Math. Nachr. **280,** No. 12, 1344 – 1353 (2007) / **DOI** 10.1002/mana.200510550

On nodal sextic fivefold

Ivan Cheltsov[*1]

[1] School of Mathematics, The University of Edinburgh, Edinburgh, EH9 3JZ, UK

Received 15 June 2005, revised 24 November 2005, accepted 13 June 2007
Published online 7 August 2007

Key words Nonrational, hypersurface, Fano variety, birationally rigid, birational automorphisms
MSC (2000) 14E05, 14E07, 14E08, 14J40, 14J45, 14J70

We prove birational superrigidity and nonrationality of every sextic fivefold with ordinary double points.

1 Introduction

All varieties are assumed to be projective, normal and defined over $\mathbf{C}$.

In many cases the only known way to prove the nonrationality of a given Fano variety is to prove its birational rigidity (cf. [16], [7] and [4]). Many counterexamples to the Lüroth problem are obtained in this way (see [13]).

Birational rigidity is proved in the following cases:

- for some smooth Fano threefolds (see [13], [12] and [14]);
- for many singular Fano threefolds (see [20], [22], [11], [9], [8] and [17]);
- for many smooth Fano n-folds (see [18], [23], [25], [2], [26], [27], [30], [10], [3] and [4]), where $n > 3$;
- for some singular Fano n-folds (see [20], [22], [28], [29] and [4]), where $n > 3$.

Let X be a hypersurface in $\mathbf{P}^6$ of degree 6 that has at most isolated ordinary double points. Then

$$-K_X \sim \mathcal{O}_{\mathbf{P}^6}(1)\big|_X,$$

the variety X has $\mathbf{Q}$-factorial terminal singularities and $\operatorname{rk}\operatorname{Pic}(X) = 1$ (see [1]). We prove the following result.

Theorem 1.1 *The hypersurface X is birationally superrigid.*

In the smooth case the assertion of Theorem 1.1 is proved in [2].

Example 1.2 The singularities of the hypersurface

$$x_0^4\big(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2\big) = x_1^6 + x_2^6 + x_3^6 + x_4^6 + x_5^6 + x_6^6 \subset \mathbf{P}^6 \cong \operatorname{Proj}\big(\mathbf{C}[x_0, \ldots, x_6]\big)$$

consist of a single ordinary double point, which implies that it is nonrational by Theorem 1.1.

Example 1.3 Let X be a hypersurface with 729 isolated ordinary double points

$$\sum_{i=0}^{2} a_i\big(x_0, \ldots, x_6\big) b_i\big(x_0, \ldots, x_6\big) = 0 \subset \mathbf{P}^6 \cong \operatorname{Proj}\big(\mathbf{C}[x_0, \ldots, x_6]\big),$$

where a_i and b_i are general homogeneous polynomials of degree 3. Then X is nonrational by Theorem 1.1.

The assertion of Theorem 1.1 is a fivefold generalization of the birational rigidity of a nodal $\mathbf{Q}$-factorial quartic threefold (see [13], [20] and [17]). The assertion of Theorem 1.1 is relevant to the results obtained in [28] and [29], which cannot be used to produce explicit examples of nonrational Fano hypersurfaces.

* e-mail: I.Cheltsov@ed.ac.uk, Phone: +44 131 650 4881, Fax: +44 131 650 6553

2 The Noether–Fano inequality

Let X be an arbitrary Fano variety having at most terminal and $\mathbf{Q}$-factorial singularities such that $\operatorname{rk}\operatorname{Pic}(X)=1$, and the variety X is not birationally superrigid. Then the following result holds (see [5]).

Theorem 2.1 *There is a linear system $\mathcal{M}$ on the variety X such that $\mathcal{M}$ does not have fixed components, and the singularities of the log pair $(X,\gamma\mathcal{M})$ are not canonical, where $\gamma\in\mathbf{Q}$ is such that $K_X+\gamma\mathcal{M}\equiv 0$.*

In the rest of the section we prove Theorem 2.1. Let $\rho : X \dashrightarrow Y$ be a birational map such that the rational map ρ is not biregular and one of the following holds:

• the variety Y has terminal $\mathbf{Q}$-factorial singularities and $\operatorname{rk}\operatorname{Pic}(Y)=1$ (the Fano case);

• the variety Y is smooth, and there is a surjective morphism $\tau : Y \to Z$ such that sufficiently general fiber of the morphism τ has negative Kodaira dimension, and $\dim(Y)\neq\dim(Z)\neq 0$ (the fibration case).

Let us consider a commutative diagram

$$\begin{array}{ccc} & W & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ X & \underset{\rho}{\dashrightarrow} & Y, \end{array}$$

such that the variety W is smooth, α and β are birational morphisms. In the Fano case let $\mathcal{D}$ be the complete linear system $|-rK_Y|$ for $r\gg 0$, in the fibration case let $\mathcal{D}$ be the linear system $|\tau^*(H)|$, where H is a very ample divisor on the variety Z. Let $\mathcal{M}$ be a proper transform of $\mathcal{D}$ on the variety X. Take a $\gamma\in\mathbf{Q}$ such that

$$K_X+\gamma\mathcal{M}\equiv 0.$$

Suppose that the singularities of the log pair $(X,\gamma\mathcal{M})$ are canonical. Let us show that this assumption leads to a contradiction. Let $\mathcal{B}$ be a proper transform on W of the linear system $\mathcal{M}$. Then

$$\sum_{i=1}^{k}a_iF_i\equiv\alpha^*\bigl(K_X+\gamma\mathcal{M}\bigr)+\sum_{i=1}^{k}a_iF_i\equiv K_W+\gamma\mathcal{B}\equiv\beta^*\bigl(K_Y+\gamma\mathcal{D}\bigr)+\sum_{i=1}^{l}b_iG_i,$$

where F_j is a β-exceptional divisor, G_i is an α-exceptional divisor, a_i is a nonnegative rational number, and b_i is a positive rational number. Let n be a sufficiently big and sufficiently divisible natural number. Then

$$1=h^0\left(\mathcal{O}_W\left(\sum_{j=1}^{k}na_jF_j\right)\right)=h^0\left(\mathcal{O}_W\bigl(\beta^*\bigl(nK_Y+n\gamma\mathcal{D}\bigr)+\sum_{i=1}^{l}nb_iG_i\bigr)\right),$$

but $h^0\bigl(\mathcal{O}_W\bigl(\beta^*(nK_Y+\gamma\mathcal{D})+\sum_{i=1}^{l}nb_iG_i\bigr)\bigr)=0$ in the fibration case. Hence, the fibration case is impossible.

