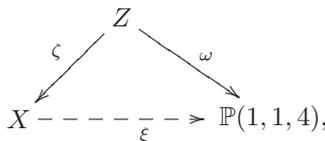
where ξ is a projection, ζ is the blow-up of O with weights $(1,1,2)$, and ω is an elliptic fibration.

Proposition 23.1. *Either there is a commutative diagram*

$$
\begin{array}{ccc}
X & \dashrightarrow^{\ \rho\ } & V \\
{\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\nu} \\
\mathbb{P}(1,1,3) & \dashrightarrow[\ \varphi\ ] & \mathbb{P}^2
\end{array}
\tag{23.1}
$$

or there is a commutative diagram

$$
\begin{array}{ccccc}
X & \dashrightarrow^{\ \theta\ } & X & \dashrightarrow^{\ \rho\ } & V \\
{\scriptstyle\xi}\downarrow & & & & \downarrow{\scriptstyle\nu} \\
\mathbb{P}(1,1,4) & & \dashrightarrow[\ \sigma\ ] & & \mathbb{P}^2,
\end{array}
\tag{23.2}
$$

where φ, θ and σ are birational maps.

Proof. Suppose that $O \in \mathbb{CS}\big(X, \tfrac{1}{k}\mathcal{M}\big)$. Then the existence of the commutative diagram (23.2) follows from Theorem 2.2. Similarly, the existence of the commutative diagram (23.1) follows from Theorem 2.2 in the case when $\mathbb{CS}\big(X, \tfrac{1}{k}\mathcal{M}\big) = \{P_1, P_2\}$. Therefore, we may assume that $\mathbb{CS}\big(X, \tfrac{1}{k}\mathcal{M}\big)$ consists of the singular point P_1 by Theorem 3.3, Proposition 3.5 and Lemma 3.11.

Let Q be the singular point of U such that $\alpha(Q) = P_1$ and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on U. Then $Q \in \mathbb{CS}\big(U, \tfrac{1}{k}\mathcal{B}\big)$ by Theorem 2.2 and Lemmas 2.1 and 2.3.

Let $\upsilon \colon \bar{U} \to U$ be the weighted blow-up of Q with weights $(1,1,2)$, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on $\bar{U}$ and S a sufficiently general surface in the pencil $\big|{-}K_{\bar{U}}\big|$. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_{\bar{U}}$ by Theorem 2.2, S is normal and the base locus of $\big|{-}K_{\bar{U}}\big|$ consists of the irreducible curve Δ such that $\alpha \circ \upsilon(\Delta)$ is the unique base curve of $|{-}K_X|$. Moreover, the inequality $\Delta^2 < 0$ holds on S. But the equivalence $\mathcal{D}|_S \sim_{\mathbb{Q}} k\Delta$ holds, which is impossible by Lemmas 2.9 and 2.7.

§ 24. The case $n = 31$: a hypersurface of degree 16 in $\mathbb{P}(1,1,4,5,6)$

We use the notation and assumptions of § 3. Let $n = 31$. Then X is a general hypersurface in $\mathbb{P}(1,1,2,3,4)$ of degree 10 and $-K_X^3 = 2/15$. The singularities of X consist of points P_1, P_2 and P_3 that are singularities of type $\tfrac{1}{2}(1,1,1)$, $\tfrac{1}{5}(1,1,4)$ and $\tfrac{1}{6}(1,1,5)$, respectively. There is a commutative diagram

where ψ is a projection, α_2 is the weighted blow-up of P_2 with weights $(1,1,4)$, α_3 is the weighted blow-up of P_3 with weights $(1,1,5)$, β_3 is the weighted blow-up with weights $(1,1,5)$ of the proper transform of P_3 on U_2, β_2 is the weighted blow-up with weights $(1,1,4)$ of the proper transform of P_2 on U_3, β_4 is the weighted blow-up with weights $(1,1,4)$ of the singular point of U_3 that is contained in the exceptional divisor of α_3, γ_2 is the weighted blow-up with weights $(1,1,4)$ of the proper transform of P_2 on U_{34}, γ_4 is the weighted blow-up with weights $(1,1,4)$ of the singular point of U_{23} that is contained in the exceptional divisor of β_3, and η is an elliptic fibration. There is a commutative diagram

$$
\begin{array}{ccc}
U_2 & \xleftarrow{\quad\beta\quad} & W \\
\alpha_2 \downarrow & & \downarrow \omega \\
X & \dashrightarrow[\xi] & \mathbb{P}(1,1,6),
\end{array}
$$

where ξ is a projection, β is the weighted blow-up with weights $(1,1,3)$ of the singular point of U_2 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$, and ω is an elliptic fibration.

Proposition 24.1. *Either there is a commutative diagram*

$$
\begin{array}{ccc}
X & \dashrightarrow[\rho] & V \\
\psi \downarrow & & \downarrow \nu \\
\mathbb{P}(1,1,4) & \dashrightarrow[\varphi] & \mathbb{P}^2
\end{array}
\tag{24.1}
$$

or there is a commutative diagram

$$
\begin{array}{ccccc}
X & \dashrightarrow[\theta] & X & \dashrightarrow[\rho] & V \\
\xi \downarrow & & & & \downarrow \nu \\
\mathbb{P}(1,1,5) & & \dashrightarrow[\sigma] & & \mathbb{P}^2,
\end{array}
\tag{24.2}
$$

where φ, θ and σ are birational maps.

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_2, P_3\}$. Let $\mathcal{D}_2$, $\mathcal{D}_3$, $\mathcal{D}_{23}$ and $\mathcal{D}_{34}$ be the proper transforms of $\mathcal{M}$ on U_2, U_3, U_{23} and U_{34}, respectively. Then it follows from Lemma 2.1 that the set $\mathbb{CS}\big(U_\mu, \frac{1}{k}\mathcal{D}_\mu\big)$ is non-empty if $\mathcal{D}_\mu \sim_{\mathbb{Q}} -kK_{U_\mu}$.

Lemma 24.2. *Suppose that the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains the point P_3. Let $\bar{P}_2$ be the proper transform of P_2 on U_3 and $\bar{P}_4$ the singular point of U_3 that is contained in the exceptional divisor of α_3. Then $\mathcal{D}_3 \sim_{\mathbb{Q}} -kK_{U_3}$ and $\mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big) \subseteq \{\bar{P}_2, \bar{P}_4\}$.*

Proof. The equivalence $\mathcal{D}_3 \sim_{\mathbb{Q}} -kK_{U_3}$ follows from Theorem 2.2. Suppose that

$$
\mathbb{CS}\left(U_3, \frac{1}{k}\mathcal{D}_3\right) \nsubseteq \{\bar{P}_2, \bar{P}_4\}.
$$

We shall show that this assumption leads to a contradiction.

Let G be the exceptional divisor of α_3. Then $G \cong \mathbb{P}(1, 1, 5)$, and it follows from Lemma 2.3 that there is a curve $C \subset G$ of the linear system $\big|\mathcal{O}_{\mathbb{P}(1, 1, 5)}(1)\big|$ that is contained in $\mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big)$, which is impossible by Lemma 2.4.

Lemma 24.3. *Let $\bar{P}_3$ be the proper transform of P_3 on U_2. Suppose that the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains the point P_2. Then either $\mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big) = \{\bar{P}_3\}$ or the commutative diagram* (24.2) *exists.*

Proof. The equivalence $\mathcal{D}_2 \sim_{\mathbb{Q}} -kK_{U_2}$ follows from Theorem 2.2. Suppose that the set $\mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big)$ does not consist of the point $\bar{P}_3$. Let $\bar{P}_5$ be the singular point of U_2 that is contained in the exceptional divisor of α_2. Then $\bar{P}_5$ is a quotient singularity of type $\frac{1}{4}(1, 1, 3)$ on U_2, and $\mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big)$ contains $\bar{P}_5$ by Lemma 2.3.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on W. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2, which implies the existence of the commutative diagram (24.2). $\square$

By Theorem 2.2, we may assume that either $\mathcal{D}_{23} \sim_{\mathbb{Q}} -kK_{U_{23}}$ or $\mathcal{D}_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$. Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on Y.

Lemma 24.4. *Suppose that $\mathcal{D}_{23} \sim_{\mathbb{Q}} -kK_{U_{23}}$. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$.*

Proof. Let F be the exceptional divisor of β_2, G the exceptional divisor of β_3, $\check{P}_4$ the singular point of G and $\check{P}_5$ the singular point of the surface F. Arguing as in the proof of Lemma 24.3, we see that $\mathbb{CS}\big(U_{23}, \frac{1}{k}\mathcal{D}_{23}\big)$ does not contain the point $\check{P}_5$. Hence, it follows from Lemmas 2.3 and 2.1 that the set $\mathbb{CS}\big(U_{23}, \frac{1}{k}\mathcal{D}_{23}\big)$ contains the point $\check{P}_4$, which implies that $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2. $\square$

Lemma 24.5. *Suppose that $\mathcal{D}_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$.*

Proof. Let G be the exceptional divisor of β_4, $\check{P}_2$ the proper transform of P_2 on U_{34} and $\check{P}_6$ the singular point of the surface G. Then G is a cone over a smooth rational cubic curve and $\check{P}_6$ is a quotient singularity of type $\frac{1}{3}(1, 1, 2)$ on U_{34}.

The set $\mathbb{CS}\big(U_{23}, \frac{1}{k}\mathcal{D}_{23}\big)$ is non-empty by Lemma 2.1. But the equivalence $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ follows from Theorem 2.2 if $\mathbb{CS}\big(U_{23}, \frac{1}{k}\mathcal{D}_{23}\big)$ contains $\check{P}_2$. Therefore, we may assume that $\mathbb{CS}\big(U_{23}, \frac{1}{k}\mathcal{D}_{23}\big)$ contains $\check{P}_6$ by Lemma 2.3.

Let $\pi \colon W \to U_{34}$ be the weighted blow-up of $\check{P}_6$ with weights $(1, 1, 3)$, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W and S a sufficiently general surface in the pencil $|-K_W|$. Then the base locus of $|-K_W|$ consists of the irreducible curve Δ such that the equivalence $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$ holds, and the inequality $\Delta^2 < 0$ holds on S. It follows from Lemma 2.9 that $\mathcal{B}|_S = k\Delta$, which is impossible by Lemma 2.7.

Hence, the equivalence $\mathcal{D} \sim_{\mathbb{Q}} -kK_Y$ holds, which implies that $\mathcal{D}$ is contained in fibres of η. But this implies the existence of the commutative diagram (24.1). The proposition is proved. $\square$

§ 25. The case $n = 32$: a hypersurface of degree 16 in $\mathbb{P}(1, 2, 3, 4, 7)$

We use the notation and assumptions of § 3. Let $n = 32$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1, 2, 3, 4, 6)$ of degree 16, and the equality $-K_X^3 = 2/21$ holds. The singularities of X consist of points P_1, P_2, P_3 and P_4 that are quotient singularities of type $\frac{1}{2}(1, 1, 1)$, a point P_5 that is a quotient singularity of type $\frac{1}{3}(1, 1, 2)$ and a point P_6 that is a singularity of type $\frac{1}{7}(1, 3, 4)$.

Proposition 25.1. *The assertion of Theorem* 1.10 *holds for* $n = 32$.

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_6\}$.

There is a commutative diagram

$$
\begin{array}{ccc}
U & \xleftarrow{\ \ \beta\ \ } & Y \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \eta} \\
X & \dashrightarrow[\psi] & \mathbb{P}(1,2,3),
\end{array}
$$

where ψ is the natural projection, α is the weighted blow-up of P_6 with weights $(1,3,4)$, β is the weighted blow-up with weights $(1,1,3)$ of the singular point of U that is a quotient singularity of type $\frac{1}{4}(1,1,3)$ contained in the exceptional divisor of α, and η is an elliptic fibration.

Let E be the exceptional divisor of α and $\mathcal{D}$ the proper transform of $\mathcal{M}$ on U. Then $E \cong \mathbb{P}(1,3,4)$, and it follows from Theorem 2.2 that $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$. But the set $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ is non-empty by Lemma 2.1.

Let P_7 and P_8 be the singular points of U contained in the divisor E that are singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{4}(1,1,3)$, respectively.

Lemma 25.2. *Suppose that* $P_8 \in \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$. *Then there is a commutative diagram*

$$
\begin{array}{ccc}
X & \dashrightarrow[\ \ \rho\ \ ] & V \\
{\scriptstyle \psi}\downarrow & & \downarrow{\scriptstyle \nu} \\
\mathbb{P}(1,2,3) & \dashrightarrow[\zeta] & \mathbb{P}^2,
\end{array}
\tag{25.1}
$$

where ζ *is a birational map.*

Proof. Let $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on Y. Then it follows from Theorem 2.2 that $\mathcal{H} \sim_{\mathbb{Q}} -kK_Y$. Hence, $\mathcal{H}$ lies in the fibres of η, which implies the existence of the commutative diagram (25.1).

We may assume that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big) = \{P_7\}$ by Lemma 2.3.

Let $\gamma\colon W \to U$ be the weighted blow-up of P_7 with weights $(1,1,2)$, F the exceptional divisor of γ, $\bar{E}$ the proper transform of the surface E on W and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W. Then $F \cong \mathbb{P}(1,1,2)$, and it follows from Theorem 2.2 that $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$.

The hypersurface X can be given by the quasi-homogeneous equation

$$
w^2 y + w f_9(x,y,z,t) + f_{16}(x,y,z,t) = 0 \subset \mathbb{P}(1,2,3,4,7) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),
$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 2$, $\mathrm{wt}(z) = 3$, $\mathrm{wt}(t) = 4$, $\mathrm{wt}(w) = 7$, and f_9 and f_{16} are quasi-homogeneous polynomials of degree 9 and 16, respectively. Let S be the unique surface in the linear system $|-K_X|$ and D a general surface in the pencil $|-2K_X|$. Then S is cut out by $x = 0$ and D is cut out by

$$
\lambda x^2 + \mu y = 0,
$$

where $(\lambda, \mu) \in \mathbb{P}^1$. The surface D is normal and the base locus of $|-2K_X|$ consists of the curve C such that $C = D \cdot S$.

In the neighbourhood of P_6, the monomials x, z and t can be regarded as weighted local coordinates on X such that $\mathrm{wt}(x) = 1$, $\mathrm{wt}(z) = 3$ and $\mathrm{wt}(z) = 4$. Then in the neighbourhood of the singular point P_6, the surface D can be given by the equation

$$\lambda x^2 + \mu(\varepsilon_1 x^9 + \varepsilon_2 zx^6 + \varepsilon_3 z^2 x^3 + \varepsilon_4 z^3 + \varepsilon_5 t^2 x + \varepsilon_6 tx^5 + \varepsilon_7 tzx^2 + \cdots) = 0,$$

where $\varepsilon_i \in \mathbb{C}$. In the neighbourhood of P_7, α can be given by the equations

$$x = \tilde{x}\tilde{z}^{\frac{1}{7}}, \qquad z = \tilde{z}^{\frac{3}{7}}, \qquad t = \tilde{t}\tilde{z}^{\frac{4}{7}},$$

where $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ are weighted local coordinates on U in the neighbourhood of the singular point P_7 such that $\mathrm{wt}(\tilde{x}) = 1$, $\mathrm{wt}(\tilde{z}) = 2$ and $\mathrm{wt}(\tilde{t}) = 1$. Let $\widetilde{D}$, $\widetilde{S}$ and $\widetilde{C}$ be the proper transforms on U of D, S and C, respectively, and let E be the exceptional divisor of α. Then in the neighbourhood of P_7, E is given by the equation $\tilde{z} = 0$, $\widetilde{D}$ is given by the vanishing of the analytic function

$$\lambda \tilde{x}^2 + \mu(\varepsilon_1 \tilde{x}^9 \tilde{z} + \varepsilon_2 \tilde{z}\tilde{x}^6 + \varepsilon_3 \tilde{z}\tilde{x}^3 + \varepsilon_4 \tilde{z} + \varepsilon_5 \tilde{t}^2 \tilde{x}\tilde{z} + \varepsilon_6 \tilde{t}\tilde{x}^5 \tilde{z} + \varepsilon_7 \tilde{t}\tilde{z}\tilde{x}^2 + \cdots),$$

and S is given by the equation $\tilde{x} = 0$.

In the neighbourhood of the singular point of F, β can be given by the equations

$$\tilde{x} = \bar{x}\bar{z}^{\frac{1}{3}}, \qquad \tilde{z} = \bar{z}^{\frac{2}{3}}, \qquad \tilde{t} = \bar{t}\bar{z}^{\frac{1}{3}},$$

where $\bar{x}$, $\bar{z}$ and $\bar{t}$ are weighted local coordinates on W in the neighbourhood of the singular point of F such that $\mathrm{wt}(\bar{x}) = \mathrm{wt}(\bar{z}) = \mathrm{wt}(\bar{t}) = 1$. The surface F is given by the equation $\bar{z} = 0$, the proper transform of D on W is given by the vanishing of the analytic function

$$\lambda \bar{x}^2 + \mu(\varepsilon_1 \bar{x}^9 \bar{z}^3 + \varepsilon_2 \bar{z}^2 \bar{x}^6 + \varepsilon_3 \bar{z}\bar{x}^3 + \varepsilon_4 + \varepsilon_5 \bar{t}^2 \bar{x}\bar{z} + \varepsilon_6 \bar{t}\bar{x}^5 \bar{z}^2 + \varepsilon_7 \bar{t}\bar{z}\bar{x}^2 + \cdots) = 0,$$

the proper transform of S on W is given by the equation $\bar{x} = 0$, and the proper transform of E on W is given by the equation $\bar{z} = 0$.

Let $\mathcal{P}$, $\bar{D}$, $\bar{S}$ and $\bar{C}$ be the proper transforms on W of the pencil $|-2K_X|$, the surface D, the surface S and the curve C, respectively, and let $\bar{H}$ be the proper transform on W of the surface that is cut out on X by the equation $y = 0$. Then $\bar{D}$ is a general surface in the pencil $\mathcal{P}$ and

$$\bar{E} \sim_{\mathbb{Q}} \gamma^*(E) - \frac{2}{3} F,$$

$$\bar{D} \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*(-2K_X) - \frac{2}{7}\gamma^*(E) - \frac{2}{3} F \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*(-2K_X) - \frac{2}{7}\bar{E} - \frac{6}{7} F,$$

$$\bar{S} \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*(-K_X) - \frac{1}{7}\gamma^*(E) - \frac{1}{3} F \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*(-K_X) - \frac{1}{7}\bar{E} - \frac{3}{7} F,$$

$$\bar{H} \sim_{\mathbb{Q}} \gamma^*\left(\alpha^*(-2K_X) - \frac{9}{7} E\right) \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*(-2K_X) - \frac{9}{7}\bar{E} - \frac{6}{7} F \sim_{\mathbb{Q}} 2\bar{S} - \bar{E}.$$

$$(25.2)$$

The curve $\bar{C}$ is contained in the base locus of $\mathcal{P}$ but is not the only such curve. To see this, let L be the curve on E that is contained in $|\mathcal{O}_{\mathbb{P}(1,3,4)}(1)|$, which means that

L is given locally by the equations $\tilde{x} = \tilde{z} = 0$, and let $\bar{L}$ be the proper transform of L on W. Then $\bar{L}$ is also contained in the base locus of $\mathcal{P}$. Moreover, it follows from local computations that the base locus of $\mathcal{P}$ does not contain curves outside the union $\bar{C} \cup \bar{L}$. The curve $\bar{C}$ is the intersection of the divisors $\bar{S}$ and $\bar{H}$ and the curve $\bar{L}$ is the intersection of the divisors $\bar{S}$ and $\bar{E}$. Moreover, we have $2\bar{C} = \bar{D} \cdot \bar{H}$ $\bar{C} + \bar{L} = \bar{S} \cdot \bar{D}$ and $2\bar{L} = \bar{D} \cdot \bar{E}$.

The curves $\bar{C}$ and $\bar{L}$ can be regarded as divisors on the normal surface $\bar{D}$. Then it follows from the equivalences (25.2) that

$$\bar{L} \cdot \bar{L} = \frac{\bar{E} \cdot \bar{E} \cdot \bar{D}}{4} = -\frac{5}{8},$$

$$\bar{C} \cdot \bar{C} = \frac{\bar{H} \cdot \bar{H} \cdot \bar{D}}{4} = \bar{S} \cdot \bar{S} \cdot \bar{D} - \bar{S} \cdot \bar{E} \cdot \bar{D} - \frac{3}{2} \bar{E} \cdot \bar{E} \cdot \bar{D} = -\frac{7}{24}, \qquad (25.3)$$

$$\bar{C} \cdot \bar{L} = \frac{\bar{H} \cdot \bar{E} \cdot \bar{D}}{4} = \frac{\bar{S} \cdot \bar{E} \cdot \bar{D}}{2} - \frac{\bar{E} \cdot \bar{E} \cdot \bar{D}}{4} = \frac{3}{8},$$

which implies that the intersection form of $\bar{C}$ and $\bar{L}$ on $\bar{D}$ is negative definite. On the other hand, the equivalence $\mathcal{B}|_{\bar{D}} \sim_{\mathbb{Q}} k\bar{C} + k\bar{L}$ holds, which is impossible by Lemmas 2.9 and 2.7. The proposition is proved.

Remark 25.3. Let $\overline{\mathbb{NE}}(W)$ be the closure in $\mathbb{R}^3$ of the cone that is generated by the effective one-dimensional cycles on W. Then the negative definiteness of the intersection form of $\bar{C}$ and $\bar{L}$ on $\bar{D}$ implies that $\bar{C}$ and $\bar{L}$ generate the two-dimensional extremal face of the cone $\overline{\mathbb{NE}}(W)$ that does not contain irreducible curves on W other than $\bar{C}$ and $\bar{L}$. This can be shown using the equivalences (25.2) without studying the geometry of $\bar{C}$ and $\bar{L}$ on $\bar{D}$.

§ 26. The case $n = 36$: a hypersurface of degree 18 in $\mathbb{P}(1,1,4,6,7)$

We use the notation and assumptions of § 3. Let $n = 36$. Then X is a sufficiently general hypersurface in $\mathbb{P}(1,1,4,6,7)$ of degree 18, the equality $-K_X^3 = 3/28$ holds, and the singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, a point P_2 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$ and a point P_3 that is a quotient singularity of type $\frac{1}{7}(1,1,6)$.

