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Double spaces with isolated singularities

I. A. Cheltsov

Abstract. The non-rationality is proved for double covers of $\mathbb{P}^n$ branched over a hypersurface $F \subset \mathbb{P}^n$ of degree $2n \geqslant 8$ with isolated singularities such that the multiplicity of each singular point of F does not exceed $2(n-2)$ and the projectivization of its tangent cone is smooth.

Bibliography: 15 titles.

§ 1. Introduction

The main method of the proof of the non-rationality of rationally connected varieties[1] consists in the proof of the birational superrigidity of certain Fano varieties.

Definition 1. A terminal $\mathbb{Q}$-factorial Fano variety X of Picard rank $\operatorname{rk}\operatorname{Pic}(X) = 1$ is called *birationally superrigid* if the following conditions hold:

- there exists no rational dominant map $\xi\colon X \dashrightarrow Z$, where Z is distinct from a point, such that the general fibre of ξ is rationally connected;
- the variety X is not birational to a terminal $\mathbb{Q}$-factorial Fano variety Y with $\operatorname{rk}\operatorname{Pic}(Y) = 1$ such that Y is not biregular to X;
- the groups $\operatorname{Bir}(V)$ and $\operatorname{Aut}(V)$ coincide.

Let $\pi\colon X \to \mathbb{P}^n$ be a double cover branched over an irreducible reduced hypersurface $F \subset \mathbb{P}^n$ of degree $2n$ with isolated ordinary singularities[2] each of multiplicity $2(n-2)$, and assume that $n \geqslant 3$. Then

$$-K_X \sim \pi^*(\mathscr{O}_{\mathbb{P}^n}(1)),$$

and the variety X is terminal $\mathbb{Q}$-factorial (see Lemma 22).

In the case when F is a general hypersurface, the non-rationality of X follows from Theorem 5.13 in [1], Ch. V. We shall prove the next result[3] in § 3.

Theorem 2. *The variety X is birationally superrigid for $n \geqslant 4$.*

It is known that $|\operatorname{Sing}(X)|$ does not exceed the number of points $(a_1, \dots, a_n) \subset \mathbb{Z}^n$ such that

$$n^2 - 2n + 2 \leqslant \sum_{i=1}^{n} a_i \leqslant n^2,$$

[1] All the varieties throughout are projective, normal, and defined over $\mathbb{C}$.

[2] A singular point O of a variety V is called an *ordinary singular point* if O is a hypersurface singularity on V and the projectivization of the tangent cone to V at O is non-singular.

[3] The result of Theorem 2 was proved in [2] in the case when F has a unique isolated ordinary singular point of even multiplicity not exceeding $2(n-2)$. For $n = 3$ the birational superrigidity of the variety X was proved in [3] under the assumption that X is factorial.

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where $a_i \in (0, 2n)$ (see [4]). For $n = 3$ the sharp bound is known: $|\mathrm{Sing}(X)| \leqslant 65$ (see [5], [6]).

Example 3. Let $n = 2k$ for positive integer k and let X be the variety given by an equation

$$y^2 = \sum_{i=1}^{k} a_i(x_0, \dots, x_{2k+1})x_i \subset \mathbb{P}(1^{2k+1}, 2k) \cong \mathrm{Proj}\bigl(\mathbb{C}[x_0, \dots, x_{2k+1}, y]\bigr),$$

where a_i is a general homogeneous polynomial of degree $4k-1$. Then X has $(4k-1)^k$ singular points, and X is birationally superrigid for $k \geqslant 2$ by Theorem 2.

Example 4. Let $n = 2k+1$ for positive integer k and let F be a hypersurface given by an equation

$$\begin{aligned} g^2(x_0, \dots, x_{2k+2}) = \sum_{i=1}^{k} a_i(x_0, \dots, x_{2k+2}) b_i(x_0, \dots, x_{2k+2}) \\ \subset \mathbb{P}^n \cong \mathrm{Proj}\bigl(\mathbb{C}[x_0, \dots, x_{2k+2}]\bigr), \end{aligned}$$

where g, the a_i, and the b_i are general homogeneous polynomials of degree $2k+1$. Then F has $(2k+1)^{2k+1}$ singular points, and X is birationally superrigid for $k \geqslant 2$ by Theorem 2.

The assumptions in Theorem 2 that $\mathrm{mult}_O(F) \leqslant 2(n-2)$ and $n \geqslant 4$ cannot be omitted.

Example 5. Let $\xi\colon V \to \mathbb{P}^n$ be a double cover branched over a general hypersurface of degree $2n$ with singular point O of multiplicity $2(n-1)$ and let

$$\gamma\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$$

be the projection from O. Then the normalization of the general fibre of $\gamma \circ \pi$ is a rational curve.

Example 6. Let $n = 3$ and let F be a Barth sextic (see [5]). Then X is rational (see [7]).

A birational classification of plane elliptic pencils was obtained in [8]. In the present paper we prove the following result (see § 4).

Theorem 7. *Let $\rho\colon X \dashrightarrow Z$ be a dominant map such that the normalization of a general fibre of it is an elliptic curve. Assume that $n \geqslant 4$. Then the diagram*

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, dashed, "\rho"'] & \mathbb{P}^n \arrow[d, "\beta"] \\ Z \arrow[r, dashed, "\gamma"'] & \mathbb{P}^{n-1} \end{tikzcd}$$

is commutative, where γ is a birational map and β is the projection from a point $O \in F$ such that $\mathrm{mult}_O(F) = 2(n-2)$.

The following result is a consequence of the proof of Theorem 7.

Theorem 8. *Assume that $n \geqslant 4$. Then X is not birational to a Fano variety with canonical singularities distinct from X.*

Theorems 7 and 8 fail for $n = 3$, even in the case of factorial X (see [3]).

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§ 2. Preliminary results

In this section we consider properties of log pairs (see [9], [10]). Throughout, X is a variety.

Theorem 9 (see [11]). *Let X be a terminal $\mathbb{Q}$-factorial Fano variety with*

$$\operatorname{rk}\operatorname{Pic}(X) = 1$$

such that a movable log pair (X, M_X) has canonical singularities once $K_X + M_X \equiv 0$ and the boundary M_X is effective. Then X is birationally superrigid.