In the Fano case the equality $h^0\bigl(\mathcal{O}_W\bigl(\beta^*(nK_Y+\gamma\mathcal{D})+\sum_{i=1}^{l}nb_iG_i\bigr)\bigr)=1$ implies that $\gamma=1/r$. Then

$$\sum_{i=1}^{k}a_iF_i\equiv\sum_{i=1}^{l}b_iG_i,$$

and $\sum_{i=1}^{k}a_iF_i=\sum_{i=1}^{l}b_iG_i$ by [15, Lemma 2.19]. Thus, the log pair $(X,\gamma\mathcal{M})$ has terminal singularities.

There is a rational number $\mu>\gamma$ such that $(X,\mu\mathcal{M})$ and $(X,\mu\mathcal{B})$ have terminal singularities. Then

$$\alpha^*\bigl(K_X+\mu\mathcal{M}\bigr)+\sum_{i=1}^{k}a_i'F_i\equiv K_W+\mu\mathcal{B}\equiv\beta^*\bigl(K_Y+\mu\mathcal{D}\bigr)+\sum_{i=1}^{l}b_i'G_i,$$

where a_i' and b_i' are positive rational numbers.

Let n be a sufficiently big and divisible natural number, and let $\psi\colon W\dashrightarrow U$ be a rational map that is given by the linear system $|nK_W+n\mu\mathcal{B}|$. Then the map $\psi\circ\beta^{-1}$ is biregular, because the divisor $n(K_Y+\mu\mathcal{D})$ is very ample. But the divisor $\sum_{i=1}^{l}nb_i'G_i$ is effective and β-exceptional. Similarly, we see that $\psi\circ\alpha^{-1}$ is biregular, which implies that ρ is biregular. The latter is a contradiction. Thus, we proved Theorem 2.1. ☐

3 The lemma of Corti

Let X be a variety with an ordinary double point $O \in X$, and let B_X be an effective **Q**-Cartier divisor on X. Let

$$\pi\colon W \longrightarrow X$$

be a blow up of the point O, E be a π-exceptional divisor, and B_W be a proper transform of B_X on W. Then

$$\pi^*(B_X) \equiv B_W + \mathrm{mult}_O(B_X)E,$$

where $\mathrm{mult}_O(B_X)$ is a nonnegative rational number.

Suppose that $\dim(X) \geqslant 3$ and the log pair (X, B_X) is not canonical at the point O. Then $\mathrm{mult}_O(B_X) > 1/2$. In the rest of the section we prove the following result, which is implied by [6, Theorem 3.10].

Lemma 3.1 *The inequality* $\mathrm{mult}_O(B_X) > 1$ *holds.*

Suppose that $\mathrm{mult}_O(B_X) \leqslant 1$. Let us show that this assumption leads to a contradiction.

Replacing the divisor B_X by $(1-\epsilon)B_X$ for some positive sufficiently small rational number ϵ, we may assume that $\mathrm{mult}_O(B_X) < 1$. Taking hyperplane sections, we may assume that $\dim(X) = 3$ by [15, Theorem 17.6].

Lemma 3.2 *Let S be a surface $\mathbf{P}^1 \times \mathbf{P}^1$, and B_S be an effective divisor on the surface S of bi-degree (a, b), where a and b are rational numbers in $[0, 1)$. Then the log pair (S, B_S) has log-terminal singularities.*

P r o o f. Suppose that the singularities of (S, B_S) are not log-terminal. Then the locus of log canonical singularities $\mathrm{LCS}(S, B_S)$ is not empty and consists of points of the surface S. Then $\mathrm{LCS}(S, F + B_S)$ is not connected, where F is a general fiber of any projection of the surface S to $\mathbf{P}^1$. The later contradicts [15, Theorem 17.4]. □

The inequality $\mathrm{mult}_O(B_X) < 1$ and the equivalence

$$K_W + B_W \equiv \pi^*\big(K_X + B_X\big) + \big(1 - \mathrm{mult}_O\big(B_X\big)\big)E,$$

imply that there is a proper subvariety $Z \subset E$ such that the log pair (W, B_W) is not canonical at general point of the variety Z. Then $(E, B_W|_E)$ is not log terminal by [15, Theorem 17.6], which is impossible by Lemma 3.2.

4 Main inequalities

Let X be a variety with an ordinary double point $O \in X$, and let $\mathcal{M}$ be a liner system on the variety X such that the linear system $\mathcal{M}$ does not have fixed components. Put $r = \dim(X)$. Suppose that $r \geqslant 4$. Let

$$\pi\colon V \longrightarrow X$$

be a blow up of the variety X at the point O, and let E be a π-exceptional divisor. Let $\mathcal{B}$ be a proper transform of the linear system $\mathcal{M}$ on the variety V. The variety E can be identified with a smooth quadric in $\mathbf{P}^r$. Then

$$\mathcal{B} \sim \pi^*(\mathcal{M}) - \mathrm{mult}_O(\mathcal{M})E,$$

where $\mathrm{mult}_O(\mathcal{M})$ is a natural number, which is different from the multiplicity of $\mathcal{M}$ at the point O.

Let S_1 and S_2 be sufficiently general divisors in the linear system $\mathcal{M}$, and H_i be a sufficiently general hyperplane section of the variety X that passes through the point O, where $i = 1, \ldots, r-2$. Put

$$\mathrm{mult}_O\big(S_1 \cdot S_2\big) = 2\mathrm{mult}^2_O\big(S_i\big) + \sum_{P \in E} \mathrm{mult}_P\big(\widehat{S}_1 \cdot \widehat{S}_2\big)\mathrm{mult}_P\big(\widehat{H}_1\big) \ldots \mathrm{mult}_P\big(\widehat{H}_{r-2}\big),$$

where $\mathrm{mult}_O(S_i)$ and $\mathrm{mult}_O(H_i)$ are natural numbers that are defined in the same way as the number $\mathrm{mult}_O(\mathcal{M})$, and $\widehat{S}_i$ and $\widehat{H}_i$ are the proper transforms on the variety V of the divisors S_i and H_i, respectively.

Remark 4.1 It follows from elementary properties of blow ups that the inequality

$$\mathrm{mult}_O\big(S_1 \cdot S_2\big) \geqslant 2\mathrm{mult}^2_O\big(S_i\big) + \mathrm{mult}_Z\big(\widehat{S}_1 \cdot \widehat{S}_2\big)$$

holds for any irreducible subvariety $Z \subset E$ of codimension one.

