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Two remarks on sextic double solids

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Abstract

We study the potential density of rational points on double solids ramified along singular reduced sextic surfaces. Also, we investigate elliptic fibration structures on nonsingular sextic double solids defined over a perfect field of characteristic 5.

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1. Introduction

In arithmetic geometry, it is an important problem to measure the size of the set of rational points of a variety defined over a number field. The most profound work in this area is Faltings' theorem that a nonsingular curve of genus at least two defined over a number field $\mathbb{F}$ has only finitely many $\mathbb{F}$-rational points (see [8]). One higher-dimensional analogue of the theorem is the conjecture that the set of rational points of a nonsingular variety of general type defined over a number field is contained in a proper Zariski closed subset of the variety. It is natural that we should expect the set of rational points of a Fano variety defined over a number field to have the opposite property as follows:

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Definition 1.1. The set of rational points of a variety V defined over a number field $\mathbb{F}$ is said to be potentially dense if there is a finite field extension $\mathbb{K}$ of the field $\mathbb{F}$ such that the set of $\mathbb{K}$-rational points of the variety V is Zariski dense in V.

One of the goals in this paper is to prove the potential density of singular sextic double solids, i.e., double covers of $\mathbb{P}^3$ ramified along singular sextic surfaces defined over number fields.

We have another reason why we expect the potential density of double solids ramified along singular sextic surfaces. Let $\tau : Y \to \mathbb{P}^2$ be a double cover branched over a reduced sextic curve $R \subset \mathbb{P}^2$ defined over a number field $\mathbb{F}$. The following result is proved in [3].[3]

Theorem 1.2. *Suppose that the curve R is singular. Then the set of rational points of the surface Y is potentially dense unless the curve R consists of six lines intersecting at a single point.*

Proof. Replacing $\mathbb{F}$ by its finite extension, we may always assume that a finite number of points and a finite number of maps we need are defined over the field $\mathbb{F}$. Let $\psi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ be the projection from a singular point p of the curve R. We then consider a birational morphism $\phi : U \to Y$ of a nonsingular surface U to Y such that the map $\psi \circ \tau \circ \phi$ is a morphism. Then the Kodaira dimension of U is at most zero because K_Y is linearly trivial.

Let L be a general fiber of $\psi \circ \tau \circ \phi$. Then L is connected if and only if the curve R does not consist of six lines passing through the point p. By our assumption, the fiber L is connected and hence it is either elliptic or rational. Moreover, in the latter case the set of rational points of the surface Y is potentially dense because Y is rational.

Suppose that the fiber L is elliptic. Then U is birational either to an elliptic $K3$ surface or to $E \times \mathbb{P}^1$, where E is a nonsingular elliptic curve. In the former case the set of rational points of Y is potentially dense due to [4], but in the latter case the set of rational points of $E \times \mathbb{P}^1$ is potentially dense by [9, Theorem 10.1]. □

In the case when R is a general sextic curve, it is unknown whether the set of rational points of the surface Y is potentially dense or not. If the curve $R \subset \mathbb{P}^2$ consists of six lines intersecting at a single point, the set of rational points on Y is not potentially dense due to [8] because Y is birational to $C \times \mathbb{P}^1$, where C is a hyperelliptic curve of genus 2.

Remark 1.3. In fact, Theorem 1.2 is valid in the case when $\mathbb{F}$ is a finitely generated extension of $\mathbb{Q}$. Indeed, the only fact we need for the proof of Theorem 1.2 is that for an elliptic curve E defined over $\mathbb{F}$ there is a finite extension $\mathbb{K}$ of the field $\mathbb{F}$ such that the rank of the Mordell–Weil group of $E(\mathbb{K})$ is not zero (see [9, Theorem 10.1]).

Now, we let $\pi : X \to \mathbb{P}^3$ be a double cover branched over a reduced sextic surface $S \subset \mathbb{P}^3$ and defined over a number field $\mathbb{F}$. When the surface S is a cone over a nonsingular plane sextic curve $E \subset \mathbb{P}^2$, the 3-fold X is birational to $D \times \mathbb{P}^1$, where D is a double cover of $\mathbb{P}^2$ branched over E, and hence the set of rational points of the 3-fold X is potentially dense if and only if the set of rational points of the surface D is potentially dense. However, the following will be proved in this paper.