Theorem 10 (see [10]). *Let X be a terminal $\mathbb{Q}$-factorial Fano variety with*

$$\operatorname{rk}\operatorname{Pic}(X) = 1$$

and let

$$\rho\colon Y \dashrightarrow X$$

be a birational map such that there exists an elliptic fibration $\tau\colon Y \to Z$. Consider a linear system $\mathscr{D} = |\tau^*(D)|$, where D is a very ample divisor on Z. Let $\mathscr{M} = \rho(\mathscr{D})$. Select a positive rational number γ such that $K_X + \gamma\mathscr{M} \equiv 0$ and let $M_X = \gamma\mathscr{M}$. Then the log pair (X, M_X) is not terminal.

Theorem 11 (see [10]). *Let x be a terminal $\mathbb{Q}$-factorial Fano variety with*

$$\operatorname{rk}\operatorname{Pic}(X) = 1$$

and let

$$\rho\colon Y \dashrightarrow X$$

be a non-biregular birational map such that Y is a Fano variety with canonical singularities. Let

$$\mathscr{D} = |-nK_Y|, \qquad \mathscr{M} = \rho(\mathscr{D}), \qquad M_X = \gamma\mathscr{M},$$

for some sufficiently large positive number n and a positive rational number γ such that $K_X + M_X \equiv 0$. Then the log pair (X, M_X) is not terminal.

Consider an arbitrary divisor

$$B_X = \sum_{i=1}^{r} a_i B_i$$

on X such that a_i is a positive rational number and B_i a prime Weil divisor on X. Assume that $K_X + B_X$ be a $\mathbb{Q}$-Cartier divisor.

Definition 12. A proper irreducible subvariety Y of a variety X is called a *centre of canonical singularities of a log pair* (X, B_X) if there exists a birational morphism $\pi\colon \overline{X} \to X$ and a π-exceptional divisor $E_1 \subset \overline{X}$ such that

$$K_{\overline{X}} + \sum_{i=1}^{r} a_i \overline{B}_i \equiv \pi^*(K_X + B_X) + \sum_{i=1}^{k} a(X, B_X, E_i)E_i,$$

where $a(X, B_X, E_i)$ is a rational number, E_i a π-exceptional divisor, $\overline{B}_i$ the proper transform of B_i on $\overline{X}$, $\pi(E_1) = Y$, and $a(X, M_X, E_1) \leqslant 0$.

Let $\mathbb{CS}(X, B_X)$ be the set of centres of canonical singularities of a log pair (X, B_X).

Definition 13. An irreducible subvariety Y of a variety X is called a *centre of log canonical singularities of a log pair* (X, B_X) if one of the following holds:

- for some $i \in \{1, \ldots, r\}$ we have $a_i \geqslant 1$ and $Y = B_i$;
- there exist a birational morphism $\pi\colon \overline{X} \to X$ and a divisor $E_1 \subset \overline{X}$ such that
$$K_W + \sum_{i=1}^{r} a_i \overline{B}_i \equiv \pi^*(K_X + B_X) + \sum_{i=1}^{k} a(X, B_X, E_i)E_i,$$
where $a(X, B_X, E_i)$ is a rational number, E_i a π-exceptional divisor, $\overline{B}_i$ the proper transform of B_i on $\overline{X}$, $\pi(E_1) = Y$, and $a(X, M_X, E_1) \leqslant -1$.

We denote the set of centres of log canonical divisors of a log pair (X, B_X) by $\mathbb{LCS}(X, B_X)$, and the union of centres of log canonical divisors, regarded as a subset of the variety X, as $\mathrm{LCS}(X, B_X)$.

Remark 14. We can introduce on the set $\mathrm{LCS}(X, B_X)$ in a natural fashion the structure of a subscheme (see [9]), which is usually denoted by $\mathscr{L}(X, B_X)$.

The next result is Shokurov's vanishing theorem.

Theorem 15 (see [9]). *Let H be a nef and big divisor on X such that*

$$K_X + B_X + H \equiv D,$$

where D is a Cartier divisor on X. Then

$$H^i\big(X, \mathscr{I}(X, B_X) \otimes \mathscr{O}_X(D)\big) = 0$$

for all $i > 0$, where $\mathscr{I}(X, B_X)$ is the ideal sheaf of the subscheme $\mathscr{L}(X, B_X)$.

Theorem 16 (see [9], Theorem 17.6). *Let S be a simple Weil divisor on X such that*

$$K_X + S + B_X$$

is a $\mathbb{Q}$-Cartier divisor. Assume that $S \not\subset \mathrm{Supp}(B_X)$ and that $a_i < 1$ for each $i \in \{1, \ldots, r\}$. Then the log pair $(X, S + B_X)$ is purely log terminal if and only if the log pair $(S, \mathrm{Diff}_S(B_X))$ is Kawamata log terminal[4].

[4] For the definitions of pure log terminality and Kawamata log terminality one can consult [9].

Corollary 17. *Let H be an effective Cartier divisor on X. Assume that there exists a subvariety $Z \subset H$ such that $Z \in \mathbb{CS}(X, B_X)$, both X and H are smooth at a generic point of Z, and $H \not\subset \operatorname{Supp}(B_X)$. Then*

$$\mathbb{LCS}\bigl(H, B_X\big|_H\bigr) \neq \varnothing.$$

We say that (X, B_X) is *movable* if $B_1, \dots, B_r$ are linear systems without fixed components. Constructions applicable to usual log pairs can also be applied to movable ones.

Theorem 18 (see [11]). *Let (X, B_X) be a movable log pair and assume that the set $\mathbb{CS}(X, B_X)$ contains a smooth point O of X and that $\dim(X) \geqslant 3$. Then*

$$\operatorname{mult}_O(B_X^2) = \operatorname{mult}_O\biggl(\biggl(\sum_{i=1}^{r} a_i B_i'\biggr) \cdot \biggl(\sum_{i=1}^{r} a_i B_i''\biggr)\biggr) \geqslant 4, \tag{$*$}$$

where B_i' and B_i'' are general divisors in B_i. Inequality ($*$) is strict for $\dim(X) \geqslant 4$.

Theorem 19. *Assume that the set $\mathbb{CS}(X, B_X)$ contains an ordinary double point O of the variety X and that $\dim(X) \geqslant 3$. Then*

$$\operatorname{mult}_O(B_X) \geqslant 1, \tag{$**$}$$

where the rational number $\operatorname{mult}_O(B_X)$ is defined by the numerical equivalence

$$\sum_{i=1}^{r} a_i \overline{B}_i \equiv \pi^*(B_X) - \operatorname{mult}_O(B_X) E;$$

here $\pi\colon \overline{X} \to X$ is the blow-up of the point O, E is a π-exceptional divisor, and $\overline{B}_i$ the proper transform of the divisor B_i on $\overline{X}$. Inequality ($**$) is strict for $\dim(X) \geqslant 4$.