There is a commutative diagram

$$
\begin{array}{ccccc}
U_3 & \xleftarrow{\;\beta_4\;} & U_{34} & \xleftarrow{\;\gamma_5\;} & Y \\
{\scriptstyle \alpha_3}\downarrow & & & & \downarrow{\scriptstyle \eta} \\
X & \dashrightarrow[\psi]{} & & & \mathbb{P}(1,1,4),
\end{array}
$$

where ψ is the natural projection, α_3 is the weighted blow-up of P_3 with weights $(1,1,7)$, β_4 is the weighted blow-up with weights $(1,1,6)$ of the singular point of U_3 that is contained in the exceptional divisor of α_3, γ_5 is the weighted blow-up with weights $(1,1,4)$ of the singular point of U_{34} that is contained in the exceptional divisor of the birational morphism β_4, and η is an elliptic fibration.

Remark 26.1. The divisors $-K_{U_3}$ and $-K_{U_{34}}$ are nef and big.

It follows from Example 1.8 that there is a commutative diagram
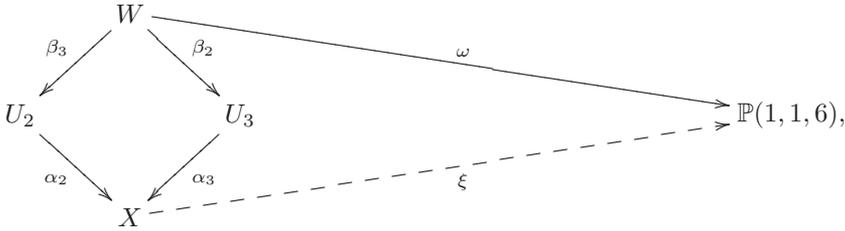
where ξ is a projection, α_2 is the blow-up of P_2 with weights $(1,1,3)$, α_3 is the weighted blow-up of P_3 with weights $(1,1,6)$, β_2 is the weighted blow-up with weights $(1,1,3)$ of the proper transform of P_2 on U_3, β_3 is the weighted blow-up with weights $(1,1,6)$ of the proper transform of P_3 on U_2, and ω is an elliptic fibration.

Remark 26.2. The divisor $-K_{U_2}$ is nef and big.

In this section we prove the following result.

Proposition 26.3. *Either there is a commutative diagram*

$$
\begin{array}{ccccc}
X & \dashrightarrow{\;\;\zeta\;\;} & X & \dashrightarrow{\;\;\rho\;\;} & V \\
{\scriptstyle\psi}\downarrow & & & & \downarrow{\scriptstyle\nu} \\
\mathbb{P}(1,1,4) & & \dashrightarrow{\;\;\varphi\;\;} & & \mathbb{P}^2
\end{array}
\tag{26.1}
$$

or there is a commutative diagram

$$
\begin{array}{ccc}
X & \dashrightarrow{\;\;\rho\;\;} & V \\
{\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle\nu} \\
\mathbb{P}(1,1,6) & \dashrightarrow{\;\;\sigma\;\;} & \mathbb{P}^2,
\end{array}
\tag{26.2}
$$

where ζ, φ and σ are birational maps.

Proof. It follows from Theorem 3.3, Proposition 3.5 and Lemma 3.11 that $\varnothing \neq \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_2, P_3\}$ and the existence of the commutative diagram (26.2) is obvious if $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \supset \{P_2, P_3\}$.

Lemma 26.4. *The set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains the point P_3.*

Proof. Suppose that $P_3 \notin \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$. Then $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_2\}$. Let $\mathcal{D}_2$ be the proper transform of $\mathcal{M}$ on U_2 and P_6 the singular point of U_2 that is contained in the exceptional divisor of α_2. Then $\mathcal{D}_2 \sim_{\mathbb{Q}} -kK_{U_2}$ by Theorem 2.2, P_6 is a quotient singularity of type $\frac{1}{3}(1,1,2)$ on U_2 and it follows from Lemmas 2.1 and 2.3 that $P_6 \in \mathbb{CS}\big(U_2, \frac{1}{k}\mathcal{D}_2\big)$.

Let $\pi\colon Z \to U_2$ be the weighted blow-up of P_6 with weights $(1,1,3)$, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on Z and S a general surface in $|-K_Z|$. Then S is normal and the base locus of $|-K_Z|$ consists of the irreducible curve Δ such that $\mathcal{B}|_S \sim_{\mathbb{Q}} k\Delta$.

The equality $\Delta^2 = 1/7$ holds on S, but this is impossible by Lemmas 2.7 and 2.9.

For the rest of the proof we may assume that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) = \{P_3\}$. Let $\mathcal{D}_3$ be the proper transform of $\mathcal{M}$ on U_3 and P_4 the singular point of U_3 contained in the exceptional divisor of α_3. Then $\mathcal{D}_3 \sim_{\mathbb{Q}} -kK_{U_3}$ and P_4 is a quotient singularity of type $\frac{1}{6}(1,1,5)$ of U_3. It follows from Lemmas 2.1 and 2.3 that $P_4 \in \mathbb{CS}\left(U_3, \frac{1}{k}\mathcal{D}_3\right)$.

Lemma 26.5. *The set $\mathbb{CS}\left(U_3, \frac{1}{k}\mathcal{D}_3\right)$ consists of the point P_4.*

Proof. Suppose that $\mathbb{CS}\left(U_3, \frac{1}{k}\mathcal{D}_3\right)$ contains a subvariety C of U_3 that is different from the point P_4. Let G be the exceptional divisor of β_4. Then G is a cone over a rational normal curve in $\mathbb{P}^6$ of degree 6. The curve C must be a ruling of G by Lemma 2.3, but this is impossible by Lemma 2.4.

Hence, $\mathbb{CS}\left(U_3, \frac{1}{k}\mathcal{D}_3\right)$ consists of the point P_4. Let $\mathcal{D}_{34}$ be the proper transform of $\mathcal{M}$ on U_{34} and P_5 the singular point of U_{34} contained in the exceptional divisor of β_4. Then $\mathcal{D}_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$ by Theorem 2.2, and P_5 is a quotient singularity of type $\frac{1}{5}(1,1,4)$ of U_{34}. But $\mathbb{CS}\left(U_{34}, \frac{1}{k}\mathcal{D}_{34}\right)$ contains P_5 by Lemmas 2.1 and 2.3. It follows from Theorem 2.2 that the proper transform of $\mathcal{M}$ on Y is contained in the fibres of η, which implies the existence of the commutative diagram (26.1). The proposition is proved.

§ 27. The case $n = 38$: a hypersurface of degree 18 in $\mathbb{P}(1,2,3,5,8)$

We use the notation and assumptions of § 3. Let $n = 38$. Then X is a general hypersurface of degree 18 in $\mathbb{P}(1,2,3,5,8)$. The singularities of X consist of points P_1 and P_2 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, a point P_3 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ and a point P_4 that is a quotient singularity of type $\frac{1}{8}(1,3,5)$. The equality $-K_X^3 = 3/40$ holds.

Proposition 27.1. *The assertion of Theorem 1.10 holds for $n = 38$.*

Proof. The singularities of the log pair $\left(X, \frac{1}{k}\mathcal{M}\right)$ are canonical (see Remark 3.2). It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) \subseteq \{P_3, P_4\}$. Arguing as in the proof of Proposition 21.1, we see that $P_4 \in \mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$.

There is a commutative diagram
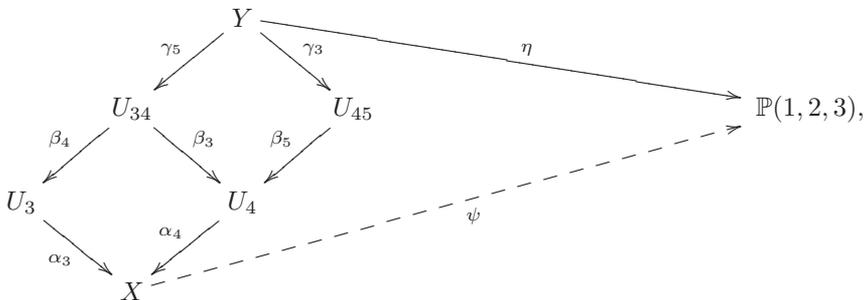
where ψ is the natural projection, α_3 is the weighted blow-up of P_3 with weights $(1,2,3)$, α_4 is the weighted blow-up of P_4 with weights $(1,3,5)$, β_4 is the

weighted blow-up with weights $(1, 3, 5)$ of the proper transform of P_4 on U_3, β_3 is the weighted blow-up with weights $(1, 2, 3)$ of the proper transform of P_3 on U_4, β_5 is the weighted blow-up with weights $(1, 2, 3)$ of the singular point of U_4 that is a quotient singularity of type $\frac{1}{5}(1, 2, 3)$ contained in the exceptional divisor of α_4, γ_3 is the weighted blow-up with weights $(1, 2, 3)$ of the proper transform of P_3 on U_{45}, γ_5 is the weighted blow-up with weights $(1, 2, 3)$ of the singular point of U_{34} that is a quotient singularity of type $\frac{1}{5}(1, 2, 3)$ contained in the exceptional divisor of β_4, and η is an elliptic fibration.

Let $\mathcal{D}_4$ be the proper transform of $\mathcal{M}$ on U_4, $\bar{P}_3$ the proper transform of P_3 on U_4, and P_5 and P_6 the singular points of U_4 that are quotient singularities of types $\frac{1}{5}(1, 2, 3)$ and $\frac{1}{3}(1, 1, 2)$, respectively, contained in the exceptional divisor of α_4. Arguing as in the proof of Proposition 25.1, we see that $\mathbb{CS}\bigl(U_4, \frac{1}{k}\mathcal{D}_4\bigr)$ contains either $\bar{P}_3$ or P_5.

Suppose that $\bar{P}_3 \in \mathbb{CS}\bigl(U_4, \frac{1}{k}\mathcal{D}_5\bigr)$. Arguing as in the proofs of Propositions 21.1 and 25.1, we see that $\mathbb{CS}\bigl(U_4, \frac{1}{k}\mathcal{D}_5\bigr)$ contains P_5. Therefore, the assertion of Theorem 2.2 implies that Theorem 1.10 holds for X. Thus, we may assume that $\mathbb{CS}\bigl(U_4, \frac{1}{k}\mathcal{D}_5\bigr)$ contains P_5.

Let $\mathcal{D}_{45}$ be the proper transform of $\mathcal{M}$ on U_{45} and P_7, P_8 the singular points of U_{45} that are quotient singularities of types $\frac{1}{2}(1, 1, 1)$, $\frac{1}{3}(1, 1, 2)$, respectively, contained in the exceptional divisor of β_5. Then it follows from Lemma 2.1 that $\mathbb{CS}\bigl(U_{45}, \frac{1}{k}\mathcal{D}_{45}\bigr) \neq \varnothing$, and Theorem 2.2 implies that Theorem 1.10 holds for X in the case when $\mathbb{CS}\bigl(U_{45}, \frac{1}{k}\mathcal{D}_{45}\bigr)$ contains the proper transform of P_3 on U_{45}.

By Lemma 2.3, we may assume that $\mathbb{CS}\bigl(U_{45}, \frac{1}{k}\mathcal{D}_{45}\bigr) \cap \{P_7, P_8\} \neq \varnothing$.

Suppose that $P_7 \in \mathbb{CS}\bigl(U_{45}, \frac{1}{k}\mathcal{D}_{45}\bigr)$. Then by considering the proper transform of the complete linear system $|-3K_X|$ on the weighted blow-up of P_7 with weights $(1, 1, 1)$, we easily obtain a contradiction as in the proof of Lemma 21.3.

Thus, the set $\mathbb{CS}\bigl(U_{45}, \frac{1}{k}\mathcal{D}_{45}\bigr)$ contains the point P_8. Let $\pi \colon W \to U_{45}$ be the weighted blow-up of P_8, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W and D a general surface in $|-2K_W|$. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2, the surface D is normal and the pencil $|-2K_W|$ is the proper transform of the pencil $|-2K_X|$. The base locus of $|-2K_W|$ consists of curves C and L such that $\alpha_4 \circ \beta_5 \circ \pi(C)$ is the unique base curve of $|-2K_X|$ and the curve $\beta_5 \circ \pi(L)$ is contained in the exceptional divisor of α_4.

The intersection form of C and L on D is negative definite. But the equivalence $\mathcal{B}|_D \sim_{\mathbb{Q}} kC + kL$ holds, which is impossible by Lemmas 2.9 and 2.7.

§ 28. The case $n = 40$: a hypersurface of degree 19 in $\mathbb{P}(1, 3, 4, 5, 7)$

We use the notation and assumptions of § 3. Let $n = 40$. Then X is a general hypersurface in $\mathbb{P}(1, 3, 4, 5, 7)$ of degree 19. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{3}(1, 1, 2)$, a point P_2 that is a quotient singularity of type $\frac{1}{4}(1, 1, 3)$, a point P_3 that is a quotient singularity of type $\frac{1}{5}(1, 2, 3)$ and a point P_4 that is a quotient singularity of type $\frac{1}{7}(1, 3, 4)$,

and $-K_X^3 = 19/420$. There is a commutative diagram
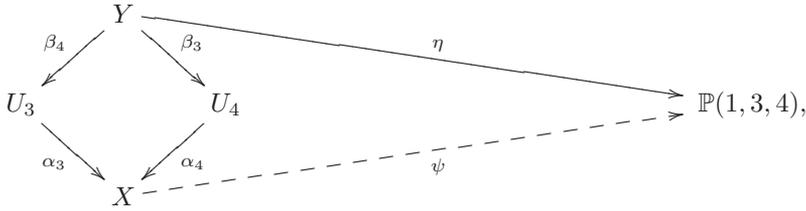
where ψ is the natural projection, α_3 is the weighted blow-up of P_3 with weights $(1, 2, 3)$, α_4 is the weighted blow-up of P_4 with weights $(1, 3, 4)$, β_3 is the weighted blow-up with weights $(1, 2, 3)$ of the proper transform of P_3 on U_4, β_4 is the weighted blow-up with weights $(1, 3, 4)$ of the proper transform of P_4 on U_3, and η is an elliptic fibration.

Proposition 28.1. *The assertion of Theorem* 1.10 *holds for* $n = 40$.

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_3, P_4\}$. The singularities of the log pair $\big(X, \frac{1}{k}\mathcal{M}\big)$ are canonical. Let $\mathcal{D}_3$ and $\mathcal{D}_4$ be the proper transforms of $\mathcal{M}$ on U_3 and U_4, respectively.

Lemma 28.2. *Suppose that* $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_3, P_4\}$. *Then there is a commutative diagram*

$$\begin{array}{ccc} X & \dashrightarrow^{\ \rho\ } & V \\ \psi \downarrow & & \downarrow \nu \\ \mathbb{P}(1, 3, 4) & \dashrightarrow_{\ \zeta\ } & \mathbb{P}^2, \end{array} \qquad (28.1)$$

where ζ *is a birational map.*

Proof. Let $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on Y. Then it follows from Theorem 2.2 that the equivalence $\mathcal{H} \sim_{\mathbb{Q}} -kK_Y$ holds, which implies that $\mathcal{H}$ lies in the fibres of η. This implies the existence of the commutative diagram (28.1).

Let P_5 and P_6 be the singular points of U_3 contained in the exceptional divisor of α_3 such that P_5 and P_6 are quotient singularities of types $\frac{1}{2}(1, 1, 1)$ and $\frac{1}{3}(1, 1, 2)$, respectively, and let P_7 and P_8 be the singular points of U_4 that are quotient singularities of types $\frac{1}{3}(1, 1, 2)$ and $\frac{1}{4}(1, 1, 3)$, respectively, contained in the exceptional divisor of α_4. Then it follows from Theorem 2.2 and Lemmas 2.1 and 2.3 that
  $\mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big)$ contains either P_5 or P_6 if $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ does not contain P_4;
  $\mathbb{CS}\big(U_4, \frac{1}{k}\mathcal{D}_4\big)$ contains either P_7 or P_8 if $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ does not contain P_3.

Lemma 28.3. *Suppose that* $P_4 \notin \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$. *Then* $P_5 \notin \mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big)$.

Proof. Suppose that $\mathbb{CS}\big(U_3, \frac{1}{k}\mathcal{D}_3\big)$ contains P_5. Let $\gamma \colon W \to U_3$ be the weighted blow-up of P_5 with weights $(1, 1, 1)$, F the exceptional divisor of γ, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on W, $\mathcal{H}$ the proper transform of the pencil $|-3K_X|$ on W, $\bar{D}$ a general surface in the linear system $\mathcal{D}$ and $\bar{H}$ a general surface in $\mathcal{H}$. Then

$$\bar{H} \sim_{\mathbb{Q}} (\alpha_3 \circ \gamma)^*(-3K_X) - \frac{3}{5}\gamma^*(E) - \frac{1}{2}F,$$

where E is the exceptional divisor of α_3. The base locus of $\mathcal{H}$ consists of a curve $\bar{C}$ such that $\alpha_3 \circ \gamma(C)$ is the unique base curve of $|-3K_X|$. On the other hand, it follows from Theorem 2.2 that $\bar{D} \sim_{\mathbb{Q}} -kK_W$. The equivalence $\bar{D}|_{\bar{H}} \sim_{\mathbb{Q}} k\bar{C}$ holds. But $\bar{C}^2 < 0$ on $\bar{H}$. It follows from Lemma 2.9 that the support of the cycle $\bar{H} \cdot \bar{D}$ consists of $\bar{C}$, which is impossible by Lemma 2.7.

Lemma 28.4. *Suppose that $P_4 \notin \mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$. Then $P_6 \notin \mathbb{CS}\left(U_3, \frac{1}{k}\mathcal{D}_3\right)$.*

Proof. We suppose that $P_6 \in \mathbb{CS}\left(U_3, \frac{1}{k}\mathcal{D}_3\right)$ and seek a contradiction.

Let $\gamma \colon W \to U_3$ be the weighted blow-up of P_6 with weights $(1, 1, 2)$, let F and G be the exceptional divisors of α_3 and γ, respectively, let $\mathcal{B}$ and $\mathcal{D}$ be the proper transforms of $\mathcal{M}$ and $|-7K_X|$, respectively, on W and let D be a general surface in $\mathcal{D}$. Then it follows from Theorem 2.2 that $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$. But the base locus of $\mathcal{D}$ does not contain curves. Moreover, we have

$$\mathcal{B} \sim_{\mathbb{Q}} -kK_W \sim_{\mathbb{Q}} (\alpha_3 \circ \gamma)^*(-kK_X) - \frac{k}{5}\gamma^*(F) - \frac{k}{3}G,$$

and the divisor D is nef because the base locus of $\mathcal{D}$ does not contain curves. But $D \cdot B_1 \cdot B_2 = -k^2/12$, where B_1 and B_2 are general surfaces in $\mathcal{B}$. This contradiction completes the proof of the lemma.

Lemma 28.5. *Suppose that $P_3 \notin \mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$. Then $P_7 \notin \mathbb{CS}\left(U_4, \frac{1}{k}\mathcal{D}_4\right)$.*

Proof. We suppose that $\mathbb{CS}\left(U_4, \frac{1}{k}\mathcal{D}_4\right)$ contains P_7 and seek a contradiction.

Let $\gamma \colon W \to U_4$ be the weighted blow-up of P_7 with weights $(1, 1, 2)$, F the exceptional divisor of γ, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on W, $\mathcal{H}$ the proper transform of $|-4K_X|$ on W, $\bar{D}$ a general surface in $\mathcal{D}$ and $\bar{H}$ a general surface in $\mathcal{H}$. Then

$$\bar{H} \sim_{\mathbb{Q}} (\alpha_3 \circ \gamma)^*(-4K_X) - \frac{4}{7}\gamma^*(E) - \frac{1}{3}F,$$

and the base locus of $\mathcal{H}$ consists of a curve $\bar{C}$ such that $\alpha_3 \circ \gamma(C)$ is the base curve of $|-4K_X|$. It follows from Theorem 2.2 that $\bar{D} \sim_{\mathbb{Q}} -kK_W$. The equality $\bar{C}^2 = -1/30$ holds on $\bar{H}$, which implies that the support of the cycle $\bar{H} \cdot \bar{D}$ consists of $\bar{C}$ because $\bar{D}|_{\bar{H}} \sim_{\mathbb{Q}} k\bar{C}$. But this is impossible by Lemma 2.7.

Lemma 28.6. *Suppose that $P_3 \notin \mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$. Then $P_8 \notin \mathbb{CS}\left(U_4, \frac{1}{k}\mathcal{D}_4\right)$.*

Proof. We suppose that $\mathbb{CS}\left(U_4, \frac{1}{k}\mathcal{D}_4\right)$ contains P_8 and seek a contradiction.

Let $\gamma \colon W \to U_4$ be the weighted blow-up of P_8 with weights $(1, 1, 3)$, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on W, $\bar{H}$ a general surface in $|-3K_W|$ and $\bar{D}$ a general surface in $\mathcal{D}$. Then $\bar{D} \sim_{\mathbb{Q}} -kK_W$. But the base locus of $|-3K_W|$ consists of an irreducible curve $\bar{C}$ such that $\alpha_4 \circ \gamma(C)$ is the base curve of $|-3K_X|$. The equality $\bar{C}^2 = -1/20$ holds on $\bar{H}$. But $\bar{D}|_{\bar{H}} \sim_{\mathbb{Q}} k\bar{C}$, which implies that the support of the cycle $\bar{H} \cdot \bar{D}$ consists of $\bar{C}$. But this is impossible by Lemma 2.7.

The proposition is proved.

§ 29. The case $n = 43$: a hypersurface of degree 20 in $\mathbb{P}(1, 2, 4, 5, 9)$

We use the notation and assumptions of § 3. Let $n = 43$. Then X is a general hypersurface in $\mathbb{P}(1, 2, 4, 5, 9)$ of degree 20 and the equality $-K_X^3 = 1/18$ holds. The singularities of X consist of points P_1 P_2, P_3, P_4 and P_5 that are quotient singularities of type $\frac{1}{2}(1, 1, 1)$ and a point P_6 that is a quotient singularity of type $\frac{1}{9}(1, 4, 5)$.

