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Extremal metrics on two Fano varieties

I. A. Cheltsov

Abstract. We prove the existence of an orbifold Kähler-Einstein metric on a general hypersurface in $\mathbb{P}(1^3,2,2)$ of degree 6 and a general hypersurface in $\mathbb{P}(1^3,2,3)$ of degree 7.

Bibliography: 50 titles.

Keywords: Fano varieties, Kähler-Einstein metric, log-canonical threshold, Tian alpha-invariant.

§ 1. Introduction

The *multiplicity* of a non-zero polynomial $\varphi\in\mathbb{C}[z_1,\dots,z_n]$ at the origin $O\in\mathbb{C}^n$ is

$$m=\min\left\{m\in\mathbb{N}\cup\{0\}\;\Big|\;\frac{\partial^m\varphi(z_1,\dots,z_n)}{\partial^{m_1}z_1\partial^{m_2}z_2\cdots\partial^{m_n}z_n}(O)\neq 0\right\},$$

which implies that $m\neq 0\iff\varphi(O)=0$. There is a similar invariant

$$c_0(\varphi)=\sup\left\{\varepsilon\in\mathbb{Q}\;\Big|\;\text{the function }\frac{1}{|\varphi|^{2\varepsilon}}\text{ is locally integrable near }O\in\mathbb{C}^n\right\}\in\mathbb{Q},$$

which is called the complex singularity *exponent* of the polynomial φ at O.

Example 1.1. Let $m_1,\dots,m_n$ be positive integers. Let $\varphi=\sum_{i=1}^n z_i^{m_i}$. Then

$$c_0(\varphi)=\min\left(1,\sum_{i=1}^n\frac{1}{m_i}\right).$$

Example 1.2. Let $m_1,\dots,m_n$ be positive integers. Let $\varphi=\prod_{i=1}^n z_i^{m_i}$. Then

$$c_0(\varphi)=\min\left(\frac{1}{m_1},\frac{1}{m_2},\dots,\frac{1}{m_n}\right).$$

Let X be a variety[1] with at most log terminal singularities, let $Z\subseteq X$ be a closed subvariety, and let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on the variety X. Then the number

$$\mathrm{lct}_Z(X,D)=\sup\bigl\{\lambda\in\mathbb{Q}\mid\text{the log pair }(X,\lambda D)\text{ is log canonical along }Z\bigr\}\in\mathbb{Q}$$

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[1]All varieties are assumed to be complex, algebraic, projective and normal.

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is called a *log canonical threshold* of the divisor D along Z. It follows from [1] that

$$\mathrm{lct}_O\big(\mathbb{C}^n, (\varphi = 0)\big) = c_0(\varphi),$$

so that $\mathrm{lct}_Z(X, D)$ is an algebraic counterpart of the number $c_0(\phi)$. One has

$$\begin{aligned}\mathrm{lct}_X(X, D) &= \inf\big\{\mathrm{lct}_P(X, D) \mid P \in X\big\}\\ &= \sup\big\{\lambda \in \mathbb{Q} \mid \text{the log pair } (X, \lambda D) \text{ is log canonical}\big\},\end{aligned}$$

and we put $\mathrm{lct}(X, D) = \mathrm{lct}_X(X, D)$ for simplicity.[2]

Example 1.3. Let $X = \mathbb{P}^2$ and $D \in |\mathscr{O}_{\mathbb{P}^2}(3)|$. Then

$$\mathrm{lct}(X, D) = \begin{cases} 1 & \text{if } D \text{ is a curve with at most ordinary double points,}\\ 5/6 & \text{if } D \text{ is a curve with one cuspidal point,}\\ 3/4 & \text{if } D \text{ consists of an irredicible conic and a line that are tangent,}\\ 2/3 & \text{if } D \text{ consists of three lines intersecting at one point,}\\ 1/2 & \text{if } \mathrm{Supp}(D) \text{ consists of two lines,}\\ 1/3 & \text{if } \mathrm{Supp}(D) \text{ consists of one line.}\end{cases}$$

Now suppose additionally that X is a Fano variety.

Definition 1.4. The *global log canonical threshold* of the Fano variety X is the quantity

$$\mathrm{lct}(X) = \inf\big\{\mathrm{lct}(X, D) \mid D \text{ is an effective } \mathbb{Q}\text{-divisor on } X \text{ such that } D \equiv -K_X\big\} \geqslant 0.$$

The number $\mathrm{lct}(X)$ is an algebraic counterpart of the α-invariant of a variety X introduced in [3]. One easily sees that

$$\mathrm{lct}(X) = \sup\big\{\lambda \in \mathbb{Q} \mid \text{the log pair } (X, \lambda D) \text{ is log canonical for every effective } \mathbb{Q}\text{-divisor } D \equiv -K_X\big\}.$$

Example 1.5. Let X be a smooth hypersurface in $\mathbb{P}^n$ of degree $m < n$. Then

$$\mathrm{lct}(X) = \frac{1}{n + 1 - m}$$

as shown in [4]. In particular, the equality $\mathrm{lct}(\mathbb{P}^n) = 1/(n + 1)$ holds.

Example 1.6. Let X be a smooth hypersurface in $\mathbb{P}(1^{n+1}, d)$ of degree $2d \geqslant 2$. Then

$$\mathrm{lct}(X) = \frac{1}{n + 1 - d}$$

in the case when $2 \leqslant d \leqslant n - 1$ (see [5]).

[2] Log canonical thresholds were introduced by Shokurov in [2].

Example 1.7. Let X be a rational homogeneous space such that

$$\mathrm{Pic}(X) = \mathbb{Z}[D],$$

where D is an ample divisor. We have

$$-K_X \sim rD$$

for some integer $r \geqslant 1$. Then $\mathrm{lct}(X) = 1/r$ (see [6]).

In general the number $\mathrm{lct}(X)$ depends on small deformations of the variety X.

Example 1.8. Let X be a smooth hypersurface in $\mathbb{P}(1,1,1,1,3)$ of degree 6. Then

$$\mathrm{lct}(X) \in \left\{\frac{5}{6}, \frac{43}{50}, \frac{13}{15}, \frac{33}{38}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10}, \frac{11}{12}, \frac{13}{14}, \frac{15}{16}, \frac{17}{18}, \frac{19}{20}, \frac{21}{22}, \frac{29}{30}, 1\right\}$$

by [7] and [8] and all these values of $\mathrm{lct}(X)$ are attained.

Example 1.9. Let X be a smooth hypersurface in $\mathbb{P}(1^{n+1}, n)$ of degree $2n$. Then the inequalities

$$1 \geqslant \mathrm{lct}(X) \geqslant \frac{2n-1}{2n}$$

hold (see [8]). Moreover, the equality $\mathrm{lct}(X) = 1$ holds if X is general and $n \geqslant 3$.

Example 1.10. Let X be a smooth hypersurface in $\mathbb{P}^n$ of degree $n \geqslant 2$. Then the inequalities

$$1 \geqslant \mathrm{lct}(X) \geqslant \frac{n-1}{n}$$

hold (see [4]). Moreover, it follows from [7] and [8] that

$$\mathrm{lct}(X) \geqslant \begin{cases} 1 & \text{if } n \geqslant 6, \\ 22/25 & \text{if } n = 5, \\ 16/21 & \text{if } n = 4, \\ 3/4 & \text{if } n = 3, \end{cases}$$

whenever X is general, but $\mathrm{lct}(X) = 1 - 1/n$ if X contains a cone of dimension $n-2$.

It is unknown in the general case whether $\mathrm{lct}(X) \in \mathbb{Q}$ or not, but many examples confirm that it is a rational number.

Example 1.11. Let X be a smooth del Pezzo surface. It follows from [9] that

$$\mathrm{lct}(X) = \begin{cases} 1 & \text{if } K_X^2 = 1 \text{ and } |-K_X| \text{ contains no cuspidal curves}, \\ 5/6 & \text{if } K_X^2 = 1 \text{ and } |-K_X| \text{ contains a cuspidal curve}, \\ 5/6 & \text{if } K_X^2 = 2 \text{ and } |-K_X| \text{ contains no tacnodal curves}, \\ 3/4 & \text{if } K_X^2 = 2 \text{ and } |-K_X| \text{ contains a tacnodal curve}, \\ 3/4 & \text{if } X \text{is a cubic in } \mathbb{P}^3 \text{ with no Eckardt point}, \\ 2/3 & \text{if } X \text{ is a cubic in } \mathbb{P}^3 \text{ with Eckardt point, or } K_X^2 = 4, \\ 1/2 & \text{if } X \cong \mathbb{P}^1 \times \mathbb{P}^1 \text{ or } K_X^2 \in \{5,6\}, \\ 1/3 & \text{in the remaining cases.} \end{cases}$$

Example 1.12. Let X be a singular cubic surface in $\mathbb{P}^3$. It follows from [10] that

$$\mathrm{lct}(X)=\begin{cases} 2/3 & \text{if } \mathrm{Sing}(X)=\{\mathbb{A}_1\},\\ 1/3 & \text{if } \mathrm{Sing}(X)\supseteq\{\mathbb{A}_4\},\\ 1/3 & \text{if } \mathrm{Sing}(X)=\{\mathbb{D}_4\},\\ 1/3 & \text{if } \mathrm{Sing}(X)\supseteq\{\mathbb{A}_2,\mathbb{A}_2\},\\ 1/4 & \text{if } \mathrm{Sing}(X)\supseteq\{\mathbb{A}_5\},\\ 1/4 & \text{if } \mathrm{Sing}(X)=\{\mathbb{D}_5\},\\ 1/6 & \text{if } \mathrm{Sing}(X)=\{\mathbb{E}_6\},\\ 1/2 & \text{in the remaining cases.}\end{cases}$$

We expect that the following holds[3] (cf. [11], Question 1).

Conjecture 1.13. There is an effective $\mathbb{Q}$-divisor $D\equiv -K_X$ on X such that

$$\mathrm{lct}(X)=\mathrm{lct}(X,D)\in\mathbb{Q}.$$

The following deep result holds (see [3], [12], [13]).

Theorem 1.14. *Suppose that X has at most quotient singularities. If*

$$\mathrm{lct}(X)>\frac{\dim(X)}{\dim(X)+1},$$

then X admits an orbifold Kähler-Einstein metric.

If a variety with quotient singularities admits an orbifold Kähler-Einstein metric, then

- either its canonical divisor is numerically trivial;
- or its canonical divisor is ample (a variety of general type);
- or its canonical divisor is antiample (a Fano variety).

Remark 1.15. Every variety with at most quotient singularities that has numerically trivial or ample canonical divisor always admits an orbifold Kähler-Einstein metric (see [14]–[16]).

If $\mathrm{Sing}(X)=\varnothing$, then X does not admit a Kähler-Einstein metric if

- either the group $\mathrm{Aut}(X)$ is not reductive (see [17]);
- or the tangent bundle of X is not polystable with respect to $-K_X$ (see [18]);
- or the Futaki character of holomorphic vector fields on X does not vanish (see [19]).

Corollary 1.16. *The following varieties admit no Kähler-Einstein metric*:

- *a blow up of $\mathbb{P}^2$ at one or two distinct points* (*see* [17]);
- *a smooth Fano threefold $\mathbb{P}(\mathscr{O}_{\mathbb{P}^2}\oplus\mathscr{O}_{\mathbb{P}^2}(1))$* (*see* [20]);
- *a smooth Fano fourfold*

$$\mathbb{P}\big(\alpha^*(\mathscr{O}_{\mathbb{P}^1}(1))\oplus\beta^*(\mathscr{O}_{\mathbb{P}^2}(1))\big),$$

  where $\alpha\colon\mathbb{P}^1\times\mathbb{P}^2\to\mathbb{P}^1$ and $\beta\colon\mathbb{P}^1\times\mathbb{P}^2\to\mathbb{P}^2$ are natural projections (*see* [19]).

[3]The assertion of Conjecture 1.13 is unknown even for del Pezzo surfaces.

There are also more subtle obstructions to the existence of a Kähler-Einstein metric.

Example 1.17. Let X be a smooth Fano threefold such that

$$\operatorname{Pic}(X) = \mathbb{Z}[-K_X]$$

and $-K_X^3 = 22$. Then

- the tangent bundle of the threefold X is stable (see [20]);
- the group $\operatorname{Aut}(X)$ is trivial if the threefold X is general;
- there exists X such that $\operatorname{Aut}(X)$ is a trivial group, but X admits no Kähler-Einstein metric (see [21]);
- if $\operatorname{Aut}(X) \cong \operatorname{PSL}(2,\mathbb{C})$, then X has a Kähler-Einstein metric (see [22]).

The problem of the existence of Kähler-Einstein metrics on smooth toric Fano varieties is completely solved. Namely, the following result holds (see [23]–[26]).

Theorem 1.18. *If X is smooth and toric, then the following conditions are equivalent*:

- *the Fano variety X admits a Kähler-Einstein metric*;
- *the Futaki character of holomorphic vector fields of X vanishes*;
- *the barycentre of the reflexive polytope of X is zero.*

However, we do not know many smooth Fano varieties that admit a Kähler-Einstein metric.

Example 1.19. By [3], [12], [27] and [28] the following varieties admit Kähler-Einstein metrics:

- every smooth del Pezzo surface whose automorphism group is reductive;
- every Fermat hypersurface in $\mathbb{P}^n$ of degree $d \leqslant n$ for $d \geqslant n/2$;
- every double cover X of $\mathbb{P}^n$ branched in a hypersurface of degree $2d$ for $n \geqslant d > (n+1)/2$;
- every smooth complete intersection in $\mathbb{P}^n$ of two quadric hypersurfaces.

The problem of the existence of orbifold Kähler-Einstein metrics on singular Fano varieties that have quotient singularities is not well studied even in dimension 2.

Example 1.20. Let X be a cubic surface in $\mathbb{P}^3$. Then

- the surface X admits a Kähler-Einstein metric if $\operatorname{Sing}(X) = \varnothing$ (see [27]);
- the surface X does not admit an orbifold Kähler-Einstein metric if X has a singular point that is not of type $\mathbb{A}_1$ or $\mathbb{A}_2$ (see [29]);
- the cubic surface given by the equation

  $$xyz + xyt + xzt + yzt = 0 \subseteq \mathbb{P}^3 \cong \operatorname{Proj}(\mathbb{C}[x,y,z,t])$$

  admits a Kähler-Einstein metric and has four singular points of type $\mathbb{A}_1$ (see [10]);
- the cubic surface given by the equation

  $$xyz = t^3 \subseteq \mathbb{P}^3 \cong \operatorname{Proj}(\mathbb{C}[x,y,z,t]),$$

  admits a Kähler-Einstein metric and has three singular points of type $\mathbb{A}_2$ (see [10]);

- it is unknown whether X admits a Kähler-Einstein metric in the remaining cases.

One can use Theorem 1.14 to construct many examples of Fano varieties with quotient singularities that admit an orbifold Kähler-Einstein metric.