[3] Paper [3] has a gap that it misses the fact that if the reduced sextic curve R has a point of multiplicity six, then the blow up of Y at the singular point is not rational. It is a $\mathbb{P}^1$-bundle over a nonsingular curve of genus two via Stein factorization.

Theorem 1.4. *Suppose that the surface S is singular. Then the set of rational points of the* 3*-fold X is potentially dense if the surface S is neither six planes intersecting at a single line nor a cone over a nonsingular plane sextic curve.*

If the surface S consists of six planes intersecting along a single line, the set of rational points of X is not potentially dense due to [8] because X is birational to $C \times \mathbb{P}^2$, where C is a hyperelliptic curve of genus 2.

In fact, it follows from the proof of Theorem 1.4 that Theorem 1.4 is valid over a finitely generated extension of the field $\mathbb{Q}$ as well (see Remark 1.3).

Let us move to the other goal of this paper that we investigate elliptic fibration structures on nonsingular sextic double solids defined over a perfect field of characteristic 5.

In papers [5,6], nonsingular sextic double solids defined over a perfect field of characteristic 0 or $p > 5$ are proved not to be birationally transformed into elliptic fibrations. In addition, paper [6] has an interesting example of a nonsingular sextic double solid defined over a perfect field of characteristic 5 that can be birationally transformed into an elliptic fibration.

Example 1.5. Suppose that the base field is $\mathbb{F}_5 = \mathbb{Z}/5\mathbb{Z}$. Let $\pi : X \to \mathbb{P}^3 = \mathrm{Proj}(\mathbb{F}_5[x, y, z, w])$ be the double cover ramified along the sextic surface S given by the equation

$$x^5y + x^4y^2 + x^2y^3z - y^5z - 2x^4z^2 + xz^5 + yz^5 + x^3y^2w + 2x^2y^3w \\ - xyz^3w - xyz^2w^2 - x^2yw^3 + xy^2w^3 + x^2zw^3 + xyw^4 + xw^5 + 2yw^5 = 0.$$

We can check that X is nonsingular. Also we see that $\mathrm{Pic}(X) \cong \mathbb{Z}$ (see [10]). Moreover, X contains the curve C given by the equations $x = y = 0$ whose image in $\mathbb{P}^3$ is a line L contained in the sextic $S \subset \mathbb{P}^3$. For a general enough point $p \in X$, there is a unique hyperplane $H_p \subset \mathbb{P}^3$ containing $\pi(p)$ and L. The residual quintic curve $Q_p \subset H_p$ given by $S \cap H_p = L \cup Q_p$ intersects L at a single point q_p with $\mathrm{mult}_{q_p}(Q_p|_L) = 5$. The two points $\pi(p)$ and q_p determine a line L_p in $\mathbb{P}^3$. We define a rational map $\Xi_L : X \dashrightarrow Gr(2, 4)$ by $\Xi_L(p) = L_p$. The image of the map Ξ_L is isomorphic to $\mathbb{P}^2$, hence we may assume that the map Ξ_L is a rational map of X onto $\mathbb{P}^2$. Obviously, the map Ξ_L is not defined over L. However, the resolution of indeterminacy of the map Ξ_L by blowing up X along the line L birationally transforms the 3-fold X into an elliptic fibration.

This example results from the fact that the characteristic of the base field is 5. Indeed, the nonsingular sextic surface S in the example has a line L that a generic hyperplane section passing through the line L consists of a line and a plane quintic curve intersecting at a single point with multiplicity 5, which is impossible in other characteristics. In the present paper, we will show that the birational transform into an elliptic fibration constructed above is a unique type that a nonsingular sextic double solid defined over a perfect field of characteristic 5 may have.

Theorem 1.6. *A double cover X of $\mathbb{P}^3$ defined over a perfect field of characteristic* 5 *and ramified along a nonsingular sextic surface S can be birationally transformed into an elliptic fibration if and only if the surface S has a line L such that for a generic hyperplane H passing through the line L, the residual quintic curve Q given by $S \cap H = Q \cup L$ intersects L at a point with multiplicity* 5*. Furthermore, the elliptic fibration is defined in the way of Example* 1.5.

2. Potential density

Let $\pi : X \to \mathbb{P}^3$ be a double cover ramified along a singular reduced sextic surface $S \subset \mathbb{P}^3$ and defined over a number field $\mathbb{F}$. Suppose that the surface S is neither the union of six planes intersecting along a single line nor a cone over a nonsingular plane sextic curve.