Proposition 29.1. *The assertion of Theorem* 1.10 *holds for* $n = 43$.

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_6\}$. There is a commutative diagram

$$
\begin{array}{ccc}
U & \xleftarrow{\ \beta\ } & Y \\
{\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \eta} \\
X & \dashrightarrow{\ \psi\ } & \mathbb{P}(1, 2, 4),
\end{array}
$$

where ψ is the natural projection, α is the weighted blow-up of P_6 with weights $(1, 4, 5)$, β is the weighted blow-up with weights $(1, 1, 4)$ of the singular point of U that is a quotient singularity of type $\frac{1}{5}(1, 1, 4)$ contained in the exceptional divisor of α, and η is an elliptic fibration.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U and P_7, P_8 the singular points of U that are quotient singularities of types $\frac{1}{4}(1, 1, 3)$, $\frac{1}{5}(1, 1, 4)$, respectively, contained in the exceptional divisor of α. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2.

We must show that the proper transform of $\mathcal{M}$ on Y is contained in the fibres of η. This is implied by Theorem 2.2 if $P_8 \in \mathbb{CS}(U, \frac{1}{k}\mathcal{D})$. Thus, we may assume that $P_8 \notin \mathbb{CS}(U, \frac{1}{k}\mathcal{D})$.

Remark 29.2. The set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains P_7 by Lemma 2.3 because $-K_U$ is nef and big.

Let $\gamma: W \to U$ be the weighted blow-up of P_7 with weights $(1, 1, 3)$, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W and P_9 the singular point of W that is a quotient singularity of type $\frac{1}{3}(1, 1, 2)$ contained in the exceptional divisor of γ. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2.

Lemma 29.3. *The set* $\mathbb{CS}(W, \frac{1}{k}\mathcal{B})$ *does not contain the point* P_9.

Proof. We suppose that $\mathbb{CS}(W, \frac{1}{k}\mathcal{B})$ contains P_9 and seek a contradiction.

Let $\pi: Z \to W$ be the weighted blow-up of P_9 with weights $(1, 1, 2)$, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on Z and $\mathcal{P}$ the proper transform of $|-5K_X|$ on Z. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_Z$ by Theorem 2.2, but the base locus of $\mathcal{P}$ consists of an irreducible curve Γ such that $\alpha \circ \gamma \circ \pi(\Gamma)$ is the base curve of $|-5K_X|$.

Let H_1 and H_2 be general surfaces in $\mathcal{H}$ and D a general surface in $\mathcal{P}$. Then $D \cdot \Gamma = 1$ and $D^3 = 6$. Therefore, the divisor D is nef and big, but elementary computations imply that $D \cdot H_1 \cdot H_2 = 0$, which is impossible by Corollary 2.6.

Therefore, Lemma 2.3 has the following consequence.

Corollary 29.4. *The singularities of the log pair* $(W, \frac{1}{k}\mathcal{B})$ *are terminal.*

The hypersurface X can be given by the following quasi-homogeneous equation of degree 20:

$$w^2 y + w f_{11}(x, y, z, t) + f_{20}(x, y, z, t) = 0 \subset \mathbb{P}(1, 2, 4, 5, 9) \cong \mathrm{Proj}\big(\mathbb{C}[x, y, z, t, w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 2$, $\mathrm{wt}(z) = 4$, $\mathrm{wt}(t) = 5$, $\mathrm{wt}(w) = 9$, and $f_i(x, y, z, t)$ is a quasi-homogeneous polynomial of degree i. Let D be a general surface in the pencil $|-2K_X|$ and S the surface cut out on X by the equation $x = 0$. Then D is cut out on X by the quasi-homogeneous equation $\lambda x^2 + \mu y = 0$, where $(\lambda, \mu) \in \mathbb{P}^1$. The base locus of $|-2K_X|$ consists of the irreducible curve C cut out on X by the equations $x = y = 0$.

In the neighbourhood of the point P_6, the monomials x, z and t can be regarded as weighted local coordinates on X such that $\mathrm{wt}(x) = 1$, $\mathrm{wt}(z) = 4$ and $\mathrm{wt}(t) = 5$. Then in the neighbourhood of P_7, the weighted blow-up α is given by the equations

$$x = \tilde{x}\tilde{z}^{\frac{1}{9}}, \qquad z = \tilde{z}^{\frac{4}{9}}, \qquad t = \tilde{t}\tilde{z}^{\frac{5}{9}},$$

where $\tilde{x}$, $\tilde{z}$ and $\tilde{t}$ are weighted local coordinates on U in the neighbourhood of P_7 such that $\mathrm{wt}(\tilde{x}) = 1$, $\mathrm{wt}(\tilde{z}) = 3$ and $\mathrm{wt}(\tilde{t}) = 1$.

Let E be the exceptional divisor of α and $\widetilde{D}$, $\widetilde{S}$, $\widetilde{C}$ the proper transforms on U of D, S, C, respectively. Then E is given by the equation $\tilde{z} = 0$ and $\widetilde{S}$ by the equation $\tilde{x} = 0$. Moreover, it follows from the local equation of $\widetilde{D}$ that $\widetilde{D} \cdot \widetilde{S} = \widetilde{C} + 2\tilde{L}_1$, where $\tilde{L}_1$ is the curve given locally by the equations $\tilde{z} = \tilde{x} = 0$. Moreover, $\widetilde{D}$ is not normal at a general point of $\tilde{L}_1$ and $\widetilde{D} \sim_{\mathbb{Q}} 2\widetilde{S}$.

In the neighbourhood of P_9, γ is given by the equations

$$\tilde{x} = \bar{x}\bar{z}^{\frac{1}{4}}, \qquad \tilde{z} = \bar{z}^{\frac{3}{4}}, \qquad \tilde{t} = \bar{t}\bar{z}^{\frac{1}{4}},$$

where $\bar{x}$, $\bar{z}$ and $\bar{t}$ are weighted local coordinates on W in the neighbourhood of P_9 such that $\mathrm{wt}(\bar{x}) = 1$, $\mathrm{wt}(\bar{z}) = 2$ and $\mathrm{wt}(\bar{t}) = 1$. In particular, the exceptional divisor of γ is given by the equation $\bar{z} = 0$ and the proper transform of S on W is given by the equation $\bar{x} = 0$.

Let F be the exceptional divisor of γ and $\bar{D}$, $\bar{S}$, $\bar{E}$, $\bar{C}$, $\bar{L}_1$ the proper transforms on W of D, S, E, C, $\tilde{L}_1$, respectively. Then

$$\bar{S} \sim_{\mathbb{Q}} -K_W \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*(-K_X) - \frac{1}{9}\gamma^*(E) - \frac{1}{4}F,$$

$$\bar{D} \sim_{\mathbb{Q}} -2K_W \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*(-2K_X) - \frac{2}{9}\gamma^*(E) - \frac{1}{2}F, \qquad (29.1)$$

$$\bar{E} \sim_{\mathbb{Q}} \gamma^*(E) - \frac{3}{4}F.$$

Let $\bar{L}_2$ be the curve on W given by the equations $\bar{z} = \bar{x} = 0$. Then

$$\bar{D} \cdot \bar{S} = \bar{C} + 2\bar{L}_1 + \bar{L}_2, \qquad \bar{D} \cdot \bar{E} = 2\bar{L}_1, \qquad \bar{D} \cdot F = 2\bar{L}_2,$$

and the base locus of $|-2K_W|$ consists of $\bar{C}$, $\bar{L}_1$ and $\bar{L}_2$. The equivalences (29.1) imply that

$$\bar{D} \cdot \bar{C} = 0, \qquad \bar{D} \cdot \bar{L}_1 = -\frac{2}{5}, \qquad \bar{D} \cdot \bar{L}_2 = \frac{2}{3}.$$

Let $\bar{H}$ and $\bar{T}$ be the proper transforms on W of the surfaces cut out on X by the equations $y = 0$ and $t = 0$, respectively. Then

$$\bar{T} \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*(-5K_X) - \frac{5}{9}\gamma^*(E) - \frac{1}{4}F \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*(-5K_X) - \frac{5}{9}\bar{E} - \frac{2}{3}F,$$
$$\bar{H} \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*(-2K_X) - \frac{11}{9}\gamma^*(E) - \frac{3}{2}F \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*(-2K_X) - \frac{11}{9}\bar{E} - \frac{5}{3}F, \tag{29.2}$$

which implies that $-14K_W \sim_{\mathbb{Q}} 14\bar{D} \sim_{\mathbb{Q}} 2\bar{T} + 2\bar{H} + 2\bar{E}$, and the support of the cycle $\bar{T} \cdot \bar{H}$ does not contain $\bar{L}_2$ or $\bar{C}$. Therefore, the base locus of $|-14K_W|$ does not contain curves other than $\bar{L}_1$. The singularities of the mobile log pair $(W, \lambda | -14K_W|)$ are log-terminal for some rational number $\lambda > 1/14$, but the divisor $K_W + \lambda|-14K_W|$ has non-negative intersection with all the curves on W except $\bar{L}_1$. It follows from [12] that the log-flip $\zeta \colon W \dashrightarrow Z$ in $\bar{L}_1$ with respect to the log pair $(W, \lambda | -14K_W|)$ exists.

Let $\mathcal{P}$ be the proper transform of $\mathcal{M}$ on Z. Then the singularities of the log pair $(Z, \frac{1}{k}\mathcal{P})$ are terminal because those of the log pair $(W, \frac{1}{k}\mathcal{B})$ are, but the rational map ζ is a log flop with respect to the log pair $(W, \frac{1}{k}\mathcal{B})$ while $-K_Z$ is numerically effective because the base locus of $|-14K_W|$ does not contain curves other than $\bar{L}_1$ and the inequality $-K_W \cdot \bar{L}_1 < 0$ holds. We claim that $-K_Z$ is big, which is impossible by Lemma 2.1.

The rational functions y/x^2 and ty/x^7 are contained in $|2S|$ and $|7S|$, respectively, but the equivalences (29.2) imply that y/x^2 and ty/x^7 are contained in $|2\bar{S}|$ and $|7\bar{S}|$, respectively.

Let $\bar{Z}$ be the proper transform on W of the irreducible surface cut out on X by the equation $z = 0$. Then the equivalences

$$\bar{Z} \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*(-4K_X) - \frac{4}{9}\gamma^*(E) \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*(-4K_X) - \frac{4}{9}\bar{E} - \frac{1}{3}F$$

hold, so that $-6K_W \sim_{\mathbb{Q}} \bar{Z} + \bar{H} + \bar{E}$. Thus, the rational function zy/x^6 is contained in the linear system $|6\bar{S}|$. Thus, the linear system $|-42K_W|$ maps W dominantly onto some three-dimensional variety, which implies that the divisor $-K_Z$ is big. Thus, the proposition is proved.

Remark 29.5. The function wy^3/x^{15} belongs to $|15\bar{S}|$. Thus, $|-210K_W|$ induces a birational map $W \dashrightarrow X'$ such that X' is a hypersurface in $\mathbb{P}(1, 2, 6, 7, 15)$ of degree 30.

§ 30. The case $n = 44$: a hypersurface of degree 20 in $\mathbb{P}(1, 2, 5, 6, 7)$

We use the notation and assumptions of § 3. Let $n = 44$. Then X is a general hypersurface in $\mathbb{P}(1, 2, 5, 6, 7)$ of degree 20, and the equality $-K_X^3 = 1/21$ holds. The singularities of X consist of points P_1, P_2 and P_3 that are quotient singularities of type $\frac{1}{2}(1, 1, 1)$, a point P_4 that is a quotient singularity of type $\frac{1}{6}(1, 1, 5)$, and a point P_5 that is a quotient singularity of type $\frac{1}{7}(1, 2, 5)$.

There is a commutative diagram

$$
\begin{array}{ccc}
& Y & \\
\beta_5 \swarrow & & \searrow \beta_4 \qquad \eta \\
U_4 & & U_5 \longrightarrow \mathbb{P}(1,2,5), \\
\alpha_4 \searrow & & \swarrow \alpha_5 \qquad \psi \\
& X & \\
\end{array}
$$
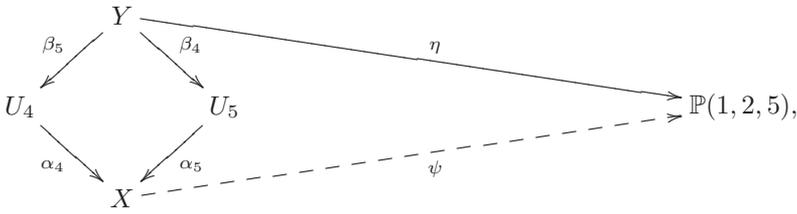
where ψ is a projection, α_4 is the weighted blow-up of P_4 with weights $(1,1,5)$, α_5 is the weighted blow-up of P_5 with weights $(1,2,5)$, β_4 is the weighted blow-up with weights $(1,1,5)$ of the proper transform of P_4 on U_5, β_5 is the weighted blow-up with weights $(1,2,5)$ of the proper transform of P_5 on U_4, and η is an elliptic fibration.
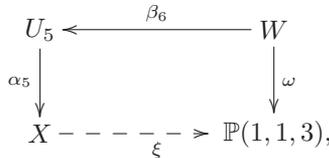
There is a commutative diagram

$$
\begin{array}{ccc}
U_5 & \xleftarrow{\;\;\beta_6\;\;} & W \\
\alpha_5 \downarrow & & \downarrow \omega \\
X & \dashrightarrow{\;\xi\;} & \mathbb{P}(1,1,3),
\end{array}
$$

where ξ is a projection, β_6 is the weighted blow-up with weights $(1,2,3)$ of the singular point of U_5 that is a singularity of type $\frac{1}{5}(1,2,3)$ contained in the α_5-exceptional divisor, and ω is an elliptic fibration.

Proposition 30.1. *Either there is a commutative diagram*

$$
\begin{array}{ccc}
X & \dashrightarrow{\;\rho\;} & V \\
\psi \downarrow & & \downarrow \nu \\
\mathbb{P}(1,2,5) & \dashrightarrow{\;\varphi\;} & \mathbb{P}^2,
\end{array}
\tag{30.1}
$$

or there is a commutative diagram

$$
\begin{array}{ccccc}
X & \dashrightarrow{\;\zeta\;} & X & \dashrightarrow{\;\rho\;} & V \\
\xi \downarrow & & & & \downarrow \nu \\
\mathbb{P}(1,1,3) & & \dashrightarrow{\;\sigma\;} & & \mathbb{P}^2,
\end{array}
\tag{30.2}
$$

where φ, ζ and σ are birational maps.

Proof. It follows from Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\bigl(X, \frac{1}{k}\mathcal{M}\bigr) \subseteq \{P_4, P_5\}$. Arguing as in the proof of Lemma 18.2, we see that $\mathbb{CS}\bigl(X, \frac{1}{k}\mathcal{M}\bigr)$ contains P_5. The existence of the commutative diagram (30.1) follows easily from Theorem 2.2 in the case when $\mathbb{CS}\bigl(X, \frac{1}{k}\mathcal{M}\bigr) = \{P_4, P_5\}$. Thus, we may assume that $\mathbb{CS}\bigl(X, \frac{1}{k}\mathcal{M}\bigr)$ consists of the point P_5.

Let $\mathcal{D}_5$ be the proper transform of $\mathcal{M}$ on U_5. Then $\mathcal{D}_5 \sim_{\mathbb{Q}} -kK_{U_5}$ by Theorem 2.2, which implies that $\mathbb{CS}\big(U_5, \frac{1}{k}\mathcal{D}_5\big)$ is non-empty by Lemma 2.1. Let G be the exceptional divisor of α_5 and $\bar{P}_6$, $\bar{P}_7$ the singular points of the surface G that are quotient singularities of types $\frac{1}{5}(1,2,3)$, $\frac{1}{2}(1,1,1)$ on U_5, respectively. In the case when $\mathbb{CS}\big(U_5, \frac{1}{k}\mathcal{D}_5\big)$ contains $\bar{P}_6$, the existence of the commutative diagram (30.2) follows from Theorem 2.2. Therefore, we may assume by Lemma 2.3 that $\mathbb{CS}\big(U_5, \frac{1}{k}\mathcal{D}_5\big)$ contains the point $\bar{P}_7$.

Remark 30.2. The linear system $|-5K_{U_5}|$ is a proper transform of $|-5K_X|$ and its base locus consists of the irreducible curve that is the fibre of the rational map $\psi \circ \alpha_5$ passing through $\bar{P}_7$.

Let $\pi\colon U \to U_5$ be the weighted blow-up of $\bar{P}_7$ with weights $(1,1,1)$, let F be the exceptional divisor of the blow-up π, let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U and let $\mathcal{H}$ be the proper transform of $|-5K_{U_5}|$ on U. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2. But

$$\mathcal{H} \sim_{\mathbb{Q}} \pi^*\big(-5K_{U_5}\big) - \frac{1}{2}F,$$

and the base locus of $\mathcal{H}$ consists of an irreducible curve Z such that $\alpha_5 \circ \pi(Z)$ is the unique curve in the base locus of $|-5K_X|$.

Let S be a general surface in $\mathcal{H}$. Then the equality $S \cdot Z = 1/3$ holds, which implies that the divisor $\pi^*(-10K_{U_5}) - F$ is nef. Let D_1 and D_2 be general surfaces in $\mathcal{D}$. Then

$$-\frac{2k^2}{3} = \big(\pi^*(-10K_{U_5})-F\big)\cdot\left(\pi^*(-kK_{U_5})-\frac{k}{2}F\right)^2 = \big(\pi^*(-10K_{U_5})-F\big)\cdot D_1\cdot D_2 \geqslant 0,$$

which is a contradiction. The proposition is proved.

§31. The case $n = 47$: a hypersurface of degree 21 in $\mathbb{P}(1,1,5,7,8)$

We use the notation and assumptions of §3. Let $n = 47$. Then X is a general hypersurface in $\mathbb{P}(1,1,5,7,8)$ of degree 21 whose singularities consist of a point P_1 that is a singularity of type $\frac{1}{5}(1,2,3)$ and a point P_2 that is a singularity of type $\frac{1}{8}(1,1,7)$.

Proposition 31.1. *The assertion of Theorem 1.10 holds for $n = 47$.*

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_1, P_2\}$.

The hypersurface X can be given by the equation

$$w^2z + \sum_{i=0}^{2} wz^i g_{13-5i}(x,y,t) + \sum_{i=0}^{3} z^i g_{21-5i}(x,y,t) = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 5$, $\mathrm{wt}(t) = 7$, $\mathrm{wt}(w) = 8$ and $g_i(x, y, t)$ is a quasi-homogeneous polynomial of degree i. There is a commutative diagram

$$
\begin{array}{ccccc}
U & \xleftarrow{\;\;\beta\;\;} & W & \xleftarrow{\;\;\gamma\;\;} & Y \\
{\scriptstyle \alpha}\downarrow & & & & \downarrow{\scriptstyle \eta} \\
X & \dashleftarrow\!\!\!\dashrightarrow & & \dashrightarrow & \mathbb{P}(1,1,5),\\
& & {\scriptstyle \psi} & &
\end{array}
$$

where α is the weighted blow-up of P_2 with weights $(1,1,7)$, β is the weighted blow-up with weights $(1,1,6)$ of the singular point of U that is a quotient singularity of type $\frac{1}{7}(1,1,6)$, γ is the weighted blow-up with weights $(1,1,5)$ of the singular point of W that is a quotient singularity of type $\frac{1}{6}(1,1,5)$, and η is an elliptic fibration.

Lemma 31.2. *The set* $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ *contains the point* P_2.

Proof. Suppose that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ does not contain P_2. Then $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_1\}$.

Let $\pi\colon Z \to X$ be the weighted blow-up of P_1 with weights $(1,2,3)$, E the exceptional divisor of π and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on Z. Then $E \cong \mathbb{P}(1,2,35)$ and $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ by Theorem 2.2.

Let $\bar{P}_3$ and $\bar{P}_4$ be the singular points of Z contained in E that are singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$, respectively. Arguing as in the proof of Proposition 22.1, we see that the singularities of the log pair $\big(Z, \frac{1}{k}\mathcal{B}\big)$ are terminal.

The base locus of the pencil $|-K_Z|$ consists of irreducible curves C and L such that the curve $\pi(C)$ is cut out on X by the equations $x = y = 0$ and L is contained in the divisor E and in the linear system $\big|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)\big|$. The inequalities $-K_Z \cdot C < 0$ and $-K_Z \cdot L > 0$ hold. It follows from [12] that there is an antiflip $\zeta\colon Z \dashrightarrow \bar{Z}$ in C. Then the divisor $-K_{\bar{Z}}$ is nef.

Let $\mathcal{P}$ be the proper transform of $\mathcal{M}$ on $\bar{Z}$. Then the singularities of the log pair $\big(\bar{Z}, \frac{1}{k}\mathcal{P}\big)$ are terminal because those of the log pair $\big(Z, \frac{1}{k}\mathcal{B}\big)$ are, and the antiflip ζ is a log flop with respect to the log pair $\big(Z, \frac{1}{k}\mathcal{B}\big)$. The rational functions y/x, zy/x^6, ty/x^8, yw/x^9 are contained in the linear system $|-aK_Z|$, where $a = 1, 6, 8, 9$, respectively. Therefore, the complete linear system $|-72K_Z|$ induces a birational map $\chi\colon Z \dashrightarrow \bar{X}$ such that $\bar{X}$ is a hypersurface of degree 24 in $\mathbb{P}(1,1,6,8,9)$. Hence, the divisor $-K_{\bar{Z}}$ is big, which is impossible by Lemma 2.1.

Let G be the exceptional divisor of α, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on U, $\bar{P}_1$ the proper transform of P_1 on U and $\bar{P}_5$ the singular point of U that is contained in G. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2.

Lemma 31.3. *The set* $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ *contains the point* $\bar{P}_5$.

Proof. Suppose that $\bar{P}_5 \notin \mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$. Then $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big) = \{\bar{P}_1\}$ by Lemmas 2.1 and 2.3.

Let $\pi\colon Z \to U$ be the weighted blow-up of $\bar{P}_1$ with weights $(1,2,3)$, E the exceptional divisor of π and $\bar{G}$, $\mathcal{B}$ the proper transforms of G, $\mathcal{M}$ on Z, respectively. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ by Theorem 2.2. Arguing as in the proof of Lemma 31.2, we see that the singularities of the log pair $\big(Z, \frac{1}{k}\mathcal{B}\big)$ are terminal.