Proof. This follows from the proof of Theorem 3.10 of [11] and the application of Theorem 16.

We now establish two results underlying the proof of Theorem 2.

Proposition 20. *Let $\tau\colon V \to \mathbb{P}^k$ be a double cover ramified in a smooth hypersurface $S \subset \mathbb{P}^k$ of degree $2d$ and B_V an effective divisor on V such that*

$$B_V \equiv \tau^*(\mathscr{O}_{\mathbb{P}^k}(\lambda)),$$

where $\lambda \in \mathbb{Q}$ and the inequalities $\lambda < 1$ and $2 \leqslant d \leqslant k-1$ hold. Then

$$\mathbb{LCS}(V, B_V) = \varnothing.$$

Proof. Let $C \subset V$ be a curve such that $\tau(C) \subset S$ and $\operatorname{mult}_C(B_V) \geqslant 1$. Take a point $O \in \tau(C)$ and a hypersurface $\Pi \subset \mathbb{P}^k$ tangent to S at O. Fix a line $L \subset \Pi$ passing through O. Let

$$\widehat{L} = \tau^{-1}(L)$$

and let $\widehat{O} = \tau^{-1}(O)$. Then the curve $\widehat{L}$ is singular at $\widehat{O}$ and a component of $\widehat{L}$ lies in $\operatorname{Supp}(B_V)$ since otherwise

$$2 > 2\lambda = \widehat{L} \cdot B_V \geqslant \operatorname{mult}_{\widehat{O}}(\widehat{L}) \operatorname{mult}_C(B_V) \geqslant 2,$$

which is a contradiction. The hyperplane Π is tangent to S at finitely many points (see [12]). Hence the curve $\widehat{L}$ spans the whole of V as O varies on the curve $\tau(C)$ and the line L varies in Π, which is a contradiction.

Assume now that the set $\mathbb{LCS}(V, B_V)$ contains a subvariety $R \subset V$ such that $\dim(R) \geqslant 2$. Then $\mathrm{mult}_R(B_V) \geqslant 1$ and R contains a curve $\widehat{C}$ such that

$$\mathrm{mult}_{\widehat{C}}(B_V) \geqslant 1$$

and $\tau(\widehat{C}) \subset S$. This is a contradiction. Hence the set $\mathbb{LCS}(V, B_V)$ contains only curves and points.

Suppose that $\mathbb{LCS}(V, B_V)$ contains a curve. Let T be the union of all curves in the set $\mathbb{LCS}(V, B_V)$. Let Y be a general divisor in the linear system $|\tau^*(\mathscr{O}_{\mathbb{P}^k}(1))|$. Let

$$\gamma = \tau\big|_Y : Y \to \mathbb{P}^{k-1}$$

and $B_Y = B_V\big|_Y$. Then the variety Y is smooth, $Y \not\subset \mathrm{Supp}(B_V)$, and γ is a double cover branched over a smooth hypersurface of degree $2d$.

It follows from the generality in the choice Y that the set $\mathbb{LCS}(Y, B_Y)$ does not contain subvarieties of positive dimension. The set $\mathbb{LCS}(Y, B_Y)$ contains all points of $T \cap Y$.

Consider a Cartier divisor F on the variety Y such that

$$F \equiv K_Y + B_Y + (1-\lambda)H \sim (d-k-1)H,$$

where $H = \gamma^*(\mathscr{O}_{\mathbb{P}^{k-1}}(1))$. Then the sequence of groups

$$H^0\big(\mathscr{O}_Y(F)\big) \to H^0\big(\mathscr{O}_{\mathscr{L}(Y,B_Y)}(F)\big) \to 0$$

is exact by Theorem 15. On the other hand, $\mathrm{Supp}(\mathscr{L}(Y, B_Y))$ consists of all the points in $T \cap Y$, therefore

$$H^0\big(\mathscr{O}_{\mathscr{L}(Y,B_Y)}(F)\big) = H^0\big(\mathscr{O}_{\mathscr{L}(Y,B_Y)}\big),$$

which is impossible for $d < k-1$. In the case $d = k-1$ we obtain $|T \cap Y| = 1$.

Thus, it follows from the assumption that $\mathbb{LCS}(V, B_V)$ contains a curve that

- $d = k-1$;
- the set $\mathbb{LCS}(V, B_V)$ contains a unique curve $\overline{C} \subset V$;
- the curve $\tau(\overline{C}) \subset \mathbb{P}^k$ is a line;
- the map $\tau\big|_{\overline{C}}$ is an isomorphism;
- the inequality $\mathrm{mult}_{\overline{C}}(B_V) \geqslant 1$ holds.

We observe that $\tau(\overline{C}) \not\subset S$. Therefore, there exists a curve $\widetilde{C} \subset V$ such that $\overline{C} \neq \widetilde{C}$ and $\tau(\overline{C}) = \tau(\widetilde{C})$.

Let $D_1, \ldots, D_{k-2}$ be general divisors in $|\tau^*(\mathscr{O}_{\mathbb{P}^k}(1))|$ passing through the curves $\overline{C}$ and $\widetilde{C}$. Let

$$D = \bigcap_{i=1}^{k-2} D_i$$

and consider the curves $\overline{C}$ and $\widetilde{C}$ on the smooth surface D. Then

$$\overline{C}^2 = \widetilde{C}^2 = 1 - d < 0$$

because $d>2$ by assumption. Consider the divisor $B_D = B_V\big|_D$. Then

$$B_D = \mathrm{mult}_{\overline{C}}(B_V)\overline{C} + \mathrm{mult}_{\widetilde{C}}(B_V)\widetilde{C} + \Delta,$$

where Δ is an effective divisor on D such that $\overline{C} \not\subset \mathrm{Supp}(\Delta) \not\supset \widetilde{C}$. On the other hand,

$$B_D \equiv \lambda(\overline{C} + \widetilde{C}),$$

so that we have the equivalence

$$\big(\lambda - \mathrm{mult}_{\widetilde{C}}(B_V)\big)\widetilde{C} \equiv \big(\mathrm{mult}_{\overline{C}}(B_V) - \lambda\big)\overline{C} + \Delta$$

and therefore $\mathrm{mult}_{\widetilde{C}}(B_V) \geqslant \lambda$ because $\widetilde{C}^2 < 0$. Now, the equivalence

$$-\Delta \equiv \big(\mathrm{mult}_{\overline{C}}(B_V) - \lambda\big)\overline{C} + \big(\mathrm{mult}_{\widetilde{C}}(B_V) - \lambda\big)\widetilde{C}$$

yields $\Delta = \varnothing$ and the equalities

$$\mathrm{mult}_{\widetilde{C}}(B_V) = \mathrm{mult}_{\overline{C}}(B_V) = \lambda,$$

which is impossible because $\mathrm{mult}_{\overline{C}}(B_V) \geqslant 1$ and $\lambda < 1$. A contradiction.