Example 4.2 Let X be a singular hypersurface in $\mathbf{P}^6$ of degree 6 that has at most isolated ordinary double points, and let O be a singular point of the variety X. It follows from [1] that

$$S_i \sim nH$$

where H is a hyperplane section of the variety X, and $n \in \mathbf{N}$. Then $\mathrm{mult}_O(S_1 \cdot S_2) \leqslant 6n^2$.

Suppose that $\big(X, \frac{1}{n}\mathcal{M}\big)$ is canonical in a punctured neighborhood of O, and $\big(X, \frac{1}{n}\mathcal{M}\big)$ is not canonical at O.

Lemma 4.3 *Suppose that* $r > 5$. *Then* $\mathrm{mult}_O(S_1 \cdot S_2) > 6n^2$.

P r o o f. We may assume that $r = 6$, because the proof in the case $r > 6$ is similar. Then

$$K_V + \frac{1}{n}\mathcal{B} \equiv \pi^*\left(K_X + \frac{1}{n}\mathcal{M}\right) + \left(4 - \frac{\mathrm{mult}_O(\mathcal{M})}{n}\right)E.$$

Put $\check{X} = \bigcap_{i=1}^{3} H_i$ and $\check{\mathcal{M}} = \mathcal{M}|_{\check{X}}$. The point O is an ordinary double point of the variety $\check{X}$, and the singularities of the log pair $\big(\check{X}, \frac{1}{n}\check{\mathcal{M}}\big)$ are not log canonical in the point O by [15, Theorem 17.6].

Let $\check{\pi} : \check{V} \to \check{X}$ be a blow up of the point O, and $\check{E}$ be an exceptional divisor of $\check{\pi}$. Then the diagram

$$\begin{array}{ccc} \check{V} & \hookrightarrow & V \\ {\scriptstyle \check{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ \check{X} & \hookrightarrow & X \end{array}$$

is commutative, where $\check{V}$ is identified with a proper transform of $\check{X}$ on the variety V. We have $\check{E} = E \cap \check{V}$. Then

$$\mathrm{mult}_O\big(\check{\mathcal{M}}\big) = \mathrm{mult}_O\big(\mathcal{M}\big),$$

and we may assume that $\mathrm{mult}_O(\mathcal{M}) < 2n$, because otherwise $\mathrm{mult}_O(S_1 \cdot S_2) > 6n^2$.

Let $\mathcal{B}$ be a proper transform of the linear system $\mathcal{M}$ on the variety V, and $\check{\mathcal{B}}$ be a proper transform of the linear system $\check{\mathcal{M}}$ on the threefold $\check{V}$. Then $\check{\mathcal{B}} = \mathcal{B}|_{\check{V}}$ and we have

$$K_V + \frac{1}{n}\mathcal{B} + \left(\frac{\mathrm{mult}_O(\mathcal{M})}{n} - 1\right)E + \widehat{H}_1 + \widehat{H}_2 + \widehat{H}_3 \equiv \pi^*\left(K_X + \frac{1}{n}\mathcal{M} + H_1 + H_2 + H_3\right)$$

and

$$K_{\check{V}} + \frac{1}{n}\check{\mathcal{B}} + \left(\frac{\mathrm{mult}_O(\mathcal{M})}{n} - 1\right)\check{E} \equiv \check{\pi}^*\left(K_{\check{X}} + \frac{1}{n}\check{\mathcal{M}}\right),$$

but $\mathrm{mult}_O(\mathcal{M}) < 2n$. Thus, there are irreducible subvarieties $\Omega \subsetneq E$ and $\check{\Omega} \subsetneq \check{E}$ such that

- the log pair $\big(V, \frac{1}{n}\mathcal{B} + (\mathrm{mult}_O(\mathcal{M})/n - 1)E\big)$ is not log canonical at general point of Ω,
- the log pair $\big(\check{V}, \frac{1}{n}\check{\mathcal{B}} + (\mathrm{mult}_O(\mathcal{M})/n - 1)\check{E}\big)$ is not log canonical at general point of $\check{\Omega}$, and $\check{\Omega} \subseteq \Omega \cap \check{V}$.

We may assume that Ω and $\check{\Omega}$ have the biggest dimensions among all subvarieties having such properties.

We have $\check{\Omega} = \Omega \cap \check{V}$ when $\dim(\check{\Omega}) > 0$. Let us show that $\check{\Omega} = \Omega \cap \check{V}$ when $\dim(\check{\Omega}) = 0$.

Applying [15, Theorem 17.4] to the log pair $\big(\check{V}, \frac{1}{n}\check{\mathcal{B}} + (\mathrm{mult}_O(\mathcal{M})/n - 1)\check{E}\big)$ and the morphism $\check{\pi}$, we see that in the case $\dim(\check{\Omega}) = 0$ the locus of log canonical singularities

$$\mathrm{LCS}\left(\check{V}, \frac{1}{n}\check{\mathcal{B}} + \big(\mathrm{mult}_O(\mathcal{M})/n - 1\big)\check{E}\right)$$

consists of a single point $\check{\Omega}$ in the neighborhood of the divisor $\check{E}$. In particular, we have $\check{\Omega} = \Omega \cap \check{V}$.

Suppose that $\dim(\check{\Omega}) = 0$. Then $\check{\Omega} = \Omega \cap \check{V}$ implies that Ω is a linear subspace in $\mathbf{P}^6$ of codimension 3 that is contained in the smooth quadric hypersurface $E \subset \mathbf{P}^6$. The latter is impossible by the Lefschetz theorem.

Hence, the inequality $\dim(\check{\Omega}) \geqslant 1$ holds, which implies $\dim(\Omega) = 4$.