Proposition 2.1. *If the surface S is a cone over a plane sextic curve, then the set of rational points of X is potentially dense.*

Proof. Suppose that the surface S is a cone over a plane sextic curve $R \subset \mathbb{P}^2$. Then the 3-fold X is birational to $D \times \mathbb{P}^1$, where D is a double cover of $\mathbb{P}^2$ branched over $R \subset \mathbb{P}^2$. Moreover, by our assumption, the reduced curve R is neither a nonsingular curve nor the union of six lines intersecting at a single point. Therefore, the set of rational points of the 3-fold X is potentially dense by Theorem 1.2. □

Therefore, we may assume that the surface S is not a cone over a plane sextic curve in order to prove Theorem 1.4.

Let p be a singular point of the sextic surface $S \subset \mathbb{P}^3$. Replacing $\mathbb{F}$ by a finite extension of $\mathbb{F}$, we may assume that the point p is defined over $\mathbb{F}$. We then let $\overline{L}$ be a general line in $\mathbb{P}^3$ that passes through the point p. We also put $\overline{C} := \pi^{-1}(\overline{L})$. The curve $\overline{C}$ is irreducible; otherwise the surface S would be a cone with vertex p because $\overline{L}$ is a general line passing through the point p.

We consider the rational map $\rho : X \dashrightarrow \mathbb{P}^2$ defined by the composition of the morphism π and the projection of $\mathbb{P}^3$ to a hyperplane $\Pi \cong \mathbb{P}^2$ centered at the point p. We resolve the indeterminacy of the rational map ρ to obtain the following commutative diagram:

$$\begin{array}{ccc} & W & \\ {}^{g}\swarrow & & \searrow^{f} \\ X & \overset{\rho}{\dashrightarrow} & \mathbb{P}^2, \end{array}$$

where the 3-fold W is nonsingular.

Proposition 2.2. *If the normalization of the curve $\overline{C}$ is a rational curve, then the set of rational points of X is potentially dense.*

Proof. The hypothesis implies that $f : W \to \mathbb{P}^2$ is a conic bundle. Let $G \subset \mathbb{P}^3$ be a sufficiently general hyperplane defined over $\mathbb{F}$ such that $p \notin G$ and G is tangent to the surface S somewhere. Put $\hat{G} = (\pi \circ g)^{-1}(G)$. Then $\hat{G}$ does not lie in fibers of the conic bundle f.

Suppose $\hat{G}$ is reducible. Then each component of $\hat{G}$ is rational, which implies the rationality of the 3-fold X. Indeed, each component of the surface $\hat{G}$ is a section of the conic bundle $f : W \to \mathbb{P}^2$. The set of rational points of a rational variety is potentially dense.

Suppose $\hat{G}$ is irreducible. The surface $\hat{G}$ is a two-section of the conic bundle f. The set of rational points of the surface $\hat{G}$ is potentially dense by Theorem 1.2. Thus we have a conic bundle $f : W \to \mathbb{P}^2$ with a two-section $\hat{G}$ such that the set of rational points of the surface $\hat{G}$ is potentially dense. In this case the set of rational points of W is potentially dense. □

The genus of the normalization of the curve $\overline{C}$ cannot exceed 1. Therefore, we may assume that $f : W \to \mathbb{P}^2$ is an elliptic fibration.

Lemma 2.3. *Suppose that there is a family $\mathcal{M}$ of divisors in $|(\pi \circ g)^*(\mathcal{O}_{\mathbb{P}^3}(1))|$ such that for a sufficiently general point $w \in W$ there is an irreducible divisor $M \in \mathcal{M}$ that contains the point w and has potentially dense set of rational points. Then the set of rational points of the 3-fold X is potentially dense.*

Proof. Let E be a sufficiently general fiber of f and w be a sufficiently general point of E. Then there is an irreducible divisor $M \in \mathcal{M}$ passing through the point w such that the set of rational points of M is potentially dense. Moreover, by [9, Theorem 10.1], we may assume that the divisor $2w - M|_E$ is not a torsion divisor. On the other hand, the intersection $M \cap E$ consists of the point w and another point $v \in W$. Hence, the divisor $w - v$ is not a torsion divisor on E. Therefore, the set of rational points of X is potentially dense by [11, Proposition 3.4]. □

Proposition 2.4. *If the singularities of the surface S is not isolated, the set of rational points of X is potentially dense.*