Let S_x, S_y, S_z, S_t and S_w be the proper transforms on Z of the surfaces cut out on X by the equations $x = 0$, $y = 0$, $z = 0$, $t = 0$ and $w = 0$, respectively. Then

$$
\begin{aligned}
S_x &\sim_{\mathbb{Q}} (\alpha \circ \pi)^*(-K_X) - \frac{1}{5} E - \frac{1}{8} \bar{G}, \\
S_y &\sim_{\mathbb{Q}} (\alpha \circ \pi)^*(-K_X) - \frac{6}{5} E - \frac{1}{8} \bar{G}, \\
S_z &\sim_{\mathbb{Q}} (\alpha \circ \pi)^*(-5K_X) - \frac{1}{5} E - \frac{13}{8} \bar{G}, \\
S_t &\sim_{\mathbb{Q}} (\alpha \circ \pi)^*(-7K_X) - \frac{2}{5} E - \frac{7}{8} \bar{G}, \\
S_w &\sim_{\mathbb{Q}} (\alpha \circ \pi)^*(-8K_X) - \frac{3}{5} E.
\end{aligned}
\tag{31.1}
$$

The base locus of $|-K_Z|$ consists of irreducible curves C and L such that the curve $\alpha \circ \pi(C)$ is cut out on X by the equations $x = y = 0$, the curve L is contained in the divisor E, and L is the unique curve of $\big|\mathcal{O}_{\mathbb{P}(1,2,3)}(1)\big|$.

It follows from [12] that there is an antiflip $\zeta \colon Z \dashrightarrow \bar{Z}$ in C such that the divisor $-K_{\bar{Z}}$ is nef.

Let $\mathcal{P}$ be the proper transform of $\mathcal{M}$ on $\bar{Z}$. Then the singularities of the log pair $\big(\bar{Z}, \frac{1}{k}\mathcal{P}\big)$ are terminal. The equivalences (31.1) imply that the functions y/x, zy/x^6, ty/x^8 and wzy^2/x^{15} are contained in the complete linear system $|aS_x|$, where $a = 1, 6, 8$ and 15, respectively. Hence, the linear system $|-120K_Z|$ induces a birational map $\chi \colon Z \dashrightarrow \bar{X}$ such that $\bar{X}$ is a hypersurface of degree 30 in $\mathbb{P}(1,1,6,8,15)$. Hence, the divisor $-K_{\bar{Z}}$ is big, which is impossible by Lemma 2.1.

Remark 31.4. It follows from Lemma 2.4 that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big) = \{\bar{P}_5\}$.

Let $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on W, F the exceptional divisor of β, $\widetilde{P}_1$ the proper transform of P_1 on W and $\widetilde{P}_6$ the singular point of W contained in F. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2, which implies by Lemma 2.1 that $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{H}\big) \neq \varnothing$.

Lemma 31.5. *Suppose that* $\widetilde{P}_6 \in \mathbb{CS}\big(W, \frac{1}{k}\mathcal{H}\big)$. *Then there is a commutative diagram*

$$
\begin{array}{ccc}
X & \dashrightarrow^{\ \rho\ } & V \\
\psi \downarrow & & \downarrow \nu \\
\mathbb{P}(1,1,5) & \dashrightarrow_{\ \zeta\ } & \mathbb{P}^2,
\end{array}
\tag{31.2}
$$

where ζ is a birational map.

Proof. Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on Y. It follows from Theorem 2.2 that $\mathcal{B} \sim_{\mathbb{Q}} -kK_Y$, which implies that $\mathcal{B}$ lies in the fibres of η. This implies the existence of the diagram (31.2).

We may assume by Lemma 2.3 that $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{H}\big)$ consists of the point $\widetilde{P}_1$.

Let $\pi \colon Z \to W$ be the weighted blow-up of $\widetilde{P}_1$ with weights $(1,2,3)$, E the exceptional divisor of π, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on Z and $\widetilde{G}$, $\widetilde{F}$ the proper transforms on Z of the surfaces G, F, respectively. Arguing as in the proof of

Lemma 31.2, we see that the singularities of the log pair $\left(Z, \frac{1}{k}\mathcal{B}\right)$ are terminal. But $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ by Theorem 2.2.

Let S_x, S_y, S_z, S_t and S_w be the proper transforms on Z of the surfaces cut out on X by the equations $x = 0$, $y = 0$, $z = 0$, $t = 0$ and $w = 0$, respectively. Then

$$
\begin{aligned}
S_x &\sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-K_X) - \frac{1}{5}E - \frac{1}{8}\widetilde{G} - \frac{1}{4}\widetilde{F}, \\
S_y &\sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-K_X) - \frac{6}{5}E - \frac{1}{8}\widetilde{G} - \frac{1}{4}\widetilde{F}, \\
S_z &\sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-5K_X) - \frac{1}{5}E - \frac{13}{8}\widetilde{G} - \frac{9}{4}\widetilde{F}, \\
S_t &\sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-7K_X) - \frac{2}{5}E - \frac{7}{8}\widetilde{G} - \frac{3}{4}\widetilde{F}, \\
S_w &\sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-8K_X) - \frac{3}{5}E.
\end{aligned}
\tag{31.3}
$$

The equivalences (31.3) imply that the functions y/x, zy/x^6, tzy^2/x^{14} and wz^2y^3/x^{21} are contained in the linear systems $|S_x|$ $|6S_x|$, $|14S_x|$ and $|21S_x|$, respectively. Therefore, the complete linear system $|-42K_Z|$ induces a birational map $\chi\colon Z \dashrightarrow \bar{X}$ such that $\bar{X}$ is a hypersurface in $\mathbb{P}(1,1,6,14,21)$ of degree 42. The base locus of $|-42K_Z|$ consists of an irreducible curve C such that the curve $\alpha \circ \beta \circ \pi(C)$ is cut out on X by the equations $x = y = 0$. Therefore, it follows from [12] that there is an antiflip $\zeta\colon Z \dashrightarrow \bar{Z}$ in C, which implies that $-K_{\bar{Z}}$ is nef and big. The rational map ζ is a log flop with respect to the log pair $\left(Z, \frac{1}{k}\mathcal{B}\right)$. Therefore, we see that the singularities of the log pair $\left(\bar{Z}, \frac{1}{k}\mathcal{P}\right)$ are terminal, where $\mathcal{P}$ is the proper transform of $\mathcal{M}$ on Z. But this is impossible by Lemma 2.1. The proposition is proved.

Remark 31.6. We have constructed birational transformations of X into hypersurfaces in $\mathbb{P}(1,1,6,8,9)$, $\mathbb{P}(1,1,6,8,15)$ and $\mathbb{P}(1,1,6,14,21)$ of degrees 24, 30 and 42, respectively. The anticanonical models of the varieties U and W are hypersurfaces in $\mathbb{P}(1,1,5,7,13)$ and $\mathbb{P}(1,1,5,12,18)$ of degrees 26 and 36, respectively. Arguing as in the proof of Proposition 31.1, we see that, up to the action of the group $\mathrm{Bir}(X)$, there are no other non-trivial birational transformations of X into Fano threefolds with canonical singularities.

§32. The case $n = 48$: a hypersurface of degree 21 in $\mathbb{P}(1,2,3,7,9)$

We use the notation and assumptions of §3. Let $n = 48$. Then X is a general hypersurface in $\mathbb{P}(1,2,3,7,9)$ of degree 21 whose singularities consist of a point P_1 that is a singularity of type $\frac{1}{2}(1,1,1)$, points P_2 and P_3 that are singularities of type $\frac{1}{3}(1,1,2)$ and a point P_4 that is a singularity of type $\frac{1}{9}(1,2,7)$. There is a commutative diagram

$$
\begin{array}{ccccc}
U & \xleftarrow{\ \beta\ } & W & \xleftarrow{\ \gamma\ } & Y \\
{\scriptstyle \alpha}\downarrow & & & & \downarrow{\scriptstyle \eta} \\
X & \dashrightarrow{\ \ \ \ \psi\ \ \ \ } & & & \mathbb{P}(1,2,3),
\end{array}
$$

where ψ is the natural projection, α is the weighted blow-up of P_4 with weights $(1, 2, 7)$, β is the weighted blow-up with weights $(1, 2, 5)$ of the singular point of U that is a quotient singularity of type $\frac{1}{7}(1, 2, 5)$ contained in the exceptional divisor of α, γ is the weighted blow-up with weights $(1, 2, 3)$ of the singular point of W that is a singularity of type $\frac{1}{5}(1, 2, 3)$ contained in the exceptional divisor of β, and η is an elliptic fibration.

Proposition 32.1. *The assertion of Theorem* 1.10 *holds for* $n = 48$.

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_4\}$. Let E be the α-exceptional divisor, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on U and P_5, P_6 the singular points of U contained in the surface E that are singularities of types $\frac{1}{2}(1, 1, 1)$, $\frac{1}{7}(1, 2, 5)$, respectively. Then $E \cong \mathbb{P}(1, 2, 7)$, but $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2.

Lemma 32.2. *The set* $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ *does not contain the point* P_5.

Proof. Suppose that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ contains P_5. Let $\pi\colon Z \to U$ be the weighted blow-up of P_5 with weights $(1, 1, 1)$, G the exceptional divisor of the blow-up π and $\mathcal{B}$, $\mathcal{P}$ the proper transforms of the linear systems $\mathcal{M}$, $|-7K_X|$ on Z, respectively. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ by Theorem 2.2, but the base locus of $\mathcal{P}$ does not contain curves. Let H be a general divisor of $\mathcal{P}$. Then H is numerically effective. But

$$
H \cdot B_1 \cdot B_2 = \left((\alpha \circ \pi)^*(-kK_X) - \frac{k}{9}\,\pi^*(E) - \frac{k}{2}\,G \right)^2
$$
$$
\times \left((\alpha \circ \pi)^*(-7K_X) - \frac{7}{9}\,\pi^*(E) - \frac{1}{2}\,G \right) = -\frac{1}{6}\,k^2,
$$

where B_1 and B_2 are general surfaces in $\mathcal{B}$. The resulting contradiction completes the proof of the lemma.

It follows from Lemmas 2.1 and 2.3 that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big) = \{P_6\}$. Let F be the exceptional divisor of the blow-up β, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on W and P_7, P_8 the singular points of W contained in F that are singularities of types $\frac{1}{2}(1, 1, 1)$, $\frac{1}{5}(1, 2, 3)$, respectively. Then $F \cong \mathbb{P}(1, 2, 5)$, but $\mathcal{H} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2.

Lemma 32.3. *Suppose that* $P_8 \in \mathbb{CS}\big(W, \frac{1}{k}\mathcal{H}\big)$. *Then there is a commutative diagram*

$$
\begin{array}{ccc}
X & \dashrightarrow^{\;\rho\;} & V \\
\psi\,\downarrow & & \downarrow\,\nu \\
\mathbb{P}(1, 2, 3) & \dashrightarrow_{\;\zeta\;} & \mathbb{P}^2,
\end{array}
\tag{32.1}
$$

where ζ *is a birational map.*

Proof. Let S be the proper transform of a general surface in $\mathcal{M}$ on Y and Γ a generic fibre of η. Then $S \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2, which implies that $S \cdot \Gamma = 0$. Therefore, S lies in the fibres of η, which implies the existence of the commutative diagram (32.1).

By Lemmas 2.1 and 2.3, we may assume that $P_7 \in \mathbb{CS}\left(W, \frac{1}{k}\mathcal{H}\right)$.

Let $\pi\colon Z \to W$ be the weighted blow-up of P_7 with weights $(1,1,1)$, G the exceptional divisor of the blow-up π and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on Z. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ by Theorem 2.2.

The hypersurface X can be given by the equation

$$w^2 z + w f_{12}(x, y, z, t) + f_{21}(x, y, z, t) = 0 \subset \mathbb{P}(1, 2, 3, 7, 9) \cong \mathrm{Proj}\left(\mathbb{C}[x, y, z, t, w]\right),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 2$, $\mathrm{wt}(z) = 3$, $\mathrm{wt}(t) = 7$, $\mathrm{wt}(w) = 9$ and $f_i(x, y, z, t)$ is a quasi-homogeneous polynomial of degree i. Let $\mathcal{P}$ be the proper transform on Z of the pencil of surfaces cut out on X by the equations $\lambda x^3 + \mu z = 0$, where $(\lambda, \mu) \in \mathbb{P}^1$. Then the base locus of $\mathcal{P}$ consists of irreducible curves C, L_1 and L_2 such that $\alpha \circ \beta \circ \pi(C)$ is the curve cut out on X by the equations $x = z = 0$, $\beta \circ \pi(L_1)$ is contained in the exceptional divisor E, $\beta \circ \pi(L_1)$ is the unique curve in the base locus of $\left|\mathcal{O}_{\mathbb{P}(1,2,7)}(1)\right|$, $\pi(L_2)$ is contained in F and $\pi(L_2)$ is the unique curve of $\left|\mathcal{O}_{\mathbb{P}(1,2,5)}(1)\right|$.

Let D be a general surface in $\mathcal{P}$, $\bar{E}$ and $\bar{F}$ the proper transforms of E and F on Z, respectively and S the proper transform on Z of the surface cut out on X by the equation $x = 0$. Then

$$S \cdot D = C + L_1 + L_2, \qquad \bar{E} \cdot D = 3L_1, \qquad \bar{F} \cdot D = 3L_2,$$

the surface D is normal, and

$$\begin{aligned}
\bar{F} &\sim_{\mathbb{Q}} \pi^*(F) - \frac{1}{2}G, \\
\bar{E} &\sim_{\mathbb{Q}} (\beta \circ \pi)^*(E) - \frac{5}{7}\pi^*(F) - \frac{1}{2}G, \\
D &\sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-3K_X) - \frac{3}{9}(\beta \circ \pi)^*(E) - \frac{3}{7}\pi^*(F) - \frac{3}{2}G, \\
S &\sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-K_X) - \frac{1}{9}(\beta \circ \pi)^*(E) - \frac{1}{7}\pi^*(F) - \frac{1}{2}G.
\end{aligned} \tag{32.2}$$

The curves C, L_1 and L_2 are divisors on $\bar{D}$. It follows from the equivalences (32.2) that

$$C \cdot C = L_1 \cdot L_1 = -\frac{1}{2}, \qquad L_2 \cdot L_2 = -\frac{2}{5}, \qquad C \cdot L_1 = C \cdot L_2 = L_1 \cdot L_2 = 0,$$

which implies that the intersection form of the curves C, L_1 and L_2 on D is negative definite. On the other hand, we have $B|_D \sim_{\mathbb{Q}} kC + kL_1 + kL_2$, where B is a general surface in $\mathcal{B}$. But this is impossible by Lemmas 2.9 and 2.7. The proposition is proved.

§ 33. The case $n = 49$: a hypersurface of degree 21 in $\mathbb{P}(1, 3, 5, 6, 7)$

We use the notation and assumptions of § 3. Let $n = 49$. Then X is a general hypersurface in $\mathbb{P}(1, 3, 5, 6, 7)$ of degree 21 whose singularities consist of points P_1, P_2 and P_3 that are singularities of type $\frac{1}{3}(1, 1, 2)$, a point P_4 that is a singularity

of type $\frac{1}{5}(1, 2, 3)$ and a point P_5 that is a singularity of type $\frac{1}{6}(1, 1, 5)$. There is a commutative diagram

$$
\begin{array}{ccc}
 & Y & \\
\alpha \swarrow & & \searrow \eta \\
X \dashrightarrow[\psi] & & \mathbb{P}(1, 3, 5),
\end{array}
$$

where ψ is a projection, α is the blow-up of P_5 with weights $(1, 1, 5)$, and η is an elliptic fibration. There is a commutative diagram

$$
\begin{array}{ccc}
 & U & \\
\beta \swarrow & & \searrow \omega \\
X \dashrightarrow[\xi] & & \mathbb{P}(1, 3, 6),
\end{array}
$$

where ξ is a projection, β is the blow-up of P_4 with weights $(1, 2, 3)$ and ω is an elliptic fibration.

Proposition 33.1. *Either there is a commutative diagram*

$$
\begin{array}{ccc}
X & \overset{\rho}{\dashrightarrow} & V \\
\psi \downarrow & & \downarrow \nu \\
\mathbb{P}(1, 3, 5) & \underset{\varphi}{\dashrightarrow} & \mathbb{P}^2
\end{array}
\tag{33.1}
$$

or there is a commutative diagram

$$
\begin{array}{ccc}
X & \overset{\rho}{\dashrightarrow} & V \\
\xi \downarrow & & \downarrow \nu \\
\mathbb{P}(1, 3, 6) & \underset{\sigma}{\dashrightarrow} & \mathbb{P}^2,
\end{array}
\tag{33.2}
$$

where φ and σ are birational maps.

Proof. It follows from Theorem 3.3, Proposition 3.5 and Lemma 3.11 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_4, P_5\}$.

Suppose that the set $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$ contains the point P_4. Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2. Intersecting a general surface in $\mathcal{D}$ with a generic fibre of ω, we see that $\mathcal{D}$ lies in the fibres of ω. This implies the existence of the commutative diagram (33.2). Similarly, we see that the commutative diagram (33.1) exists in the case when $P_5 \in \mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big)$.

§34. The case $n = 51$: a hypersurface of degree 22 in $\mathbb{P}(1, 1, 4, 6, 11)$

We use the notation and assumptions of §3. Let $n = 51$. Then X is a hypersurface of degree 22 in $\mathbb{P}(1, 1, 4, 6, 11)$ and the equality $-K_X^3 = 1/12$ holds. The

singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, a point P_2 that is a quotient singularity of type $\frac{1}{3}(1,1,4)$ and a point P_3 that is a quotient singularity of type $\frac{1}{6}(1,1,5)$.

There is a commutative diagram

$$
\begin{array}{ccc}
U & \xleftarrow{\quad\beta\quad} & Y \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\eta} \\
X & \dashrightarrow[\psi] & \mathbb{P}(1,1,4),
\end{array}
$$

where ψ is the natural projection, α is the weighted blow-up of P_3 with weights $(1,1,5)$, β is the weighted blow-up with weights $(1,1,4)$ of the singular point of U that is contained in the exceptional divisor of α, and η is an elliptic fibration.

There is a commutative diagram

$$
\begin{array}{ccc}
& W & \\
{\scriptstyle\gamma}\swarrow & & \searrow{\scriptstyle\omega} \\
X & \dashrightarrow[\xi] & \mathbb{P}(1,1,6),
\end{array}
$$

where ξ is a projection, γ is the blow-up of P_2 with weights $(1,1,3)$ and ω is an elliptic fibration.

Proposition 34.1. *Either there is a commutative diagram*

$$
\begin{array}{ccc}
X & \dashrightarrow{\rho} & V \\
{\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\nu} \\
\mathbb{P}(1,1,4) & \dashrightarrow[\varphi] & \mathbb{P}^2
\end{array}
\tag{34.1}
$$

or there is a commutative diagram

$$
\begin{array}{ccc}
X & \dashrightarrow{\rho} & V \\
{\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle\nu} \\
\mathbb{P}(1,1,6) & \dashrightarrow[\sigma] & \mathbb{P}^2,
\end{array}
\tag{34.2}
$$

where φ and σ are birational maps.

Proof. The fact that X is birationally superrigid implies that the singularities of the log pair $\left(X, \frac{1}{k}\mathcal{M}\right)$ are canonical.

Suppose that P_2 is contained in $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$. Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on W. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2, and this implies the existence of the commutative diagram (34.2).

We may assume that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) = \{P_3\}$ by Theorem 3.3, Proposition 3.5 and Lemma 3.11.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2, but the anticanonical divisor $-K_U$ is nef and big. It follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains the singular point of U that is contained in the exceptional divisor of α.

Let $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on Y. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.2. Intersecting a general surface in $\mathcal{H}$ with a generic fibre of η, we see that $\mathcal{H}$ lies in the fibres of η. This implies the existence of the commutative diagram (34.1).

§35. The case $n = 56$: a hypersurface of degree 24 in $\mathbb{P}(1, 2, 3, 8, 11)$

We use the notation and assumptions of §3. Let $n = 56$. Then X is a general hypersurface in $\mathbb{P}(1, 2, 3, 8, 11)$ of degree 24 whose singularities consist of points P_1, P_2 and P_3 that are singularities of type $\frac{1}{2}(1, 1, 1)$ and a point P_4 that is a singularity of type $\frac{1}{11}(1, 3, 8)$. The equality $-K_X^3 = 1/22$ holds.

There is a commutative diagram

$$
\begin{array}{ccccc}
U & \xleftarrow{\ \beta\ } & W & \xleftarrow{\ \gamma\ } & Y \\
{\scriptstyle \alpha}\downarrow & & & & \downarrow{\scriptstyle \eta} \\
X & \dashrightarrow\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\!\underset{\psi}{} & & & \mathbb{P}(1, 2, 3),
\end{array}
$$

where ψ is the natural projection, α is the weighted blow-up of P_4 with weights $(1, 3, 8)$, β is the weighted blow-up with weights $(1, 3, 5)$ of the point of U that is a quotient singularity of type $\frac{1}{8}(1, 3, 5)$ contained in the exceptional divisor of α, γ is the weighted blow-up with weights $(1, 2, 3)$ of the point of W that is a quotient singularity of type $\frac{1}{5}(1, 2, 3)$ contained in the exceptional divisor of β, and η is an elliptic fibration.

Proposition 35.1. *The assertion of Theorem 1.10 holds for $n = 56$.*

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_4\}$.

Let E be the exceptional divisor of α, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on U and P_5, P_6 the singular points of U contained in E that are singularities of types $\frac{1}{3}(1, 1, 2)$, $\frac{1}{8}(1, 3, 5)$, respectively. Then $E \cong \mathbb{P}(1, 3, 8)$, and $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2. The divisor $-K_U$ is nef and big. It follows from Lemmas 2.1 and 2.3 that one of the following assertions holds:

  (i) the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains the point P_5;
  (ii) the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ consists of the point P_6.

Lemma 35.2. *The set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ does not contain the point P_5.*

Proof. We suppose that $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains P_5 and seek a contradiction.