We see that the singularities of the log pair $(V, \frac{1}{n}\mathcal{B} + (\mathrm{mult}_O(\mathcal{M})/n - 1)E)$ are not log canonical at general point of the irreducible subvariety $\Omega \subset E$ that has dimension 4. Therefore, we can apply [6, Theorem 3.1] to the log pair $\big(V, \frac{1}{n}\mathcal{B} + (\mathrm{mult}_O(\mathcal{M})/n - 1)E\big)$ in the general point of the subvariety Ω. The latter gives

$$\mathrm{mult}_\Omega\big(\widehat{S}_1 \cdot \widehat{S}_2\big) > 4\big(2n^2 - n\mathrm{mult}_O(\mathcal{M})\big),$$

where $\widehat{S}_i$ is a proper transform of S_i on the variety V. Hence, the inequalities

$$\mathrm{mult}_O\big(S_1 \cdot S_2\big) \geqslant 2\mathrm{mult}_O\big(\mathcal{M}\big)^2 + \mathrm{mult}_\Omega\big(\widehat{S}_1 \cdot \widehat{S}_2\big) > 6n^2 + 2\big(n - \mathrm{mult}_O(\mathcal{M})\big)^2 \geqslant 6n^2$$

hold, which is exactly what we need to proof. □

Let Δ be an effective divisor on the variety X passing through the point O and $\hat{\Delta}$ be its proper transform on the variety V. Suppose that Δ does not contain irreducible components of the cycle $S_1 \cdot S_2$, and $\hat{\Delta}$ does not contain irreducible components of the cycle $\widehat{S}_1 \cdot \widehat{S}_2$. Then we can put

$$\begin{aligned}\mathrm{mult}_O\big(S_1 \cdot S_2 \cdot \Delta\big) &= 2\mathrm{mult}^2_O(S_i)\mathrm{mult}_O(\Delta)\\ &\quad + \sum_{P \in E} \mathrm{mult}_P\big(\widehat{S}_1 \cdot \widehat{S}_2 \cdot \hat{\Delta}\big)\mathrm{mult}_P\big(\widehat{H}_1\big) \ldots \mathrm{mult}_P\big(\widehat{H}_{r-3}\big),\end{aligned}$$

which implies $\mathrm{mult}_O(S_1 \cdot S_2 \cdot \Delta) = \mathrm{mult}_O(S_1|_\Delta \cdot S_2|_\Delta)$ if O is an isolated ordinary double point of Δ.

Lemma 4.4 *Suppose that $r = 4$. Then there is a line $\Lambda \subset E \subset \mathbf{P}^4$ such that*

$$\mathrm{mult}_O\big(S_1 \cdot S_2 \cdot \Delta\big) > 6n^2$$

in the case when O is an ordinary double point of the divisor Δ, and $\Lambda \subset \hat{\Delta}$.

Proof. We have $\mathrm{mult}_O(\mathcal{M}) > n$ by Lemma 3.1, but

$$K_V + \frac{1}{n}\mathcal{B} \equiv \pi^*\bigg(K_X + \frac{1}{n}\mathcal{M}\bigg) + \bigg(2 - \frac{\mathrm{mult}_O(\mathcal{M})}{n}\bigg)E.$$

Suppose that O is an ordinary double point on Δ. Put $\bar{S}_i = S_i|_\Delta$ and $\overline{\mathcal{M}} = \mathcal{M}|_\Delta$. Then the log pair $\big(\Delta, \frac{1}{n}\overline{\mathcal{M}}\big)$ is not log canonical in the point O by [15, Theorem 17.6].

Let $\bar{\pi} : \tilde{\Delta} \to \Delta$ be a blow up of O, and $\widetilde{E}$ is a $\bar{\pi}$-exceptional divisor. Then the diagram

$$\begin{array}{ccc} \tilde{\Delta} & \hookrightarrow & V \\ \downarrow{\scriptstyle \bar{\pi}} & & \downarrow{\scriptstyle \pi} \\ \Delta & \hookrightarrow & X \end{array}$$

is commutative, where we can identify $\tilde{\Delta}$ with $\hat{\Delta}$, and $\widetilde{E} = E \cap \tilde{\Delta}$ can be considered as a nonsingular quadric hypersurface in $\mathbf{P}^3$. The inequality $\mathrm{mult}_O\big(\,\overline{\mathcal{M}}\,\big) \geqslant 2n$ gives

$$\mathrm{mult}_O\big(S_1 \cdot S_2 \cdot \Delta\big) = \mathrm{mult}_O\big(\bar{S}_1 \cdot \bar{S}_2\big) \geqslant 8n^2,$$

hence, we may assume that $\mathrm{mult}_O\big(\,\overline{\mathcal{M}}\,\big) < 2n$.

Let $\widetilde{\mathcal{M}}$ be a proper transform of the linear system $\overline{\mathcal{M}}$ on the variety $\tilde{\Delta}$. Then $\mathrm{mult}_O\big(\overline{\mathcal{M}}\big) < 2n$ implies that there is an irreducible subvariety $\Xi \subsetneq \widetilde{E}$ such that the singularities of the log pair

$$\bigg(\tilde{\Delta}, \frac{1}{n}\widetilde{\mathcal{M}} + \big(\mathrm{mult}_O\big(\,\overline{\mathcal{M}}\,\big)/n - 1\big)\widetilde{E}\bigg).$$

are not log canonical in the general point of Ξ.

Suppose that Ξ is a curve. Let $\widetilde{S}_i$ be a proper transform of $\bar{S}_i$ on the variety $\tilde{\Delta}$. Then the inequality

$$\mathrm{mult}_O\big(\bar{S}_1 \cdot \bar{S}_2\big) \geqslant 2\mathrm{mult}_O(\mathcal{M})^2 + \mathrm{mult}_\Xi\big(\tilde{S}_1 \cdot \tilde{S}_2\big)$$

holds. Applying [6, Theorem 3.1] to $\big(\tilde{\Delta}, \frac{1}{n}\widetilde{\mathcal{M}} + \big(\mathrm{mult}_O\big(\overline{\mathcal{M}}\big)/n - 1\big)\widetilde{E}\big)$ at the general point of Ξ, we see that

$$\mathrm{mult}_\Xi\big(\widetilde{S}_1 \cdot \widetilde{S}_2\big) > 4\big(2n^2 - n\mathrm{mult}_O\big(\overline{\mathcal{M}}\big)\big),$$

which immediately implies that

$$\mathrm{mult}_O\big(\bar{S}_1 \cdot \bar{S}_2\big) > 2\mathrm{mult}^2_O\big(\overline{\mathcal{M}}\big) + 4\big(2n^2 - n\mathrm{mult}_O\big(\overline{\mathcal{M}}\big)\big) \geqslant 6n^2.$$

To conclude the proof we may assume that Ξ is a point.