Proof. Let H be a sufficiently general hyperplane in $\mathbb{P}^3$. We consider $F = (\pi \circ g)^{-1}(H)$. We easily see that $F \in |(\pi \circ g)^*(\mathcal{O}_{\mathbb{P}^3}(1))|$ and

$$\pi \circ g|_F : F \to H \cong \mathbb{P}^2$$

is a double cover branched over a singular sextic curve $H \cap S$. Replacing the field $\mathbb{F}$ by its finite extension, we may assume that H and F are defined over the field $\mathbb{F}$. Hence the set of rational points of F is potentially dense by Theorem 1.2. Therefore, the statement immediately follows from Lemma 2.3. □

From now on, we suppose that the surface S has only isolated singularities.

Let T be the set of points $q \in S \setminus \mathrm{Sing}(S)$ such that the hyperplane $D \subset \mathbb{P}^3$ tangent to the surface S at the point q satisfies the following:

(1) the intersection $S \cap D$ is reduced;
(2) the intersection $S \cap D$ does not contain the point p;
(3) the intersection $S \cap D$ does not consist of six lines passing through the point q.

Lemma 2.5. *The set T contains a nonempty Zariski open subset of the sextic $S \subset \mathbb{P}^3$.*

Proof. Let q be a general point on the sextic $S \subset \mathbb{P}^3$ and D be the hyperplane tangent to the surface S at the point q in $\mathbb{P}^3$.

Suppose that there is a line L on the surface S passing through the point q. Then the line L does not pass through any singular point of S because the point q is sufficiently general and the singularities of the surface S are isolated, but we are assuming that the surface S is not a cone. Hence, the self-intersection L^2 of the line L is negative by the adjunction formula, which contradicts the fact that L moves at least in a one-dimensional family on S. Hence, there is no line on S passing through q. In particular, the intersection $D \cap S$ does not consist of six lines passing through the point q. Moreover, the Gauss map of the surface S is finite at the point $q \in S$ by [15, Theorem 2.3], which implies that $D \cap S$ is reduced and does not contain the point p. □

Let w be a sufficiently general point on W. Then it follows from Lemma 2.5 that there is a hyperplane $D \subset \mathbb{P}^3$ such that $\pi \circ g(w) \in D$, $p \notin D$, the intersection $D \cap S$ is reduced, and the hyperplane D is tangent to the surface S at some point of the surface S.

Let $F = (\pi \circ g)^{-1}(D)$. Then the morphism

$$f|_F : F \to D \cong \mathbb{P}^2$$

is the double cover branched over the reduced sextic curve $D \cap S$ which is singular. Moreover, the set of rational points of F is potentially dense by Theorem 1.2. Hence, the set of rational points of X is potentially dense by Lemma 2.3. We have completed our proof of Theorem 1.4.

Let us conclude this section with the following result.

Theorem 2.6. *Let $\tau : V \to \mathbb{P}^r$ be a double cover ramified along a singular reduced sextic hypersurface $S \subset \mathbb{P}^r$ defined over a number field. Suppose that S is not a cone over a nonsingular sextic hypersurface in $\mathbb{P}^k$, where $k < r$. Then the set of rational points of the variety V is potentially dense for every $r \geqslant 2$.*

Proof. The claim follows from Theorem 1.2 and the proof of Theorem 1.4. Indeed, we can prove the claim by induction on r. Every step of the proof of Theorem 1.4 is valid in the case $r > 3$. Moreover, in the case $r > 3$ the claim of Lemma 2.5 is implied by the finiteness of the Gauss map of a nonsingular hypersurface in $\mathbb{P}^{r-1}$ (see [15]). □

Because of the following result proved in [7], it should be pointed out that the unirationality of a variety V defined over a number field implies that the set of rational points of V are potentially dense.

Theorem 2.7. *Let $\tau : V \to \mathbb{P}^r$ be a double cover ramified along a sufficiently general hypersurface of degree d. Then there is a natural number $r(d)$ such that V is unirational if $r \geqslant r(d)$.*

However, when the number r is relatively small, it is unknown whether the variety V of Theorem 2.7 is unirational or not.

3. Elliptic fibrations

Throughout this section, all varieties are always assumed to be defined over a perfect field k of characteristic 5. Because the field k is perfect, we may assume that it is algebraically closed.

For the problem of elliptic fibration structures on nonsingular sextic double solids over a perfect field of characteristic 5, we use mobile log pairs introduced in [2]. Before we proceed, we briefly overview their properties.