Example 1.21. Let X be a quasismooth hypersurface in $\mathbb{P}(a_0, a_1, a_2, a_3)$ of degree $\sum_{i=0}^{3} a_i - 1$, where $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3$. Then $\mathrm{lct}(X) > 2/3$ if X is general and singular (see [13], [30]–[32]).

Example 1.22. Let X be a quasismooth hypersurface in $\mathbb{P}(a_0, \ldots, a_4)$ of degree $\sum_{i=0}^{4} a_i - 1$, where $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$. Then it follows from [33] that

- $\mathrm{lct}(X) > 3/4$ for at least 1936 values of the quintuple $(a_0, a_1, a_2, a_3, a_4)$;
- $\mathrm{lct}(X) \geqslant 1$ for at least 1605 values of the quintuple $(a_0, a_1, a_2, a_3, a_4)$.

It is clear from Examples 1.9–1.11, 1.21 and 1.22 that the number $\mathrm{lct}(X)$ is important in Kähler geometry. It also plays an important role in birational geometry.

Example 1.23. Let V and $\overline{V}$ be varieties with at most terminal and $\mathbb{Q}$-factorial singularities and let Z be a smooth curve. Suppose that there is a commutative diagram

$$\begin{array}{ccc} V & \overset{\rho}{\dashrightarrow} & \overline{V} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\bar{\pi}} \\ Z & = & Z \end{array}$$

such that π and $\bar{\pi}$ are flat morphisms and ρ is a birational map inducing an isomorphism

$$V \setminus X \cong \overline{V} \setminus \overline{X},$$

where X and $\overline{X}$ are scheme fibres of π and $\bar{\pi}$ over a point $O \in Z$, respectively. Suppose that

- the fibres X and $\overline{X}$ are irreducible and reduced;
- the divisors $-K_V$ and $-K_{\overline{V}}$ are π-ample and $\bar{\pi}$-ample, respectively;
- the varieties X and $\overline{X}$ have at most log terminal singularities;

and ρ is not an isomorphism. Then it follows from [34] and [10] that

$$\mathrm{lct}(X) + \mathrm{lct}(\overline{X}) \leqslant 1, \qquad (*)$$

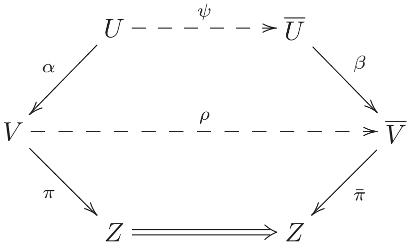
where X and $\overline{X}$ are Fano varieties by the adjunction formula.

In general inequality $(*)$ is easily seen to be sharp.

Example 1.24. Let $\pi \colon V \to Z$ be a surjective flat morphism such that

- the variety V is smooth and $\dim(V) = 3$;
- the variety Z is a smooth curve;
- the divisor K_V is π-ample;

let X be a scheme fibre of the morphism π over a point $O \in Z$ such that X is a smooth cubic surface in $\mathbb{P}^3$, and let L_1, L_2, L_3 be lines in X passing through a point $P \in V$. Then it follows from [35] that there is a commutative diagram



such that α is a blow up of the point P, the map ψ is an antiflip in the proper transforms of the lines L_1, L_2, L_3 and β is a contraction of the proper transform of the fibre X. Then

- the birational map ρ is not an isomorphism;
- the threefold $\overline{V}$ has terminal and $\mathbb{Q}$-factorial singularities;
- the divisor $-K_{\overline{V}}$ is a Cartier $\bar{\pi}$-ample divisor;
- the map ρ induces an isomorphism $V \setminus X \cong \overline{V} \setminus \overline{X}$, where $\overline{X}$ is a scheme fibre of $\bar{\pi}$ over the point O.

Then $\overline{X}$ is a cubic surface with a singular point of type $\mathbb{D}_4$, which implies that $\mathrm{lct}(X) = 2/3$ and $\mathrm{lct}(\overline{X}) = 1/3$ (see Examples 1.11 and 1.12).

We now describe another application of $\mathrm{lct}(X)$. Suppose additionally that X has at most $\mathbb{Q}$-factorial terminal singularities and $\mathrm{rk}\,\mathrm{Pic}(X) = 1$.

Definition 1.25. The Fano variety X is said to be *birationally superrigid*[4] if for every linear system $\mathscr{M}$ on the variety X that has no fixed components the log pair $(X, \mathscr{M})$ has canonical singularities, where λ is a rational number such that $K_X + \lambda\mathscr{M} \equiv 0$.

If the variety X is birationally superrigid, then

- there is no rational dominant map $\rho\colon X \dashrightarrow Y$ such that the general fibre of the map ρ is rationally connected and $\dim(Y) \geqslant 1$;
- there is no non-biregular map $\rho\colon X \dashrightarrow Y$ such that Y has terminal $\mathbb{Q}$-factorial singularities and $\mathrm{rk}\,\mathrm{Pic}(Y) = 1$;
- the variety X is non-rational.

Example 1.26. The following smooth Fano varieties are birationally superrigid:

- a general hypersurface in $\mathbb{P}^n$ of degree $n \geqslant 4$ (see [38], [39]);
- a smooth hypersurface in $\mathbb{P}(1^{n+1}, n)$ of degree $2n \geqslant 6$ (see [40], [41]).

Let $X_1, \dots, X_r$ be Fano varieties with at most $\mathbb{Q}$-factorial terminal singularities such that $\mathrm{rk}\,\mathrm{Pic}(X_i) = 1$ for every $i = 1, \dots, r$. The following result was proved in [7].

Theorem 1.27. *If* X_i *is birationally superrigid and* $\mathrm{lct}(X_i) \geqslant 1$ *for all* $i = 1, \dots, r$, *then*

$$\mathrm{Bir}(X_1 \times \dots \times X_r) = \mathrm{Aut}(X_1 \times \dots \times X_r),$$

[4] There are several definitions of birational superrigidity (see [36], [37]).

the variety $X_1 \times \cdots \times X_r$ *is non-rational and for every rational dominant map* $\rho\colon X_1 \times \cdots \times X_r \dashrightarrow Y$ *whose general fibre is rationally connected there is a commutative diagram*

$$\begin{array}{ccc} X_1 \times \cdots \times X_r & & \\ \downarrow{\scriptstyle \pi} & \overset{\rho}{\searrow} & \\ X_{i_1} \times \cdots \times X_{i_k} & \overset{\xi}{\dashrightarrow} & Y \end{array}$$

for some $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, r\}$, *where* ξ *is a birational map and* π *is the projection.*

Fano varieties satisfying the hypotheses of Theorem 1.27 do exist (see Examples 1.9, 1.10 and 1.26).

Definition 1.28. The variety X is said to be *birationally rigid*[5] if for every non-empty linear system $\mathscr{M}$ on X that has no fixed components there exists $\xi \in \mathrm{Bir}(X)$ such that the log pair

$$(X, \lambda\xi(\mathscr{M}))$$

has canonical singularities, where λ is a rational number such that $K_X + \lambda\xi(\mathscr{M}) \equiv 0$.

If X is birationally rigid, then

- there is no rational dominant map $\rho\colon X \dashrightarrow Y$ such that a general fibre of the map ρ is rationally connected and $\dim(Y) \geqslant 1$;
- there is no birational map $\rho\colon X \dashrightarrow Y$ such that $Y \not\cong X$, the variety Y has terminal $\mathbb{Q}$-factorial singularities and $\mathrm{rk}\,\mathrm{Pic}(Y) = 1$;
- the variety X is non-rational.

Example 1.29. The following Fano threefolds are birationally rigid, but not birationally superrigid:

- a general complete intersection of a quadric and a cubic in $\mathbb{P}^5$ (see [42]);
- a smooth threefold that is a double cover of a smooth three-dimensional quadric in $\mathbb{P}^4$ branched over a surface of degree 8 (see [40]).

One usually seeks the birational automorphism from Definition 1.28 among a given set of birational automorphisms. This leads to the following definition.

Definition 1.30. A subset Γ of $\mathrm{Bir}(X)$ *untwists all maximal singularities* on the variety X if for each linear system $\mathscr{M}$ on X that has no fixed components there exists $\xi \in \Gamma$ such that the log pair

$$(X, \lambda\xi(\mathscr{M}))$$

has canonical singularities, where λ is a rational number such that $K_X + \lambda\xi(\mathscr{M}) \equiv 0$.

If there is a subset $\Gamma \subset \mathrm{Bir}(X)$ that untwists all maximal singularities, then the group $\mathrm{Bir}(X)$ is generated by Γ and the biregular automorphisms.

[5] There are several definitions of birational rigidity (see [36], [37]).

Example 1.31. Let X be a general hypersurface in $\mathbb{P}^n$ of degree $n \geqslant 5$ that has one singular point O, which is an ordinary singular point of multiplicity $n-2$. Then the projection

$$\psi\colon X \dashrightarrow \mathbb{P}^{n-1}$$

from the point O induces an involution that untwists all maximal singularities (see [43]).

We now show how Theorem 1.27 can be generalized for birationally rigid Fano varieties.

Definition 1.32. The variety X is *universally birationally rigid* if for any variety U the variety

$$X \otimes \operatorname{Spec}(\mathbb{C}(U))$$

is birationally rigid over a field of rational functions $\mathbb{C}(U)$ of the variety U.

It should be pointed out that Definition 1.28 makes sense also for Fano varieties defined over an arbitrary perfect field.

Definition 1.33. A subset Γ of $\operatorname{Bir}(X)$ *universally untwists all maximal singularities* if for every variety U the induced subgroup

$$\Gamma \subset \operatorname{Bir}(X) \subseteq \operatorname{Bir}\bigl(X \otimes \operatorname{Spec}(\mathbb{C}(U))\bigr)$$

untwists all maximal singularities on the variety $X \otimes \operatorname{Spec}(\mathbb{C}(U))$ defined over the field of rational functions $\mathbb{C}(U)$ of U.

One can easily verify that any subset of $\operatorname{Aut}(X)$ universally untwists all maximal singularities if the Fano variety X is birationally superrigid.

Remark 1.34. As Kollár pointed out [44], if $\dim(X) \geqslant 2$, then a subset Γ of $\operatorname{Bir}(X)$ universally untwists all maximal singularities if and only if Γ untwists all maximal singularities and $\operatorname{Bir}(X)$ is countable.

Let $X_1, \dots, X_r$ be Fano varieties with terminal $\mathbb{Q}$-factorial singularities and assume that $\operatorname{rk}\operatorname{Pic}(X_i) = 1$ for every $i = 1, \dots, r$. Consider the natural projection

$$\pi_i\colon X_1 \times \dots \times X_{i-1} \times X_i \times X_{i+1} \times \dots \times X_r \longrightarrow X_1 \times \dots \times X_{i-1} \times \widehat{X_i} \times X_{i+1} \times \dots \times X_r$$

and let $\beth_i$ be a general fibre of π_i in the scheme sense.

Remark 1.35. $\beth_i$ is a Fano variety defined over the field of rational functions of the variety

$$X_1 \times \dots \times X_{i-1} \times \widehat{X_i} \times X_{i+1} \times \dots \times X_r.$$

There are natural embeddings of groups

$$\prod_{i=1}^{r} \operatorname{Bir}(X_i) \subseteq \bigl\langle \operatorname{Bir}(\beth_1), \dots, \operatorname{Bir}(\beth_r) \bigr\rangle \subseteq \operatorname{Bir}(X_1 \times \dots \times X_r),$$

and the following result was proved in [45].

Theorem 1.36. *If $X_1, \ldots, X_r$ are universally birationally rigid and* $\operatorname{lct}(X_i) \geqslant 1$ *for all $i = 1, \ldots, r$, then*

$$\operatorname{Bir}(X_1 \times \cdots \times X_r) = \big\langle \operatorname{Bir}(\beth_1), \ldots, \operatorname{Bir}(\beth_r), \operatorname{Aut}(X_1 \times \cdots \times X_r) \big\rangle,$$

the variety $X_1 \times \cdots \times X_r$ is non-rational and for every map $\rho\colon X_1 \times \cdots \times X_r \dashrightarrow Y$ whose general fibre is rationally connected there are a subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, r\}$ and a commutative diagram

$$\begin{array}{ccc} X_1 \times \cdots \times X_r & \overset{\sigma}{\dashrightarrow} & X_1 \times \cdots \times X_r \\ \Big\downarrow{\scriptstyle\pi} & & \searrow{\scriptstyle\rho} \\ X_{i_1} \times \cdots \times X_{i_k} & \underset{\xi}{\dashrightarrow} & Y \end{array}$$

where π is the natural projection and ξ and σ are birational maps.

Corollary 1.37. *Suppose that there exist subgroups $\Gamma_i \subseteq \operatorname{Bir}(X_i)$ universally untwisting all maximal singularities and that $\operatorname{lct}(X_i) \geqslant 1$ for every $i = 1, \ldots, r$. Then*

$$\operatorname{Bir}(X_1 \times \cdots \times X_r) = \left\langle \prod_{i=1}^{r} \Gamma_i, \operatorname{Aut}(X_1 \times \cdots \times X_r) \right\rangle.$$

Let X be a general well-formed quasismooth hypersurface in $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ of degree $\sum_{i=1}^4 a_i$, that has at most terminal singularities, where $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$. Then

$$-K_X \equiv \mathscr{O}_{\mathbb{P}(1,a_1,a_2,a_3,a_4)}(1),$$

and the group $\operatorname{Cl}(X)$ is generated by the divisor $-K_X$. We see that X is a Fano variety.

Remark 1.38. There are precisely 95 values of the quadruple (a_1, a_2, a_3, a_4) (see [33], [46]).

It follows from [47] that there are finitely many birational involutions $\tau_1, \ldots, \tau_k \in \operatorname{Bir}(X)$ and that the following result holds.

Theorem 1.39. *The group $\langle \tau_1, \ldots, \tau_k \rangle$ untwists universally maximal singularities.*

Corollary 1.40. *The variety X is universally birationally rigid.*

The relations between $\tau_1, \ldots, \tau_k$ were found in [48]. By [14] there is an exact sequence of groups

$$1 \longrightarrow \langle \tau_1, \ldots, \tau_k \rangle \longrightarrow \operatorname{Bir}(X) \longrightarrow \operatorname{Aut}(X) \longrightarrow 1,$$

and by [45] and [49] we have the following result.

Theorem 1.41. *Suppose that $-K_X^3 \leqslant 1$. Then* $\operatorname{lct}(X) = 1$.

In particular, there do exist varieties satisfying the hypotheses of Theorem 1.36 and Corollary 1.37 that are not birationally superrigid.

Example 1.42. Let X be a general hypersurface of degree 20 in $\mathbb{P}(1,1,4,5,10)$. Then there is an exact sequence of groups

$$1 \longrightarrow \prod_{i=1}^{m}(\mathbb{Z}_2 * \mathbb{Z}_2) \longrightarrow \mathrm{Bir}(\underbrace{X \times \cdots \times X}_{m \text{ factors}}) \longrightarrow \mathrm{S}_m \longrightarrow 1,$$

where $\mathbb{Z}_2 * \mathbb{Z}_2$ is the infinite dihedral group.