Thus, we have shown that the set $\mathbb{LCS}(V, B_V)$ contains only points.

Assume that $\mathbb{LCS}(V, B_V)$ contains a point $O \in V$. Let E be a Cartier divisor such that

$$E \equiv K_V + B_V + (1-\lambda)H,$$

where $H = \tau^*(\mathscr{O}_{\mathbb{P}^k}(1))$. Then $H^0(\mathscr{O}_V(E)) = 0$. The sequence of cohomology groups

$$H^0(\mathscr{O}_V(E)) \to H^0\big(\mathscr{O}_{\mathscr{L}(V,B_V)}(E)\big) \to 0$$

is exact by Theorem 15, but on the other hand, $\mathrm{Supp}(\mathscr{L}(V, B_V))$ consists of finitely many points. Hence

$$0 = H^0\big(\mathscr{O}_{\mathscr{L}(V,B_V)}(E)\big) = H^0(\mathscr{O}_{\mathscr{L}(V,B_V)}) \neq 0,$$

which is a contradiction. Thus, the set $\mathbb{LCS}(V, B_V)$ is empty.

Proposition 21. *Let S be a smooth hypersurface of degree d in $\mathbb{P}^n$ and B an effective divisor on $\mathbb{P}^n$ such that $B \equiv \mathscr{O}_{\mathbb{P}^n}(\lambda)$ for $\lambda \in \mathbb{Q}$, where $\lambda < 1$ and $2 \leqslant d \leqslant 2(n-1)$. Then $\mathbb{LCS}(\mathbb{P}^n,\, B + \frac{1}{2}S) = \varnothing$.*

Proof. Let Z be a variety of highest dimension in $\mathbb{LCS}(\mathbb{P}^n,\, B + \frac{1}{2}S)$. Then

$$\lambda + \frac{1}{2} \geqslant \mathrm{mult}_Z(B) + \frac{1}{2}\,\mathrm{mult}_Z(S) \geqslant \mathrm{mult}_Z\left(B + \frac{1}{2}\,S\right) \geqslant 1,$$

which implies that $Z \subset S$ and $Z \neq S$. In particular, $\dim(Z) < n-1$.

Assume that Z is a point. Let E be the Cartier divisor such that

$$E = K_{\mathbb{P}^n} + B + \frac{1}{2}\,S + \left(n - \frac{d}{2} - \lambda\right)H,$$

where $H \sim \mathscr{O}_{\mathbb{P}^n}(1)$. Then $n > d/2+\lambda$ and $E \sim -H$, so that $H^0(\mathscr{O}_{\mathbb{P}^n}(E)) = 0$. The sequence

$$H^0(\mathscr{O}_{\mathbb{P}^n}(E)) \to H^0\bigl(\mathscr{O}_{\mathscr{L}(\mathbb{P}^n,\, B+\frac{1}{2}S)}(E)\bigr) \to 0$$

is exact by Theorem 15. However, $\operatorname{Supp}\bigl(\mathscr{L}(\mathbb{P}^n,\, B+\frac{1}{2}S)\bigr)$ consists of finitely many points, therefore

$$H^0\bigl(\mathscr{O}_{\mathscr{L}(\mathbb{P}^n,\, B+\frac{1}{2}S)}(E)\bigr) = H^0(\mathscr{O}_{\mathscr{L}(\mathbb{P}^n,\, B+\frac{1}{2}S)}),$$

which is a contradiction. Thus, $\dim(Z) > 0$.

Let $B + \frac{1}{2}S = D + \beta S$, where D is an effective divisor and β a positive rational number such that $S \not\subset \operatorname{Supp}(D)$. Then $\beta < 1$ and $D \equiv \mu H$ for some positive rational number $\mu < 1$, where $H \sim \mathscr{O}_{\mathbb{P}^n}(1)$. In particular,

$$Z \in \mathbb{LCS}(\mathbb{P}^n,\, D+S),$$

so that $\mathbb{LCS}\bigl(S, D\big|_S\bigr) \neq \varnothing$ by Theorem 16 because $Z \subset S$.

Theorem 16 ensures the existence of a proper subvariety $T \subset S$ such that

$$T \in \mathbb{LCS}\bigl(S, D\big|_S\bigr)$$

and $Z \subseteq T$. We have $\dim(T) \geqslant 1$ and $\operatorname{mult}_T\bigl(D\big|_S\bigr) \geqslant 1$, which is impossible (see [12]).

§ 3. Proof of Theorem 2

Let X be a weighted complete intersection in $\mathbb{P}(a_0, \dots, a_n)$ such that

- the complete intersection $X \subset \mathbb{P}(a_0, \dots, a_n)$ is well formed (see [13]);
- the complete intersection X has at most isolated singularities;
- $\dim(X) \geqslant 4$.

Let

$$L = \mathscr{O}_{\mathbb{P}(a_0,\dots,a_n)}(1)\big|_X.$$

Lemma 22. *The group* $\operatorname{Cl}(X)$ *is generated by the divisor* L.

Proof. Consider a Weil divisor D on the variety X. For the proof it is sufficient to show that $D \sim rL$ for some $r \in \mathbb{Z}$.

Let H be a general divisor in $|kL|$ for $k \gg 0$. Then H is a smooth weighted complete intersection (see [13]) in $\mathbb{P}(a_0, \dots, a_n)$ such that H is well formed and $\dim(H) \geqslant 3$.