Definition 3.1. On a variety X a mobile boundary $\mathcal{M}_X = \sum_{i=1}^{n} a_i \mathcal{M}_i$ is a formal finite $\mathbb{Q}$-linear combination of linear systems $\mathcal{M}_i$ on X such that each $\mathcal{M}_i$ has no fixed component and each coefficient a_i is nonnegative. A mobile log pair $(X, \mathcal{M}_X)$ is a variety X with a mobile boundary $\mathcal{M}_X$.

Remark 3.2. Throughout this section, we always assume that every 3-fold is nonsingular. To implement theory of mobile log pairs to the fullest, we need the resolutions of indeterminacy of rational maps. However, what we need for this paper is the resolutions of indeterminacy of birational maps of nonsingular 3-folds defined over a perfect field of characteristic 5, which is proved by Abhyankar (see [1]). Because the sextic double solid in Theorem 1.6 is nonsingular, we are free to use the tools described in what follows.

To understand a mobile log pair, it is convenient that we consider a mobile log pair as a usual log pair by replacing each linear system by its general element. To be precise, for a mobile boundary $\mathcal{M}_X = \sum_{i=1}^n a_i \mathcal{M}_i$ on a variety X, we take a general member M_i from each linear system $\mathcal{M}_i$ and then we can handle the mobile boundary $\mathcal{M}_X$ with the effective $\mathbb{Q}$-divisor $\sum_{i=1}^n a_i M_i$.

By taking the scheme-theoretic intersection of two general elements of $\mathcal{M}_X$, we can also consider the self-intersection $\mathcal{M}_X^2$ of $\mathcal{M}_X$ as a well-defined effective codimension-two cycle when X is $\mathbb{Q}$-factorial.

The notions such as discrepancies, (log) terminality, and (log) canonicity can be defined for mobile log pairs as for usual log pairs (see [13]).

Let $\mathcal{M}_X = \sum_{i=1}^n a_i \mathcal{M}_i$ be a mobile boundary on a variety X. For a birational morphism $f: W \to X$, the pullback $f^*(\mathcal{M}_i)$ of $\mathcal{M}_i$ may obtain fixed components from the exceptional divisors of f. However, the proper transform $f^{-1}(\mathcal{M}_i)$ of $\mathcal{M}_i$ has no fixed component. Therefore, we may write

$$f^*(\mathcal{M}_i) = f^{-1}(\mathcal{M}_i) + \sum_{E:\ f\text{-exceptional divisor}} m_{i,E} E,$$

for each i, and hence

$$f^*(\mathcal{M}_X) = \sum_{i=1}^n a_i f^*(\mathcal{M}_i) = f^{-1}(\mathcal{M}_X) + \sum_{E:\ f\text{-exceptional divisor}} \left(\sum_{i=1}^n a_i m_{i,E} \right) E,$$

where $f^{-1}(\mathcal{M}_X) = \sum_{i=1}^n a_i f^{-1}(\mathcal{M}_i)$. On the other hand, we have

$$K_W = f^*(K_X) + \sum_{E:\ f\text{-exceptional divisor}} d_E E.$$

For an exceptional divisor E of f, we define the discrepancy of E with respect to the mobile log pair $(X, \mathcal{M}_X)$ by the rational number

$$a(X, \mathcal{M}_X, E) = d_E - \sum_{i=1}^n a_i m_{i,E}.$$

Definition 3.3. A mobile log pair $(X, \mathcal{M}_X)$ has canonical (terminal, respectively) singularities if for every birational morphism $f: W \to X$ each exceptional divisor has nonnegative (positive, respectively) discrepancy. A proper irreducible subvariety $Y \subset X$ is called a center of the canonical singularities of a mobile log pair $(X, \mathcal{M}_X)$ if there are a birational morphism $f: W \to X$ and an f-exceptional divisor $E \subset W$ such that the discrepancy $a(X, \mathcal{M}_X, E) \leqslant 0$ and $f(E) = Y$. The set of all the centers of the canonical singularities of the mobile log pair $(X, \mathcal{M}_X)$ will be denoted by $\mathbb{CS}(X, \mathcal{M}_X)$.

Note that a log pair $(X, \mathcal{M}_X)$ is terminal if and only if $\mathbb{CS}(X, \mathcal{M}_X) = \emptyset$.