Suppose that Δ is a general hyperplane section of X such that $O \in \Delta$. We can apply [15, Theorem 17.4] to the morphism $\tilde{\pi}$ and the log pair $\big(\tilde{\Delta}, \frac{1}{n}\widetilde{\mathcal{M}} + \big(\mathrm{mult}_O\big(\overline{\mathcal{M}}\big)/n - 1\big)\widetilde{E}\big)$. We see that

- either $\big(V, \frac{1}{n}\mathcal{B} + (\mathrm{mult}_O(\mathcal{M})/n - 1)E\big)$ is not log canonical at general point of a surface contained in E,
- or $\big(V, \frac{1}{n}\mathcal{B} + (\mathrm{mult}_O(\mathcal{M})/n - 1)E\big)$ is not log canonical at general point of a line $\Lambda \subset E$ and $\Xi = \Lambda \cap \hat{\Delta}$.

In the case when the log pair $\big(V, \frac{1}{n}\mathcal{B} + (\mathrm{mult}_O(\mathcal{M})/n - 1)E\big)$ is not log canonical at general point of a surface contained in E, the previous arguments implies the inequality $\mathrm{mult}_O\big(\bar{S}_1 \cdot \bar{S}_2\big) > 6n^2$.

We may assume that there is a line $\Lambda \subset E$ such that $\Xi = \Lambda \cap \tilde{\Delta}$ and the singularities of the log pair

$$\left(V,\ \frac{1}{n}\mathcal{B} + \big(\mathrm{mult}_O(\mathcal{M})/n - 1\big)E\right)$$

are not log canonical at general point of the curve Λ.

The line Λ does not depend on the choice of Δ. So, we may assume that $\Lambda \subset \hat{\Delta}$, where $\hat{\Delta} = \tilde{\Delta}$. Then

$$\left(\tilde{\Delta},\ \frac{1}{n}\widetilde{\mathcal{M}} + \big(\mathrm{mult}_O\big(\overline{\mathcal{M}}\big)/n - 1\big)\widetilde{E}\right)$$

is not log canonical at the general point of Λ by [15, Theorem 17.6], because $\mathrm{mult}_O(\mathcal{M}) > n$.

Now we can apply [6, Theorem 3.1] to the log pair $\big(\tilde{\Delta}, \frac{1}{n}\widetilde{\mathcal{M}} + \big(\mathrm{mult}_O\big(\overline{\mathcal{M}}\big)/n - 1\big)\widetilde{E}\big)$ at general point of the curve Λ to obtain the inequalities

$$\mathrm{mult}_O\big(\bar{S}_1 \cdot \bar{S}_2\big) > 2\mathrm{mult}^2_O\big(\overline{\mathcal{M}}\big) + 4\big(2n^2 - n\mathrm{mult}_O\big(\overline{\mathcal{M}}\big)\big) \geqslant 6n^2,$$

which conclude the proof. □

Finally, let us prove the following result.

Lemma 4.5 *Suppose that* $r = 5$. *Then* $\mathrm{mult}_O\big(S_1 \cdot S_2\big) > 6n^2$.

P r o o f. Put $\check{X} = H_1 \cap H_2$ and $\check{\mathcal{M}} = \mathcal{M}|_{\check{X}}$. Then $\big(\check{X}, \frac{1}{n}\check{\mathcal{M}}\big)$ is not log canonical at O by [15, Theorem 17.6], and O is an ordinary double point of the threefold $\check{X}$. Let $\breve{\pi} : \check{V} \to \check{X}$ be a blow up of O, and $\check{E}$ be an exceptional divisor of the morphism $\breve{\pi}$. Then we can identify $\check{V}$ with a proper transform of $\check{X}$ on the variety V. Because

$$\mathrm{mult}_O(S_1 \cdot S_2) \geqslant 2\mathrm{mult}^2_O(\mathcal{M}) > 6n^2$$

in the case when $\mathrm{mult}_O(\mathcal{M}) \geqslant 2n$, we may assume that the inequality $\mathrm{mult}_O(\mathcal{M}) < 2n$ holds.

Let $\check{\mathcal{B}}$ be a proper transform of the linear system $\check{\mathcal{M}}$ on the variety $\check{V}$. Then $\check{\mathcal{B}} = \mathcal{B}|_{\check{V}}$. We have

$$K_V + \frac{1}{n}\mathcal{B} + \left(\frac{\mathrm{mult}_O(\mathcal{M})}{n} - 1\right)E + \widehat{H}_1 + \widehat{H}_2 \equiv \pi^*\left(K_X + \frac{1}{n}\mathcal{M} + H_1 + H_2\right)$$

and $K_{\check{V}} + \frac{1}{n}\check{\mathcal{B}} + (\mathrm{mult}_O(\mathcal{M})/n - 1)\check{E} \equiv \breve{\pi}^*\big(K_{\check{X}} + \frac{1}{n}\check{\mathcal{M}}\big)$. So, there are subvarieties $\Omega \subsetneq E$ and $\check{\Omega} \subsetneq \check{E}$ such that

- both subvarieties Ω and $\breve{\Omega}$ are irreducible and $\breve{\Omega} \subseteq \Omega \cap \breve{V}$,
- the log pair $\big(V, \frac{1}{n}\mathcal{B} + (\mathrm{mult}_O(\mathcal{M})/n - 1)E\big)$ is not log canonical at general point of Ω;
- the log pair $\big(\breve{V}, \frac{1}{n}\breve{\mathcal{B}} + (\mathrm{mult}_O(\mathcal{M})/n - 1)\breve{E}\big)$ is not log canonical at general point of $\breve{\Omega}$.

We may assume that the subvarieties Ω and $\breve{\Omega}$ have the biggest dimensions among all subvarieties with such properties. Then $\breve{\Omega} = \Omega \cap \breve{V}$ in the case when $\dim(\breve{\Omega}) \geqslant 1$.

Suppose that $\dim(\breve{\Omega}) \geqslant 1$ holds. Then $\dim(\Omega) = 3$. Therefore, the inequality

$$\mathrm{mult}_\Omega\big(\hat{S}_1 \cdot \hat{S}_2\big) > 4\big(2n^2 - n\mathrm{mult}_O(\mathcal{M})\big)$$

holds by [6, Theorem 3.1]. Therefore, the inequalities

$$\mathrm{mult}_O(S_1 \cdot S_2) \geqslant 2\,\mathrm{mult}_O^2(\mathcal{M}) + \mathrm{mult}_\Omega\big(\hat{S}_1 \cdot \hat{S}_2\big) > 6n^2$$

hold. Thus, we may assume that $\dim\big(\breve{\Omega}\big) = 0$.