The aim of this paper is to prove the following two results.

Theorem 1.43. *Let* $(a_1, a_2, a_3, a_4) = (1,1,2,2)$. *Then* $\mathrm{lct}(X) \geqslant 4/5$.

Theorem 1.44. *Let* $(a_1, a_2, a_3, a_4) = (1,1,2,3)$. *Then* $\mathrm{lct}(X) \geqslant 6/7$.

It follows from [49] that $\mathrm{lct}(X) \geqslant 7/9$ for $(a_1, a_2, a_3, a_4) = (1,1,1,2)$, but

$$-K_X^3 > 1 \quad \Longleftrightarrow \quad (a_1, a_2, a_3, a_4) \in \big\{(1,1,1,1),(1,1,1,2),(1,1,1,3), (1,1,2,2),(1,1,2,3)\big\},$$

which, in particular, implies the following result (see Examples 1.10 and 1.9).

Corollary 1.45. *General well-formed quasismooth hypersurfaces in* $\mathbb{P}(1, a_1, \dots, a_4)$ *of degree* $\sum_{i=1}^{4} a_i$ *that have terminal singularities admit Kähler-Einstein metrics.*

We prove Theorem 1.43 in § 3 and Theorem 1.44 in § 4.

§ 2. Preliminaries

Let V be a variety with at most quotient singularities.

Remark 2.1. Let H be a nef divisor on V and let B and T, $B \neq T$, be effective and irreducible divisors on V. Let $\dim(V) = 3$ and let

$$B \cdot T = \sum_{i=1}^{r} \varepsilon_i L_i + \Delta,$$

where L_i is an irreducible curve, ε_i is a non-negative integer and Δ is an effective cycle whose support does not contain the curves $L_1, \dots, L_r$. Then

$$\sum_{i=1}^{r} \varepsilon_i H \cdot L_i \leqslant B \cdot T \cdot H.$$

Let D be an effective $\mathbb{Q}$-divisor on V such that the log pair (V, D) is not log canonical.

Remark 2.2. Let B be an effective $\mathbb{Q}$-divisor on the variety V such that the singularities of the log pair (V, B) are log canonical. Then the singularities of the log pair

$$\left(V, \frac{1}{1-\alpha}(D - \alpha B)\right)$$

are not log canonical for all $\alpha \in \mathbb{Q}$ such that $0 \leqslant \alpha < 1$.

Let P be a point in V such that the log pair (V, D) is not log canonical at P.

Remark 2.3. Suppose that P is a singular point of V of type $\frac{1}{r}(1, a, r-a)$, where a and r are positive integers such that $(a, r) = 1$ and $r > 2a$. Let $\alpha\colon U \to V$ be a weighted blow up of the point P with weights $(1, a, r-a)$. There exists a rational number μ such that

$$\overline{D} \equiv \alpha^*(D) - \mu E,$$

where $\overline{D}$ is the proper transform of the divisor D on the variety U and E is the α-exceptional divisor. Then $\mu > 1/r$ by [1], Lemma 8.12.

It is clear that $\mathrm{mult}_P(D) > 1$ in the case when $P \notin \mathrm{Sing}(V)$.

Remark 2.4. Suppose that $P \notin \mathrm{Sing}(V)$ and $\dim(V) = 2$. Let

$$D = mC + \Omega$$

for an irreducible curve C, a non-negative rational number m and an effective $\mathbb{Q}$-divisor Ω on the surface V whose support does not contain the curve C. Then

$$C \cdot \Omega \geqslant \mathrm{mult}_P\bigl(\Omega\big|_C\bigr) > 1$$

by [1], Theorem 7.5 in the case when $P \in C \setminus \mathrm{Sing}(C)$ and $m \leqslant 1$.

Suppose additionally that $\dim(V) = 3$ and that P is a smooth point of the variety V. Let $\pi\colon U \to V$ be a blow up of the point P. Then

$$\overline{D} \equiv \alpha^*(D) - \mathrm{mult}_P(D)E,$$

where E is the α-exceptional divisor and $\overline{D}$ is the proper transform of D on U.

Lemma 2.5. *Either* $\mathrm{mult}_P(D) > 2$, *or there is a line* $L \subset E \cong \mathbb{P}^2$ *such that*

$$\mathrm{mult}_L(\overline{D}) + \mathrm{mult}_P(D) > 2.$$

Proof. Let H be a sufficiently general hyperplane section of the variety V passing through the point P and let $\overline{H}$ be the proper transform of the divisor H on the variety U. Then

$$\overline{H} \equiv \alpha^*(D) - E,$$

and we can assume that $\overline{H}$ is very ample. From

$$K_U + \overline{D} + (\mathrm{mult}_P(D) - 2)E \equiv \alpha^*(K_V + D)$$

it follows that $\bigl(U, \overline{D} + (\mathrm{mult}_P(D) - 2)E\bigr)$ is not log canonical in a neighbourhood of E. The log pair

$$\bigl(U, \overline{D} + (\mathrm{mult}_P(D) - 1)E\bigr)$$

is not log canonical in a neighbourhood of divisor E either. Finally, the log pair

$$\bigl(U, \overline{D} + (\mathrm{mult}_P(D) - 1)E + \overline{H}\bigr)$$

is not log canonical in a neighbourhood of E as well. We point out that $\mathrm{mult}_P(D) > 1$.

Let $\beta = \alpha\big|_{\overline{H}}\colon \overline{H} \to H$ and $\overline{E} = E\big|_{\overline{H}}$. Then

$$K_{\overline{H}} + \overline{D}\big|_{\overline{H}} + (\mathrm{mult}_P(D) - 1)\overline{E} \equiv \beta^*\bigl(K_H + D\big|_H\bigr),$$

and the support of the divisor $\overline{D}\big|_H$ does not contain the curve $\overline{E}$ because of the generality in the choice of H. Then

$$\mathrm{mult}_P\bigl(D\big|_H\bigr) = \mathrm{mult}_P(D),$$

and the proper transform of the divisor $D\big|_H$ on the surface $\overline{H}$ is the divisor $\overline{D}\big|_H$.

The log pair $(H, D\big|_H)$ is not log canonical at the point P by [1], Theorem 7.5. Then

$$\bigl(\overline{H},\, \overline{D}\big|_H + (\mathrm{mult}_P(D) - 1)\overline{E}\bigr)$$

is not log canonical in a neighbourhood of the curve $\overline{E}$.

Suppose that $\mathrm{mult}_P(D) < 2$. Then it follows from the connectedness principle ([1], Theorem 7.5) that there is a unique point $Q_{\overline{H}} \in \overline{E}$ such that the log pair

$$\bigl(\overline{H},\, \overline{D}\big|_H + (\mathrm{mult}_P(D) - 1)\overline{E}\bigr)$$

is not log terminal at $Q_{\overline{H}}$, but is log terminal outside $Q_{\overline{H}}$ in a neighbourhood of $\overline{E}$. By the generality of the surface H we may assume that $\overline{H}$ is a general hyperplane section of U. Hence there is a curve $L \subset E$ such that $L \cap \overline{H} = Q_{\overline{H}}$, and the log pair

$$\bigl(U,\, \overline{D} + (\mathrm{mult}_P(D) - 1)E\bigr)$$

is not log terminal at a general point of the curve L, but is log terminal outside L in a neighbourhood of $Q_{\overline{H}}$.

The curve L is a line in $\mathbb{P}^2$ because the intersection $L \cap \overline{H}$ consists of a single point. Then

$$\mathrm{mult}_L(\overline{D}) + (\mathrm{mult}_P(D) - 1)\,\mathrm{mult}_L(E) \geqslant 1,$$

which implies that $\mathrm{mult}_L(\overline{D}) + \mathrm{mult}_P(D) \geqslant 2$.

Hence we see that either $\mathrm{mult}_P(D) \geqslant 2$ or there is a line $L \subset E$ such that

$$\mathrm{mult}_L(\overline{D}) + \mathrm{mult}_P(D) \geqslant 2,$$

but $(V, \lambda D)$ is not log canonical at P for some positive rational number $\lambda < 1$. Applying the last assertion to the log pair $(V, \lambda D)$ we obtain the required strict inequality and complete the proof.

The assertion of Lemma 2.5 is an easy generalization of Corollary 3.5 in [36].

§ 3. Fano threefold of degree 3/2

Let X be a general hypersurface in $\mathbb{P}(1,1,1,2,2)$ of degree 6. Then X has three singular points O_1, O_2, O_3, which are singular points of type $\frac{1}{2}(1,1,1)$. Let D be an arbitrary divisor in the linear system $|-nK_X|$, where n is a positive integer. We set $\lambda = 4/(5n)$.

Remark 3.1. To prove Theorem 1.43 it is sufficient to show that the log pair $(X, \lambda D)$ is log canonical because D is an arbitrary divisor in $|-nK_X|$.

Suppose that the log pair $(X, \lambda D)$ is not log canonical. We shall show that this leads to a contradiction. We can assume that D is irreducible (see Remark 2.2).

Lemma 3.2. *The inequality $n \neq 1$ holds.*

Proof. Let $n = 1$. Then the log pair (X, D) is log canonical at every singular point of the hypersurface X by [1], Lemma 8.12 and Proposition 8.14. We have $a_1 = 1$.

Suppose that the log pair (X, D) is not log canonical at some smooth point P of the hypersurface X. We shall show that this assumption leads to a contradiction.

Consider the set of pairs

$$\mathscr{S} = \big\{(O, F) \bigm| O \in \mathbb{P}(1,1,1,2,2),\ F \in H^0(\mathbb{P}(1,1,1,2,2),\ \mathscr{O}_{\mathbb{P}(1,1,1,2,2)}(6))\big\}$$

with projections

$$\pi\colon \mathscr{S} \to H^0(\mathbb{P}(1,1,1,2,2), \mathscr{O}_{\mathbb{P}(1,1,1,2,2)}(6)) \quad \text{and} \quad \zeta\colon \mathscr{S} \to \mathbb{P}(1,1,1,2,2).$$

Let

$$\mathscr{I} = \big\{(O, F) \in \mathscr{S} \bigm| F(O) = 0,\ \text{the hypersurface } F = 0 \text{ is quasismooth and is smooth at } O\big\}.$$

Suppose that the point O is given by the equations $x = y = w = t = 0$ in

$$\mathbb{P}(1,1,1,2,2) \cong \mathrm{Proj}(\mathbb{C}[x, y, z, t, w]),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = \mathrm{wt}(z) = 1$ and $\mathrm{wt}(t) = \mathrm{wt}(w) = 2$. Then

$$\begin{aligned} F = {} & z^5 q_1(x, y) + z^4 q_2(x, y, t, w) + z^3 q_3(x, y, t, w) + z^2 q_4(x, y, t, w) \\ & + z q_5(x, y, t, w) + q_6(x, y, t, w), \end{aligned}$$

where $q_i(x, y, t, w)$ is a quasihomogeneous polynomial of degree i.

We say that O is a *bad* point of $F = 0$ if $q_2(0, 0, t, w) = 0$ and the surface cut out on $F = 0$ by the equation $q_1(x, y) = 0$ has non-canonical singularities at O.

Let Q be a point in $\mathbb{P}(1,1,1,2,2)$ and let Ω be the fibre of π over the point Q. Then

$$\dim(\Omega) = \dim\big(H^0(\mathbb{P}(1,1,1,2,2), \mathscr{O}_{\mathbb{P}(1,1,1,2,2)}(6))\big),$$

and we can put

$$\mathscr{Y} = \big\{(O, F) \in \mathscr{I} \bigm|\ O \text{ is a bad point of the hypersurface } F = 0\big\}.$$

The restriction $\pi\big|_{\mathscr{Y}} \colon \mathscr{Y} \to \mathbb{P}(1,1,1,2,2)$ is surjective. Easy computations show that

$$\dim(\Omega \cap \mathscr{Y}) \leqslant \dim(\Omega) - 5,$$

which implies that the restriction

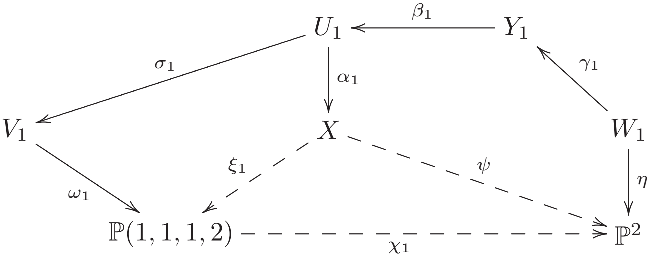
$$\zeta\big|_{\mathscr{Y}} \colon \mathscr{Y} \longrightarrow H^0\big(\mathbb{P}(1,1,1,2,2), \mathscr{O}_{\mathbb{P}(1,1,1,2,2)}(6)\big)$$

is not surjective. Thus, a general hypersurface in $\mathbb{P}(1,1,1,2,2)$ of degree 6 has no bad points.

By assumption, the log pair (X, D) is not log canonical at the point P, which is a smooth point of the hypersurface X. In particular, the surface D is singular at the point P. However, we may assume that the surface D has canonical singularities at the point P.

Singularities of the surface D are not log canonical at P by [1], Theorem 7.5, which is a contradiction because D has canonical singularities at the point P. The proof is complete.

It follows form [50] that there is a commutative diagram



where ξ_1, ψ and χ_1 are projections, α_1 is a blow up of O_1 with weights $(1,1,1)$, β_1 is a blow up with weights $(1,1,1)$ of the point dominating O_2, γ_1 is a blow up with weights $(1,1,1)$ of the point dominating O_3, η is an elliptic fibration, ω_1 is a double cover and σ_1 is a birational morphism contracting 24 curves $\overline{C}_1^1, \dots, \overline{C}_{24}^1$.

Remark 3.3. The curves $\overline{C}_1^1, \dots, \overline{C}_{24}^1$ are smooth, irreducible and rational.

We set $C_i^1 = \alpha_1(\overline{C}_i^1)$ for every $i = 1, \dots, 24$. The rational map ξ_1 is undefined only at the point O_1 and contracts the curves $C_1^1, \dots, C_{24}^1$. Note that ψ is a natural projection.

Remark 3.4. The fibre of the projection ψ over the point $\psi(C_i^1)$ consists of the smooth rational curve C_i^1 and another irreducible smooth rational curve Z_i^1 such that

$$C_i^1 \ni O_1 \notin Z_i^1, \qquad Z_i^1 \ni O_2 \notin C_i^1, \qquad Z_i^1 \ni O_3 \notin C_i^1,$$

the curves C_i^1 and Z_i^1 intersect transversally at two points and

$$-K_X \cdot Z_i^1 = -2K_X \cdot C_i^1 = 1.$$

In a similar way we can construct maps $\xi_2 \colon X \dashrightarrow \mathbb{P}(1,1,1,2)$ and $\xi_3 \colon X \dashrightarrow \mathbb{P}(1,1,1,2)$, which are undefined only at the points O_2 and O_3, respectively. These rational maps ξ_2 and ξ_3 contract precisely 48 curves $C_1^2, \dots, C_{24}^2$ and $C_1^3, \dots, C_{24}^3$, respectively.