It is known that $\operatorname{Pic}(H)$ is generated by the divisor $L\big|_H$ (see [13]). Hence there exists an integer $r \in \mathbb{Z}$ such that $D\big|_H \sim rL\big|_H$. Let $\Delta = D - rL$. The sequence of sheaves

$$0 \to \mathscr{O}_X(\Delta) \otimes \mathscr{O}_X(-H) \to \mathscr{O}_X(\Delta) \to \mathscr{O}_H \to 0$$

is exact because the sheaf $\mathscr{O}_X(\Delta)$ is locally free in the neighbourhood of H. Therefore, the sequence of cohomology groups

$$0 \to H^0(\mathscr{O}_X(\Delta)) \to H^0(\mathscr{O}_H) \to H^1\bigl(\mathscr{O}_X(\Delta) \otimes \mathscr{O}_X(-H)\bigr)$$

is exact. On the other hand, the sheaf $\mathscr{O}_X(\Delta)$ is reflexive (see [14]). Consequently, there exists an exact sequence of sheaves

$$0 \to \mathscr{O}_X(\Delta) \to \mathscr{E} \to \mathscr{F} \to 0,$$

where $\mathscr{E}$ is a locally free sheaf and $\mathscr{F}$ is torsion free. Hence the sequence of groups

$$H^0\big(\mathscr{F} \otimes \mathscr{O}_X(-H)\big) \to H^1\big(\mathscr{O}_X(\Delta - H)\big) \to H^1\big(\mathscr{E} \otimes \mathscr{O}_X(-H)\big)$$

is exact. However, $H^0\big(\mathscr{F} \otimes \mathscr{O}_X(-H)\big)$ is trivial because the sheaf $\mathscr{F}$ has no torsion and $H^1\big(\mathscr{E} \otimes \mathscr{O}_X(-H)\big)$ is trivial by the lemma of Enriques–Severi–Zariski (see [15]) since X is a normal variety. Thus, we have

$$H^1\big(\mathscr{O}_X(\Delta) \otimes \mathscr{O}_X(-H)\big) = 0$$

and $H^0(\mathscr{O}_X(\Delta)) = \mathbb{C}$. The same method yields $H^0(\mathscr{O}_X(-\Delta)) = \mathbb{C}$, so that the divisor Δ is rationally equivalent to zero, that is, $D \sim rL$.

Now let $\pi\colon X \to \mathbb{P}^n$ be a double cover ramified in an irreducible reduced hypersurface $F \subset \mathbb{P}^n$ of degree $2n \geqslant 8$ with isolated ordinary singular points of multiplicities at most $2(n-2)$. Then

$$\mathrm{Cl}(X) = \mathrm{Pic}(X) = \mathbb{Z}[-K_X]$$

by Lemma 22. Assuming that X is not birationally superrigid we shall bring our assumptions to a contradiction.

On the Fano variety X there exists a movable log pair (X, M_X) with effective boundary M_X such that

$$\mathbb{CS}(X, M_X) \neq \varnothing$$

and $M_X \equiv -rK_X$ for some positive rational number $r < 1$ (see Theorem 9).

Let $Z \subset X$ be an element of the set $\mathbb{CS}(X, M_X)$.

Lemma 23. *The subvariety Z is not a smooth point of X.*

Proof. Let Z be a smooth point of X. Then

$$\mathrm{mult}_Z(M_X^2) > 4$$

by Theorem 18. Let $H_1, \dots, H_{n-2}$ be general divisors in $|\pi^*(\mathscr{O}_{\mathbb{P}^n}(1))|$ passing through Z. Then

$$2 > M_X^2 \cdot H_1 \cdots H_{n-2} \geqslant \mathrm{mult}_Z(M_X^2)\,\mathrm{mult}_Z(H_1) \cdots \mathrm{mult}_Z(H_{n-2}) > 4,$$

which is a contradiction.

Lemma 24. *The subvariety Z is not a singular point of X.*

Proof. Let $Z \in \mathrm{Sing}(X)$. Then $O = \pi(Z)$ is a singular point of a hypersurface $F \subset \mathbb{P}^n$. There exist two possible cases: $\mathrm{mult}_O(F)$ is either even or odd. We handle these cases separately. In the first case the proof is based on Proposition 20 and in the second on Proposition 21.

We point out that X can be defined as a hypersurface

$$y^2 = f_{2n}(x_0, \dots, x_n) \subset \mathbb{P}(1^{n+1}, n) \cong \mathrm{Proj}\big(\mathbb{C}[x_0, \dots, x_n, y]\big),$$

where f_{2n} is a homogeneous polynomial of degree $2n$.

Let $\mathrm{mult}_O(F) = 2m \geqslant 2$ for some $m \in \mathbb{N}$. Then $m \leqslant n-2$ and there exists a weighted blow-up

$$\beta \colon U \to \mathbb{P}(1^{n+1}, n)$$

of the point Z with weights $(m, 1^n)$ such that the proper transform $V \subset U$ of the variety X is non-singular in the neighbourhood of the β-exceptional divisor E. The birational morphism β induces a birational morphism $\alpha \colon V \to X$ with exceptional divisor $G \subset V$.

We observe that $E\big|_V = G$ and G is a smooth hypersurface in $E \cong \mathbb{P}(1^n, m)$, which can be given by the equation

$$z^2 = g_{2m}(t_1, \ldots, t_n) \subset \mathbb{P}(1^n, m) \cong \mathrm{Proj}\big(\mathbb{C}[t_1, \ldots, t_n, z]\big),$$

where $g_{2m}(t_1, \ldots, t_n)$ is a homogeneous polynomial of degree $2m$.

Let $\mathrm{mult}_Z(M_X)$ be a positive rational number such that

$$M_V \equiv \alpha^*(M_X) - \mathrm{mult}_Z(M_X)G,$$

where M_V is the proper transform of M_X on V. Then

$$K_V + M_V \equiv \alpha^*(K_X + M_X) + \big(n - 1 - m - \mathrm{mult}_Z(M_X)\big)G.$$

However, the linear system $|\alpha^*(-K_X) - G|$ yields a morphism $\psi \colon V \to \mathbb{P}^{n-1}$ such that the diagram

$$\begin{tikzcd} V \arrow[rr, "\psi"] \arrow[d, "\alpha"'] & & \mathbb{P}^{n-1} \\ X \arrow[r, "\pi"'] & \mathbb{P}^n \arrow[ur, dashed, "\chi"'] & \end{tikzcd}$$

is commutative, where χ is the projection from O. Let C be a general fibre of ψ. Then

$$0 \leqslant M_V \cdot C = 2\big(1 - \mathrm{mult}_Z(M_X)\big) + \alpha^*(K_X + M_X) \cdot C < 2\big(1 - \mathrm{mult}_Z(M_X)\big)$$

because $-(K_X + M_X)$ is ample. Thus, $\mathrm{mult}_Z(M_X) < 1$.