Applying [15, Theorem 17.4] to the log pair $\big(\breve{V}, \frac{1}{n}\breve{\mathcal{B}} + (\mathrm{mult}_O(\mathcal{M})/n - 1)\breve{E}\big)$ and $\breve{\pi}$, we see that the locus

$$\mathrm{LCS}\bigg(\breve{V},\ \frac{1}{n}\breve{\mathcal{B}} + \big(\mathrm{mult}_O(\mathcal{M})/n - 1\big)\breve{E}\bigg)$$

consists of a single point $\breve{\Omega}$ in the neighborhood of the divisor $\breve{E}$. Hence, the subvariety Ω is a plane in $\mathbf{P}^5$.

The referee pointed out to the author that Ω cannot be a plane. We follow the arguments of the referee to complete the proof. Let us use the arguments of the original proof of Lemma 3.1 (see [6, Theorem 3.10]).

Let $\breve{X}$ be a general hyperplane section of X passing through the point O that is locally given as

$$xy + zt = 0 \subset \mathbf{C}^5 \cong \mathrm{Spec}\big(\mathbf{C}[x, y, z, t, u]\big)$$

in the neighborhood of the point O, which is given by $x = y = z = t = u = 0$. Then $\breve{X}$ has non-isolated singularities. But we can apply the previous arguments to the variety $\breve{X}$.

Let $\breve{V}$ be the proper transform of $\breve{X}$ on the variety V, and let $\breve{\pi} : \breve{V} \to \breve{X}$ be the induced morphism. Then

$$K_{\breve{V}} + \frac{1}{n}\breve{\mathcal{B}} + \big(\mathrm{mult}_O(\mathcal{M})/n - 2\big)\breve{E} \equiv \breve{\pi}^*\left(K_{\breve{X}} + \frac{1}{n}\mathcal{M}\Big|_{\breve{X}}\right),$$

where $\breve{\mathcal{B}} = \mathcal{B}|_{\breve{V}}$, and $\breve{E}$ is the exceptional divisor of the morphism $\breve{\pi}$, which is a cone over $\mathbf{P}^1 \times \mathbf{P}^1$.

Let $\breve{S}_x$ and $\breve{S}_y$ be Weil divisors on $\breve{X}$ that are given by the equations $x = t = 0$ and $y = t = 0$, respectively. Then $\breve{S}_x$ and $\breve{S}_y$ are not $\mathbf{Q}$-Cartier divisors, but the divisor $\breve{S}_x + \breve{S}_y$ is Cartier. We have

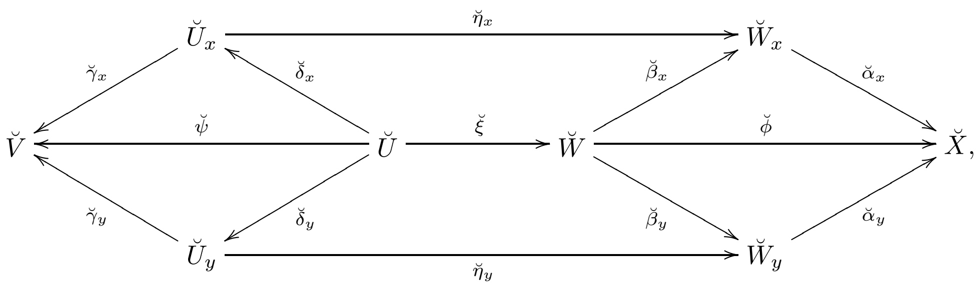
$$K_{\breve{V}} + \frac{1}{n}\breve{\mathcal{B}} + \big(\mathrm{mult}_O(\mathcal{M})/n - 1\big)\breve{E} + \breve{H}_x + \breve{H}_y \equiv \breve{\pi}^*\bigg(K_{\breve{X}} + \frac{1}{n}\mathcal{M}\Big|_{\breve{X}} + \breve{S}_x + \breve{S}_y\bigg),$$

where $\breve{H}_x$ and $\breve{H}_y$ are proper transforms of the subvarieties $\breve{S}_x$ and $\breve{S}_y$ on the variety $\breve{V}$, respectively. Then

$$\mathrm{LCS}\bigg(\breve{V},\ \frac{1}{n}\breve{\mathcal{B}} + \big(\mathrm{mult}_O(\mathcal{M})/n - 1\big)\breve{E}\bigg) = \breve{\Omega},$$

where $\breve{\Omega} = \Omega|_{\breve{V}}$ is a line on $\breve{E} \subset \mathbf{P}^4$. Indeed, we can apply the previous arguments to $\big(\breve{X}, \frac{1}{n}\mathcal{M}\big|_{\breve{X}} + \breve{S}_x + \breve{S}_y\big)$.

There are natural ways to desingularize the varieties $\breve{X}$ and $\breve{V}$. There is a commutative diagram



where we use the following notation:

- $\breve{\phi}$ is a blow up of the ideal sheaf of the curve $x = y = z = t = 0$;
- $\breve{\alpha}_x$ and $\breve{\alpha}_y$ are blow ups of the ideal sheaves of $\breve{S}_x$ and $\breve{S}_y$, respectively;
- $\breve{\beta}_x$ and $\breve{\beta}_y$ are blow ups of the exceptional surfaces of $\breve{\alpha}_x$ and $\breve{\alpha}_y$, respectively;
- $\breve{\xi}$, $\breve{\beta}_x$, $\breve{\beta}_y$ are blow ups of the fibers of ϕ, $\breve{\alpha}_x$, $\breve{\alpha}_y$ over the point O, respectively;
- $\breve{\psi}$ is a blow up of the ideal sheaf of the proper transform of $x = y = z = t = 0$;
- $\breve{\gamma}_x$ and $\breve{\gamma}_y$ are blow ups of the ideal sheaves of $\breve{H}_x$ and $\breve{H}_y$, respectively;
- $\breve{\delta}_x$ and $\breve{\delta}_y$ are blow ups of the exceptional surfaces of $\breve{\gamma}_x$ and $\breve{\gamma}_y$, respectively.

The varieties $\breve{W}$, $\breve{W}_x$, $\breve{W}_y$, $\breve{U}$, $\breve{U}_x$, $\breve{U}_y$ are smooth, the morphisms $\breve{\alpha}_x$, $\breve{\alpha}_y$, $\breve{\gamma}_x$, $\breve{\gamma}_y$ are small, and $\breve{\pi} \circ \breve{\psi} = \breve{\phi} \circ \breve{\xi}$. Let $\breve{F}$ be the exceptional divisor of the birational morphism $\breve{\xi}$. Then

$$\breve{F} \cong \mathbf{P}\big(\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(1)\big),$$

where $\mathcal{O}_{\mathbf{P}^1\times\mathbf{P}^1}(1)$ is a hyperplane section of $\mathbf{P}^1 \times \mathbf{P}^1$ with respect to the natural embedding into $\mathbf{P}^3$.