Remark 3.5. Let Z be a curve on the variety X such that $-K_X \cdot Z = 1/2$. Then

$$Z \in \{C_1^1, \dots, C_{24}^1, C_1^2, \dots, C_{24}^2, C_1^3, \dots, C_{24}^3\}.$$

In a similar way we see that there are smooth irreducible rational curves $Z_1^2, \dots, Z_{24}^2$ and $Z_1^3, \dots, Z_{24}^3$ that are components of the fibres of the rational map ψ over the points $\psi(C_1^2), \dots, \psi(C_{24}^2)$ and $\psi(C_1^3), \dots, \psi(C_{24}^3)$, respectively.

Remark 3.6. Let F be a reducible fibre of the map ψ. Then

$$F \in \{C_1^1 \cup Z_1^1, \ldots, C_{24}^1 \cup Z_{24}^1,\ C_1^2 \cup Z_1^2, \ldots, C_{24}^2 \cup Z_{24}^2,\ C_1^3 \cup Z_1^3, \ldots, C_{24}^3 \cup Z_{24}^3\}.$$

Let P be a point in the variety V such that the log pair $(X, \lambda D)$ is not log canonical at P, and let F be a scheme fibre of the projection ψ that passes through the point P.

Remark 3.7. If $P \notin \mathrm{Sing}(X)$, then F is uniquely defined.

Note that F is reduced. Let S be a general surface in $|-K_X|$ such that $P \in S$.

Lemma 3.8. *Suppose that* $\mathrm{Sing}(X) \not\ni P \notin \mathrm{Sing}(F)$. *Then F is reducible.*

Proof. Suppose that F is irreducible. Let $\pi\colon \overline{X} \to X$ be a blow up of the point P. Then

$$\overline{D} \equiv \pi^*(D) - \mathrm{mult}_P(D)E,$$

where E is the π-exceptional divisor and $\overline{D}$ is the proper transform of the divisor D on $\overline{X}$.

We point out that $\mathrm{mult}_P(D) > n$. Suppose that $\mathrm{mult}_P(D) > 3n/2$ and let

$$D\big|_S = mF + \Omega,$$

where m is a non-negative rational number and Ω is an effective $\mathbb{Q}$-divisor on S whose support does not contain the curve F. Then

$$\frac{3n}{2} = F \cdot (mF + \Omega) = \frac{3m}{2} + F \cdot \Omega \geqslant \frac{3m}{2} + \mathrm{mult}_P(\Omega) > \frac{3m}{2} + \frac{3n}{2} - m = \frac{3n}{2} + \frac{m}{2},$$

which is a contradiction. We see that $\mathrm{mult}_P(D) \leqslant 3n/2$.

It follows from Lemma 2.5 that there is a line $L \subset E \cong \mathbb{P}^2$ such that

$$\mathrm{mult}_L(\overline{D}) + \mathrm{mult}_P(D) > \frac{2}{\lambda} = \frac{5n}{2}.$$

It follows from the smoothness of the curve F at P that $|-K_X|$ does not contain surfaces singular at the point P. Hence we see that

$$H^0\big(\mathscr{O}_{\overline{X}}(\pi^*(-2K_X) - 2E)\big) \cong \mathbb{C}^4,$$

and it follows from the standard exact sequence

$$H^0\big(\mathscr{O}_{\overline{X}}(\pi^*(-2K_X) - 3E)\big) \longrightarrow H^0\big(\mathscr{O}_{\overline{X}}(\pi^*(-2K_X) - 2E)\big) \longrightarrow H^0\big(\mathscr{O}_E(-2E\big|_E)\big) \cong \mathbb{C}^5$$

that either there is a surface $T \in |-2K_X|$ such that $\mathrm{mult}_P(T) \geqslant 3$ or there is a surface $R \in |-2K_X|$ such that $\mathrm{mult}_P(R) = 2$ and $L \subset \overline{R}$, where $\overline{R}$ is the proper transform of the surface R on the variety $\overline{X}$. The parameter count (see the proof of Lemma 3.2) shows that the former case is impossible.

We see that there exists a (possibly reducible) surface $R \in |-2K_X|$ such that $\mathrm{mult}_P(R) = 2$ and $L \subset \overline{R}$, where $\overline{R}$ is the proper transform of this surface R on the variety $\overline{X}$. Then $D \not\subseteq \mathrm{Supp}(R)$ because $\mathrm{mult}_P(D) > n$. We have

$$\begin{aligned}\mathrm{mult}_P(R \cdot D) &\geqslant \mathrm{mult}_L(\overline{D})\,\mathrm{mult}_L(\overline{R}) + \mathrm{mult}_P(D)\,\mathrm{mult}_P(R)\\ &\geqslant \mathrm{mult}_L(\overline{D}) + 2\,\mathrm{mult}_P(D) > 3n.\end{aligned}$$

Let $R \cdot D = \varepsilon F + \Delta$, where $\varepsilon \in \mathbb{Q}$ and Δ is an effective 1-cycle whose support does not contain the curve F. Then $\Delta \not\subset \operatorname{Supp}(S)$ and $\operatorname{mult}_P(\Delta) > 3n - \varepsilon$. We have

$$3n = S \cdot R \cdot D = \frac{3\varepsilon}{2} + S \cdot \Delta > \frac{3\varepsilon}{2} - 3n - \varepsilon = 3n + \frac{\varepsilon}{2},$$

which is a contradiction completing the proof.

Lemma 3.9. *Suppose that $P \notin \operatorname{Sing}(X)$. Then F is reducible.*

Proof. Suppose that F is irreducible. Then F is singular at the point P by Lemma 3.8, which implies that there is $T \in |-K_X|$ such that $\operatorname{mult}_P(T) \geqslant 2$. Then $T \neq D$ by Lemma 3.2. Now the generality of the hypersurface X implies that $\operatorname{mult}_P(F) = 2$.

Now let $T \cdot D = \varepsilon F + \Delta$, where $\varepsilon \in \mathbb{Q}$ and Δ is an effective 1-cycle whose support does not contain the curve F. Then $\Delta \not\subset \operatorname{Supp}(S)$ and $\operatorname{mult}_P(\Delta) > 2n - 2\varepsilon$. We have

$$\frac{3n}{2} = S \cdot T \cdot D = \frac{3\varepsilon}{2} + S \cdot \Delta > \frac{3\varepsilon}{2} + 2n - 2\varepsilon = 2n - \frac{\varepsilon}{2},$$

which implies that $\varepsilon > n$, and this is impossible by Remark 2.1.

Lemma 3.10. *P is a singular point of the hypersurface X.*

Proof. Suppose that P is a smooth point of X. Then F is reducible by Lemma 3.9, and it follows from Remark 3.6 that

$$F \in \left\{C_1^1 \cup Z_1^1, \ldots, C_{24}^1 \cup Z_{24}^1,\, C_1^2 \cup Z_1^2, \ldots, C_{24}^2 \cup Z_{24}^2,\, C_1^3 \cup Z_1^3, \ldots, C_{24}^3 \cup Z_{24}^3\right\}.$$

Without loss of generality we may assume that $F = C_1^1 \cup Z_1^1$. Let

$$D\big|_S = m_1 C_1^1 + m_2 Z_1^1 + \Omega \equiv -nK_X\big|_S,$$

where m_1 and m_2 are non-negative rational numbers and Ω is an effective $\mathbb{Q}$-divisor on the surface S whose support does not contain the curves C_1^1 and Z_1^1. Then the log pair

$$(S,\, \lambda m_1 C_1^1 + \lambda m_2 Z_1^1 + \lambda\Omega)$$

is not log canonical at the point P by [1], Theorem 7.5. We shall show that this contradicts the numerical equivalence $m_1 C_1^1 + m_2 Z_1^1 + \Omega \equiv -nK_X\big|_S$.

The singularities of the log pair $(S, C_1^1 + Z_1^1)$ are log canonical at the point P by the generality of the hypersurface X. Hence it follows from the numerical equivalence

$$C_1^1 + Z_1^1 \equiv -K_X\big|_S$$

and Remark 2.2 that we may assume that either $m_1 = 0$ or $m_2 = 0$.

Let $m_1 = 0$. Then it follows from

$$\frac{n}{2} = C_1^1 \cdot (m_2 Z_1^1 + \Omega) = 2m_2 + C_1^1 \cdot \Omega \geqslant 2m_2$$

that $m_2 \leqslant n/4$. We have $P \notin C_1^1$ because otherwise

$$\frac{n}{2} = C_1^1 \cdot (m_2 Z_1^1 + \Omega) = 2m_2 + C_1^1 \cdot \Omega > 2m_2 + \frac{1}{\lambda} \geqslant \frac{5n}{4}$$

by Remark 2.4. We see that $P \in Z_1^1$. Then

$$n = Z_1^1 \cdot (m_2 Z_1^1 + \Omega) = -m_2 + Z_1^1 \cdot \Omega > -m_2 + \frac{1}{\lambda} \geqslant -m_2 + \frac{5n}{4}$$

by Remark 2.4, so that $m_2 > n/4$, although we have $m_2 \leqslant n/4$, which is a contradiction.

Hence we see that $m_2 = 0$. Arguing as above we obtain

$$n = Z_1^1 \cdot (m_1 C_1^1 + \Omega) = 2m_1 + Z_1^1 \cdot \Omega \geqslant 2m_1,$$

which implies that $m_1 \leqslant n/2$. Then $P \notin Z_1^1$ because otherwise

$$n = Z_1^1 \cdot (m_1 C_1^1 + \Omega) = 2m_1 + Z_1^1 \cdot \Omega > 2m_1 + \frac{1}{\lambda} \geqslant \frac{5n}{4}$$

by Remark 2.4. We see that $P \in C_1^1$. Then

$$\frac{n}{2} = C_1^1 \cdot (m_1 C_1^1 + \Omega) = -\frac{3m_1}{2} + C_1^1 \cdot \Omega > -\frac{3m_1}{2} + \frac{1}{\lambda} \geqslant -\frac{3m_1}{2} + \frac{5n}{4}$$

by Remark 2.4. We see that $m_1 > n/2$, but $m_1 \leqslant n/2$, which is a contradiction completing the proof.

Without loss of generality we may assume that $P = O_1$. Then $-K_{U_1}^3 = 1$ and

$$\overline{D} \equiv \alpha_1^*(D) - \mu E_1,$$

where E_1 is the α_1-exceptional divisor, $\overline{D}$ is the proper transform of the divisor D on the variety U_1, and $\mu \in \mathbb{Q}$. Then $\mu > n/(2\lambda)$ by Remark 2.3. We have

$$K_{U_1} + \lambda\overline{D} + \left(\lambda\mu - \frac{1}{2}\right)E_1 \equiv \alpha_1^*(K_X + \lambda D).$$

Lemma 3.11. $\mu \leqslant 3n/4$.

Proof. The point O_1 can be given by $x = y = z = t = 0$ and X can be given by

$$w^2 t + w f_4(x, y, z, t) + f_6(x, y, z, t) = 0 \subset \mathbb{P}(1, 1, 1, 2, 2) \cong \operatorname{Proj}(\mathbb{C}[x, y, z, t, w]),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = \mathrm{wt}(z) = 1$, $\mathrm{wt}(t) = \mathrm{wt}(w) = 2$ and f_4, f_6 are quasihomogeneous polynomials of degrees 4 and 6, respectively. In these coordinates the curves $C_1^1, \dots, C_{24}^1$ are cut out on the hypersurface X by the equations

$$t = f_4(x, y, z, t) = f_6(x, y, z, t) = 0.$$

Let R be a surface on X that is cut out by the equation $t = 0$ and let $\overline{R}$ be the proper transform of the surface R on the variety U_1. The surface R is irreducible and

$$\overline{R} \equiv \alpha_1^*(-2K_X) - 2E;$$

but $\left(X, \frac{1}{2}R\right)$ is log canonical at the point O_1 by [1], Lemma 8.12 and Proposition 8.14 because we may assume that the hypersurface X is sufficiently general.

The log pair $(X, \lambda D)$, where $\lambda = 4/5$, is not log canonical at the point P. Hence $R \neq D$ and

$$0 \leqslant -K_{U_1} \cdot \overline{R} \cdot \overline{D} = 3n - 4\mu$$

because $-K_{U_1}$ is nef. Thus, $\mu \leqslant 3n/4$ and the proof is complete.

In particular, there is a point $Q \in E$ such that the log pair

$$\left(U_1,\, \lambda\overline{D} + \left(\lambda\mu - \frac{1}{2}\right)E_1\right)$$

is not log canonical at Q. Let $\overline{S}$ be a general surface in $|-K_{U_1}|$ such that $Q \in \overline{S}$.

Remark 3.12. The proper transform of the surface E_1 on the variety W_1 is a section of the elliptic fibration η. In particular, the surface $\overline{S}$ is smooth at Q.

Let $\overline{Z}_i^k$ be the proper transform of Z_i^k on the threefold U_1, where $k = 1, 2, 3$ and $i = 1, \ldots, 24$.

Lemma 3.13. *The point Q is not contained in $\bigcup_{i=1}^{24} \overline{C}_i^1$.*

Proof. Suppose that $Q \in \bigcup_{i=1}^{24} \overline{C}_i^1$. We can assume that $Q \in \overline{C}_1^1$. Let

$$\overline{D}\big|_{\overline{S}} + \left(\mu - \frac{n}{2}\right)E\big|_{\overline{S}} = m_1\overline{C}_1^1 + m_2\overline{Z}_1^1 + \Omega \equiv -nK_{U_1}\big|_{\overline{S}},$$

where m_1 and m_2 are non-negative rational numbers and Ω is an effective $\mathbb{Q}$-divisor on the surface $\overline{S}$ whose support does not contain the curves $\overline{C}_1^1$ and $\overline{Z}_1^1$. The log pair

$$\left(\overline{S},\, \frac{m_1}{n}\,\overline{C}_1^1 + \frac{m_2}{n}\,\overline{Z}_1^1 + \frac{1}{n}\,\Omega\right)$$

is not log canonical at the point Q by [1], Theorem 7.5. We claim that this is impossible.

The log pair $(\overline{S}, \overline{C}_1^1 + \overline{Z}_1^1)$ is log canonical at the point Q. Thus, it follows from the equivalence

$$\overline{C}_1^1 + \overline{Z}_1^1 \equiv -K_{U_1}\big|_{\overline{S}}$$

and Remark 2.2 that we may assume that either $m_1 = 0$ or $m_2 = 0$.