Let $\pi\colon Z \to U$ be the weighted blow-up of P_5 with weights $(1, 1, 2)$, G the exceptional divisor of the blow-up π and $\mathcal{B}$, $\mathcal{P}$ the proper transforms of $\mathcal{M}$, $|-8K_X|$

on Z, respectively. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$, but the base locus of $\mathcal{P}$ does not contain curves. In particular, a sufficiently general surface H in $\mathcal{P}$ is nef, but

$$H \cdot B_1 \cdot B_2 = \left((\alpha \circ \pi)^*(-kK_X) - \frac{k}{11}\pi^*(E) - \frac{k}{3}G\right)^2$$
$$\times \left((\alpha \circ \pi)^*(-8K_X) - \frac{8}{11}\pi^*(E) - \frac{2}{3}G\right) = 0,$$

where B_1 and B_2 are general surfaces in $\mathcal{B}$. But this is impossible by Corollary 2.6 because $H^3 > 0$.

Hence, the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ consists of the point P_6. Let F be the exceptional divisor of β, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on W and P_7, P_8 the singular points of W contained in F that are singularities of types $\frac{1}{3}(1,1,2)$, $\frac{1}{5}(1,2,3)$, respectively. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2. It follows from Lemmas 2.1 and 2.3 that $\mathbb{CS}(W, \frac{1}{k}\mathcal{H}) = \{P_7\}$ or $P_8 \in \mathbb{CS}(W, \frac{1}{k}\mathcal{H})$.

Arguing as in the proof of Lemma 32.3, we obtain the existence of a commutative diagram

$$
\begin{array}{ccc}
X & \dashrightarrow^{\rho} & V \\
\psi \downarrow & & \downarrow \nu \\
\mathbb{P}(1,2,3) & \dashrightarrow_{\zeta} & \mathbb{P}^2
\end{array}
$$

in the case when $P_8 \in \mathbb{CS}(W, \frac{1}{k}\mathcal{H})$, where ζ is a birational map. Thus, we may assume that $\mathbb{CS}(W, \frac{1}{k}\mathcal{H}) = \{P_7\}$.

Let $\pi \colon Z \to W$ be the weighted blow-up of P_7 with weights $(1,1,2)$, G the exceptional divisor of π and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on Z. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ by Theorem 2.2.

Lemma 35.3. *The singularities of the log pair $\left(Z, \frac{1}{k}\mathcal{B}\right)$ are terminal.*

Proof. Suppose that the set $\mathbb{CS}(Z, \frac{1}{k}\mathcal{B})$ is non-empty. Let P_9 be the singular point of the surface G. Then P_9 is a singularity of type $\frac{1}{2}(1,1,1)$ of Z, the set $\mathbb{CS}(Z, \frac{1}{k}\mathcal{B})$ contains P_9 by Lemma 2.3, and $G \cong \mathbb{P}(1,1,2)$.

Let $\bar{\pi} \colon \bar{Z} \to Z$ be the weighted blow-up of P_9 with weights $(1,1,1)$ and $\bar{G}$ the exceptional divisor of $\bar{\pi}$. Take any divisor D on $\bar{Z}$ such that

$$D \sim_{\mathbb{Q}} -2K_{\bar{Z}} - (\beta \circ \pi \circ \bar{\pi})^*(16K_U) - (\pi \circ \bar{\pi})^*(18K_W).$$

Analyzing the base locus of the pencil $\left|-2K_{\bar{Z}}\right|$, we see that D is nef, but $D^3 > 0$. Thus, the divisor D is nef and big. But $D \cdot \bar{H}_1 \cdot \bar{H}_2 = 0$, where $\bar{H}_1$ and $\bar{H}_2$ are the proper transforms on $\bar{Z}$ of general surfaces in $\mathcal{M}$. But this is impossible by Corollary 2.6.

The hypersurface X can be given by the equation

$$w^2 y + w f_{13}(x,y,z,t) + f_{24}(x,y,z,t) = 0 \subset \mathbb{P}(1,2,3,8,11) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 2$, $\mathrm{wt}(z) = 3$, $\mathrm{wt}(t) = 8$, $\mathrm{wt}(w) = 11$ and $f_i(x,y,z,t)$ is a sufficiently general quasi-homogeneous polynomial of degree i. Let $\bar{E}$ and $\bar{F}$ be

the proper transforms on Z of E and F, respectively, and let S_1, S_2, S_3 and S_8 be the proper transforms on Z of the surfaces cut out on X by the equations $x = 0$, $y = 0$, $z = 0$ and $t = 0$, respectively. Then

$$
\begin{aligned}
\overline{F} &\sim_{\mathbb{Q}} \pi^*(F) - \frac{1}{3}\,G, \\
\overline{E} &\sim_{\mathbb{Q}} (\beta \circ \pi)^*(E) - \frac{5}{8}\,\pi^*(F) - \frac{2}{3}\,G, \\
S_1 &\sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-K_X) - \frac{1}{11}\,(\beta \circ \pi)^*(E) - \frac{1}{8}\,\pi^*(F) - \frac{1}{3}\,G, \\
S_2 &\sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-2K_X) - \frac{13}{11}\,(\beta \circ \pi)^*(E) - \frac{5}{8}\,\pi^*(F) - \frac{2}{3}\,G, \\
S_3 &\sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-3K_X) - \frac{3}{11}\,(\beta \circ \pi)^*(E) - \frac{3}{8}\,\pi^*(F) - \frac{1}{3}\,G, \\
S_8 &\sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-8K_X) - \frac{8}{11}\,(\beta \circ \pi)^*(E).
\end{aligned}
\tag{35.1}
$$

The base locus of $|-2K_Z|$ consists of irreducible curves C, L_1, L_2 and L_3 such that the curve $\alpha \circ \beta \circ \pi(C)$ is cut out on X by the equations $x = y = 0$, the curve $\beta \circ \pi(L_1)$ is contained in E, the curve $\beta \circ \pi(L_1)$ is contained in $|\mathcal{O}_{\mathbb{P}(1,3,8)}(1)|$, the curve $\pi(L_2)$ is contained in F, the curve $\pi(L_2)$ is contained in $|\mathcal{O}_{\mathbb{P}(1,3,5)}(1)|$ and L_3 is contained in G and in $|\mathcal{O}_{\mathbb{P}(1,1,2)}(1)|$. Then

$$
S_1 \cdot D = C + 2L_1 + 2L_2 + L_3, \qquad \overline{E} \cdot D = 2L_1, \qquad \overline{F} \cdot D = 2L_2, \qquad G \cdot D = 2L_3,
$$

where D is a general surface in $|-2K_Z|$. It follows from the equivalences (35.1) that

$$
-K_Z \cdot C = \frac{1}{10}, \qquad -K_Z \cdot L_1 = -\frac{1}{3}, \qquad -K_Z \cdot L_2 = -\frac{1}{10}, \qquad -K_Z \cdot L_3 = \frac{1}{2}.
$$

The singularities of the log pair $(Z, \lambda|-2K_Z|)$ are log-terminal for some rational number $\lambda > 1/2$ because X is birationally rigid. But the divisor $K_Z + \lambda|-2K_Z|$ has non-negative intersection with all curves on Z except L_1 and L_2. It follows from [12] that there is a birational map $\zeta \colon Z \dashrightarrow \overline{Z}$ which is an isomorphism in codimension 1, and the divisor $-K_{\overline{Z}}$ is numerically effective.

Let $\mathcal{P}$ be the proper transform of $\mathcal{M}$ on $\overline{Z}$. Then the singularities of the log pair $(\overline{Z}, \frac{1}{k}\mathcal{P})$ are terminal because those of the log pair $(Z, \frac{1}{k}\mathcal{B})$ are, and ζ is a log flop with respect to the log pair $(Z, \frac{1}{k}\mathcal{B})$. It follows from the equivalences (35.1) that $-K_Z \sim_{\mathbb{Q}} S_1$ and

$$
\begin{aligned}
S_1 &\sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-K_X) - \frac{1}{11}\,\overline{E} - \frac{2}{11}\,\overline{F} - \frac{5}{11}\,G, \\
S_2 &\sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-2K_X) - \frac{13}{11}\,\overline{E} - \frac{15}{11}\,\overline{F} - \frac{21}{11}\,G, \\
S_3 &\sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-3K_X) - \frac{3}{11}\,\overline{E} - \frac{6}{11}\,\overline{F} - \frac{4}{11}\,G, \\
S_8 &\sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-8K_X) - \frac{8}{11}\,\overline{E} - \frac{5}{11}\,\overline{F} - \frac{7}{11}\,G.
\end{aligned}
\tag{35.2}
$$

The equivalences (35.2) imply that the rational functions y/x^2, zy/x^5 and ty^3/x^{14} are contained in the linear systems $|2S_1|$, $|5S_1|$ and $|14S_1|$, respectively. In particular, the complete linear system $|-70K_Z|$ induces a dominant rational map $Z \dashrightarrow \mathbb{P}(1,2,5,14)$, which implies that the divisor $-K_{\bar{Z}}$ is nef and big. But this is impossible by Lemma 2.1. The proposition is proved.

§ 36. The case $n = 58$: a hypersurface of degree 24 in $\mathbb{P}(1,3,4,7,10)$

We use the notation and assumptions of § 3. Let $n = 58$. Then X is a hypersurface of degree 24 in $\mathbb{P}(1,3,4,7,10)$ and the equality $-K_X^3 = 1/35$ holds. The singularities of X consist of points P_1, P_2, P_3 of types $\frac{1}{2}(1,1,1)$, $\frac{1}{7}(1,3,4)$, $\frac{1}{10}(1,3,7)$, respectively.

Proposition 36.1. *The assertion of Theorem 1.10 holds for $n = 58$.*

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_2, P_3\}$.

There is a commutative diagram
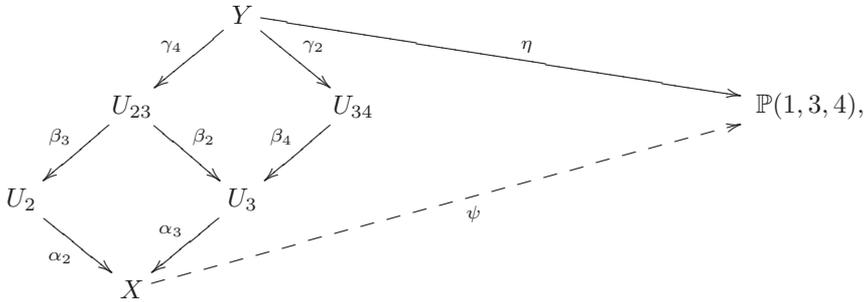
where ψ is the natural projection, α_2 is the weighted blow-up of P_2 with weights $(1,3,4)$, α_3 is the weighted blow-up of P_3 with weights $(1,3,7)$, β_3 is the weighted blow-up with weights $(1,3,7)$ of the proper transform of P_3 on U_2, β_2 is the weighted blow-up with weights $(1,3,4)$ of the proper transform of P_2 on U_2, β_4 is the weighted blow-up with weights $(1,3,4)$ of the singular point of U_3 that is a quotient singularity of type $\frac{1}{7}(1,3,4)$ contained in the exceptional divisor of α_3, γ_2 is the weighted blow-up with weights $(1,3,4)$ of the proper transform of P_2 on U_{34}, γ_4 is the weighted blow-up with weights $(1,3,4)$ of the singular point of U_{23} that is a quotient singularity of type $\frac{1}{7}(1,3,4)$ contained in the exceptional divisor of β_3, and η is an elliptic fibration.

Lemma 36.2. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains the point P_3.*

Proof. Suppose that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ does not contain P_3. Let $\mathcal{D}_2$ be the proper transform of $\mathcal{M}$ on U_2 and O, Q the singular points of U_2 contained in the exceptional divisor of α_2 that are singularities of types $\frac{1}{3}(1,1,2)$, $\frac{1}{4}(1,1,3)$, respectively. Then $\mathcal{D}_2 \sim_{\mathbb{Q}} -kK_{U_2}$ by Theorem 2.2.

Suppose that $O \in \mathbb{CS}(U_2, \frac{1}{k}\mathcal{D}_2)$. Let $\pi \colon W \to U_2$ be the weighted blow-up of O with weights $(1,1,2)$, $\mathcal{B}$ and $\mathcal{P}$ the proper transforms of $\mathcal{M}$ and $|-4K_X|$ on W, respectively, and S a general surface in $\mathcal{P}$. Then the base locus of $\mathcal{P}$ consists of an irreducible curve C such that $\alpha_2(C)$ is the base curve of $|-4K_X|$. Then

$\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2. The surface S is normal, the inequality $C^2 < 0$ holds on S and $\mathcal{B}|_S \sim_{\mathbb{Q}} kC$. Thus, it follows from Lemma 2.9 that

$$\mathrm{Supp}(S) \cap \mathrm{Supp}(B) = \mathrm{Supp}(C),$$

where B is a general surface in $\mathcal{B}$. But this is impossible by Lemma 2.7 because $\mathcal{B}$ is not composed of a pencil.

Thus, we see that $Q \in \mathbb{CS}\left(U_2, \frac{1}{k}\mathcal{D}_2\right)$ by Lemma 2.3 because $\mathbb{CS}\left(U_2, \frac{1}{k}\mathcal{D}_2\right) \neq \varnothing$ by Lemma 2.1.

Let $\zeta \colon U \to U_2$ be the weighted blow-up of Q with weights $(1,1,3)$, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on U, H a general surface in $\mathcal{H}$ and D a general surface in $|-3K_U|$. Then D is normal and the base locus of $|-3K_U|$ consists of an irreducible curve Z such that $\alpha_2(Z)$ is the unique base curve of $|-3K_X|$. The equivalence $\mathcal{H}|_D \sim_{\mathbb{Q}} kZ$ holds by Theorem 2.2 and the inequality $Z^2 < 0$ holds on D. But this is impossible by Lemmas 2.7 and 2.9.

Let $\mathcal{D}_2$ and $\mathcal{D}_{23}$ be the proper transforms of $\mathcal{M}$ on U_2 and U_{23}, respectively. Arguing as in the proof of Lemma 36.2, we obtain the following corollaries.

Corollary 36.3. *Suppose that $\mathcal{D}_2 \sim_{\mathbb{Q}} -kK_{U_2}$. Then the set $\mathbb{CS}\left(U_2, \frac{1}{k}\mathcal{D}_2\right)$ does not contain subvarieties of U_2 that are contained in the exceptional divisor of α_2.*

Corollary 36.4. *Suppose that $\mathcal{D}_{23} \sim_{\mathbb{Q}} -kK_{U_{23}}$. Then the set $\mathbb{CS}\left(U_{23}, \frac{1}{k}\mathcal{D}_{23}\right)$ does not contain subvarieties of U_{23} that are contained in the exceptional divisor of β_2.*

Let $\mathcal{D}_3$ and $\mathcal{D}_{23}$ be the proper transforms of $\mathcal{M}$ on U_3 and U_{34}, respectively. Then $\mathcal{D}_3 \sim_{\mathbb{Q}} -kK_{U_3}$ by Theorem 2.2.

Lemma 36.5. *The set $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$ contains the point P_2.*

Proof. Suppose that $P_2 \notin \mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$. Then $\mathcal{D}_3 \sim_{\mathbb{Q}} -kK_{U_2}$ by Theorem 2.2, and this implies that the set $\mathbb{CS}\left(U_3, \frac{1}{k}\mathcal{D}_3\right)$ is non-empty by Lemma 2.1.

Let P_4 and P_5 be the singular points of U_3 contained in the exceptional divisor of α_3 that are quotient singularities of types $\frac{1}{7}(1,3,4)$ and $\frac{1}{3}(1,1,2)$, respectively. Then the set $\mathbb{CS}\left(U_3, \frac{1}{k}\mathcal{D}_3\right)$ contains either P_4 or P_5 by Lemma 2.3. It follows from Lemma 36.2 that $\mathbb{CS}\left(U_3, \frac{1}{k}\mathcal{D}_3\right)$ does not contain P_5.

Thus, $\mathbb{CS}\left(U_3, \frac{1}{k}\mathcal{D}_3\right)$ contains P_4. Then $\mathcal{D}_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$ by Theorem 2.2, and the set $\mathbb{CS}\left(U_{34}, \frac{1}{k}\mathcal{D}_{34}\right)$ is non-empty by Lemma 2.1 because $-K_{U_{34}}$ is nef and big.

Let P_6 and P_7 be the singular points of U_{34} contained in the exceptional divisor of β_4 that are quotient singularities of types $\frac{1}{3}(1,1,2)$ and $\frac{1}{4}(1,1,3)$, respectively. Then the set $\mathbb{CS}\left(U_{34}, \frac{1}{k}\mathcal{D}_{34}\right)$ contains either P_6 or P_7 by Lemma 2.3.

Suppose that $\mathbb{CS}\left(U_{34}, \frac{1}{k}\mathcal{D}_{34}\right)$ contains P_7. Let $\zeta \colon U \to U_{34}$ be the weighted blow-up of P_7 with weights $(1,1,3)$, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on U, H a general surface in $\mathcal{H}$ and D a general surface in $|-3K_U|$. Then D is normal and the base locus of $|-3K_U|$ consists of an irreducible curve Z such that $\alpha_3 \circ \beta_4(Z)$ is the unique base curve of $|-3K_X|$. Moreover, the equivalence $\mathcal{H}|_D \sim_{\mathbb{Q}} kZ$ holds and $Z^2 < 0$ on D. But this is is impossible by Lemmas 2.7 and 2.9.

Therefore, the set $\mathbb{CS}\left(U_{34}, \frac{1}{k}\mathcal{D}_{34}\right)$ contains the point P_6.

The hypersurface X can be given by the quasi-homogeneous equation

$$w^2z + wf_{14}(x,y,z,t) + f_{24}(x,y,z,t) = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 3$, $\mathrm{wt}(z) = 4$, $\mathrm{wt}(t) = 7$, $\mathrm{wt}(w) = 10$ and $f_i(x, y, z, w)$ is a quasi-homogeneous polynomial of degree i. Let $\mathcal{P}$ be a pencil consisting of the surfaces cut out on X by the equations $\lambda x^4 + \mu z = 0$, where $(\lambda, \mu) \in \mathbb{P}^1$. Then the base locus of $\mathcal{P}$ consists of the irreducible curve cut out on X by the equations $x = z = 0$.

Let $\pi \colon W \to U_{34}$ be the weighted blow-up of P_6 with weights $(1, 1, 2)$, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W, $\mathcal{H}$ the proper transform of $\mathcal{P}$ on W, H a general surface in $\mathcal{H}$ and E, F, G the exceptional divisors of α_3, β_4, π, respectively. Then

$$H \sim_{\mathbb{Q}} (\alpha_3 \circ \beta_4 \circ \pi)^*(-4K_X) - \frac{1}{5}(\beta_4 \circ \pi)^*(E) - \frac{4}{7}(\pi)^*(F) - \frac{1}{3}G,$$

the surface H is normal and the base locus of $\mathcal{H}$ consists of curves C and L such that $\alpha_3 \circ \beta_4 \circ \pi(C)$ is the unique base curve of $\mathcal{P}$ and $\beta_4 \circ \pi(L)$ is the unique curve on the surface $E \cong \mathbb{P}(1, 3, 7)$ that is contained in $\left|\mathcal{O}_{\mathbb{P}(1,3,7)}(1)\right|$.

Let S be a surface in $|-K_W|$ and $\bar{E}$ the proper transform of E on W. Then $S \cdot H = C + L$ and $\bar{E} \cdot H = 4L$. But

$$\bar{E} \sim_{\mathbb{Q}} (\beta_4 \circ \pi)^*(E) - \frac{4}{7}(\pi)^*(F) - \frac{1}{3}G,$$

which implies that the intersection form of L and C on H is negative definite. On the other hand, the equivalence $\mathcal{B}|_H \sim_{\mathbb{Q}} kC + kL$ holds, and this is impossible by Lemmas 2.9 and 2.7.

Hence, we see that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) = \{P_2, P_3\}$, and $\mathcal{D}_{23} \sim_{\mathbb{Q}} -kK_{U_{23}}$ by Theorem 2.2. It follows easily from Lemmas 2.1 and 2.3, the proof of Lemma 36.5 and Corollary 36.4 that the set $\mathbb{CS}\left(U_{23}, \frac{1}{k}\mathcal{D}_{23}\right)$ contains the singular point of U_{23} that is a quotient singularity of type $\frac{1}{7}(1, 3, 4)$ contained in the exceptional divisor of β_3. This implies that the proper transform of $\mathcal{M}$ on Y lies in the fibres of η by Theorem 2.2. The proposition is proved.

§37. The case $n = 64$: a hypersurface of degree 26 in $\mathbb{P}(1, 2, 5, 6, 13)$

We use the notation and assumptions of §3. Let $n = 64$. Then X is a general hypersurface in $\mathbb{P}(1, 2, 5, 6, 13)$ of degree 26. The singularities of X consist of points P_1, P_2, P_3, P_4 that are singularities of type $\frac{1}{2}(1, 1, 1)$, a point P_5 that is a singularity of type $\frac{1}{5}(1, 2, 3)$ and a point P_6 that is a singularity of type $\frac{1}{6}(1, 1, 5)$. There is a commutative diagram

$$
\begin{array}{ccc}
 & Y & \\
\alpha \swarrow & & \searrow \eta \\
X \dashrightarrow_{\psi} & & \mathbb{P}(1, 2, 5),
\end{array}
$$
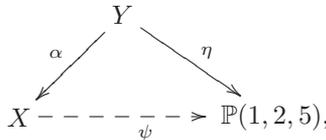
where ψ is a projection, α is the blow-up of P_6 with weights $(1, 1, 5)$ and η is an elliptic fibration. There is a commutative diagram

$$
\begin{array}{ccc}
 & U & \\
\beta \swarrow & & \searrow \omega \\
X \dashrightarrow_{\xi} & & \mathbb{P}(1, 1, 3),
\end{array}
$$
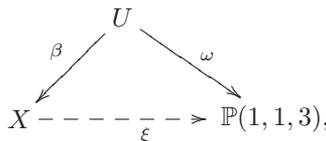
where ξ is a projection, β is the blow-up of the point P_5 with weights $(1, 2, 3)$ and ω is an elliptic fibration.