By the inequality $\mathrm{mult}_Z(M_X) < 1$ and Theorem 19 we obtain $m > 1$. The inequality

$$n - 1 - m - \mathrm{mult}_Z(M_X) > 0$$

implies the existence of a proper subvariety $\Delta \subset G$ that is a centre of canonical singularities of the log pair (V, M_V). Hence

$$\mathbb{LCS}\big(G, M_V\big|_G\big) \neq \varnothing$$

by Corollary 17, which contradicts Proposition 20.

Thus, we have shown that $\mathrm{mult}_O(F) = 2k+1 \geqslant 3$ for $k \in \mathbb{N}$ such that $k \leqslant n-3$.

Let $\alpha \colon W \to \mathbb{P}^n$ be a blow-up of O, Λ the exceptional divisor of a birational morphism α, and $\widetilde{F} \subset W$ the proper transform of the hypersurface F. Then $\widetilde{F}$ is smooth in the neighbourhood of Λ because O is an ordinary singular point. Let $S = \widetilde{F} \cap \Lambda$. Then

$$S \subset \Lambda \cong \mathbb{P}^{n-1}$$

is a smooth hypersurface of degree $2k+1$.

Let $\widetilde{\pi}\colon \widetilde{X} \to W$ be a double cover ramified in the divisor

$$\widetilde{F} \cup \Lambda \sim 2\big(\alpha^*(\mathscr{O}_{\mathbb{P}^n}(n)) - k\Lambda\big),$$

which is smooth only in S. Let $\widetilde{S} = \widetilde{\pi}^{-1}(S)$. Then W is smooth outside $\widetilde{S} \subset W$ and W has an ordinary double point along $\widetilde{S}$.

Let Ξ be the proper transform of Λ on $\widetilde{X}$. Then $\Xi \cong \mathbb{P}^{n-1}$ and there exists a birational morphism $\xi\colon \widetilde{X} \to X$ contracting Ξ into the point Z so that the diagram

$$\begin{tikzcd}
V \arrow[r, "\widetilde{\pi}"] \arrow[d, "\xi"'] & W \arrow[dr, dashed, "\alpha"] & \\
X \arrow[rr, "\pi"'] & & \mathbb{P}^n
\end{tikzcd}$$

is commutative. It is easy to see that ξ is the restriction to X of the weighted blow-up of the weighted projective space $\mathbb{P}(1^{n+1}, n)$ at the smooth point Z with weights $(2k+1,\, 2^n)$.

Let $\mathrm{mult}_Z(M_X)$ be a positive rational number such that

$$M_{\widetilde{X}} \equiv \xi^*(M_X) - \mathrm{mult}_Z(M_X)\Xi,$$

where $M_{\widetilde{X}}$ is the proper transform of the divisor M_X on $\widetilde{X}$. Then

$$K_{\widetilde{X}} + M_{\widetilde{X}} \equiv \xi^*(K_X + M_X) + \big(2(n-1-k) - \mathrm{mult}_Z(M_X)\big)\Xi,$$

and the linear system $|\xi^*(-K_X) - 2\Xi|$ yields a fibration $\omega\colon \widetilde{X} \to \mathbb{P}^{n-1}$ such that the diagram

$$\begin{tikzcd}
V \arrow[rr, "\omega"] \arrow[d, "\xi"'] & & \mathbb{P}^{n-1} \\
X \arrow[r, "\pi"'] & \mathbb{P}^n \arrow[ur, dashed, "\chi"'] &
\end{tikzcd}$$

is commutative; here χ is the projection from O.

Intersecting $M_{\widetilde{X}}$ with a general fibre of ω we see that $\mathrm{mult}_Z(M_X) < 2$. Then

$$2(n-1-k) - \mathrm{mult}_Z(M_X) > 0$$

which implies the existence of a centre of canonical singularities

$$\nabla \in \mathbb{CS}\big(\widetilde{X},\, M_{\widetilde{X}} - \big(2(n-1-k) - \mathrm{mult}_Z(M_X)\big)\Xi\big)$$

such that $\nabla \subset G$. Thus,

$$\nabla \in \mathbb{LCS}\big(\widetilde{X},\, M_{\widetilde{X}} - \big(2(n-1-k) - \mathrm{mult}_Z(M_X)\big)\Xi + 2\Xi\big)$$

because 2Ξ is a Cartier divisor. However,

$$\mathbb{LCS}\big(\widetilde{X},\, M_{\widetilde{X}} - \big(2(n-2-k) - \mathrm{mult}_Z(M_X)\big)\Xi\big) \subset \mathbb{LCS}(\widetilde{X},\, M_{\widetilde{X}} + \Xi)$$

since $2k+1 \leqslant 2(n-2)$, which means that

$$\mathbb{LCS}\big(\Xi, \mathrm{Diff}_{\Xi}(M_{\widetilde{X}})\big) = \mathbb{LCS}\big(\Xi,\, M_{\widetilde{X}}\big|_{\Xi} + \mathrm{Diff}_{\Xi}(0)\big) \neq \varnothing$$

by Theorem 16. In the present case $\mathrm{Diff}_{\Xi}(0) = \frac{1}{2}\widetilde{S}$ (see [9]) and

$$M_{\widetilde{X}}\big|_{\Xi} \equiv -\operatorname{mult}_Z(M_X)\Xi\big|_{\Xi} \equiv \frac{\operatorname{mult}_Z(M_X)}{2}\,H,$$

where H is a hyperplane section of the hypersurface $\Xi \cong \mathbb{P}^{n-1}$. We now see that

$$\mathbb{LCS}\left(\Xi,\, M_{\widetilde{X}}\big|_{\Xi} + \frac{1}{2}\,\widetilde{S}\right) \neq \varnothing,$$

which contradicts Proposition 21.

Lemma 25. $\operatorname{codim}(Z \subset X) = 2$.

Proof. Assume that $\operatorname{codim}(Z \subset X) > 2$. Then $\dim(Z) \neq 0$ by Lemmas 23 and 24, but

$$\operatorname{mult}_Z(M_X^2) \geqslant 4$$

by Theorem 18. Take a generic point O on Z and sufficiently general divisors $H_1, \ldots, H_{n-2}$ in the linear system $|\pi^*(\mathscr{O}_{\mathbb{P}^n}(1))|$ that pass through the point O. Then

$$2 > M_X^2 \cdot H_1 \cdots H_{n-2} \geqslant \operatorname{mult}_Z(M_X^2) \geqslant 4,$$

which is a contradiction.