It follows from Remark 2.4 that

$$0 = \overline{C}_1^1 \cdot (m_1\overline{C}_1^1 + m_2\overline{Z}_1^1 + \Omega) = 2m_2 + \overline{C}_1^1 \cdot \Omega > 2m_2 + n \geqslant n$$

in the case $m_1 = 0$. Hence we may assume that $m_2 = 0$. Then

$$n = \overline{Z}_1^1 \cdot (m_1\overline{C}_1^1 + \Omega) = 2m_1 + \overline{Z}_1^1 \cdot \Omega \geqslant 2m_1,$$

which implies that $m_1 \leqslant n/2$. We see that

$$0 = \overline{C}_1^1 \cdot (m_1\overline{C}_1^1 + \Omega) = -2m_1 + \overline{C}_1^1 \cdot \Omega > -2m_1 + n \geqslant -2m_1 + n$$

by Remark 2.4, so that $m_1 > n/2$, although we have $m_1 \leqslant n/2$. This is a contradiction completing the proof.

Let $\overline{C}_i^k$ be the proper transform of C_i^k on the threefold U_1, where $k = 2, 3$ and $i = 1, \ldots, 24$.

Lemma 3.14. *The point Q is not contained in $\bigcup_{i=1}^{24} \overline{Z}_i^2$ or $\bigcup_{i=1}^{24} \overline{Z}_i^3$.*

Proof. Suppose that $Q \in \bigcup_{i=1}^{24} \overline{Z}_i^2$ or $Q \in \bigcup_{i=1}^{24} \overline{Z}_i^3$. We shall show that this leads to a contradiction. We may assume without loss of generality that $Q \in \overline{Z}_1^2$. Then $Q \notin \overline{C}_1^2$. Let

$$\overline{D}\big|_{\overline{S}} + \left(\mu - \frac{n}{2}\right)E\big|_{\overline{S}} = m_1\overline{C}_1^2 + m_2\overline{Z}_1^2 + \Omega \equiv -nK_{U_1}\big|_{\overline{S}},$$

where m_1 and m_2 are non-negative rational numbers and Ω is an effective $\mathbb{Q}$-divisor on the surface $\overline{S}$ whose support does not contain the curves $\overline{C}_1^2$ and $\overline{Z}_1^2$.

It follows from [1], Theorem 7.5 that the log pair

$$\left(\overline{S},\, \frac{m_1}{n}\,\overline{C}_1^2 + \frac{m_2}{n}\,\overline{Z}_1^2 + \frac{1}{n}\,\Omega\right)$$

is not log canonical at the point Q. We claim that this is impossible.

The log pair $(\overline{S}, \overline{C}_1^2 + \overline{Z}_1^2)$ is log canonical at Q, but

$$\overline{C}_1^2 + \overline{Z}_1^2 \equiv -K_{U_1}\big|_{\overline{S}},$$

which implies that we can assume that either $m_1 = 0$ or $m_2 = 0$ (see Remark 2.2).

Let $m_2 = 0$. Then it follows from Remark 2.4 that

$$\frac{n}{2} = \overline{Z}_1^2 \cdot (m_1\overline{C}_1^2 + \Omega) = 2m_1 + \overline{Z}_1^2 \cdot \Omega > 2m_1 + n \geqslant \frac{5n}{4}\,,$$

which is a contradiction. Hence we may assume that $m_1 = 0$. Then

$$\frac{n}{2} = \overline{C}_1^2 \cdot (m_2\overline{Z}_1^2 + \Omega) = 2m_2 + \overline{C}_1^2 \cdot \Omega \geqslant 2m_2,$$

which implies that $m_2 \leqslant n/4$. We see that

$$\frac{n}{2} = \overline{Z}_1^2 \cdot (m_2\overline{Z}_1^2 + \Omega) = -\frac{3m_2}{2} + \overline{Z}_1^2 \cdot \Omega > -\frac{3m_2}{2} + n$$

by Remark 2.4, so that $m_2 > n/3$, although we have $m_2 \leqslant n/4$. This is a contradiction completing the proof.

Let $\overline{F}$ be a scheme fibre of $\psi \circ \alpha_1$ passing through the point Q. Then $\overline{F}$ is irreducible and the fibre $\overline{F}$ is smooth at the point Q. Let

$$\overline{D}\big|_{\overline{S}} + \left(\mu - \frac{n}{2}\right)E\big|_{\overline{S}} = m\overline{F} + \Omega,$$

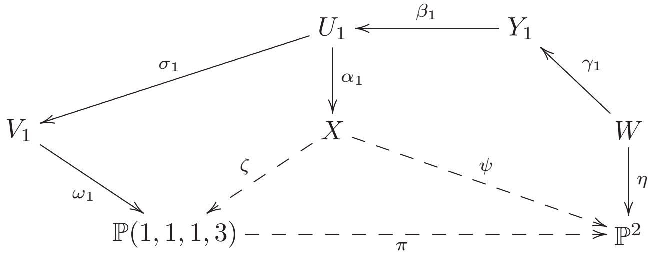
where m is a non-negative rational number and Ω is an effective $\mathbb{Q}$-divisor on $\overline{S}$ whose support does not contain the curve $\overline{F}$. Then

$$n = \overline{F} \cdot (m\overline{F} + \Omega) = m + \overline{F} \cdot \Omega \geqslant m + \mathrm{mult}_Q(\Omega) > m + n - m = n,$$

which is a contradiction. The proof of Theorem 1.43 is complete.

§ 4. Fano threefold of degree 7/6

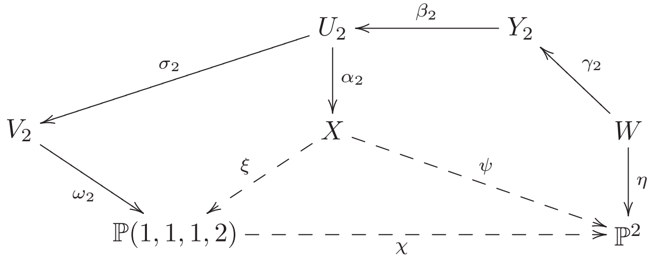
Let X be a general hypersurface in $\mathbb{P}(1,1,1,2,3)$ of degree 7. Then X has two singular points O_1 and O_2, which are singular points of type $\frac{1}{2}(1,1,1)$ and $\frac{1}{3}(1,1,2)$, respectively. There is a commutative diagram



where π, ψ and ζ are projections, α_1 is a blow up of O_1 with weights $(1,1,1)$, β_1 is a blow up with weights $(1,1,2)$ of the singular point dominating O_2, γ_1 is a blow up with weights $(1,1,1)$ of the singular point dominating O_2, η is an elliptic fibration, ω_1 is a double cover and σ_1 is a birational morphism contracting 35 curves $\overline{C}^1_1,\ldots,\overline{C}^1_{35}$.

Remark 4.1. The curves $\overline{C}^1_1,\ldots,\overline{C}^1_{35}$ are smooth, irreducible and rational.

It follows from [50] that there is a commutative diagram



where ξ, ψ and χ are projections, α_2 is a blow up of O_2 with weights $(1,1,2)$, β_2 is a blow up with weights $(1,1,1)$ of the singular point of U_2 dominating the point O_2, γ_2 is the blow up with weights $(1,1,1)$ of the point dominating O_1, η is an elliptic fibration, ω_2 is a double cover and σ_2 is a birational morphism contracting 14 curves $\overline{C}^2_1,\ldots,\overline{C}^2_{14}$.

Remark 4.2. The curves $\overline{C}^2_1,\ldots,\overline{C}^2_{14}$ are smooth, irreducible and rational.

Let $C^1_i=\alpha_1(\overline{C}^1_i)$ for all $i=1,\ldots,35$.

Remark 4.3. The fibre of the projection ψ over the point $\psi(C^1_i)$ consists of the smooth rational curve C^1_i and a smooth irreducible rational curve Z^1_i such that

$$C^1_i \ni O_1 \notin Z^1_i \quad \text{and} \quad Z^1_i \ni O_2 \notin C^1_i,$$

where C^1_i and Z^1_i intersect transversally at two points, but $-K_X\cdot Z^1_i=2/3$ and $-K_X\cdot C^1_i=1/2$.

We set $C_i^2 = \alpha_2(\overline{C}_i^2)$ for all $i = 1, \dots, 14$.

Remark 4.4. The fibre of the projection ψ over the point $\psi(C_i^2)$ consists of the smooth rational curve C_i^2 and a smooth irreducible rational curve Z_i^2 such that

$$C_i^2 \ni O_2 \in Z_i^1 \quad \text{and} \quad Z_i^2 \ni O_1 \notin C_i^2,$$

where C_i^1 and Z_i^1 intersect at O_2, the curves C_i^1 and Z_i^1 intersect transversally at a smooth point of X, and we have $-K_X \cdot Z_i^1 = 5/6$ and $-K_X \cdot C_i^1 = 1/3$.

Let D be a divisor in $|-nK_X|$, where $n \in \mathbb{N}$. We set $\mu = 6/(7n)$ and $\lambda = 1/n$.

Remark 4.5. To prove Theorem 1.44 it is sufficient to show that the log pair $(X, \mu D)$ has at most log canonical singularities because D is an arbitrary divisor in $|-nK_X|$.

To prove Theorem 1.44 we describe reducible fibres of ψ first.

Lemma 4.6. *Let F be a reducible fibre of the rational map ψ. Then*

$$F \in \left\{C_1^1 \cup Z_1^1, \dots, C_{35}^1 \cup Z_{35}^1,\ C_1^2 \cup Z_1^2, \dots, C_{14}^2 \cup Z_{14}^2\right\}.$$

Proof. Let C be an irreducible curve on the hypersurface X. Then

$$C \in \{C_1^1, \dots, C_{35}^1\}$$

if $-K_X \cdot C = 1/2$ because the proper transform of the curve C on the variety U_1 has trivial intersection with $-K_{U_1}$ in the case when $-K_X \cdot C = 1/2$.

Note that the equality $-K_X \cdot C = 1/6$ is impossible because otherwise the proper transform of the curve C on the variety U_1 has negative intersection with $-K_{U_1}$, which is nef.

Suppose that $-K_X \cdot C = 1/3$. Let $\bar{C}$ be the proper transform of the curve C on the variety U_2. Then

$$0 \leqslant -K_{U_2} \cdot \overline{C} = \left(\alpha_2^*(-K_X) - \frac{1}{3}\,E\right) \cdot \overline{C} = \frac{1}{3} - \frac{1}{3}\,E_2 \cdot \overline{C},$$

where E_2 is the exceptional divisor of α_2. On the other hand, $2E_2 \cdot \overline{C}$ is a positive integer, so that $E_2 \cdot \overline{C} = 1/2$ or $E_2 \cdot \overline{C} = 1$. The equality $E_2 \cdot \overline{C} = 1/2$ implies that

$$-K_{U_2} \cdot \overline{C} = \left(\alpha_2^*(-K_X) - \frac{1}{3}\,E\right) \cdot \overline{C} = \frac{1}{3} - \frac{1}{3}\,E_2 \cdot \overline{C} = \frac{1}{6}\,,$$

which is a contradiction because $-2K_{U_2}$ is Cartier. Hence $E_2 \cdot \overline{C} = 1$, and therefore $-K_{U_2} \cdot \overline{C} = 0$. Thus, we see that

$$C \in \{C_1^2, \dots, C_{14}^2\}$$

because the irreducible rational curves $\overline{C}_1^2, \dots, \overline{C}_{14}^2$ are the only curves on U_1 that have trivial intersection with $-K_{U_2}$.

Note that $-K_X \cdot F = 7/6$. Let C be an irreducible component of F such that $-K_X \cdot C$ is minimal. Then either $-K_X \cdot C = 1/2$ or $-K_X \cdot C = 1/3$ because $-6K_X \cdot C \in \mathbb{N}$. Then we must have

$$C \in \left\{C_1^1, \dots, C_{35}^1,\ C_1^2, \dots, C_{14}^2\right\},$$

which immediately yields the required result.

Suppose that the log pair $(X, \mu D)$ is not log canonical. We shall show that this leads to a contradiction. We may assume that D is irreducible (see Remark 2.2).

Lemma 4.7. $n \neq 1$.

Proof. Arguing as in the proof of Lemma 3.2 we obtain the required result.

Let P be a point of the variety V such that the log pair $(X, \mu D)$ is not log canonical at P, and let F be a scheme fibre of the projection ψ that passes through the point P.

Remark 4.8. If $P \notin \mathrm{Sing}(X)$, then the fibre F is uniquely defined.

The fibre F is reduced. Let S be a general surface in $|-K_X|$ such that $P \in S$.

Lemma 4.9. *Suppose that* $\mathrm{Sing}(X) \not\ni P \notin \mathrm{Sing}(F)$. *Then F is reducible.*

Proof. Suppose that F is irreducible. Let $\pi \colon \overline{X} \to X$ be a blow up of the point P. Then

$$\overline{D} \equiv \pi^*(D) - \mathrm{mult}_P(D)E,$$

where E is the π-exceptional divisor and $\overline{D}$ is the proper transform of D on the threefold $\overline{X}$.

Note that $\mathrm{mult}_P(D) > 1/\mu = 7n/6$. Let

$$D\big|_S = mF + \Omega,$$

where m is a non-negative rational number and Ω is an effective $\mathbb{Q}$-divisor on the surface S whose support does not contain the curve F. Then

$$\frac{7n}{6} = F \cdot (mF + \Omega) = \frac{7m}{6} + F \cdot \Omega \geqslant \frac{7m}{6} + \mathrm{mult}_P(\Omega) > \frac{7m}{6} + \frac{7n}{6} - m = \frac{7n}{6} + \frac{m}{6},$$

which is a contradiction completing the proof.

The log pair $(X, \lambda D)$ is also not log canonical at the point P. In the remaining part of this section we show that the last assumption also leads to a contradiction.

Lemma 4.10. *Suppose that* $P \notin \mathrm{Sing}(X)$. *Then F is reducible.*

Proof. Suppose that the fibre F is reducible. Then $\mathrm{mult}_P(F) \neq 1$ by Lemma 4.9 and it follows from the generality of the hypersurface X that $\mathrm{mult}_P(F) = 2$.

One can easily see that there exists a surface $T \in |-K_X|$ such that $\mathrm{mult}_P(T) \geqslant 2$. Let

$$T \cdot D = \varepsilon F + \Delta,$$

where ε is a non-negative rational number and Δ is an effective 1-cycle whose support does not contain the curve F. Then $\Delta \not\subseteq \mathrm{Supp}(S)$ and $\mathrm{mult}_P(\Delta) > 2n - 2\varepsilon$. We have

$$\frac{7n}{6} = S \cdot T \cdot D = \frac{7\varepsilon}{6} + S \cdot \Delta > \frac{7\varepsilon}{6} + 2n - 2\varepsilon,$$

which implies that $\varepsilon > n$. However, this is impossible by Remark 2.1 and the proof is complete.

Lemma 4.11. *P is a singular point of the hypersurface X.*

Proof. Let P be a smooth point of X. Then F is reducible by Lemma 4.10, and it follows from Lemma 4.6 that

$$F \in \{C_1^1 \cup Z_1^1, \dots, C_{35}^1 \cup Z_{35}^1,\, C_1^2 \cup Z_1^2, \dots, C_{14}^2 \cup Z_{14}^2\}.$$

Without loss of generality we may assume that either $F = C_1^1 \cup Z_1^1$ or $F = C_1^2 \cup Z_1^2$.