Proposition 37.1. *Either there is a commutative diagram*

$$
\begin{array}{ccc}
X & \overset{\rho}{-\,-\,-\,-\,-\,-} \!\!\!\!\!\!> & V \\
\downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\nu} \\
\mathbb{P}(1,2,5) & \underset{\varphi}{-\,-\,-\,-\,-} \!\!\!\!\!\!> & \mathbb{P}^2
\end{array}
\tag{37.1}
$$

or there is a commutative diagram

$$
\begin{array}{ccc}
X & \overset{\rho}{-\,-\,-\,-\,-\,-} \!\!\!\!\!\!> & V \\
\downarrow{\scriptstyle\xi} & & \downarrow{\scriptstyle\nu} \\
\mathbb{P}(1,1,3) & \underset{\sigma}{-\,-\,-\,-\,-} \!\!\!\!\!\!> & \mathbb{P}^2,
\end{array}
\tag{37.2}
$$

where φ and σ are birational maps.

Proof. See the proof of Proposition 33.1. ∎

§ 38. The case $n = 65$: a hypersurface of degree 27 in $\mathbb{P}(1, 2, 5, 9, 11)$

We use the notation and assumptions of § 3. Let $n = 65$. Then X is a general hypersurface of degree 27 in $\mathbb{P}(1, 2, 5, 9, 11)$ and the equality $-K_X^3 = 3/110$ holds. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{2}(1, 1, 1)$, a point P_2 that is a quotient singularity of type $\frac{1}{5}(1, 1, 4)$ and a point P_3 that is a quotient singularity of type $\frac{1}{11}(1, 2, 9)$. There is a commutative diagram

$$
\begin{array}{ccccc}
U & \overset{\beta}{\longleftarrow} & W & \overset{\gamma}{\longleftarrow} & Y \\
\downarrow{\scriptstyle\alpha} & & & & \downarrow{\scriptstyle\eta} \\
X & -\,-\,-\,-\,-\,-\underset{\psi}{-\,-\,-\,-\,-\,-}\!\!\!\!\!\!> & & & \mathbb{P}(1,2,5),
\end{array}
$$

where ψ is the natural projection, α is the weighted blow-up of P_3 with weights $(1, 2, 9)$, β is the weighted blow-up with weights $(1, 2, 7)$ of the point of U that is a singularity of type $\frac{1}{9}(1, 2, 7)$ contained in the α-exceptional divisor, γ is the weighted blow-up with weights $(1, 2, 5)$ of the singular point of type $\frac{1}{7}(1, 2, 5)$ contained in the β-exceptional divisor, and η is an elliptic fibration.

Proposition 38.1. *The assertion of Theorem 1.10 holds for $n = 65$.*

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_3\}$.

Let E be the exceptional divisor of α, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on U and P_4, P_5 the singular points of U contained in the surface E that are singularities of types $\frac{1}{2}(1, 1, 1)$, $\frac{1}{9}(1, 2, 7)$, respectively. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2. It follows from Lemmas 2.1 and 2.3 and the proof of Lemma 32.2 that $\mathbb{CS}(U, \frac{1}{k}\mathcal{D}) = \{P_5\}$.

Let F be the exceptional divisor of β, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on W and P_6, P_7 the singular points of W contained in F that are singularities of types $\frac{1}{2}(1,1,1)$, $\frac{1}{7}(1,2,5)$, respectively. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2. It follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains either P_6 or P_7.

Remark 38.2. In the case when $\mathbb{CS}(W, \frac{1}{k}\mathcal{H})$ contains P_7, it follows from Theorem 2.2 that there is a commutative diagram

$$
\begin{array}{ccc}
X & \dashrightarrow{\ \ \rho\ \ } & V \\
\psi \downarrow & & \downarrow \nu \\
\mathbb{P}(1,2,5) & \dashrightarrow{\ \ \zeta\ \ } & \mathbb{P}^2,
\end{array}
$$

where ζ is a birational map.

We may assume that $P_6 \in \mathbb{CS}(W, \frac{1}{k}\mathcal{H})$. Note, that $E \cong \mathbb{P}(1,2,9)$ and $F \cong \mathbb{P}(1,2,7)$.

Let $\pi\colon Z \to W$ be the weighted blow-up of P_6 with weights $(1,1,1)$, G the exceptional divisor of π and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on Z. Then $G \cong \mathbb{P}^2$, and $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ by Theorem 2.2. The hypersurface X can be given by the equation

$$
w^2 z + w f_{16}(x,y,z,t) + f_{27}(x,y,z,t) = 0 \subset \mathbb{P}(1,2,5,9,11) \cong \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),
$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 2$, $\mathrm{wt}(z) = 5$, $\mathrm{wt}(t) = 9$, $\mathrm{wt}(w) = 11$ and $f_i(x,y,z,t)$ is a quasi-homogeneous polynomial of degree i. Let $\bar{E}$ and $\bar{F}$ be the proper transforms of E and F on Z, respectively, and let $\mathcal{P}$ be the proper transform on Z of the pencil of surfaces cut out on X by the pencils $\lambda x^5 + \mu z = 0$, where $(\lambda, \mu) \in \mathbb{P}^1$.

The base locus of $\mathcal{P}$ consists of irreducible curves C, L_1, L_2, Δ_1, Δ_2 and Δ such that $\alpha \circ \beta \circ \pi(C)$ is cut out on X by the equations $x = z = 0$, $\beta \circ \pi(L_1)$ is contained in E, $\beta \circ \pi(L_1)$ is the unique curve in $\big|\mathcal{O}_{\mathbb{P}(1,2,9)}(1)\big|$, $\pi(L_1)$ is contained in F and in $\big|\mathcal{O}_{\mathbb{P}(1,2,7)}(1)\big|$, Δ_1 and Δ_2 are the lines on G cut out by $\bar{E}$ and $\bar{F}$, respectively, and Δ is a line on G different from Δ_1 and Δ_2.

Let D be a general surface in $\mathcal{P}$ and S the proper transform on Z of the surface cut out on X by the equation $x = 0$. Then

$$
S \cdot D = C + L_1 + L_2, \qquad \bar{E} \cdot D = 5L_1 + \Delta_1, \qquad \bar{F} \cdot D = 5L_2 + \Delta_2
$$

and the surface D is normal, and smooth in the neighbourhood of G. Then

$$
\Delta_1 \cdot \Delta_2 = \Delta_1 \cdot L_2 = \Delta_2 \cdot L_1 = 1, \quad \Delta_1 \cdot C = \Delta_2 \cdot C = 0, \quad \Delta_1^2 = \Delta_2^2 = -4 \quad (38.1)
$$

on D. But

$$
\bar{F} \sim_{\mathbb{Q}} \pi^*(F) - \frac{1}{2}G,
$$

$$
\bar{E} \sim_{\mathbb{Q}} (\beta \circ \pi)^*(E) - \frac{7}{9}\pi^*(F) - \frac{1}{2}G,
$$

$$
D \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-5K_X) - \frac{5}{11}(\beta \circ \pi)^*(E) - \frac{5}{9}\pi^*(F) - \frac{3}{2}G,
$$

$$
S \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-K_X) - \frac{1}{11}(\beta \circ \pi)^*(E) - \frac{1}{9}\pi^*(F) - \frac{1}{2}G.
$$

$$(38.2)$$

It follows from (38.1) and (38.2) that

$$C \cdot C = L_1 \cdot L_1 = -\frac{1}{2}, \qquad L_2 \cdot L_2 = -\frac{3}{7}, \qquad C \cdot L_1 = C \cdot L_2 = L_1 \cdot L_2 = 0$$

on D. The intersection form of the curves C, L_1 and L_2 on D is negative definite. But $\mathcal{B}|_D \sim_{\mathbb{Q}} kC + kL_1 + kL_2$, which is impossible by Lemmas 2.9 and 2.7. The proposition is proved.

§ 39. The case $n = 68$: a hypersurface of degree 28 in $\mathbb{P}(1, 3, 4, 7, 14)$

We use the notation and assumptions of § 3. Let $n = 68$. Then X is a hypersurface of degree 28 in $\mathbb{P}(1, 3, 4, 7, 14)$. The singularities of X consist of a point P_1 that is a singularity of type $\frac{1}{2}(1, 1, 1)$, a point P_2 that is a singularity of type $\frac{1}{3}(1, 1, 2)$ and points P_3 and P_4 that are singularities of type $\frac{1}{7}(1, 3, 4)$. The equality $-K_X^3 = 1/42$ holds. There is a commutative diagram
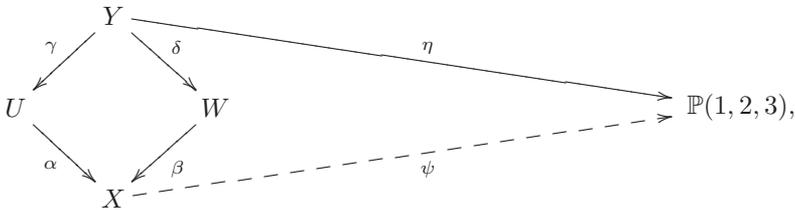
where ψ is a projection, α is the weighted blow-up of P_3 with weights $(1, 3, 4)$, β is the weighted blow-up of P_4 with weights $(1, 3, 4)$, γ is the weighted blow-up with weights $(1, 3, 4)$ of the proper transform of P_4 on U, δ is the weighted blow-up with weights $(1, 3, 4)$ of the proper transform of P_3 on W, and η is an elliptic fibration.

Proposition 39.1. *The assertion of Theorem 1.10 holds for $n = 68$.*

Proof. It follows from Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) \subseteq \{P_3, P_4\}$. The desired assertion is obvious in the case when $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) = \{P_3, P_4\}$. Hence, we may assume that $P_4 \notin \mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right)$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U and P_5, P_6 the singular points of U that are singularities of types $\frac{1}{3}(1, 1, 2)$, $\frac{1}{4}(1, 1, 3)$, respectively, contained in the exceptional divisor of α. It follows from Lemma 2.3 that the set $\mathbb{CS}\left(U, \frac{1}{k}\mathcal{B}\right)$ contains either P_5 or P_6.

Suppose that $P_6 \in \mathbb{CS}\left(U, \frac{1}{k}\mathcal{B}\right)$. Let $\pi \colon W \to U$ be the weighted blow-up of P_6 with weights $(1, 1, 3)$ and $\mathcal{B}$ and $\mathcal{P}$ the proper transforms on W of $\mathcal{M}$ and $|-3K_X|$, respectively. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2 and the base locus of $\mathcal{P}$ consists of an irreducible curve Z such that $\alpha \circ \pi(Z)$ is the base curve of $|-3K_X|$.

Let S be a general surface in $\mathcal{P}$ and B a general surface in $\mathcal{B}$. Then S is normal and $Z^2 < 0$ on S. But $\mathcal{B}|_S \sim_{\mathbb{Q}} kZ$. Therefore, the support of the cycle $B \cdot S$ is contained in Z by Lemma 2.9, which is impossible by Lemma 2.7.

Hence, $\mathbb{CS}\left(U, \frac{1}{k}\mathcal{B}\right)$ contains P_5. Let $\zeta \colon Z \to U$ be the weighted blow-up of P_5 with weights $(1, 1, 2)$ and $\mathcal{D}$ and $\mathcal{H}$ the proper transforms on Z of $\mathcal{M}$ and $|-4K_X|$, respectively. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_Z$ by Theorem 2.2 and the base locus of $\mathcal{H}$ consists of an irreducible curve C such that $\alpha \circ \zeta(C)$ is the base curve of $|-4K_X|$.

Let H be a general surface in $\mathcal{H}$. Then $H \cdot C = 0$ and $H^3 > 0$. Thus, the divisor H is nef and big. On the other hand, the equality $H \cdot D_1 \cdot D_2 = 0$ holds, where D_1 and D_2 are general surfaces in $\mathcal{D}$. But this is impossible by Corollary 2.6.

§ 40. The case $n = 74$: a hypersurface of degree 30 in $\mathbb{P}(1, 3, 4, 10, 13)$

We use the notation and assumptions of § 3. Let $n = 74$. Then X is a hypersurface of degree 30 in $\mathbb{P}(1, 3, 4, 10, 13)$, the equality $-K_X^3 = 1/52$ holds and the singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{2}(1, 1, 1)$, a point P_2 that is a quotient singularity of type $\frac{1}{4}(1, 1, 3)$ and a point P_3 that is a quotient singularity of type $\frac{1}{13}(1, 3, 10)$.

There is a commutative diagram

$$
\begin{array}{ccccc}
U & \xleftarrow{\ \beta\ } & W & \xleftarrow{\ \gamma\ } & Y \\
{\scriptstyle\alpha}\big\downarrow & & & & \big\downarrow{\scriptstyle\eta} \\
X & \dashrightarrow[\ \psi\ ] & & & \mathbb{P}(1, 3, 4),
\end{array}
$$

where ψ is the natural projection, α is the weighted blow-up of P_3 with weights $(1, 3, 10)$, β is the weighted blow-up with weights $(1, 3, 7)$ of the singular point of U that is a quotient singularity of type $\frac{1}{10}(1, 3, 7)$ contained in the exceptional divisor of α, γ is the weighted blow-up with weights $(1, 3, 4)$ of the singular point of W that is a quotient singularity of type $\frac{1}{7}(1, 3, 4)$ contained in the exceptional divisor of β, and η is an elliptic fibration.

Proposition 40.1. *The assertion of Theorem 1.10 holds for $n = 74$.*

Proof. It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) = \{P_3\}$.

Let E be the exceptional divisor of α, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on U and P_4, P_5 the singular points of U contained in the divisor E that are quotient singularities of types $\frac{1}{3}(1, 1, 2)$, $\frac{1}{10}(1, 3, 7)$, respectively. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2.

Lemma 40.2. *The set $\mathbb{CS}\left(U, \frac{1}{k}\mathcal{D}\right)$ does not contain the point P_4.*

Proof. Suppose that $\mathbb{CS}\left(U, \frac{1}{k}\mathcal{D}\right)$ contains P_4. Let $\pi\colon Z \to U$ be the weighted blow-up of P_4 with weights $(1, 1, 2)$, G the exceptional divisor of π and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on Z. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ by Theorem 2.2.

Let D be a divisor on Z such that the equivalence $D \sim_{\mathbb{Q}} -4K_Z - \pi^*(36K_U)$ holds. Analyzing the base locus of $|-4K_Z|$, we see that D is nef and big. But

$$
D \cdot B_1 \cdot B_2 = \left((\alpha \circ \pi)^*(-kK_X) - \frac{k}{13}\, \pi^*(E) - \frac{k}{3}\, G \right)^2 \left(-4K_Z - \pi^*(36K_U) \right) = 0,
$$

where B_1 and B_2 are general surfaces in $\mathcal{B}$. But this is impossible by Corollary 2.6.

Corollary 40.3. *The set $\mathbb{CS}\left(U, \frac{1}{k}\mathcal{D}\right)$ consists of the point P_5 by Lemmas 2.1 and 2.3.*

Let F be the exceptional divisor of β, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on W and P_6, P_7 the singular points of W contained in F that are quotient singularities of types $\frac{1}{3}(1,1,2)$, $\frac{1}{7}(1,3,4)$, respectively. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2.

Suppose that the set $\mathbb{CS}\left(W, \frac{1}{k}\mathcal{H}\right)$ contains the point P_7. Arguing as in the proof of Lemma 32.3, we see that the assertion of Theorem 1.10 holds for X. Hence, we may assume that $\mathbb{CS}\left(W, \frac{1}{k}\mathcal{H}\right)$ does not contain P_7. Thus, $\mathbb{CS}\left(W, \frac{1}{k}\mathcal{H}\right)$ consists of the singular point P_6 by Lemmas 2.1 and 2.3.

Let $\pi\colon Z \to W$ be the weighted blow-up of P_6 with weights $(1,1,2)$, G the exceptional divisor of π and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on Z. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ by Theorem 2.2.

Let D be a divisor on Z such that $D \sim_{\mathbb{Q}} -4K_Z - (\beta \circ \pi)^*(20K_U) - \pi^*(24K_W)$. Then D is nef and big. But

$$D \cdot B_1 \cdot B_2 = 0,$$

where B_1 and B_2 are general surfaces in $\mathcal{B}$. But this is impossible by Corollary 2.6. The proposition is proved.

§41. The case $n = 79$: a hypersurface of degree 33 in $\mathbb{P}(1,3,5,11,14)$

We use the notation and assumptions of §3. Let $n = 79$. Then X is a general hypersurface of degree 33 in $\mathbb{P}(1,3,5,11,14)$ and the equality $-K_X^3 = 1/70$ holds. The singularities of X consist of points P_1 and P_2 that are singularities of types $\frac{1}{5}(1,1,4)$ and $\frac{1}{14}(1,3,11)$, respectively.

Proposition 41.1. *The assertion of Theorem 1.10 holds for $n = 79$.*

Proof. There is a commutative diagram

where ψ is the natural projection, α is the weighted blow-up of P_2 with weights $(1,3,11)$, β is the weighted blow-up with weights $(1,3,8)$ of the singular point of type $\frac{1}{11}(1,3,8)$ contained in the exceptional divisor of α, γ is the weighted blow-up with weights $(1,3,5)$ of the singular point of type $\frac{1}{8}(1,3,5)$ contained in the exceptional divisor of β, and η is an elliptic fibration.

It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) = \{P_2\}$.

Let E be the exceptional divisor of α, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on U and P_3, P_4 the singular points of U contained in E that are quotient singularities of types $\frac{1}{3}(1,1,2)$, $\frac{1}{11}(1,3,8)$, respectively. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$. It follows from Lemmas 2.1 and 2.3 and the proof of Lemma 40.2 that $\mathbb{CS}\left(U, \frac{1}{k}\mathcal{D}\right) = \{P_4\}$.

Let F be the exceptional divisor of β, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on W and P_5, P_6 the singular points of W contained in F that are quotient singularities of types $\frac{1}{3}(1,1,2)$, $\frac{1}{8}(1,3,4)$, respectively. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2.

In the case when the set $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{H}\big)$ contains the point P_6, it follows easily from Theorem 2.2 that the assertion of Theorem 1.10 holds for X. Therefore, we may assume that $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{H}\big)$ consists of the point P_5 by Lemmas 2.1 and 2.3.

Let $\pi\colon Z \to W$ be the weighted blow-up of P_5 with weights $(1, 1, 2)$, G the exceptional divisor of π and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on Z. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_Z$ by Theorem 2.2. The hypersurface X can be given by the equation

$$w^2 z + w f_{19}(x, y, z, t) + f_{33}(x, y, z, t) = 0 \subset \mathbb{P}(1, 2, 3, 8, 11) \cong \operatorname{Proj}\big(\mathbb{C}[x, y, z, t, w]\big),$$

where $\operatorname{wt}(x) = 1$, $\operatorname{wt}(y) = 3$, $\operatorname{wt}(z) = 5$, $\operatorname{wt}(t) = 11$, $\operatorname{wt}(w) = 14$ and $f_i(x, y, z, t)$ is a quasi-homogeneous polynomial of degree i. Let $\mathcal{P}$ be the linear system on X generated by the monomials x^{30}, y^{10}, z^6, t^2x^8, $t^2y^2x^2$, ty^6x and wtz, let $\mathcal{R}$ be the proper transform of $\mathcal{P}$ on Z and let R be a general surface in $\mathcal{R}$. Then R is nef and big because the base locus of $\mathcal{R}$ does not contain curves. But

$$R \sim_{\mathbb{Q}} (\alpha \circ \beta \circ \pi)^*(-30K_X) - \frac{30}{11}(\beta \circ \pi)^*(E) - \frac{8}{11}\pi^*(F) - \frac{2}{3}G,$$

which implies that $R \cdot B_1 \cdot B_2 = 0$, where B_1 and B_2 are general surfaces in $\mathcal{B}$. This contradicts Corollary 2.6.

§ 42. The case $n = 80$: a hypersurface of degree 34 in $\mathbb{P}(1, 3, 4, 10, 17)$

We use the notation and assumptions of § 3. Let $n = 80$. Then X is a general hypersurface of degree 34 in $\mathbb{P}(1, 3, 4, 10, 17)$ whose singularities consist of a point P_1 that is a singularity of type $\frac{1}{2}(1, 1, 1)$, a point P_2 that is a singularity of type $\frac{1}{3}(1, 1, 2)$, a point P_3 that is a singularity of type $\frac{1}{4}(1, 1, 3)$ and a point P_4 that is a singularity of type $\frac{1}{10}(1, 3, 7)$. The equality $-K_X^3 = 1/60$ holds.

Proposition 42.1. *The assertion of Theorem* 1.10 *holds for* $n = 80$.

Proof. There is a commutative diagram

$$
\begin{array}{ccc}
U & \xleftarrow{\quad \beta \quad} & Y \\
{\scriptstyle \alpha} \downarrow & & \downarrow {\scriptstyle \eta} \\
X & \dashrightarrow{\; \psi \;} & \mathbb{P}(1, 3, 4),
\end{array}
$$

where ψ is the natural projection, α is the weighted blow-up of P_4 with weights $(1, 3, 7)$, β is the weighted blow-up with weights $(1, 3, 4)$ of the singular point of U that is a quotient singularity of type $\frac{1}{7}(1, 3, 4)$ contained in the exceptional divisor of α, and η is an elliptic fibration.

It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_4\}$.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U and P_5, P_6 the singular points of U that are quotient singularities of types $\frac{1}{3}(1, 1, 2)$, $\frac{1}{4}(1, 1, 3)$, respectively, contained in the exceptional divisor of α. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2. It follows from Lemmas 2.1 and 2.3 that either $P_5 \in \mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$ or $P_6 \in \mathbb{CS}\big(U, \frac{1}{k}\mathcal{D}\big)$. In the latter case, the assertion of Theorem 1.10 holds for X by Theorem 2.2.

We may assume $P_5 \in \mathbb{CS}(U, \frac{1}{k}\mathcal{D})$. The hypersurface X can be given by the quasi-homogeneous equation

$$t^3 z + t^2 f_{14}(x, y, z, w) + t f_{24}(x, y, z, w) + f_{34}(x, y, z, w) = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x, y, z, t, w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 3$, $\mathrm{wt}(z) = 4$, $\mathrm{wt}(t) = 10$, $\mathrm{wt}(w) = 17$ and $f_i(x, y, z, w)$ is a general quasi-homogeneous polynomial of degree i. Let $\mathcal{P}$ be the pencil consisting of the surfaces cut out on X by the equations $\lambda x^4 + \mu z = 0$, where $(\lambda, \mu) \in \mathbb{P}^1$. Then the base locus of $\mathcal{P}$ consists of the irreducible curve cut out on X by the equations $x = z = 0$.