Let $H_1, \ldots, H_{n-2}$ be general divisors in $|\pi^*(\mathscr{O}_{\mathbb{P}^n}(1))|$. Then

$$2 > M_X^2 \cdot H_1 \cdots H_{n-2} \geqslant \operatorname{mult}_Z^2(M_X) Z \cdot H_1 \cdots H_{n-2} \geqslant Z \cdot H_1 \cdots H_{n-2}$$

because $-(K_X + M_X)$ is ample, so that $\pi(Z)$ is a linear subspace of $\mathbb{P}^n$ of dimension $n-2$, and $\pi\big|_Z \colon Z \to \pi(Z)$ is an isomorphism.

Let $V = \bigcap_{i=1}^{n-3} H_i$, $C = Z \cap V$, $M_V = M_X\big|_V$, and $\tau = \pi\big|_V$. Then

- V is a smooth 3-fold;
- $C \subset V$ is an irreducible curve;
- M_V is an effective movable boundary;
- $\tau \colon V \to \mathbb{P}^3$ is a double cover;
- τ is branched over a surface $S \subset \mathbb{P}^3$ of degree $2n$;
- $\tau(C)$ is a line in $\mathbb{P}^3$;
- $\tau\big|_C$ is an isomorphism;
- $\tau^*(\mathscr{O}_{\mathbb{P}^3}(1)) - M_V$ is an ample divisor;
- $\operatorname{mult}_C(M_V) = \operatorname{mult}_Z(M_X)$.

Assume that $\tau(C) \not\subset S$. Then there exists a curve $\widetilde{C} \subset V$ such that

$$\tau(C) = \tau(\widetilde{C})$$

and $C \neq \widetilde{C}$. Consider a general divisor $D \in |\tau^*(\mathscr{O}_{\mathbb{P}^3}(1))|$ passing through C. Then D is a smooth surface, and C, $\widetilde{C}$ are smooth rational curves on this surface. We have

$$C^2 = \widetilde{C}^2 = 1 - n < 0$$

on the surface D. Let $M_D = M_V\big|_D$. Then

$$M_D = \mathrm{mult}_C(M_V)C + \mathrm{mult}_{\widetilde{C}}(M_V)\widetilde{C} + \Delta,$$

where Δ is a movable boundary on D. On the other hand, $M_V \equiv rD$ for a positive rational number $r < 1$. Hence

$$\bigl(r - \mathrm{mult}_{\widetilde{C}}(M_V)\bigr)\widetilde{C} \equiv \bigl(\mathrm{mult}_C(M_V) - r\bigr)C + \Delta,$$

and the inequality $\widetilde{C}^2 < 0$ implies that $\mathrm{mult}_{\widetilde{C}}(M_V) \geqslant r$. Let H be a sufficiently general divisor in the linear system $|\tau^*(\mathscr{O}_{\mathbb{P}^3}(1))|$. Then

$$2r^2 = M_V^2 \cdot H \geqslant \mathrm{mult}^2_C(M_V) + \mathrm{mult}^2_{\widetilde{C}}(M_V) \geqslant 1 + r^2,$$

which contradicts the inequality $r < 1$. Thus, we have shown that $\tau(C) \subset S$.

Let O be a general point on $\tau(C)$ and T a hyperplane in $\mathbb{P}^3$ tangent to S at O. Let $\breve{L}$ be an irreducible curve in V such that $O \in \tau(\breve{L}) \subset T$. Then

$$\breve{L} \subset \mathrm{Supp}(M_V)$$

because otherwise we have the incompatible inequalities

$$2 > \breve{L} \cdot M_V \geqslant 2\,\mathrm{mult}_C(M_V) \geqslant 2$$

since $\breve{L}$ is singular at the point dominating O.

The curve $\breve{L}$ spans a divisor on V as the line $\tau(\breve{L}) \subset T$ varies. This contradicts the movability of M_V, which completes the proof of Theorem 2.

§ 4. Proof of Theorems 7 and 8

Let $\pi\colon X \to \mathbb{P}^n$, $n \geqslant 4$, be a double cover branched over an irreducible reduced hypersurface $F \subset \mathbb{P}^n$ of degree $2n$ that has only isolated ordinary singularities of multiplicity at most $2(n-2)$.

Let $\rho\colon Y \dashrightarrow X$ be a birational map such that Y carries the structure of an elliptic fibration $\tau\colon Y \to Z$. Consider a general very ample divisor D on the variety Z and the linear system $\mathscr{D} = |\tau^*(D)|$. Let $\mathscr{M} = \rho(\mathscr{D})$.

Remark 26. The linear system $\mathscr{M}$ is not composed of a pencil.

By Lemma 22 there exists a positive rational number r such that the rational equivalence $K_X + r\mathscr{M} \equiv 0$ holds. Let $M_X = r\mathscr{M}$. Then

$$\mathbb{CS}(X, M_X) \neq \varnothing$$

by Theorem 10. Let Z be an element of the set $\mathbb{CS}(X, M_X)$ having the highest dimension.

Lemma 27. *The subvariety $Z \subset X$ is not a smooth point of X.*

Proof. See the proof of Lemma 23.

Lemma 28. *Let Z be a singular point of X. Then the diagram*

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, dashed, "\tau\circ\rho"'] & \mathbb{P}^n \arrow[d, "\chi"] \\ Z \arrow[r, dashed, "\gamma"'] & \mathbb{P}^{n-1} \end{tikzcd}$$

is commutative, where γ is a birational map and χ the projection from a point $\pi(Z)\in F$ such that $\mathrm{mult}_{\pi(Z)}(F)=2(n-2)$.

Proof. Let $O=\pi(Z)$. Then O is an ordinary singular point of $F\subset\mathbb{P}^n$ such that $\mathrm{mult}_O(F)\leqslant 2(n-2)$.

Let $\mathrm{mult}_O(F)=2m\geqslant 2$ for some positive integer m. The variety X is a hypersurface in $\mathbb{P}(1^{n+1},n)$ of degree $2n$ and there exists a weighted blow-up

$$\beta\colon U\to\mathbb{P}(1^{n+1},n)$$

of the point Z with weights $(m,1^n)$ such that the proper transform $V\subset U$ of X is smooth in the neighbourhood of the exceptional divisor E of β.

The morphism β induces a birational morphism $\alpha\colon V\to X$. Let G be the exceptional divisor of α. Then $E\big|_V=G$ and G is a double cover of $\mathbb{P}^{n-1}$ branched over a smooth hypersurface of degree $2m$.