Let $F = C_1^1 \cup Z_1^1$. Then

$$C_1^1 \cdot C_1^1 = -\frac{3}{2}, \qquad C_1^1 \cdot Z_1^1 = 2, \qquad Z_1^1 \cdot Z_1^1 = -\frac{4}{3}$$

on the surface S. Let

$$D\big|_S = m_1 C_1^1 + m_2 Z_1^1 + \Omega \equiv -nK_X\big|_S,$$

where m_1 and m_2 are non-negative rational numbers and Ω is an effective $\mathbb{Q}$-divisor on the surface S whose support does not contain the curves C_1^1 and Z_1^1. Then the log pair

$$(S,\, \lambda m_1 C_1^1 + \lambda m_2 Z_1^1 + \lambda\Omega)$$

is not log canonical at the point P by [1], Theorem 7.5. We claim that this contradicts the numerical effectiveness of

$$m_1 C_1^1 + m_2 Z_1^1 + \Omega \equiv -nK_X\big|_S$$

bearing in mind that $C_1^1 + Z_1^1 \equiv -K_X\big|_S$ on the surface S. The log pair $(S,\, C_1^1 + Z_1^1)$ is log canonical at the point P in view of the generality of the choice of X. Thus, we may assume that $m_1 = 0$ or $m_2 = 0$ by Remark 2.2.

Suppose that $m_1 = 0$. Then

$$\frac{n}{2} = C_1^1 \cdot (m_2 Z_1^1 + \Omega) = 2m_2 + C_1^1 \cdot \Omega \geqslant 2m_2,$$

which implies that $m_2 \leqslant n/4$. We have $P \notin C_1^1$ because otherwise

$$\frac{n}{2} = C_1^1 \cdot (m_2 Z_1^1 + \Omega) = 2m_2 + C_1^1 \cdot \Omega > 2m_2 + \frac{1}{\lambda} \geqslant n$$

by Remark 2.4. Hence we see that $P \in Z_1^1$. Then

$$\frac{2n}{3} = Z_1^1 \cdot (m_2 Z_1^1 + \Omega) = -\frac{4m_2}{3} + Z_1^1 \cdot \Omega > -\frac{4m_2}{3} + \frac{1}{\lambda} \geqslant -\frac{4m_2}{3} + n$$

by Remark 2.4, so that $m_2 > n/4$. However, we have $m_2 \leqslant n/4$, which is a contradiction.

Suppose that $m_2 = 0$. Arguing as in the previous case we see that it follows from Remark 2.4 and the equality

$$\frac{2n}{3} = Z_1^1 \cdot (m_1 C_1^1 + \Omega) = 2m_1 + Z_1^1 \cdot \Omega$$

that $m_1 \leqslant n/3$ and $P \notin Z_1^1$. Then $P \in C_1^1$ and

$$\frac{n}{2} = C_1^1 \cdot (m_1 C_1^1 + \Omega) = -\frac{3m_1}{2} + C_1^1 \cdot \Omega > -\frac{3m_1}{2} + \frac{1}{\lambda} \geqslant -\frac{3m_1}{2+n}$$

by Remark 2.4. We see that $m_1 > n/3$, although we have $m_1 \leqslant n/3$, which is a contradiction.

Thus, $F = C_1^2 \cup Z_1^2$. Then

$$C_1^2 \cdot C_1^2 = -\frac{4}{3}, \qquad C_1^2 \cdot Z_1^2 = \frac{5}{3}, \qquad Z_1^2 \cdot Z_1^2 = -\frac{5}{6}$$

on the surface S. As in the previous case, let

$$D\big|_S = n_1 C_1^2 + n_2 Z_1^2 + \Delta \equiv -nK_X\big|_S,$$

where n_1 and n_2 are non-negative rational numbers and Δ is an effective $\mathbb{Q}$-divisor on S whose support does not contain the curves C_1^2 and Z_1^2. Then the singularities of the log pair

$$(S,\, \lambda n_1 C_1^2 + \lambda n_2 Z_1^2 + \lambda\Delta)$$

are not log canonical at the point P by [1], Theorem 7.5. We claim that this contradicts the numerical effectiveness of

$$n_1 C_1^2 + n_2 Z_1^2 + \Delta \equiv n(C_1^2 + Z_1^2) \equiv -nK_X\big|_S$$

on S. We may assume that $n_1 n_2 = 0$ by Remark 2.2 because the log pair $(S, C_1^2 + Z_1^2)$ is log canonical at the point P.

Suppose that $n_1 = 0$. Then

$$\frac{n}{3} = C_1^2 \cdot (n_2 Z_1^2 + \Delta) = \frac{5n_2}{3} + C_1^2 \cdot \Delta \geqslant \frac{5n_2}{3},$$

which implies that $n_2 \leqslant n/5$. We have $P \notin C_1^2$ because otherwise

$$\frac{n}{3} = C_1^2 \cdot (n_2 Z_1^2 + \Delta) = \frac{5n_2}{3} + C_1^2 \cdot \Delta > \frac{5n_2}{3} + \frac{1}{\lambda} \geqslant n$$

by Remark 2.4. Hence we see that $P \in Z_1^2$. Then

$$\frac{5n}{6} = Z_1^2 \cdot (n_2 Z_1^2 + \Delta) = -\frac{5n_2}{6} + Z_1^2 \cdot \Delta > -\frac{5n_2}{6} + \frac{1}{\lambda} \geqslant -\frac{5n_2}{6} + n$$

by Remark 2.4. Thus, $n_2 > n/5$. However, we have $n_2 \leqslant n/5$, which is a contradiction.

Let $n_2 = 0$. Arguing as in the previous case, we see that it follows from Remark 2.4 and the equality

$$\frac{5n}{6} = Z_1^1 \cdot (n_1 C_1^2 + \Delta) = \frac{5n_1}{3} + Z_1^2 \cdot \Delta$$

that $n_1 \leqslant n/2$ and $P \notin Z_1^2$. Then $P \in C_1^2$ and

$$\frac{n}{3} = C_1^2 \cdot (n_1 C_1^2 + \Delta) = -\frac{4n_1}{3} + C_1^2 \cdot \Delta > -\frac{4n_1}{3} + \frac{1}{\lambda} \geqslant -\frac{4n_1}{3} + n$$

by Remark 2.4. We see that $n_1 > n/2$. However, we have $n_1 \leqslant n/2$, which is a contradiction completing the proof.

Hence we see that either $P = O_1$ or $P = O_2$. Suppose that $P = O_1$. Then

$$D_1 \equiv \alpha_1^*(D) - \mu_1 E_1,$$

where E_1 is the α_1-exceptional divisor, D_1 is the proper transform of the divisor D on the variety U_1, and μ_1 is a rational number. Then $\mu_1 > n/2$ by Remark 2.3, and we have

$$K_{U_1} + \lambda D_1 + \left(\lambda\mu_1 - \frac{1}{2}\right)E_1 \equiv \alpha_1^*(K_X + \lambda D).$$

Lemma 4.12. $\mu_1 \leqslant 7n/10$.

Proof. The point O_1 can be given by $x = y = z = w = 0$, and X can be given by the equation

$$t^2w + tf_5(x, y, z, w) + f_7(x, y, z, w) = 0 \subset \mathbb{P}(1, 1, 1, 2, 3) \cong \operatorname{Proj}(\mathbb{C}[x, y, z, t, w]),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = \mathrm{wt}(z) = 1$, $\mathrm{wt}(t) = 2$, $\mathrm{wt}(w) = 2$, and f_5, f_7 are quasihomogeneous polynomials of degrees 5 and 7, respectively. In these coordinates the curves $C_1^1, \dots, C_{35}^1$ are cut out on the hypersurface X by the equations $w = f_5(x, y, z, w) = f_7(x, y, z, w) = 0$.

Let R be a surface on X cut out by the equation $w = 0$, and let $\overline{R}$ be the proper transform of R on the variety U_1. Then R is irreducible and

$$\overline{R} \equiv \alpha_1^*(-3K_X) - \frac{5}{2}E_1,$$

but $\left(X, \frac{1}{3}R\right)$ is log canonical at O_1 by [1], Lemma 8.12 and Proposition 8.14 because we may assume that X is sufficiently general.

The log pair $(X, \lambda D)$, where $\lambda = 1/n$, is not log canonical at the point P. Then $R \neq D$ and

$$0 \leqslant -K_{U_1} \cdot \overline{R} \cdot D_1 = \frac{7n}{2} - 5\mu_1$$

because $-K_{U_1}$ is nef. Hence $\mu_1 \leqslant 7n/10$.

In particular, there is a point $Q_1 \in E_1$ such that the log pair

$$\left(U_1,\, \lambda D_1 + \left(\lambda\mu_1 - \frac{1}{2}\right)E_1\right)$$

is not log canonical at Q_1. Let S_1 be a general surface in $|-K_{U_1}|$ such that $Q_1 \in \overline{S}$.

Remark 4.13. The proper transform of the surface E_1 on the variety W_1 is a section of the elliptic fibration η. In particular, the surface S_1 is smooth at the point Q_1.

Let $\overline{Z}_i^1$ be the proper transform of the curve Z_i^1 on the variety U_1, where $i = 1, \dots, 35$.

Lemma 4.14. *The point Q_1 is not contained in $\bigcup_{i=1}^{35} \overline{C}_i^1$.*

Proof. Suppose that $Q_1 \in \bigcup_{i=1}^{35} \overline{C}_i^1$. We may assume that $Q_1 \in \overline{C}_1^1$. Let

$$D_1\big|_{S_1} + \left(\mu_1 - \frac{n}{2}\right)E_1\big|_{S_1} = m_1\overline{C}_1^1 + m_2\overline{Z}_1^1 + \Omega \equiv -nK_{U_1}\big|_{S_1},$$

where m_1 and m_2 are non-negative rational numbers and Ω is an effective $\mathbb{Q}$-divisor on the surface S whose support does not contain the curves $\overline{C}_1^1$ and $\overline{Z}_1^1$. Then the log pair

$$(S_1,\, \lambda m_1\overline{C}_1^1 + \lambda m_2\overline{Z}_1^1 + \lambda\Omega)$$

is not log canonical at Q_1 by [1], Theorem 7.5. We claim that this is impossible.

The log pair $(S_1,\, \overline{C}_1^1 + \overline{Z}_1^1)$ is log canonical at the point Q_1. It follows from Remark 2.2 that we may assume that either $m_1 = 0$ or $m_2 = 0$ because $\overline{C}_1^1 + \overline{Z}_1^1 \equiv -K_{U_1}\big|_{S_1}$.

It follows from Remark 2.4 that

$$0 = \overline{C}_1^1 \cdot (m_1\overline{C}_1^1 + m_2\overline{Z}_1^1 + \Omega) = 2m_2 + \overline{C}_1^1 \cdot \Omega > 2m_2 + n$$

if $m_1 = 0$. Hence we may assume that $m_2 = 0$. Then

$$\frac{2n}{3} = \overline{Z}_1^1 \cdot (m_1\overline{C}_1^1 + \Omega) = 2m_1 + \overline{Z}_1^1 \cdot \Omega \geqslant 2m_1,$$

which implies that $m_1 \leqslant n/3$. We see that

$$0 = \overline{C}_1^1 \cdot (m_1\overline{C}_1^1 + \Omega) = -2m_1 + \overline{C}_1^1 \cdot \Omega > -2m_1 + n$$

by Remark 2.4. Hence $m_1 > n/2$. However, we have $m_1 \leqslant n/3$, which is a contradiction completing the proof.

Let $\grave{C}_i^2$ and $\grave{Z}_i^2$ be the proper transforms of C_i^2 and Z_i^2 on U_1, respectively, where $i = 1, \ldots, 14$.

Lemma 4.15. *The point Q_1 is not contained in $\bigcup_{i=1}^{14} \grave{Z}_i^2$.*

Proof. Suppose that Q_1 is contained in $\bigcup_{i=1}^{14} \grave{Z}_i^2$. We shall show that this leads to a contradiction. We may assume that $Q_1 \in \grave{Z}_1^2$. Then

$$\grave{C}_1^2 \cdot \grave{C}_1^2 = \grave{Z}_1^2 \cdot \grave{Z}_1^2 = -\frac{4}{3}, \qquad \grave{C}_1^2 \cdot \grave{Z}_1^2 = \frac{5}{3}$$

on the surface S_1. Note that $Q_1 \notin \grave{C}_1^2$. Let

$$D_1\big|_{S_1} + \left(\mu_1 - \frac{n}{2}\right)E_1\big|_{S_1} = m_1\grave{C}_1^2 + m_2\grave{Z}_1^2 + \Omega \equiv -nK_{U_1}\big|_{S_1},$$

where m_1 and m_2 are non-negative rational numbers and Ω is an effective $\mathbb{Q}$-divisor on the surface S_1 whose support does not contain the curves $\grave{C}_1^2$ and $\grave{Z}_1^2$.

It follows from [1], Theorem 7.5 that the log pair

$$(S_1,\, \lambda m_1\grave{C}_1^2 + \lambda m_2\grave{Z}_1^2 + \lambda\Omega)$$

is not log canonical at the point Q_1. We claim that this is impossible.

The log pair $(S_1, \grave{C}_1^2 + \grave{Z}_1^2)$ is log canonical at the point Q_1. By Remark 2.2 we may assume that either $m_1 = 0$ or $m_2 = 0$ because $\grave{C}_1^2 + \grave{Z}_1^2 \equiv -K_{U_1}\big|_{S_1}$.

Suppose that $m_2 = 0$. Then it follows from Remark 2.4 that

$$\frac{n}{3} = \grave{Z}_1^2 \cdot (m_1\grave{C}_1^2 + \Omega) = \frac{5m_1}{3} + \grave{Z}_1^2 \cdot \Omega > \frac{5m_1}{3} + \frac{1}{\lambda} \geqslant n,$$

which is a contradiction. Hence we may assume that $m_1 = 0$. Then

$$\frac{n}{3} = \grave{C}_1^2 \cdot (m_2\grave{Z}_1^2 + \Omega) = \frac{5m_2}{3} + \grave{C}_1^2 \cdot \Omega \geqslant \frac{5m_2}{3},$$

which implies that $m_2 \leqslant n/5$. We see that

$$\frac{n}{3} = \grave{Z}_1^2 \cdot (m_2\grave{Z}_1^2 + \Omega) = -\frac{4m_2}{3} + \grave{Z}_1^2 \cdot \Omega > -\frac{4m_2}{3} + \frac{1}{\lambda} \geqslant -\frac{4m_2}{3} + n$$

by Remark 2.4. We obtain $m_2 > n/2$. However, we have $m_2 \leqslant n/5$, which is a contradiction completing the proof.

Let F_1 be the scheme fibre of the rational map $\psi \circ \alpha_1$ that passes through the point Q_1. Then F_1 is irreducible by Lemmas 4.6, 4.14 and 4.15 (see Remark 4.13).