Let $(X, \mathcal{M}_X)$ be a mobile log pair and $Z \subset X$ be a proper irreducible subvariety such that X is nonsingular along the subvariety Z. Then elementary properties of blow ups along nonsingular subvarieties of nonsingular varieties imply that

$$Z \in \mathbb{CS}(X, \mathcal{M}_X) \quad \Longrightarrow \quad \mathrm{mult}_Z(\mathcal{M}_X) \geqslant 1$$

and in the case when $\mathrm{codim}(Z \subset X) = 2$ we have

$$Z \in \mathbb{CS}(X, \mathcal{M}_X) \quad \Longleftrightarrow \quad \mathrm{mult}_Z(\mathcal{M}_X) \geqslant 1.$$

The following result is a key to our proof of Theorem 1.6.

Theorem 3.4. *Let X be a terminal $\mathbb{Q}$-factorial Fano variety with* $\mathrm{Pic}(X) \cong \mathbb{Z}$, *$\rho : X \dashrightarrow Y$ a birational map, and $\pi : Y \to Z$ a fibration. Suppose that a general fiber of π is a nonsingular variety of Kodaira dimension zero. Then the singularities of the mobile log pair $(X, \mathcal{M}_X)$ is not terminal, where $\mathcal{M}_X = r\rho^{-1}(|\pi^*(H)|)$ for a very ample divisor H on Z and $r \in \mathbb{Q}_{>0}$ such that $K_X + \mathcal{M}_X \sim_{\mathbb{Q}} 0$.*

Proof. Because of Remark 3.2, the proof follows the same way as in [6]. For the detail, see [6]. □

We will investigate the singularities of certain mobile log pairs on Fano varieties. It requires us to study the multiplicities of certain mobile boundaries or their self-intersections. The following result is Corollary 7.8 of [14], which holds even over fields of positive characteristic and implicitly goes back to the classical paper [12].

Theorem 3.5. *Let X be a 3-fold and $\mathcal{M}_X$ a mobile boundary on X. Suppose that a nonsingular point p on X belongs to $\mathbb{CS}(X, \mathcal{M}_X)$. Then the inequality* $\mathrm{mult}_p(\mathcal{M}_X^2) \geqslant 4$ *holds and the equality holds only when* $\mathrm{mult}_p(\mathcal{M}_X) = 2$.

Proof. See [14]. □

From now on, let $\pi : X \to \mathbb{P}^3$ be a double solid ramified along a nonsingular sextic hypersurface $S \subset \mathbb{P}^3$.

Suppose that the surface S contains a line L such that for a general hyperplane H passing through the line L the residual quintic curve Q given by $S \cap H = Q \cup L$ intersects L at a single point with multiplicity 5. For a general enough point $p \in X$ there is a unique hyperplane $H_p \subset \mathbb{P}^3$ containing $\pi(p)$ and L. The residual quintic curve $Q_p \subset H_p$ given by $S \cap H_p = L \cup Q_p$ intersects L at a single point q_p with $\mathrm{mult}_{q_p}(Q_p|_L) = 5$. The two points $\pi(p)$ and q_p determine a line L_p in $\mathbb{P}^3$. We then define a rational map $\Xi_L : X \dashrightarrow \mathbb{P}^2$ in the way described in Example 1.5. Therefore, the 3-fold X can be birationally transformed into an elliptic fibration.

For Theorem 1.6 we have to prove that the elliptic fibration described above is the only type that a nonsingular sextic double solid may have. We consider a fibration $\tau : Y \to Z$ whose general fiber is a nonsingular elliptic curve. Suppose that we have a birational map ρ of X onto Y. We then put $\mathcal{M}_X = \frac{1}{n}\mathcal{M}$ with $\mathcal{M} = \rho^{-1}(|\tau^*(H_Z)|)$, where H_Z is a very ample divisor on Z and n is the natural number such that $\mathcal{M} \subset |-nK_X|$.

By Theorem 3.4 the set $\mathbb{CS}(X, \mathcal{M}_X)$ must be nonempty. However, it does not contain a point of X. Indeed, if it contains a point p on X, from Theorem 3.5 we would obtain a contradictory inequality

$$2 = H \cdot K_X^2 = H \cdot \mathcal{M}_X^2 \geqslant \operatorname{mult}_p(\mathcal{M}_X^2) \geqslant 4,$$

where H is a general enough effective anticanonical divisor passing through the point p. Therefore, the set $\mathbb{CS}(X, \mathcal{M}_X)$ must contain a curve $C \subset X$.