The morphism $\breve{\xi}\,\big|_{\breve{F}}$ is a projection to $\mathbf{P}^1 \times \mathbf{P}^1$, the morphisms $\breve{\eta}_x \circ \breve{\delta}_x\big|_{\breve{F}}$ and $\breve{\eta}_y \circ \breve{\delta}_y\big|_{\breve{F}}$ are projections to $\mathbf{P}^1$, the morphisms $\breve{\delta}_x\big|_{\breve{F}}$ and $\breve{\delta}_y\big|_{\breve{F}}$ are contractions of the exceptional section of $\breve{F}$ to curves, and $\breve{\psi}\big|_{\breve{F}}$ is the contraction of the exceptional section of the surface $\breve{F}$ to the vertex of the cone $\breve{E}$, where $\breve{E} = \breve{\psi}\big(\breve{F}\big)$.

The subvariety $\breve{\Omega}$ is a line on the cone $\breve{E} \subset \mathbf{P}^4$ that does not pass through its vertex. But $\big(\breve{H}_x + \breve{H}_y\big) \cdot \breve{\Omega} = 1$, which implies that we may assume that $\breve{H}_x \cdot \breve{\Omega} = 0$ and $\breve{H}_y \cdot \breve{\Omega} = 1$.

Let $\breve{D}_x$ and $\breve{D}_y$ be the proper transforms of $\breve{H}_x$ and $\breve{H}_y$ on the variety $\breve{U}_y$, respectively, and $\breve{\Gamma}$ be the proper transform of $\breve{\Omega}$ on the variety $\breve{U}_y$. Then $\breve{D}_x \cdot \breve{\Gamma} = 0$ and $\breve{D}_y \cdot \breve{\Gamma} = 1$. Moreover, we have

$$K_{\breve{U}_y} + \frac{1}{n}\breve{\mathcal{D}} + \big(\mathrm{mult}_O(\mathcal{M})/n - 1\big)\breve{G} + \breve{D}_x + \breve{D}_y \equiv \big(\breve{\pi} \circ \breve{\gamma}_y\big)^* \left(K_{\breve{X}} + \frac{1}{n}\mathcal{M}\Big|_{\breve{X}} + \breve{S}_x + \breve{S}_y\right),$$

where $\breve{\mathcal{D}}$ and $\breve{G}$ are proper transforms of $\breve{\mathcal{B}}$ and $\breve{E}$ on the variety $\breve{U}_y$, respectively.

The morphism $\breve{\eta}_y$ contracts the divisor $\breve{G}$. But the morphism $\breve{\eta}_y|_{\breve{G}}$ is a $\mathbf{P}^2$-bundle.

Let $\breve{Y}$ be a general fiber of $\breve{\eta}_y\big|_{\breve{G}}$. Then $\breve{Y} \cap \breve{D}_x$ is a line in $\breve{Y} \cong \mathbf{P}^2$, the intersection $\breve{\Gamma} \cap \breve{Y}$ is a point that is not contained in $\breve{Y} \cap \breve{D}_x$, and $\breve{Y} \cap \breve{D}_y = \varnothing$. So, in the neighborhood of the fiber Y of the morphism $\breve{\eta}_y$ the locus

$$\mathrm{LCS}\left(\breve{U}_y,\ \frac{1}{n}\breve{\mathcal{D}} + \big(\mathrm{mult}_O(\mathcal{M})/n - 1\big)\breve{G} + \breve{D}_x + \breve{D}_y\right)$$

consists of $\breve{\Gamma}$ and $\breve{D}_x$, which is impossible by [15, Theorem 17.4], because $\breve{\Gamma} \cap \breve{D}_x = \varnothing$. □

5 Main result

Let X be a hypersurface in $\mathbf{P}^6$ of degree 6 with isolated ordinary double points. Suppose that X is not birationally superrigid. Let us show that this assumption leads to a contradiction.

It follows from Theorem 2.1 that there is a linear system $\mathcal{M}$ on the hypersurface X that does not have fixed components such that the log pair $\big(X, \frac{1}{m}\mathcal{M}\big)$ is not canonical, where $m \in \mathbb{N}$ such that $\mathcal{M} \sim -mK_X$.

Let Z be a proper irreducible subvariety of X such that $\big(X, \frac{1}{m}\mathcal{M}\big)$ is not canonical at general point of Z, and the subvariety Z has the biggest dimension among such subvarieties. Then $\dim(Z) \leqslant 1$ by [21, Theorem 2].

Suppose that either $\dim(Z) \neq 0$ or Z is a smooth point of X. Let P be a general point of Z, and V be a general hyperplane section of X that contains P. Put $\mathcal{B} = \mathcal{M}|_V$. Then V is a smooth hypersurface in $\mathbf{P}^5$ of degree 6, and the singularities of $\big(V, \frac{1}{m}\mathcal{B}\big)$ are not canonical at the point P by [15, Theorem 17.6].

Let S_1 and S_2 be sufficiently general divisors in $\mathcal{B}$, and $F = S_1 \cdot S_2$. Then

$$\dim\big\{O \in F \bigm| \mathrm{mult}_O\big(F\big) > m\big\} \leqslant 1$$

by [27, Proposition 5]. Let Y be a general hyperplane section of V that contains P. Put $\mathcal{P} = \mathcal{B}|_Y$. Then

$$\dim\big\{O \in F \cap Y \bigm| \mathrm{mult}_O\big(F|_Y\big) > m\big\} \leqslant 0 \tag{5.1}$$

by [10, Proposition 4.5], because Y is a smooth hypersurface in $\mathbf{P}^4$ of degree 6.