The curve F_1 is smooth at the point Q_1 by Remark 4.13. Let

$$D_1\big|_{S_1} + \left(\mu_1 - \frac{n}{2}\right)E_1\big|_{S_1} = mF_1 + \Omega,$$

where m is a non-negative rational number and Ω is an effective $\mathbb{Q}$-divisor on S_1 whose support does not contain the curve F_1. Then

$$\frac{2n}{3} = F_1 \cdot (mF_1 + \Omega) = \frac{2m}{3} + F_1 \cdot \Omega \geqslant \frac{2m}{3} + \mathrm{mult}_{Q_1}(\Omega) > \frac{2m}{3} + n - m,$$

which implies that $m > n$. This is impossible by Remark 2.1. We see that the assumption $P = O_1$ leads to a contradiction.

Remark 4.16. The equality $P = O_2$ holds.

Let D_2 be the proper transform of the divisor D on the variety U_2. Then

$$D_2 \equiv \alpha_2^*(D) - \mu_2 E_2,$$

where E_2 is the α_2-exceptional divisor and μ_2 is a rational number. We have

$$K_{U_2} + \lambda D_2 + \left(\lambda\mu - \frac{1}{3}\right)E_2 \equiv \alpha_2^*(K_X + \lambda D),$$

where $\lambda\mu - 1/3 > 0$ by Remark 2.3.

The hypersurface X can be given by the equation

$$w^2x + wf_4(x, y, z, t) + f_7(x, y, z, t) = 0 \subset \mathbb{P}(1, 1, 1, 2, 3) \cong \mathrm{Proj}(\mathbb{C}[x, y, z, t, w]),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = \mathrm{wt}(z) = 1$, $\mathrm{wt}(t) = 2$, $\mathrm{wt}(w) = 3$ and f_4, f_7 are quasi-homogeneous polynomials of degrees 4 and 7, respectively. Then O_2 is given by $x = y = z = t = 0$.

Remark 4.17. The curves $C_1^2, \dots, C_{14}^2$ are cut out on X by $x = f_4 = f_7 = 0$.

Let R be a surface on X cut out by the equation $x = 0$, and let $\overline{R}$ be the proper transform of the surface R on the variety U_2. Then R is irreducible and the equivalence

$$\overline{R} \equiv \alpha_2^*(-K_X) - \frac{4}{3}\,E_2$$

holds. The surface $\overline{R}$ is smooth in a neighbourhood of E_2 because X is general.

Lemma 4.18. $\mu_2 \leqslant 7n/12$.

Proof. By Lemma 4.7 we obtain $R \neq D$. Then

$$0 \leqslant -K_{U_2} \cdot \overline{R} \cdot D_2 = \frac{7n}{6} - 2\mu_2,$$

because the divisor $-K_{U_2}$ is nef. Hence $\mu_2 \leqslant 7n/12$.

In particular, there is a point $Q_2 \in E_2$ such that the log pair

$$\left(U_2,\, \lambda D_2 + \left(\lambda\mu_2 - \frac{1}{3}\right) E_2\right)$$

is not log canonical at Q_2. Let S_2 be a general surface in $|-K_{U_2}|$ such that $Q_2 \in S_2$.

Remark 4.19. The map ψ is induced by the embedding of graded algebras

$$\mathbb{C}[x, y, z] \subset \mathbb{C}[x, y, z, t, w],$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = \mathrm{wt}(z) = 1$, $\mathrm{wt}(t) = 2$ and $\mathrm{wt}(w) = 3$. Both E_2 and $\overline{R}$ are contracted by

$$\psi \circ \alpha_2 \colon U_2 \dashrightarrow \mathbb{P}^2$$

to the line in $\mathbb{P}^2 \cong \mathrm{Proj}(\mathbb{C}[x, y, z])$ given by the equation $x = 0$.

Let $\overline{Z}_i^2$ be the proper transform of the curve Z_i^2 on the variety U_2, where $i = 1, \dots, 14$.

Lemma 4.20. *The point Q_2 is not contained in $\bigcup_{i=1}^{14} \overline{C}_i^2$ or $\bigcup_{i=1}^{14} \overline{Z}_i^2$.*

Proof. Let $Q_2 \in \bigcup_{i=1}^{14} \overline{C}_i^2$ or $Q_2 \in \bigcup_{i=1}^{14} \overline{Z}_i^2$. Without loss of generality we may assume that $Q_2 \in \overline{C}_1^2 \cup \overline{Z}_1^2$. The surface $\overline{R}$ contains the curves $\overline{C}_1^2$ and $\overline{Z}_1^2$. Let

$$D_1\big|_{\overline{R}} + \left(\mu_2 - \frac{n}{3}\right) E_2\big|_{\overline{R}} = m_1\overline{C}_1^2 + m_2\overline{Z}_1^2 + \Omega \equiv -nK_{U_2}\big|_{\overline{R}},$$

where m_1 and m_2 are non-negative rational numbers and Ω is an effective $\mathbb{Q}$-divisor on the surface $\overline{R}$ whose support does not contain the curves $\overline{C}_1^2$ and $\overline{Z}_1^2$. The log pair

$$(\overline{R},\, \lambda m_1\overline{C}_1^2 + \lambda m_2\overline{Z}_1^2 + \lambda\Omega)$$

is not log canonical at Q_2 by [1], Theorem 7.5. We claim that this is impossible.

The log pair $(\overline{R},\, \overline{C}_1^2+\overline{Z}_1^2)$ is log canonical at the point Q_2 and $\overline{C}_1^2+\overline{Z}_1^2 \equiv -K_{U_2}\big|_{\overline{R}}$, so we may assume that either $m_1 = 0$ or $m_2 = 0$ (see Remark 2.2).

On the surface $\overline{R}$ we have

$$\overline{C}_1^2 \cdot \overline{C}_1^2 = -1, \qquad \overline{Z}_1^2 \cdot \overline{C}_1^2 = 1, \qquad \overline{Z}_1^2 \cdot \overline{Z}_1^2 = -\frac{1}{2}.$$

Let $m_1 = 0$. Then $m_2 = 0$ because

$$0 = \overline{C}_1^2 \cdot (m_2\overline{Z}_1^2 + \Omega) = m_2 + \overline{C}_1^2 \cdot \Omega \geqslant m_2,$$

and it follows from Remark 2.4 that $0 = \overline{C}_1^2 \cdot \Omega > n$ if $Q_2 \in \overline{C}_1^2$. We see that $Q_2 \in \overline{Z}_1^2$. Then

$$\frac{n}{2} = \overline{Z}_1^2 \cdot \Omega > \frac{1}{\lambda} = n$$

by Remark 2.4. The contradiction obtained implies that $m_1 \neq 0$.

Hence we may assume that $m_2 = 0$. Then

$$\frac{n}{2} = \overline{Z}_1^2 \cdot (m_1\overline{C}_1^2 + \Omega) = m_1 + \overline{Z}_1^1 \cdot \Omega \geqslant m_1,$$

which implies that $m_1 \leqslant n/2$. By Remark 2.4 we obtain

$$\frac{n}{2} = \overline{Z}_1^2 \cdot (m_1\overline{C}_1^2 + \Omega) = m_1 + \overline{Z}_1^1 \cdot \Omega > m_1 + \frac{1}{\lambda} \geqslant n$$

in the case when $Q_2 \in \overline{Z}_1^2$, which shows that $Q_2 \in \overline{C}_1^2$. Then

$$0 = \overline{C}_1^2 \cdot (m_1\overline{C}_1^1 + \Omega) = -m_1 + \overline{C}_1^1 \cdot \Omega > -m_1 + n$$

by Remark 2.4. We see that $m_1 > n$. However, $m_1 \leqslant n/2$. which is a contradiction completing the proof.

Note that the surface $\overline{R}$ does not contain the singular point of the surface E_2.

Lemma 4.21. *The surface $\overline{R}$ does not contain Q_2.*

Proof. Suppose that $Q_2 \in \overline{R}$. Then it follows from Lemma 4.20 that

$$S_2\big|_{\overline{R}} = Z \equiv -K_{U_2}\big|_{\overline{R}},$$

where Z is a smooth curve such that $Q_2 \in Z$. Then $Z \cdot Z = 1/2$ on the surface $\overline{R}$. Let

$$D_1\big|_{\overline{R}} + \left(\mu_2 - \frac{n}{3}\right) E_2\big|_{\overline{R}} = mZ + \Omega \equiv -nK_{U_2}\big|_{\overline{R}},$$

where m is a non-negative rational number and Ω is an effective $\mathbb{Q}$-divisor on $\overline{R}$ whose support does not contain the curve Z. Then the log pair

$$(\overline{R},\, \lambda m Z + \lambda\Omega)$$

is not log canonical at Q_2 by [1], Theorem 7.5. We claim that this is impossible.

The log pair $(\overline{R}, Z)$ is log canonical at Q_2. By Remark 2.2 we may assume that $m = 0$. Then $n/2 = Z \cdot \Omega > n$, which is a contradiction completing the proof.

Let O_3 be the singular point of the surface $E_2 \cong \mathbb{P}(1,1,2)$, let $\grave{C}_i^1$ and $\grave{Z}_i^1$ be the proper transforms of the curves C_i^2 and Z_i^2 on the variety U_2, respectively, where $i = 1, \dots, 14$. Then

$$\grave{Z}_1^2 \cap E_2 = \cdots = \grave{Z}_{14}^2 \cap E_2 = O_3, \qquad \grave{C}_1^2 \cap E_2 = \cdots = \grave{C}_{14}^2 \cap E_2 = \varnothing.$$

Lemma 4.22. $Q_2 = O_3$.

Proof. Suppose that $Q_2 \neq O_3$. Let F_2 be the scheme fibre of the rational map $\psi \circ \alpha_2$ that passes through the point Q_2. Then either

$$F_2 = L + \overline{C}_i^2 + \overline{Z}_i^2$$

for some $i = 1, \dots, 14$ or $F_1 = L + Z$, where L is an irreducible curve contained in the divisor E_2 and Z is an irreducible curve not contained in the divisor E_2.

Suppose that $F_1 = L + Z$. Then on the surface S_2 we have

$$L \cdot L = Z \cdot Z = -\frac{3}{2}, \qquad L \cdot Z = 2,$$

and it follows from Lemma 4.21 that $Q_2 \in L$ and $Q_2 \notin Z$ because $Z = \overline{R} \cap S_2$. Let

$$D_2\big|_{S_2} + \left(\mu_2 - \frac{n}{3}\right) E_2\big|_{S_2} = m_1 L + m_2 Z + \Omega \equiv -nK_{U_2}\big|_{S_2},$$

where m_1 and m_2 are non-negative rational numbers and Ω is an effective $\mathbb{Q}$-divisor on the surface S_2 whose support does not contain the curves L and Z.

By [1], Theorem 7.5 the log pair

$$(S_2,\, \lambda m_1 L + \lambda m_2 Z + \lambda \Omega)$$

is not log canonical at the point Q_2. We claim that this is impossible.

The log pair $(S_2,\, L + Z)$ is log canonical at the point Q_2. On the surface S_2 we have

$$L + Z \equiv -K_{U_2}\big|_{S_2},$$

which implies that we may assume that either $m_1 = 0$ or $m_2 = 0$ (see Remark 2.2).

Suppose that $m_1 = 0$. Then it follows from Remark 2.4 that

$$\frac{n}{2} = L \cdot (m_2 Z + \Omega) = 2m_2 + L \cdot \Omega > 2m_2 + \frac{1}{\lambda} \geqslant n,$$

which is a contradiction. Hence we may assume that $m_2 = 0$. Then

$$\frac{n}{2} = Z \cdot (m_1 L + \Omega) = 2m_1 + Z \cdot \Omega \geqslant 2m_1,$$

which implies that $m_1 \leqslant n/4$. We see that

$$\frac{n}{2} = L \cdot (m_1 L + \Omega) = -\frac{3m_1}{2} + L \cdot \Omega > -\frac{3m_1}{2} + \frac{1}{\lambda} \geqslant -\frac{3m_1}{2} + n$$

by Remark 2.4. Thus, $m_1 > n/3$. However, $m_1 \leqslant n/4$, which is a contradiction.

We see that $F_2 = L + \overline{C}_i^2 + \overline{Z}_i^2$ for some $i = 1, \ldots, 14$, where L is an irreducible curve contained in the exceptional divisor E_2 such that

$$\overline{R}\big|_{S_2} = L + \overline{C}_i^2 + \overline{Z}_i^2 \equiv -K_{U_2}\big|_{S_2}.$$

We may assume that $F_2 = L + \overline{C}_1^2 + \overline{Z}_1^2$. Then

$$L \cdot \overline{C}_1^2 = L \cdot \overline{Z}_1^2 = \overline{C}_1^2 \cdot \overline{Z}_1^2 = 1, \qquad \overline{C}_1^2 \cdot \overline{C}_1^2 = -2 \quad \text{and} \quad \overline{Z}_1^2 \cdot \overline{Z}_1^2 = L \cdot L = -\frac{3}{2}$$

on the surface S_2. From Lemma 4.21 we see that $Q_2 \in L$ and $\overline{C}_1^2 \not\ni Q_2 \notin \overline{Z}_1^2$. Let

$$D_2\big|_{S_2} + \left(\mu_2 - \frac{n}{3}\right)E_2\big|_{S_2} = m_1 L + m_2\overline{C}_1^2 + m_3\overline{Z}_1^2 + \Omega \equiv -nK_{U_2}\big|_{S_2},$$

where m_1, m_2 and m_3 are non-negative rational numbers and Ω is an effective $\mathbb{Q}$-divisor on S_2 whose support does not contain the curves L, $\overline{C}_1^2$ and $\overline{Z}_1^2$.

By [1], Theorem 7.5 the log pair

$$(S_2,\, \lambda m_1 L + \lambda m_2\overline{C}_1^2 + \lambda\overline{Z}_1^2 + \lambda\Omega)$$

is not log canonical at the point Q_2. We shall show that this leads to a contradiction.

The log pair $(S_2,\, L + \overline{C}_1^2 + \overline{Z}_1^2)$ is log canonical at Q_2. In view of the equivalence

$$L + \overline{C}_1^2 + \overline{Z}_1^2 \equiv -K_{U_2}\big|_{S_2}$$

and Remark 2.2, we may assume that $m_1 m_2 m_3 = 0$.

Suppose that $m_1 = 0$. Then it follows from Remark 2.4 that

$$\frac{n}{2} = L \cdot (m_2\overline{C}_1^2 + m_2\overline{Z}_1^2 + \Omega) = m_2 + m_3 + L \cdot \Omega > m_2 + m_3 + \frac{1}{\lambda} \geqslant n,$$

which is a contradiction. Hence we may assume that $m_1 \neq 0$.

Suppose that $m_2 = 0$. Then

$$0 = \overline{C}_1^2 \cdot (m_1 L + m_3\overline{Z}_1^2 + \Omega) = m_1 + m_3 + \overline{C}_1^2 \cdot \Omega \geqslant m_1 + m_3,$$

which implies that $m_1 = m_3 = 0$. However, we know that $m_1 \neq 0$, which is a contradiction.