Lemma 3.6. *The intersection number $-K_X \cdot C$ is* 1. *In particular, the curve $\pi(C) \subset \mathbb{P}^3$ is a line and $C \cong \mathbb{P}^1$.*

Proof. Let H be a general enough divisor in the linear system $|-K_X|$. Then we have

$$2 = H \cdot K_X^2 = H \cdot \mathcal{M}_X^2 \geqslant \operatorname{mult}_C(\mathcal{M}_X^2) H \cdot C \geqslant -K_X \cdot C,$$

which implies $-K_X \cdot C \leqslant 2$.

Suppose $-K_X \cdot C = 2$. Then $\operatorname{Supp}(\mathcal{M}_X^2) = C$ and $\operatorname{mult}_C(\mathcal{M}_X^2) = \operatorname{mult}_C^2(\mathcal{M}_X) = 1$, which means that for two different divisors M_1 and M_2 in the linear system $\mathcal{M}$ we have

$$\operatorname{mult}_C(M_1 \cdot M_2) = n^2, \qquad \operatorname{mult}_C(M_1) = \operatorname{mult}_C(M_2) = n,$$

and set-theoretically $M_1 \cap M_2 = C$. For a general enough point $p \notin C$ the linear subsystem $\mathcal{D}$ of $\mathcal{M}$ that consists of members of $\mathcal{M}$ passing through the point p has no base components because the linear system $\mathcal{M}$ is not composed from a pencil. Let D_1 and D_2 be general enough divisors in $\mathcal{D}$. Then we obtain set-theoretically

$$p \in D_1 \cap D_1 = M_1 \cap M_2 = C,$$

which is a contradiction. Consequently, the intersection number $-K_X \cdot C$ must be 1 and hence the curve $\pi(C) \subset \mathbb{P}^3$ is a line and $C \cong \mathbb{P}^1$. □

Because the proof above have not used the irreducibility of the curve C, it is clear that the set $\mathbb{CS}(X, \mathcal{M}_X)$ consists of a single curve C.

Lemma 3.7. *The line $\pi(C)$ is contained in the sextic surface S.*

Proof. Suppose $\pi(C) \not\subset S$. Let $\mathcal{H} \subset |-K_X|$ be the linear system of surfaces containing the curve C. The base locus of $\mathcal{H}$ consists of the curve C and the curve $\widetilde{C}$ such that $\pi(C) = \pi(\widetilde{C})$. Choose a general enough surface D in the pencil $\mathcal{H}$. The restriction $\mathcal{M}_X|_D$ is not mobile, but

$$\mathcal{M}_X|_D = \operatorname{mult}_C(\mathcal{M}_X)C + \operatorname{mult}_{\widetilde{C}}(\mathcal{M}_X)\widetilde{C} + \mathcal{R}_D,$$

where $\mathcal{R}_D$ is a mobile boundary. The surface D is nonsingular along $C \cup \widetilde{C}$. Thus, on the surface D, we have

$$C^2 = \widetilde{C}^2 = -2.$$

Immediately, the inequality

$$\big(1-\mathrm{mult}_{\widetilde{C}}(\mathcal{M}_X)\big)\widetilde{C}^2 \geqslant \big(\mathrm{mult}_C(\mathcal{M}_X)-1\big)C\cdot\widetilde{C}+\mathcal{R}_D\cdot\widetilde{C}\geqslant 0$$

implies $\widetilde{C}\in\mathbb{CS}(X,\mathcal{M}_X)$. It contradicts $\mathbb{CS}(X,\mathcal{M}_X)=\{C\}$. $\square$

A hyperplane section of the nonsingular sextic surface S passing though the line $\pi(C)$ consists of the line $\pi(C)$ and a quintic plane curve. For a general hyperplane section containing the line $\pi(C)$, the residual quintic curve and the line $\pi(C)$ intersect at five distinct points if the characteristic of the base field is bigger than 5. However, over a field of characteristic 5, it may happen that they intersect only at a single point.

Lemma 3.8. *For a general enough hyperplane H passing through the line $L:=\pi(C)$, the residual quintic curve Q given by $S\cap H=L\cup Q$ intersects the line L at a single point q with* $\mathrm{mult}_q(Q|_L)=5$.