Let $\gamma \colon W \to U$ be the weighted blow-up of P_5 with weights $(1, 1, 2)$, $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W, $\mathcal{H}$ the proper transform of $\mathcal{P}$ on W, D a sufficiently general surface in $\mathcal{H}$ and E, F the exceptional divisors of α, γ, respectively. Then the surface D is normal, the equivalences

$$D \sim_{\mathbb{Q}} -4K_W \sim_{\mathbb{Q}} (\alpha \circ \gamma)^*(-4K_X) - \frac{2}{5}\gamma^*(E) - \frac{4}{3}F$$

hold and the base locus of $\mathcal{H}$ consists of curves C and L such that $\alpha \circ \gamma(C)$ is the base curve of $\mathcal{P}$ and $\gamma(L)$ is the curve on $E \cong \mathbb{P}(1, 3, 7)$ that is contained in $\big|\mathcal{O}_{\mathbb{P}(1,3,7)}(1)\big|$.

Let S be the unique surface in $|-K_W|$ and $\bar{E}$ the proper transform of E on W. Then $S \cdot D = C + L$ and $\bar{E} \cdot D = 4L$. The equivalence $\bar{E} \sim_{\mathbb{Q}} \gamma^*(E) - \frac{2}{3}F$ holds, which implies that

$$C \cdot C = -\frac{1}{3}, \qquad C \cdot L = 0, \qquad L \cdot L = -\frac{2}{7},$$

on D. Hence, the intersection form of C and L on D is negative definite. But $\mathcal{B}|_D \sim_{\mathbb{Q}} kC + kL$, which is impossible by Lemmas 2.9 and 2.7.

§ 43. The case $n = 82$: a hypersurface of degree 36 in $\mathbb{P}(1, 1, 5, 12, 18)$

We use the notation and assumptions of § 3. Let $n = 82$. Then X is a general hypersurface of degree 36 in $\mathbb{P}(1, 1, 5, 12, 18)$, the equality $-K_X^3 = 1/30$ holds and the singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{5}(1, 2, 3)$ and a point P_2 that is a quotient singularity of type $\frac{1}{6}(1, 1, 5)$.

Proposition 43.1. *The assertion of Theorem* 1.10 *holds for* $n = 82$.

Proof. Suppose that $P_2 \in \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. There is a commutative diagram

$$
\begin{array}{ccc}
 & U & \\
{\scriptstyle \alpha}\swarrow & & \searrow{\scriptstyle \eta} \\
X & \dashrightarrow{\,\psi\,} & \mathbb{P}(1, 1, 5),
\end{array}
$$
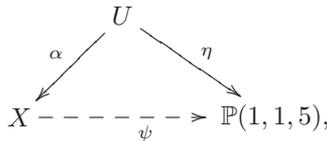
where ψ is a projection, α is the weighted blow-up of P_2 with weights $(1, 1, 6)$ and η is an elliptic fibration. Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$

by Theorem 2.2, which implies the existence of a commutative diagram

$$
\begin{array}{ccc}
X & \dashrightarrow{\;\;\rho\;\;} & V \\
\psi\downarrow & & \downarrow\nu \\
\mathbb{P}(1,1,5) & \dashrightarrow{\;\;\sigma\;\;} & \mathbb{P}^2,
\end{array}
$$

where σ is a birational map. Thus, it follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that we may assume that the set $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$ consists of the point P_1.

Let $\pi\colon W \to X$ be the weighted blow-up of P_1 with weights $(1,2,3)$ and $\mathcal{B}$ the proper transform of $\mathcal{M}$ on W. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_W$ by Theorem 2.2. The singularities of the exceptional divisor of π consist of the points Q and O that are quotient singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$ on W, respectively.

Suppose that the set $\mathbb{CS}\big(W,\frac{1}{k}\mathcal{B}\big)$ is non-empty. Then it contains either O or Q. Arguing as in the proof of Proposition 22.1, we see that $\mathbb{CS}\big(W,\frac{1}{k}\mathcal{B}\big)$ can contain neither O nor Q because $\mathcal{B}$ is not composed of a pencil.

Hence, the singularities of the log pair $(W,\frac{1}{k}\mathcal{B})$ are terminal. Arguing as in the proof of Proposition 29.1, we see that there is a birational map $\gamma\colon W \dashrightarrow Y$ such that γ is an antiflip, the divisor $-K_Y$ is nef and big and $|-rK_Y|$ induces a birational map $Y \dashrightarrow X'$ such that X' is a hypersurface in $\mathbb{P}(1,1,6,14,21)$ of degree 42 with canonical singularities (see [2], proof of Theorem 5.5.1), where $r \gg 0$.

Let $\mathcal{H}$ be the proper transform of $\mathcal{M}$ on Y. Then $\mathcal{H} \sim_{\mathbb{Q}} -kK_Y$ because γ is an isomorphism in codimension one. The map γ is a log flip with respect to $(W,\lambda\mathcal{B})$ for some rational number $\lambda > 1/k$. Thus, the singularities of the mobile log pair $\big(Y,\frac{1}{k}\mathcal{H}\big)$ are terminal, which is impossible by Lemma 2.1.

§ 44. Conclusion of the proof of Theorem 1.10

We use the notation and assumptions of § 3. In this section we complete the proof of Theorem 1.10.

Proposition 44.1. *Suppose that $n \in \{21, 24, 33, 35, 41, 42, 46, 50, 54, 55, 61, 62, 63,$ $67, 69, 71, 76, 77, 83, 85, 91\}$. Then there is a commutative diagram*

$$
\begin{array}{ccc}
X & \dashrightarrow{\;\;\rho\;\;} & V \\
\psi\downarrow & & \downarrow\nu \\
\mathbb{P}(1,a_1,a_2) & \dashrightarrow{\;\;\sigma\;\;} & \mathbb{P}^2,
\end{array}
\tag{44.1}
$$

where ψ is the natural projection and σ is a birational map.

Proof. It follows from Theorem 3.3 and Lemma 3.11 that the set $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big)$ consists of singular points of X. We shall prove the existence of the diagram (44.1) case by case.

Case $n = 21$. The variety X is a general hypersurface in $\mathbb{P}(1,1,2,4,7)$ of degree 14, the equality $-K_X^3 = 1/4$ holds and the singularities of X consist of points O_1 and O_2

that are quotient singularities of type $\frac{1}{2}(1,1,1)$ and a point P that is a quotient singularity of type $\frac{1}{4}(1,1,3)$. It follows from Proposition 3.5 that $\mathbb{CS}\left(X,\frac{1}{k}\mathcal{M}\right)=\{P\}$. Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2, and it follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\left(U,\frac{1}{k}\mathcal{D}\right)$ contains the singular point of U that is contained in the exceptional divisor of α. The existence of the commutative diagram (44.1) now follows from Theorem 2.2 because there is a commutative diagram

$$
\begin{array}{ccc}
U & \xleftarrow{\ \beta\ } & W \\
\alpha\downarrow & & \downarrow\eta \\
X & \dashrightarrow[\psi] & \mathbb{P}(1,1,2),
\end{array}
$$

where ψ is the natural projection, α is the weighted blow-up of P with weights $(1,1,3)$, β is the weighted blow-up with weights $(1,1,2)$ of the singular point of U that is a singularity of type $\frac{1}{3}(1,1,2)$, and η is an elliptic fibration.

Case $n = 24$. The variety X is a general hypersurface in $\mathbb{P}(1,1,2,5,7)$ of degree 15. The equality $-K_X^3 = 3/14$ holds. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$ and a point P_2 that is a quotient singularity of type $\frac{1}{7}(1,2,5)$. There is a commutative diagram

$$
\begin{array}{ccccc}
U & \xleftarrow{\ \beta\ } & W & \xleftarrow{\ \gamma\ } & Y \\
\alpha\downarrow & & & & \downarrow\eta \\
X & & \dashrightarrow[\psi] & & \mathbb{P}(1,1,2),
\end{array}
$$

where α is the weighted blow-up of P with weights $(1,2,5)$, β is the weighted blow-up with weights $(1,2,3)$ of the singular point of U that is a singularity of type $\frac{1}{5}(1,2,3)$, γ is the weighted blow-up with weights $(1,1,2)$ of the singular point of W that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\left(X,\frac{1}{k}\mathcal{M}\right) = \{P_2\}$.

Arguing as in the proof of Proposition 22.1, we see that $\mathcal{D} \sim_{\mathbb{Q}} -kK_W$, where $\mathcal{D}$ is the proper transform of $\mathcal{M}$ on W and the set $\mathbb{CS}\left(W,\frac{1}{k}\mathcal{D}\right)$ does not contain subvarieties of W that are not contained in the exceptional divisor of β. We can apply the arguments in the proof of Proposition 22.1 to the log pair $\left(W,\frac{1}{k}\mathcal{D}\right)$ to prove that the set $\mathbb{CS}\left(W,\frac{1}{k}\mathcal{D}\right)$ contains the singular point of W that is a singularity of type $\frac{1}{3}(1,1,2)$. This implies the existence of the commutative diagram (44.1) by Theorem 2.2.

Case $n = 33$. The variety X is a general hypersurface in $\mathbb{P}(1,2,3,5,7)$ of degree 17. The equality $-K_X^3 = 17/210$ holds. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, a point P_2 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$, a point P_3 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ and

a point P_4 that is a singularity of type $\frac{1}{7}(1,2,5)$. There is a commutative diagram
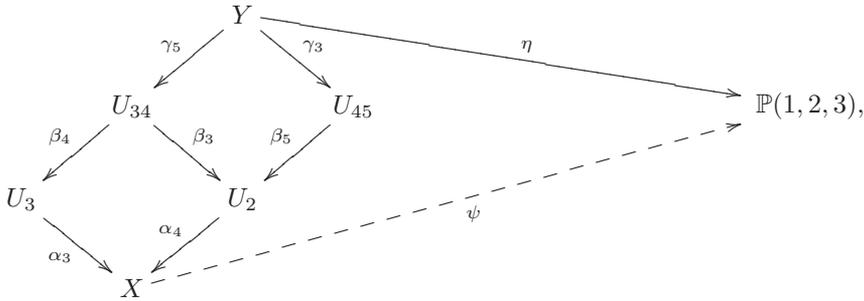
where α_3 is the weighted blow-up of P_3 with weights $(1,2,3)$, α_4 is the weighted blow-up of P_4 with weights $(1,2,5)$, β_4 is the weighted blow-up with weights $(1,2,5)$ of the proper transform of P_4 on U_3, β_3 is the weighted blow-up with weights $(1,2,3)$ of the proper transform of P_3 on U_4, β_5 is the weighted blow-up with weights $(1,2,3)$ of the singular point of the variety U_4 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of α_4, γ_3 is the weighted blow-up with weights $(1,2,3)$ of the proper transform of P_3 on U_{45}, γ_5 is the weighted blow-up with weights $(1,2,3)$ of the singular point of U_{34} that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ contained in the exceptional divisor of β_4, and η is an elliptic fibration.

Arguing as in the proof of Proposition 18.1, we see that $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big) = \{P_3, P_4\}$.

Let $\mathcal{D}_{34}$ be the proper transform of $\mathcal{M}$ on U_{34} and let $\bar{P}_5$ and $\bar{P}_6$ be the singular points of U_{34} that are quotient singularities of types $\frac{1}{5}(1,2,3)$ and $\frac{1}{2}(1,1,1)$, respectively, contained in the exceptional divisor of β_4. Then $\mathcal{D}_{34} \sim_{\mathbb{Q}} -kK_{U_{34}}$ by Theorem 2.2. It follows from Lemma 2.3 and the proof of Proposition 18.1 that $\bar{P}_5 \in \mathbb{CS}\big(U_{34},\frac{1}{k}\mathcal{D}_{34}\big)$. The existence of the commutative diagram (44.1) now follows from Theorem 2.2.

Case $n = 35$. The variety X is a general hypersurface in $\mathbb{P}(1,1,3,5,9)$ of degree 18 and the equality $-K_X^3 = 2/15$ holds. The singularities of X consist of points P_1 and P_2 that are singularities of type $\frac{1}{3}(1,1,2)$ and a point P_3 that is a singularity of type $\frac{1}{5}(1,1,4)$. There is a commutative diagram
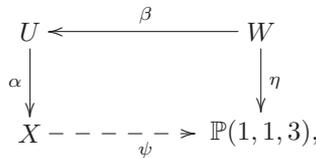
where α is the blow-up of P_3 with weights $(1,1,4)$, β is the blow-up with weights $(1,1,3)$ of the point of U that is a singularity of type $\frac{1}{4}(1,1,3)$, and η is an elliptic fibration.

It follows from Theorem 3.3, Lemma 3.11 and Proposition 3.5 that $\mathbb{CS}\big(X,\frac{1}{k}\mathcal{M}\big) = \{P_3\}$.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2. It follows from Lemma 2.3 that the set $\mathbb{CS}\big(U,\frac{1}{k}\mathcal{D}\big)$ contains the singular point of U

that is contained in the exceptional divisor of α. The existence of the diagram (44.1) follows from Theorem 2.2.

Case $n = 41$. The variety X is a general hypersurface in $\mathbb{P}(1,1,4,5,10)$ of degree 20 whose singularities consist of a point O that is a singularity of type $\frac{1}{2}(1,1,1)$ and points P_1 and P_2 that are singularities of type $\frac{1}{5}(1,1,4)$.

There is a commutative diagram
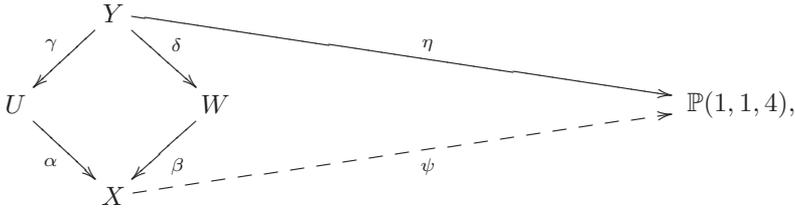
where α is the weighted blow-up of P_1 with weights $(1,1,4)$, β is the weighted blow-up of P_2 with weights $(1,1,4)$, γ is the weighted blow-up with weights $(1,1,4)$ of the proper transform of P_2 on U, δ is the weighted blow-up with weights $(1,1,4)$ of the proper transform of P_1 on W, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\left(X,\frac{1}{k}\mathcal{M}\right) \subseteq \{P_1, P_2\}$. Arguing as in the proof of Proposition 13.1, we see that the diagram (44.1) exists.

Case $n = 42$. The variety X is a hypersurface in $\mathbb{P}(1,2,3,5,10)$ of degree 20. The equality $-K_X^3 = 1/15$ holds. The singularities of X consist of points P_1, P_2, P_3 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, a point P_4 that is a singularity of type $\frac{1}{3}(1,1,2)$ and points P_5, P_6 that are singularities of type $\frac{1}{5}(1,2,3)$.

There is a commutative diagram
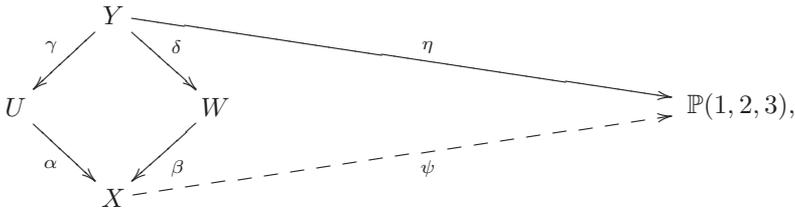
where α is the weighted blow-up of P_5 with weights $(1,2,3)$, β is the weighted blow-up of P_6 with weights $(1,2,3)$, γ is the weighted blow-up of the proper transform of P_6 on U with weights $(1,2,3)$, δ is the weighted blow-up of the proper transform of P_5 on W with weights $(1,2,3)$, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\left(X,\frac{1}{k}\mathcal{M}\right) \subseteq \{P_5, P_6\}$. The existence of the diagram (44.1) follows from Theorem 2.2 in the case when $\mathbb{CS}\left(X,\frac{1}{k}\mathcal{M}\right) = \{P_5, P_6\}$. Thus, we may assume that $\mathbb{CS}\left(X,\frac{1}{k}\mathcal{M}\right) = \{P_5\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U and O, Q the singular points of U that are quotient singularities of types $\frac{1}{3}(1,1,2)$, $\frac{1}{2}(1,1,1)$, respectively, contained in the exceptional divisor of α. Arguing as in the proof of Lemma 21.3, we see that $\mathbb{CS}\left(U,\frac{1}{k}\mathcal{B}\right) = \{Q\}$.

Let $\zeta\colon Z \to U$ be the weighted blow-up of Q with weights $(1,1,1)$, $\mathcal{D}$ the proper transform of $\mathcal{M}$ on Z and $\mathcal{H}$ the proper transform of $|-3K_X|$ on Z. Then

$\mathcal{D} \sim_{\mathbb{Q}} -kK_Z$ by Theorem 2.2. The base locus of $\mathcal{H}$ consists of an irreducible curve C such that $\alpha \circ \zeta(C)$ is the base curve of $|-3K_X|$.

Let S be a general surface in $\mathcal{H}$. Then S is normal and $C^2 < 0$ on S. But the equivalence $\mathcal{D}|_S \sim_{\mathbb{Q}} kC$ holds, which is impossible by Lemmas 2.7 and 2.9.

Case $n = 45$. The variety X is a general hypersurface in $\mathbb{P}(1, 3, 4, 5, 8)$ of degree 20 whose singularities consist of a point P_1 of type $\frac{1}{3}(1, 1, 2)$, points P_2 and P_3 of type $\frac{1}{4}(1, 1, 3)$ and a point P_4 of type $\frac{1}{8}(1, 3, 5)$.

There is a commutative diagram

$$
\begin{array}{ccc}
U & \xleftarrow{\quad \beta \quad} & Y \\
{\scriptstyle \alpha} \downarrow & & \downarrow {\scriptstyle \eta} \\
X & \dashrightarrow{\;\psi\;} & \mathbb{P}(1, 3, 4),
\end{array}
$$

where α is the weighted blow-up of P_4 with weights $(1, 3, 5)$, β is the weighted blow-up with weights $(1, 2, 3)$ of the singular point of U that is a quotient singularity of type $\frac{1}{5}(1, 2, 3)$ contained in the exceptional divisor of α, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_4\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U, $\bar{P}_5$ the singular point of U that is a quotient singularity of type $\frac{1}{5}(1, 2, 3)$ contained in the exceptional divisor of α and $\bar{P}_6$ the singular point of U that is a quotient singularity of type $\frac{1}{2}(1, 1, 1)$ contained in the exceptional divisor of α. Then it follows from Lemma 2.3 that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big) \cap \{\bar{P}_5, \bar{P}_6\} \neq \varnothing$.

Arguing as in the proof of Proposition 30.1, we see that $\bar{P}_5 \in \mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$, which implies the existence of the commutative diagram (44.1) by Theorem 2.2.

Case $n = 46$. The variety X is a general hypersurface in $\mathbb{P}(1, 1, 3, 7, 10)$ of degree 21. The equality $-K_X^3 = 1/10$ holds. The singularities of X consist of a point P that is a singularity of type $\frac{1}{10}(1, 3, 7)$. There is a commutative diagram

$$
\begin{array}{ccccc}
U & \xleftarrow{\quad \beta \quad} & W & \xleftarrow{\quad \gamma \quad} & Y \\
{\scriptstyle \alpha} \downarrow & & & & \downarrow {\scriptstyle \eta} \\
X & \dashrightarrow{\;\psi\;} & & & \mathbb{P}(1, 1, 3),
\end{array}
$$

where α is the blow-up of P with weights $(1, 3, 7)$, β is the blow-up with weights $(1, 3, 4)$ of the singular point of U that is a singularity of type $\frac{1}{7}(1, 3, 4)$, γ is the blow-up with weights $(1, 1, 3)$ of the singular point of W that is a singularity of type $\frac{1}{4}(1, 1, 3)$, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P\}$.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on W. Arguing as in the proof of Proposition 22.1, we see that $\mathcal{D} \sim_{\mathbb{Q}} -kK_W$ and the set $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{D}\big)$ does not contain subvarieties of W that are not contained in the exceptional divisor of β. The set $\mathbb{CS}\big(W, \frac{1}{k}\mathcal{D}\big)$ is non-empty because the divisor $-K_W$ is nef and big. Therefore, we can apply the arguments in the proof of Proposition 22.1 to the log pair $\big(W, \frac{1}{k}\mathcal{D}\big)$,

which implies that the set $\mathbb{CS}\left(W, \frac{1}{k}\mathcal{D}\right)$ contains the singular point of W that is a singularity of type $\frac{1}{4}(1,1,3)$. The existence of the commutative diagram (44.1) is implied by Theorem 2.2.

Case $n = 50$. The variety X is a general hypersurface in $\mathbb{P}(1,1,3,7,11)$ of degree 22 and the inequality $-K_X^3 = 2/21$ holds. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{3}(1,1,2)$ and a point P_2 that is a quotient singularity of type $\frac{1}{7}(1,3,4)$. There is a commutative diagram

$$
\begin{array}{ccc}
U & \xleftarrow{\quad\beta\quad} & W \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\eta} \\
X & \dashrightarrow[\psi] & \mathbb{P}(1,1,3),
\end{array}
$$

where α is the blow-up of P_2 with weights $(1,3,7)$, β is the blow-up with weights $(1,1,3)$ of the singular point of U that is a singularity of type $\frac{1}{4}(1,1,3)$ and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) = \{P_2\}$. Arguing as in the proof of Proposition 22.1, we see that the diagram (44.1) exists.

Case $n = 54$. The variety X is a general hypersurface in $\mathbb{P}(1,1,6,8,9)$ of degree 24 whose singularities consist of a point P_1 type $\frac{1}{2}(1,1,1)$, a point P_2 of type $\frac{1}{3}(1,1,2)$ and a point P_3 of type $\frac{1}{9}(1,1,8)$. The equality $-K_X^3 = 1/18$ holds.