The log pair $\big(Y, \frac{1}{m}\mathcal{P}\big)$ is not log canonical at P by [15, Theorem 17.6]. Let $\eta : \mathbf{P}^4 \dashrightarrow \mathbf{P}^2$ be a general projection, and L be a general line in $\mathbb{P}^2$. Then it follows from [10, Theorem 1.1] that

$$\eta(P) \in \mathrm{LCS}\left(\mathbf{P}^2,\ L + \frac{1}{4m^2}\eta_*\Big[F\Big|_Y\Big]\right) \ni L,$$

but it follows from [10, Proposition 4.7] and the inequality 5.1 that the log pair $\big(\mathbf{P}^2, \frac{1}{4m^2}\eta_*\big[F|_Y\big]\big)$ is log terminal in a punctured neighborhood of the point $\eta(P)$. The latter is impossible by [15, Theorem 17.4], because

$$K_{\mathbf{P}^2} + L + \frac{1}{4m^2}\eta_*\Big[F\Big|_Y\Big] \equiv -\frac{1}{2}L.$$

We see that Z is a singular point of the variety X. Let $\pi : U \to X$ be a blow up of Z, and E be a π-exceptional divisor. Then $\mathrm{mult}_Z(\mathcal{M}) > m$ by Lemma 3.1. But

$$K_U + \frac{1}{m}\mathcal{H} \equiv \pi^*\left(K_X + \frac{1}{m}\mathcal{M}\right) + \left(3 - \frac{1}{m}\mathrm{mult}_Z(\mathcal{M})\right)E,$$

where $\mathcal{H}$ is a proper transform of $\mathcal{M}$ on the variety U. Let M_1 and M_2 be general divisors in $\mathcal{M}$. Then

$$\mathrm{mult}_Z\big(M_1 \cdot M_2\big) > 6m^2$$

by Lemma 4.5. Let H_1, H_2, H_3 be general hyperplane sections of X that pass through the point Z. Then

$$6m^2 = M_1 \cdot M_2 \cdot H_1 \cdot H_2 \cdot H_3 \geqslant \mathrm{mult}_Z\big(M_1 \cdot M_2\big) > 6m^2,$$

which is a contradiction. The obtained contradiction completes the proof of Theorem 1.1. □

Acknowledgements The author would like to thank the referee who helped to improve the original assertion of Lemma 4.5.

References

[1] F. Call and G. Lyubeznik, A simple proof of Grothendieck's theorem on the parafactoriality of local rings, Contemp. Math. **159**, 15–18 (1994).
[2] I. Cheltsov, On smooth quintic 4-fold, Mat. Sb. **191**, 139–162 (2000).
[3] I. Cheltsov, Nonrationality of a four-dimensional smooth complete intersection of a quadric and a quartic not containing a plane, Mat. Sb. **194**, 95–116 (2003).
[4] I. Cheltsov, Birationally superrigid cyclic triple spaces, Izv. Math. **68**, 157–208 (2004).
[5] A. Corti, Factorizing birational maps of threefolds after Sarkisov, J. Algebraic Geom. **4**, 223–254 (1995).
[6] A. Corti, Singularities of linear systems and threefold birational geometry, London Math. Soc. Lecture Note Ser. **281**, 259–312 (2000).
[7] A. Corti, J. Kollár, and K. Smith, Rational and Nearly Rational Varieties (Cambridge University Press, Cambridge, 2003).
[8] A. Corti and M. Mella, Birational geometry of terminal quartic threefolds I, Am. J. Math. **126**, 739–761 (2004).
[9] A. Corti, A. Pukhlikov, and M. Reid, Fano threefold hypersurfaces, London Math. Soc. Lecture Note Ser. **281**, 175–258 (2000).
[10] T. de Fernex, L. Ein, and M. Mustata, Bounds for log canonical thresholds with applications to birational rigidity, Math. Res. Lett. **10**, 219–236 (2003).

[11] M. Grinenko, Birational automorphisms of a three-dimensional dual quadric with the simplest singularity, Mat. Sb. **189**, 101–118 (1998).
[12] V. Iskovskikh, Birational automorphisms of three-dimensional algebraic varieties, Soviet Math. **13**, 815–868 (1980).
[13] V. Iskovskikh and Yu. Manin, Three-dimensional quartics and counterexamples to the Lüroth problem, Mat. Sb. **86**, 140–166 (1971).
[14] V. Iskovskikh and A. Pukhlikov, Birational automorphisms of multidimensional algebraic manifolds, J. Math. Sci. (New York) **82**, 3528–3613 (1996).
[15] J. Kollár et al., Flips and Abundance for Algebraic Threefolds, Astérisque **211** (1992).
[16] J. Kollár, Rational Curves on Algebraic Varieties (Springer-Verlag, Berlin, 1996).
[17] M. Mella, Birational geometry of quartic threefolds II: the importance of being $\mathbf{Q}$-factorial, Math. Ann. **330**, 107–126 (2004).
[18] A. Pukhlikov, Birational isomorphisms of four-dimensional quintics, Invent. Math. **87**, 303–329 (1987).
[19] A. Pukhlikov, Birational automorphisms of a double space and a double quartic, Izv. Akad. Nauk SSSR Ser. Mat. **52**, 229–239 (1988).
[20] A. Pukhlikov, Birational automorphisms of a three-dimensional quartic with a simple singularity, Mat. Sb. **177**, 472–496 (1988).
[21] A. Pukhlikov, Notes on theorem of V. A. Iskovskikh and Yu. I. Manin about threefold quartic, Proceedings of Steklov Inst. Math. **208**, 278–289 (1995).
[22] A. Pukhlikov, Birational automorphisms of double spaces with sigularities, J. Math. Sci. (New York) **85**, 2128–2141 (1997).
[23] A. Pukhlikov, Birational automorphisms of Fano hypersurfaces, Invent. Math. **134**, 401–426 (1998).
[24] A. Pukhlikov, Essentials of the method of maximal singularities, London Math. Soc. Lecture Note Ser. **281**, 73–100 (2000).
[25] A. Pukhlikov, Birationally rigid Fano double hypersurfaces, Sb. Math. **191**, No. 6, 883–908 (2000).
[26] A. Pukhlikov, Birationally rigid Fano complete intersections, J. Reine Angew. Math. **541**, 55–79 (2001).
[27] A. Pukhlikov, Birationally rigid Fano hypersurfaces, Izv. Math. **66**, No. 6, 1243–1269 (2002).
[28] A. Pukhlikov, Birationally rigid Fano hypersurfaces with isolated singularities, Sb. Math. **193**, 445–471 (2002).
[29] A. Pukhlikov, Birationally rigid singular Fano hypersurfaces, J. Math. Sci., New York **115**, 2428–2436 (2003).
[30] A. Pukhlikov, Birationally rigid iterated Fano double covers, Izv. Math. **67**, No. 3, 555–596 (2003).