Hence we see that $m_1 \neq 0$ and $m_2 \neq 0$, which implies that $m_3 = 0$. Then

$$\frac{n}{2} = \overline{Z}_1^2 \cdot (m_1 L + m_2\overline{C}_1^2 + \Omega) = m_1 + m_2 + \overline{Z}_1^2 \cdot \Omega \geqslant m_1 + m_2$$

because $\overline{Z}_1^2 \cdot \Omega \geqslant 0$. On the other hand, it follows from Remark 2.4 that

$$\frac{n}{2} = L \cdot (m_1 L + m_2\overline{C}_1^2 + \Omega) = -\frac{3m_1}{2} + m_2 + L \cdot \Omega > -\frac{3m_1}{2} + m_2 + n$$

because $m_1 \leqslant n/2$. These relations are not yet contradictory, but

$$0 = \overline{C}_1^2 \cdot (m_1 L + m_2\overline{C}_1^2 + \Omega) = m_1 - 2m_2 + \overline{C}_1^2 \cdot \Omega \geqslant m_1 - 2m_2,$$

which implies that $m_2 \geqslant m_1/2$. The inequalities obtained are inconsistent, which completes the proof.

We see that $Q_2 = O_3$. Let $\breve{D}$ be the proper transform of D on the variety Y_2. Then

$$\breve{D} \equiv (\alpha_2 \circ \beta_2)^*(D) - \mu_2 \alpha_2^*(E_2) - \varepsilon G,$$

where G is the β_2-exceptional divisor and ε is a rational number. Now,

$$K_{Y_2} + \lambda \breve{D} + \left(\lambda\mu_2 - \frac{n}{3}\right)\breve{E}_2 + \left(\lambda\varepsilon + \frac{\lambda\mu_2}{2} - \frac{2}{3}\right)G \equiv (\alpha_2 \circ \beta_2)^*(K_X + \lambda D) \equiv 0,$$

where $\breve{E}_2$ is the proper transform of the surface E_2 on the variety Y. Then

$$\varepsilon + \frac{\mu_2}{2} > \frac{2n}{3}$$

by Remark 2.3. We now find an upper bound for $\varepsilon + \mu_2/2$.

Lemma 4.23. $\varepsilon + \mu_2/2 \leqslant 7n/6$.

Proof. Let F be a sufficiently general fibre of the map $\psi \circ \alpha_2 \circ \beta_2$. Then

$$0 \leqslant \breve{D} \cdot F = \left((\alpha_2 \circ \beta_2)^*(D) - \mu_2 \breve{E}_2 - \left(\varepsilon + \frac{\mu_2}{2}\right)G\right) \cdot F = \frac{7n}{6} - \varepsilon - \frac{\mu_2}{2},$$

which yields the required inequality and completes the proof.

Thus, there is a point $Q \in G$ such that the log pair

$$\left(Y_2,\, \lambda\breve{D} + \left(\lambda\mu_2 - \frac{n}{3}\right)\breve{E}_2 + \left(\lambda\varepsilon + \frac{\lambda\mu_2}{2} - \frac{2}{3}\right)G\right)$$

is not log canonical at Q. Let $\breve{S}$ be a general surface in $|-K_{Y_2}|$ such that $Q \in \breve{S}$.

Remark 4.24. The surface $\breve{S}$ is smooth at the point Q.

Let $\breve{F}$ be the fibre of the map $\psi \circ \alpha_2 \circ \beta_2$ passing through the point Q. Then $Q \notin \mathrm{Sing}(\breve{F})$.

Lemma 4.25. *The fibre $\breve{F}$ is reducible.*

Proof. Suppose that $\breve{F}$ is irreducible. Let

$$\overline{D}\big|_{\breve{S}} + \left(\mu_2 - \frac{n}{3}\right)\breve{E}_2\big|_{\breve{S}} + \left(\varepsilon + \frac{\mu_2}{2} - \frac{2n}{3}\right)G\big|_{\breve{S}} = m\breve{F} + \Omega \equiv -nK_{Y_2}\big|_{\breve{S}},$$

where m is a non-negative rational number and Ω is an effective $\mathbb{Q}$-divisor on $\breve{S}$ whose support does not contain the curve $\breve{F}$.

By [1], Theorem 7.5 the log pair

$$(\breve{S},\, \lambda m\breve{F} + \lambda\Omega)$$

is not log canonical at the point Q_2. We claim that this is impossible.

Note that $m \leqslant n$ because

$$m\breve{F} + \Omega \equiv n\breve{F} \equiv -nK_{Y_2}\big|_{\breve{S}}$$

on the surface $\breve{S}$. By Remark 2.2 we may assume that $m = 0$. Then

$$\frac{n}{2} = \breve{F} \cdot \Omega > \frac{1}{\lambda} = n$$

by Remark 2.4, which is a contradiction. The proof is complete.

Let $\breve{C}_i^1$ and $\breve{Z}_i^1$ be the proper transforms of C_i^1 and Z_i^1 on Y_2, respectively, where $i = 1, \dots, 35$.

Lemma 4.26. *The fibre $\breve{F}$ does not contain any curve among*

$$\breve{C}_1^1, \dots, \breve{C}_{35}^1, \breve{Z}_1^1, \dots, \breve{Z}_{35}^1.$$

Proof. Suppose that the support of the curve $\breve{F}$ contains one of the curves listed above. We shall show that this assumption leads to a contradiction.

Without loss of generality we may assume that the support of the curve $\breve{F}$ contains either the curve $\breve{C}_1^1$ or the curve $\breve{Z}_1^1$. Then $\breve{F} = \breve{C}_1^1 + \breve{Z}_1^1$. On the surface $\breve{S}$,

$$\breve{C}_1^1 \cdot \breve{Z}_1^2 = 2, \qquad \breve{C}_1^1 \cdot \breve{C}_1^1 = -\frac{3}{2}, \qquad \breve{Z}_1^1 \cdot \breve{Z}_1^1 = -2$$

We have $\breve{C}_1^1 \not\ni Q \in \breve{Z}_1^1$. As usual, let

$$\breve{D}\big|_{\breve{S}} + \left(\mu_2 - \frac{n}{3}\right)\breve{E}_2\big|_{\breve{S}} + \left(\varepsilon + \frac{\mu_2}{2} - \frac{2n}{3}\right)G\big|_{\breve{S}} = m_1\breve{C}_1^1 + m_2\breve{Z}_1^1 + \Omega \equiv n\breve{C}_1^1 + n\breve{Z}_1^1,$$

where m_1 and m_2 are non-negative rational numbers and Ω is an effective $\mathbb{Q}$-divisor on $\breve{S}$ whose support does not contain the curves $\breve{C}_1^1$ and $\breve{Z}_1^1$.

By [1], Theorem 7.5 the log pair

$$(\breve{S},\, \lambda m_1\breve{C}_1^1 + \lambda m_2\breve{Z}_1^1 + \lambda\Omega)$$

is not log canonical at the point Q. We shall show that this leads to a contradiction.

The log pair $(\breve{S}, \breve{C}_1^1 + \breve{Z}_1^1)$ is log canonical at Q. Hence we may assume by Remark 2.2 that $m_1 = 0$ or $m_2 = 0$.

Suppose that $m_1 = 0$. Then

$$\frac{n}{2} = \breve{C}_1^1 \cdot (m_2\breve{Z}_1^1 + \Omega) = 2m_2 + \breve{C}_1^1 \cdot \Omega \geqslant 2m_2,$$

which implies that $m_2 \leqslant n/2$. By Remark 2.4 we obtain

$$0 = \breve{Z}_1^1 \cdot (m_2\breve{Z}_1^1 + \Omega) = -2m_2 + \breve{Z}_1^1 \cdot \Omega > -2m_2 + n,$$

which implies that $m_2 > n/2$. This inequality contradicts the relation $m_2 \leqslant n/2$.

Thus, to complete the proof we may assume that $m_1 \neq 0$ and $m_2 = 0$. Then

$$0 = \breve{Z}_1^1 \cdot (m_1\breve{C}_1^1 + \Omega) = 2m_1 + \breve{Z}_1^1 \cdot \Omega \geqslant 2m_1,$$

which is impossible because $m_1 \neq 0$. The proof is complete.

Let $\breve{C}_i^2$ and $\breve{Z}_i^2$ be the proper transforms of C_i^2 and Z_i^2 on Y_2, respectively, where $i = 1, \dots, 14$.

Lemma 4.27. *The fibre $\breve{F}$ does not contain any curve among*

$$\breve{C}_1^2, \dots, \breve{C}_{14}^2, \breve{Z}_1^2, \dots, \breve{Z}_{14}^2.$$

Proof. Suppose that the support of the curve $\breve{F}$ contains one of the curves listed above. We shall show that this leads to a contradiction.

We may assume that $\breve{F}$ contains $\breve{C}_1^2$ or $\breve{Z}_1^2$. Then

$$\breve{F} = \breve{L} + \breve{C}_1^2 + \breve{Z}_1^2,$$

where $\breve{L}$ is an irreducible curve such that $\breve{L} \subset \breve{E}_2$. Then

$$\breve{L} \cdot \breve{C}_1^2 = \breve{L} \cdot \breve{Z}_1^2 = \breve{C}_1^2 \cdot \breve{Z}_1^2 = 1, \qquad \breve{C}_1^2 \cdot \breve{C}_1^2 = \breve{L} \cdot \breve{L} = -2, \qquad \breve{Z}_1^2 \cdot \breve{Z}_1^2 = -\frac{3}{2}$$

on the surface $\breve{S}$. We know that $Q \in \breve{L}$ and $\breve{C}_1^2 \not\ni Q \notin \breve{Z}_1^2$. Let

$$\begin{aligned} \breve{D}\big|_{\breve{S}} + \left(\mu_2 - \frac{n}{3}\right)\breve{E}_2\big|_{\breve{S}} + \left(\varepsilon + \frac{\mu_2}{2} - \frac{2n}{3}\right)G\big|_{\breve{S}} \\ = m_1\breve{L} + m_2\breve{C}_1^2 + m_3\breve{Z}_1^2 + \Omega \equiv n\breve{L} + n\breve{C}_1^2 + n\breve{Z}_1^2, \end{aligned}$$

where m_1, m_2 and m_3 are non-negative rational numbers and Ω is an effective $\mathbb{Q}$-divisor on $\breve{S}$ whose support does not contain the curves $\breve{L}$, $\breve{C}_1^2$ or $\breve{Z}_1^2$.

By [1], Theorem 7.5 the log pair

$$(\breve{S},\, \lambda m_1\breve{L} + \lambda m_2\breve{C}_1^2 + \lambda m_3\breve{Z}_1^2 + \lambda\Omega)$$

is not log canonical at Q. We shall show that this leads to a contradiction.

The log pair $(\breve{S},\, \breve{L} + \breve{C}_1^2 + \breve{Z}_1^2)$ is log canonical at Q, so we may assume that either $m_1 = 0$, or $m_2 = 0$, or $m_3 = 0$ (see Remark 2.2).

Suppose that $m_1 = 0$. Then it follows from Remark 2.4 that

$$0 = \breve{L} \cdot (m_2\breve{C}_1^2 + m_3\breve{Z}_1^2 + \Omega) = m_2 + m_3 + \breve{L} \cdot \Omega > m_2 + m_3 + n,$$

which is a contradiction. Thus, we may assume that $m_1 \neq 0$.

Suppose that $m_2 = 0$. Then

$$0 = \breve{C}_1^2 \cdot (m_1\breve{L} + m_3\breve{Z}_1^2 + \Omega) = m_1 + m_3 + \breve{C}_1^2 \cdot \Omega \geqslant m_1 + m_3,$$

which implies that $m_1 = m_3 = 0$. However, $m_1 \neq 0$, which is a contradiction.

Hence we see that $m_1 \neq 0$ and $m_2 \neq 0$. We may assume that $m_3 = 0$. Then

$$\frac{n}{2} = \breve{Z}_1^2 \cdot (m_1\breve{L} + m_2\breve{C}_1^2 + \Omega) = m_1 + m_2 + \breve{Z}_1^2 \cdot \Omega \geqslant m_1 + m_2,$$

which implies, in particular, that $m_1 \leqslant n/2$. By Remark 2.4 we obtain

$$0 = \breve{L} \cdot (m_1\breve{L} + m_2\breve{C}_1^2 + \Omega) = -2m_1 + m_2 + \breve{L} \cdot \Omega > -2m_1 + m_2 + n,$$

which means that $m_1 > n/2$. This contradicts the inequality $m_1 \leqslant n/2$ and completes the proof.

By Lemmas 4.25–4.27 we have $\breve{F}=\breve{L}+\breve{Z}$, where $\breve{L}$ and $\breve{Z}$ are irreducible curves such that $\breve{L}\subset\breve{E}_2$ and $\breve{Z}\not\subset\breve{E}_2$. Note that $\breve{Z}\not\ni Q\in\breve{L}$ because $\breve{Z}\cap G=\varnothing$. Then

$$\breve{L}\cdot\breve{Z}=2,\qquad \breve{Z}\cdot\breve{Z}=-\frac{3}{2}\quad\text{and}\quad \breve{L}\cdot\breve{L}=-2$$

on the surface $\breve{S}$. As usual, let

$$\breve{D}\big|_{\breve{S}}+\biggl(\mu_2-\frac{n}{3}\biggr)\breve{E}_2\big|_{\breve{S}}+\biggl(\varepsilon+\frac{\mu_2}{2}-\frac{2n}{3}\biggr)G\big|_{\breve{S}}=m_1\breve{L}+m_2\breve{Z}+\Omega\equiv n\breve{L}+n\breve{Z},$$

where m_1 and m_2 are non-negative rational numbers and Ω is an effective $\mathbb{Q}$-divisor on the surface $\breve{S}$ whose support does not contain the curves $\breve{L}$ and $\breve{Z}$.

By [1], Theorem 7.5 the log pair

$$(\breve{S},\,\lambda m_1\breve{L}+\lambda m_2\breve{Z}+\lambda\Omega)$$

is not log canonical at the point Q. We shall show that this leads to a contradiction.

By Remark 2.2 we may assume that $m_1=0$ or $m_2=0$ because the singularities of the log pair $(\breve{S},\breve{L}+\breve{Z})$ are log canonical at the point Q.

Suppose that $m_1=0$. Then it follows from Remark 2.4 that

$$0=\breve{L}\cdot(m_2\breve{Z}+\Omega)=2m_2+\breve{L}\cdot\Omega>2m_2+n,$$

which is a contradiction. Hence we may assume that $m_2=0$. Then

$$\frac{n}{2}=\breve{Z}\cdot(m_1\breve{L}+\Omega)=2m_1+\breve{Z}\cdot\Omega\geqslant 2m_1,$$

which implies that $m_1\leqslant n/2$. By Remark 2.4 we obtain

$$0=\breve{L}\cdot(m_1\breve{L}+\Omega)=-2m_1+\breve{L}\cdot\Omega>-2m_1+n,$$

which implies that $m_1>n/2$—a contradiction. The proof of Theorem 1.44 is complete.

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