There is a commutative diagram

$$
\begin{array}{ccccc}
U & \xleftarrow{\quad\beta\quad} & W & \xleftarrow{\quad\gamma\quad} & Y \\
{\scriptstyle\alpha}\downarrow & & & & \downarrow{\scriptstyle\eta} \\
X & & \dashrightarrow[\psi] & & \mathbb{P}(1,1,6),
\end{array}
$$

where α is the blow-up of P_3 with weights $(1,1,8)$, β is the blow-up with weights $(1,1,7)$ of the singular point of U that is a singularity of type $\frac{1}{8}(1,1,7)$, γ is the blow-up with weights $(1,1,3)$ of the singular point of W that is a singularity of type $\frac{1}{7}(1,1,6)$, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) = \{P_3\}$. Arguing as in the proof of Proposition 14.1, we see that the diagram (44.1) exists.

Case $n = 55$. The variety X is a general hypersurface in $\mathbb{P}(1,2,3,7,12)$ of degree 24 whose singularities consist of points P_1, P_2 of type $\frac{1}{2}(1,1,1)$, points P_3, P_4 of type $\frac{1}{3}(1,1,2)$ and a point P_5 of type $\frac{1}{7}(1,2,5)$. There is a commutative diagram

$$
\begin{array}{ccc}
U & \xleftarrow{\quad\beta\quad} & W \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\eta} \\
X & \dashrightarrow[\psi] & \mathbb{P}(1,2,3),
\end{array}
$$

where α is the blow-up of P_5 with weights $(1,2,5)$, β is the blow-up with weights $(1,2,3)$ of the singular point of U that is a singularity of type $\frac{1}{5}(1,2,3)$ and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_5\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U and let P_6 and P_7 be the singular points of U that are singularities of types $\frac{1}{2}(1,1,1)$ and $\frac{1}{5}(1,2,3)$, respectively, contained in the exceptional divisor of α. Then $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big) \subseteq \{P_6, P_7\}$ by Lemmas 2.1, 2.3 and 2.4.

The existence of the diagram (44.1) follows from Theorem 2.2 in the case when $P_7 \in \mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$. Hence, we may assume that $\mathbb{CS}\big(U, \frac{1}{k}\mathcal{B}\big)$ consists of the point P_6.

The hypersurface X can be given by the equation

$$t^3 z + t^2 f_{10}(x,y,z,w) + t f_{17}(x,y,z,w) + f_{24}(x,y,z,w) = 0 \subset \mathrm{Proj}\big(\mathbb{C}[x,y,z,t,w]\big),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 2$, $\mathrm{wt}(z) = 3$, $\mathrm{wt}(t) = 7$, $\mathrm{wt}(w) = 12$ and $f_i(x,y,z,w)$ is a sufficiently general quasi-homogeneous polynomial of degree i. Let $\gamma\colon W \to U$ be the weighted blow-up of P_6 with weights $(1,1,1)$, $\mathcal{H}$ the proper transform of $\mathcal{M}$ on W, $\mathcal{P}$ the proper transform on W of the pencil of surfaces cut out on X by the equations $\lambda x^3 + \mu z = 0$, where $(\lambda, \mu) \in \mathbb{P}^1$, and D a sufficiently general surface in $\mathcal{P}$. Then the base locus of $\mathcal{P}$ consists of irreducible curves C, L and Δ such that $\alpha \circ \gamma(C)$ is the base curve of $|-3K_X|$, $\gamma(L)$ is contained in the exceptional divisor of α and Δ is contained in the exceptional divisor of γ.

The surface D is normal. The intersection form of the curves L and C on the surface D is negative definite. But $\mathcal{H}|_D \sim_{\mathbb{Q}} kC + kL$, which is impossible by Lemmas 2.9 and 2.7.

Case $n = 61$. The variety X is a general hypersurface in $\mathbb{P}(1,4,5,7,9)$ of degree 25. The equality $-K_X^3 = 5/252$ holds. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{4}(1,1,3)$, a point P_2 that is a quotient singularity of type $\frac{1}{7}(1,2,5)$ and a point P_3 that is a quotient singularity of type $\frac{1}{9}(1,4,5)$. There is a commutative diagram
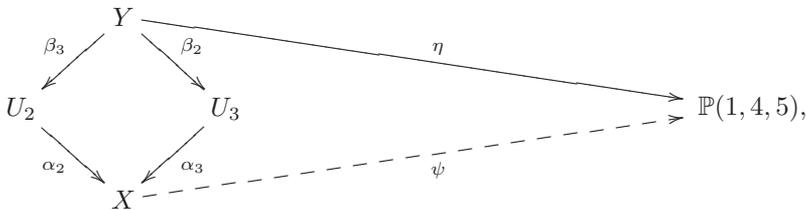
where α_2 is the weighted blow-up of P_2 with weights $(1,2,5)$, α_3 is the weighted blow-up of P_3 with weights $(1,4,5)$, β_2 is the weighted blow-up with weights $(1,2,5)$ of the proper transform of P_2 on U_3, β_3 is the weighted blow-up with weights $(1,4,5)$ of the proper transform of P_3 on U_2, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) \subseteq \{P_2, P_3\}$. Arguing as in the proof of Lemma 36.2, we see that $\mathbb{CS}\big(X, \frac{1}{k}\mathcal{M}\big) = \{P_2, P_3\}$. The existence of the diagram (44.1) follows from Theorem 2.2.

Case $n = 62$. The variety X is a hypersurface in $\mathbb{P}(1, 1, 5, 7, 13)$ of degree 26. The equality $-K_X^3 = 2/35$ holds. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{5}(1, 2, 3)$ and a point P_2 that is a singularity of type $\frac{1}{7}(1, 1, 6)$.

There is a commutative diagram

$$
\begin{array}{ccc}
U & \xleftarrow{\;\;\beta\;\;} & Y \\
{\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\eta} \\
X & \dashrightarrow[\psi] & \mathbb{P}(1, 1, 5),
\end{array}
$$

where α is the weighted blow-up of P_2 with weights $(1, 1, 6)$, β is the weighted blow-up with weights $(1, 1, 5)$ of the singular point of U that is a quotient singularity of type $\frac{1}{6}(1, 1, 5)$, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) \subseteq \{P_1, P_2\}$. Arguing as in the proof of Proposition 31.1, we see that the commutative diagram (44.1) exists.

Case $n = 63$. The variety X is a general hypersurface in $\mathbb{P}(1, 2, 3, 8, 13)$ of degree 26. The singularities of X consist of points P_1, P_2, P_3 that are quotient singularities of type $\frac{1}{2}(1, 1, 1)$, a point P_4 that is a quotient singularity of type $\frac{1}{3}(1, 1, 2)$ and a point P_5 that is a singularity of type $\frac{1}{8}(1, 3, 5)$. The equality $-K_X^3 = 1/24$ holds. There is a commutative diagram

$$
\begin{array}{ccc}
U & \xleftarrow{\;\;\beta\;\;} & Y \\
{\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\eta} \\
X & \dashrightarrow[\psi] & \mathbb{P}(1, 2, 3),
\end{array}
$$

where α is the weighted blow-up of P_5 with weights $(1, 3, 5)$, β is the weighted blow-up with weights $(1, 2, 3)$ of the singular point of U that is a quotient singularity of type $\frac{1}{5}(1, 2, 3)$ contained in the exceptional divisor of α, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_5\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U and P_6, P_7 the singular points of U that are quotient singularities of types $\frac{1}{5}(1, 2, 3)$, $\frac{1}{2}(1, 1, 1)$, respectively, contained in the exceptional divisor of α. Then it follows from Lemma 2.3 that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains either P_6 or P_7.

Suppose that $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains P_7. Let $\gamma\colon W \to U$ be the weighted blow-up of P_7 with weights $(1, 1, 1)$ and let $\mathcal{H}$ and $\mathcal{P}$ be the proper transforms of the linear system $\mathcal{M}$ and the pencil $|-2K_X|$ on W, respectively. Then the base locus of $\mathcal{P}$ consists of irreducible curves C and L such that $\alpha \circ \gamma(C)$ is the unique curve in the base locus of $|-2K_X|$ and $\gamma(L)$ is contained in the exceptional divisor of α. The intersection form of L and C on D is negative definite. On the other hand, the equivalence $\mathcal{H}|_D \sim_{\mathbb{Q}} kC + kL$ holds, which is impossible by Lemmas 2.9 and 2.7.

Therefore, the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains the point P_6. The assertion of Theorem 2.2 implies the existence of the commutative diagram (44.1).

Case $n = 67$. The variety X is a general hypersurface in $\mathbb{P}(1, 1, 4, 9, 14)$ of degree 28. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{2}(1, 1, 1)$ and a point P_2 that is a quotient singularity of type $\frac{1}{9}(1, 4, 5)$. The equality $-K_X^3 = 1/18$ holds. It follows from Proposition 3.5 that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) = \{P_2\}$. There is a commutative diagram

$$
\begin{array}{ccc}
U & \xleftarrow{\ \ \beta\ \ } & W \\
\alpha \downarrow & & \downarrow \eta \\
X & \dashrightarrow[\psi] & \mathbb{P}(1, 1, 4),
\end{array}
$$

where α is the weighted blow-up of P_2 with weights $(1, 4, 5)$, β is the weighted blow-up with weights $(1, 1, 4)$ of the singular point of U that is a quotient singularity of type $\frac{1}{5}(1, 1, 4)$, and η is an elliptic fibration. Arguing as in the proof of Proposition 22.1, we see that the commutative diagram (44.1) exists.

Case $n = 69$. The variety X is a general hypersurface in $\mathbb{P}(1, 4, 6, 7, 11)$ of degree 28. The singularities of X consist of points P_1, P_2 that are quotient singularities of type $\frac{1}{2}(1, 1, 1)$ and points P_3, P_4 that are quotient singularities of types $\frac{1}{6}(1, 1, 5)$, $\frac{1}{11}(1, 4, 7)$, respectively. The equality $-K_X^3 = 1/66$ holds. There is a commutative diagram

$$
\begin{array}{ccc}
U & \xleftarrow{\ \ \beta\ \ } & W \\
\alpha \downarrow & & \downarrow \eta \\
X & \dashrightarrow[\psi] & \mathbb{P}(1, 3, 4),
\end{array}
$$

where α is the weighted blow-up of P_4 with weights $(1, 4, 7)$, β is the weighted blow-up with weights $(1, 3, 4)$ of the singular point of type $\frac{1}{7}(1, 3, 4)$, and η is an elliptic fibration. It follows from Proposition 3.5 that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) = \{P_4\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U and let P_5 and P_6 be the singular points of U that are quotient singularities of types $\frac{1}{7}(1, 3, 4)$ and $\frac{1}{4}(1, 1, 3)$, respectively, contained in the exceptional divisor of α. Then it follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\left(U, \frac{1}{k}\mathcal{B}\right)$ contains either P_5 or P_6. Arguing as in the proof of Lemma 28.5, we see that $\mathbb{CS}\left(U, \frac{1}{k}\mathcal{B}\right)$ does not contain P_6. Thus, it contains P_5. The existence of the commutative diagram (44.1) follows from Theorem 2.2.

Case $n = 71$. The variety X is a general hypersurface in $\mathbb{P}(1, 1, 6, 8, 15)$ of degree 30 whose singularities consist of points P_1, P_2 and P_3 that are singularities of types $\frac{1}{2}(1, 1, 1)$, $\frac{1}{3}(1, 1, 2)$ and $\frac{1}{8}(1, 1, 7)$, respectively. The equality $-K_X^3 = 1/24$ holds.

It follows from Proposition 3.5 that $\mathbb{CS}\left(X, \frac{1}{k}\mathcal{M}\right) = \{P_3\}$. There is a commutative diagram

$$
\begin{array}{ccc}
U & \xleftarrow{\ \ \beta\ \ } & W \\
\alpha \downarrow & & \downarrow \eta \\
X & \dashrightarrow[\psi] & \mathbb{P}(1, 1, 6),
\end{array}
$$

where α is the weighted blow-up of P_3 with weights $(1,1,7)$, β is the weighted blow-up with weights $(1,1,6)$ of the singular point of U that is contained in the exceptional divisor of α, and η is an elliptic fibration.

Let $\mathcal{D}$ be the proper transform of $\mathcal{M}$ on U. Then $\mathcal{D} \sim_{\mathbb{Q}} -kK_U$ by Theorem 2.2 and it follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{D})$ contains the singular point of U that is contained in the exceptional divisor of α. The existence of the commutative diagram (44.1) is implied by Theorem 2.2.

Case $n = 76$. The variety X is a general hypersurface in $\mathbb{P}(1,5,6,8,11)$ of degree 30 whose singularities consist of points P_1, P_2 and P_3 that are singularities of types $\frac{1}{2}(1,1,1)$, $\frac{1}{8}(1,3,5)$ and $\frac{1}{11}(1,5,6)$, respectively. The equality $-K_X^3 = 1/88$ holds. There is a commutative diagram

$$
\begin{array}{ccccc}
 & & Y & & \\
 & \overset{\beta_3}{\swarrow} & & \overset{\beta_2}{\searrow} & \\
U_2 & & & & U_3 \qquad\qquad \overset{\eta}{\longrightarrow} \quad \mathbb{P}(1,5,6), \\
 & \overset{\alpha_2}{\searrow} & & \overset{\alpha_3}{\swarrow} & \\
 & & X & & \quad \overset{\psi}{\dashrightarrow}
\end{array}
$$
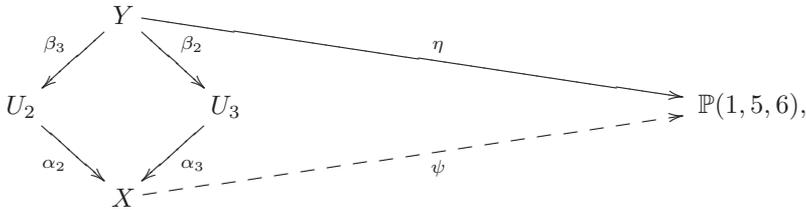
where α_2 is the weighted blow-up of P_2 with weights $(1,3,5)$, α_3 is the weighted blow-up of P_3 with weights $(1,5,6)$, β_2 is the weighted blow-up of the proper transform of P_2 on U_3 with weights $(1,3,5)$, β_3 is the weighted blow-up of the proper transform of P_3 on U_2 with weights $(1,5,6)$, and η is an elliptic fibration.

Arguing as in the proofs of Lemmas 29.3 and 36.2, we see that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_2, P_3\}$. The existence of the commutative diagram (44.1) follows from Theorem 2.2.

Case $n = 77$. The variety X is a general hypersurface in $\mathbb{P}(1,2,5,9,16)$ of degree 32. The singularities of X consist of points P_1, P_2 that are quotient singularities of type $\frac{1}{2}(1,1,1)$, a point P_3 that is a quotient singularity of type $\frac{1}{5}(1,1,4)$ and a point P_4 that is a quotient singularity of type $\frac{1}{9}(1,2,7)$. The equality $-K_X^3 = 1/45$ holds. There is a commutative diagram
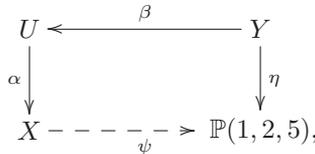
$$
\begin{array}{ccc}
U & \overset{\beta}{\longleftarrow} & Y \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\eta} \\
X & \overset{\psi}{\dashrightarrow} & \mathbb{P}(1,2,5),
\end{array}
$$

where α is the blow-up of P_4 with weights $(1,2,7)$, β is the blow-up with weights $(1,2,5)$ of the singular point of U that is a singularity of type $\frac{1}{7}(1,2,5)$, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_4\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U and let P_5 and P_6 be the singular points of U that are singularities of types $\frac{1}{7}(1,2,5)$ and $\frac{1}{2}(1,1,1)$, respectively, contained in the exceptional divisor of α. It follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains either P_5 or P_6.

Suppose that $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains P_6. Let $\gamma\colon W \to U$ be the weighted blow-up of P_5 with weights $(1, 1, 1)$, let $\mathcal{H}$ and $\mathcal{D}$ be the proper transforms of $\mathcal{M}$ and $|-16K_X|$ on W, respectively, let D be a general surface in $\mathcal{D}$ and let H_1 and H_2 be general surfaces in $\mathcal{H}$. Then the base locus of $\mathcal{D}$ does not contain curves. In particular, the divisor D is nef. But $D \cdot H_1 \cdot H_1 < 0$.

Therefore, the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains the point P_5. The assertion of Theorem 2.2 now implies the existence of the commutative diagram (44.1).

Case $n = 83$. The variety X is a general hypersurface in $\mathbb{P}(1, 3, 4, 11, 18)$ of degree 36. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{2}(1, 1, 1)$, points P_2 and P_3 that are quotient singularities of type $\frac{1}{3}(1, 1, 2)$ and a point P_4 that is a quotient singularity of type $\frac{1}{11}(1, 4, 7)$. The equality $-K_X^3 = 1/66$ holds. There is a commutative diagram

$$
\begin{array}{ccc}
U & \xleftarrow{\;\;\;\beta\;\;\;} & W \\
{\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\eta} \\
X & \dashrightarrow{\;\;\psi\;\;} & \mathbb{P}(1, 3, 4),
\end{array}
$$

where α is the weighted blow-up of P_4 with weights $(1, 4, 7)$, β is the weighted blow-up with weights $(1, 3, 4)$ of the singular point of U that is a quotient singularity of type $\frac{1}{7}(1, 3, 4)$, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_4\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U and let P_5 and P_6 be the singular points of U that are singularities of types $\frac{1}{7}(1, 3, 4)$ and $\frac{1}{4}(1, 1, 3)$, respectively, contained in the exceptional divisor of α. Then it follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ contains either P_5 or P_6.

Arguing as in the proof of Proposition 25.1, we see that $\mathbb{CS}(U, \frac{1}{k}\mathcal{B})$ does not contain P_6. Thus, it contains the point P_5. The existence of the diagram (44.1) follows from Theorem 2.2.

Case $n = 85$. The variety X is a general hypersurface in $\mathbb{P}(1, 3, 5, 11, 19)$ of degree 38. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{3}(1, 1, 2)$, a point P_2 that is a quotient singularity of type $\frac{1}{5}(1, 1, 4)$ and a point P_3 that is a quotient singularity of type $\frac{1}{11}(1, 3, 8)$. The equality $-K_X^3 = 2/165$ holds.

There is a commutative diagram

$$
\begin{array}{ccc}
U & \xleftarrow{\;\;\;\beta\;\;\;} & Y \\
{\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\eta} \\
X & \dashrightarrow{\;\;\psi\;\;} & \mathbb{P}(1, 3, 5),
\end{array}
$$

where α is the weighted blow-up of P_3 with weights $(1, 3, 8)$, β is the weighted blow-up with weights $(1, 3, 5)$ of the singular point of U that is a singularity of type $\frac{1}{8}(1, 3, 5)$, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M}) = \{P_3\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U and let P_4 and P_5 be the singular points of U that are quotient singularities of types $\frac{1}{8}(1,3,5)$ and $\frac{1}{3}(1,1,2)$, respectively, contained in the exceptional divisor of α. Then it follows from Lemmas 2.1 and 2.3 that the set $\mathbb{CS}\left(U,\frac{1}{k}\mathcal{B}\right)$ contains either P_4 or P_5.

Suppose that $\mathbb{CS}\left(U,\frac{1}{k}\mathcal{B}\right)$ contains P_5. Let $\gamma\colon W\to U$ be the weighted blow-up of P_5 with weights $(1,1,2)$, let $\mathcal{H}$ and $\mathcal{D}$ be the proper transforms of $\mathcal{M}$ and $|-19K_X|$ on W, respectively, let D be a general surface in $\mathcal{D}$ and let H_1 and H_2 be general surfaces in $\mathcal{H}$. Then the base locus of $\mathcal{D}$ does not contain curves. Thus, the divisor D is nef. In particular, the inequality $D\cdot H_1\cdot H_1\geqslant 0$ holds. But $D\cdot H_1\cdot H_1=-2k^2/15$, which is a contradiction.

Hence, the set $\mathbb{CS}\left(U,\frac{1}{k}\mathcal{B}\right)$ contains the singular point P_4. The existence of the commutative diagram (44.1) follows from Theorem 2.2.

Case $n=91$. The variety X is a general hypersurface in $\mathbb{P}(1,4,5,13,22)$ of degree 44. The singularities of X consist of a point P_1 that is a quotient singularity of type $\frac{1}{2}(1,1,1)$, a point P_2 that is a quotient singularity of type $\frac{1}{5}(1,2,3)$ and a point P_3 that is a quotient singularity of type $\frac{1}{13}(1,4,9)$. The equality $-K_X^3=1/130$ holds.

There is a commutative diagram

$$
\begin{array}{ccc}
U & \xleftarrow{\quad\beta\quad} & W \\
{\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\eta} \\
X & \dashrightarrow{\ \ \psi\ \ } & \mathbb{P}(1,4,5),
\end{array}
$$

where α is the weighted blow-up of P_3 with weights $(1,4,9)$, β is the weighted blow-up with weights $(1,4,5)$ of the singular point of U that is a quotient singularity of type $\frac{1}{9}(1,4,5)$, and η is an elliptic fibration.

It follows from Proposition 3.5 that $\mathbb{CS}\left(X,\frac{1}{k}\mathcal{M}\right)=\{P_3\}$.

Let $\mathcal{B}$ be the proper transform of $\mathcal{M}$ on U and let P_4 and P_5 be the singular points of U that are quotient singularities of types $\frac{1}{9}(1,4,5)$ and $\frac{1}{4}(1,1,3)$, respectively, contained in the exceptional divisor of α. Then it follows from Lemmas 2.1 and 2.3 that $\mathbb{CS}\left(U,\frac{1}{k}\mathcal{B}\right)$ contains either P_4 or P_5.

Arguing as in the proof of Proposition 25.1, we see that $\mathbb{CS}\left(U,\frac{1}{k}\mathcal{B}\right)$ does not contain P_5. Thus, it contains the singular point P_4. The existence of the commutative diagram (44.1) follows from Theorem 2.2.

This completes the proof of Theorem 1.10.

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I. A. Cheltsov
Steklov Mathematical Institute;
University of Edinburgh
E-mail: `cheltsov@yahoo.com`