Let M_V be the proper transform of M_X on V. Then

$$M_V\equiv\alpha^*(M_X)-\mathrm{mult}_Z(M_X)G,$$

where $\mathrm{mult}_Z(M_X)$ is a positive rational number. We now have

$$K_V+M_V\equiv\alpha^*(K_X+M_X)+\big(n-1-m-\mathrm{mult}_Z(M_X)\big)G,$$

and on the other hand, the linear system $|\alpha^*(-K_X)-G|$ defines a fibration

$$\psi\colon V\to\mathbb{P}^{n-1}$$

such that $\psi=\chi\circ\pi\circ\alpha$, where $\chi\colon\mathbb{P}^n\dashrightarrow\mathbb{P}^{n-1}$ is the projection from the point O. Let C be a general fibre of ψ. Then

$$M_V\cdot C=2\big(1-\mathrm{mult}_Z(M_Z)\big)$$

and $g(C)=n-m+1$. Hence $\mathrm{mult}_Z(M_X)\leqslant 1$.

The equality $\mathrm{mult}_Z(M_X)=1$ implies that ψ and τ are birationally equivalent fibrations, which is impossible in the case when $m<n-2$ because then $g(C)\neq 1$. In the case when $m=n-2$ the birational equivalence of the fibrations τ and ψ yields the required result.

We can assume that $\mathrm{mult}_Z(M_X)<1$. Proceeding as in the proof of Lemma 24 we now arrive at a contradiction.

Thus, we can assume that O is a singular point of odd multiplicity of the hypersurface F. In this case the above arguments in combination with the proof of Lemma 24 bring us to a contradiction.

Lemma 29. $\mathrm{codim}(Z\subset X)=2$.

Proof. See the proof of Lemma 25.

Lemma 30. *The equality* $\mathrm{codim}(Z \subset X) = 2$ *is impossible.*

Proof. Let $\mathrm{codim}(Z \subset X) = 2$. Consider sufficiently general divisors $H_1, \dots, H_{n-2}$ in $|\pi^*(\mathscr{O}_{\mathbb{P}^n}(1))|$. Then

$$2 = M_X^2 \cdot H_1 \cdots H_{n-2} \geqslant \mathrm{mult}_Z^2(M_X) Z \cdot H_1 \cdots H_{n-2} \geqslant Z \cdot H_1 \cdots H_{n-2},$$

and the integer $k = Z \cdot H_1 \cdots H_{n-2}$ is equal to 1 or 2.

Let $k = 2$. Then for two sufficiently general divisors D_1 and D_2 in the linear system $\mathscr{M}$ their intersection $D_1 \cap D_2$ coincides with Z in the set-theoretic sense. Let P be a sufficiently general point of X not lying in Z and let $\mathscr{D}$ be a linear subsystem of $\mathscr{M}$ consisting of the divisors passing through P. Then $\mathscr{D}$ has no base components because $\mathscr{M}$ is not composed of a pencil. Assume that both divisors D_1 and D_2 are from $\mathscr{D}$. Then $D_1 \cap D_2 = Z$, which is a contradiction.

Thus, $k = 1$. That is, $\pi(Z)$ is a linear subspace of $\mathbb{P}^n$ of dimension $n-2$ and $\pi\big|_Z$ is an isomorphism.

Assume that $\pi(Z) \not\subset F$. Then there exists a subvariety $\widetilde{Z} \subset X$ of codimension 2 such that $\pi(\widetilde{Z}) = \pi(Z)$ and $\widetilde{Z} \neq Z$. The proof of Theorem 2 now immediately yields

$$\mathrm{mult}_{\widetilde{Z}}(M_X) = \mathrm{mult}_Z(M_X) = 1,$$

and we can obtain a contradiction as in the case $k = 2$.

Thus, $\pi(Z) \subset F$. Consider the smooth 3-fold

$$V = \bigcap_{i=1}^{n-3} H_i$$

and the curve $C = Z \cap V$. Let $M_V = M_X\big|_V$, $\mathscr{D} = \mathscr{M}\big|_V$, and $\tau = \pi\big|_V$. Then

$$\tau \colon V \to \mathbb{P}^3$$

is a double cover branched over a smooth surface $S \subset \mathbb{P}^3$ of degree $2n$.

The curve $\tau(C)$ is a line lying in S, and $\tau\big|_C$ is an isomorphism. Moreover, we have the equivalence

$$M_V \equiv \tau^*(\mathscr{O}_{\mathbb{P}^3}(1)),$$

and $\mathrm{mult}_C(M_V) = \mathrm{mult}_Z(M_X) \geqslant 1$.

Let O be a general point on $\tau(C)$ and T a hyperplane in $\mathbb{P}^3$ tangent to the hypersurface S at O. Let $\breve{L}$ be an irreducible curve on V such that $O \in \tau(\breve{L}) \subset T$. Then

$$2 \geqslant \breve{L} \cdot M_V \geqslant 2\,\mathrm{mult}_C(M_V) \geqslant 2,$$

and therefore $\mathrm{mult}_C(M_V) = 1$ because $\breve{L}$ spans a divisor as the line $\tau(\breve{L}) \subset T$ varies.

Let $\psi \colon U \to V$ be the blow-up of C, G the ψ-exceptional divisor, D a general divisor in

$$\big|(\tau \circ \psi)^*(\mathscr{O}_{\mathbb{P}^3}(1)) - G\big|,$$

and M_U the proper transform of M_V on the variety U. We set $M_D = M_U\big|_D$. Then

$$M_D = \mathrm{mult}_{\widetilde{C}}(M_U)\widetilde{C} + \Delta,$$

where $\widetilde{C} \subset G$ is a base curve of the linear system $\big|(\tau \circ f)^*(\mathscr{O}_{\mathbb{P}^3}(1)) - G\big|$ and Δ is a movable boundary on D. We now have $\widetilde{C}^2 = 1 - n$, but on the other hand,

$$M_D \equiv \widetilde{C},$$

which immediately implies that $\mathrm{mult}_{\widetilde{C}}(M_U) = 1$ and $\Delta = \varnothing$.

Blowing up the curve $\widetilde{C}$ we see that the linear system $\mathscr{D}$ lies in the fibres of the rational map given by the pencil $\big|(\tau \circ f)^*(\mathscr{O}_{\mathbb{P}^3}(1)) - G\big|$, which is impossible because $\mathscr{D}$ is not composed of a pencil.

Thus, the proof of Theorem 7 is complete. The proof of Theorem 8 is almost identical to the proof of Theorem 7. The only difference is the use of Theorem 11 instead of Theorem 10.

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