Proof. Let H be a general enough hyperplane in $\mathbb{P}^3$ passing through the line L. We then consider the curve

$$D=H\cap S=L\cup Q,$$

where Q is the residual quintic curve. The curve L cannot be contained in the curve Q; otherwise the hyperplane H would be tangent to S along the line L. The curve D is singular at which the line L intersects the quintic curve Q. Note that the intersection number $L\cdot Q$ on H is 5.

Suppose that the curve Q intersects the line L at two distinct points p_1 and p_2. Then the hyperplane H is tangent to the sextic S at the points p_1 and p_2. Let L_1 and L_2 be general enough lines in H passing through the points p_1 and p_2, respectively. Then each line L_j is tangent to S at the point p_j. Therefore, the proper transform $\widetilde{L_j}\subset X$ of the curve L_j is an irreducible curve with $-K_X\cdot\widetilde{L_j}=2$. It is singular at the point $\widetilde{p_j}=\pi^{-1}(p_j)$. Consider the proper transform $\widetilde{H}$ of the surface H on X and a general surface M in the linear system $\mathcal{M}$. Then

$$M|_{\widetilde{H}}=\mathrm{mult}_C(\mathcal{M})C+R,$$

where R is an effective divisor on $\widetilde{H}$ such that $C\not\subset\mathrm{Supp}(R)$. Moreover,

$$2n=M\cdot\widetilde{L_j}\geqslant\mathrm{mult}_{\widetilde{p_j}}(\widetilde{L_j})\,\mathrm{mult}_C(M)+\sum_{p\in(M\setminus C)\cap\widetilde{L_j}}\mathrm{mult}_p(M)\cdot\mathrm{mult}_p(\widetilde{L_j})\geqslant 2n,$$

which implies $M\cap\widetilde{L_j}\subset C$ set-theoretically. However, the proper transforms of lines passing through the point p_j form a pencil on $\widetilde{H}$ whose base locus consists of the point $\widetilde{p_j}$. Therefore, the divisor R must be zero due to the generality in the choice of two curves L_1 and L_2.

Hence, set-theoretically $M\cap\widetilde{H}=C$ for a general divisor $\widetilde{H}\in|-K_X|$ passing through the curve C and a divisor $M\in\mathcal{M}$ with $\widetilde{H}\not\subset\mathrm{Supp}(M)$. Let $\tilde{p}$ be a general point on the surface $\widetilde{H}$ and $\mathcal{M}_{\tilde{p}}$ be the linear system of surfaces in $\mathcal{M}$ containing the point $\tilde{p}$. Then $\mathcal{M}_{\tilde{p}}$ has no base components because the linear system is not composed from a pencil. Therefore, for a general divisor $\widetilde{M}$ in $\mathcal{M}_{\tilde{p}}$

$$\tilde{p} \in \widetilde{M} \cap \widetilde{H} = C$$

because $\widetilde{H} \not\subset \mathrm{Supp}(\widetilde{M})$, which contradicts the generality of the point $\tilde{p} \in \widetilde{H}$. □

Lemma 3.9. *There is a birational map $\alpha : \mathbb{P}^2 \to Z$ such that the diagram*

$$\begin{array}{ccc} X & \overset{\rho}{\dashrightarrow} & Y \\ {\scriptstyle \Xi_L}\downarrow & & \downarrow{\scriptstyle \tau} \\ \mathbb{P}^2 & \overset{\alpha}{\dashrightarrow} & Z \end{array}$$

is commutative, where Ξ_L is the rational map defined in Example 1.5.

Proof. Let $g : W \to X$ be the blow up along the curve C. We then get

$$-K_W = g^*(-K_X) - E,$$

where E is the g-exceptional divisor. Let L' be a curve on W such that $\pi \circ g(L')$ is a line tangent to S at some general point of $\pi(C)$. Then

$$\mathcal{M}_W \cdot L' \leqslant 2 - 2\,\mathrm{mult}_C(\mathcal{M}_X) \leqslant 0,$$

where $\mathcal{M}_W = g^{-1}(\mathcal{M}_X)$. Because such curves as L' sweep out a Zariski dense subset in W, we obtain $\mathrm{mult}_C(\mathcal{M}_X) = 1$. Each elliptic curve L' is a fiber of the elliptic fibration $\Xi_L \circ g : W \to \mathbb{P}^2$. Thus $\mathcal{M}_W$ lies in fibers of $\Xi_L \circ g$, which implies the claim. □

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