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Log canonical thresholds of three-dimensional Fano hypersurfaces

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Abstract. We study global log canonical thresholds of generic hypersurfaces in $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ of degree $\sum_{i=1}^4 a_i$ that have at most terminal singularities.

Keywords: Fano variety, log canonical threshold, Tian's alpha-invariant, Kähler–Einstein metric.

§ 1. Introduction

Let X be a Fano variety[1] with at most log terminal singularities.

Definition 1.1. The *global log canonical threshold* of X is the number

$$\operatorname{lct}(X) = \sup\left\{\lambda \in \mathbb{Q} \;\middle|\; \begin{array}{c}\text{the log pair } (X, \lambda D) \text{ has log canonical singularities}\\ \text{for every effective } \mathbb{Q}\text{-divisor } D \equiv -K_X\end{array}\right\} \geqslant 0.$$

The number $\operatorname{lct}(X)$ plays an important role in Kähler geometry.

Example 1.2. If X has at most quotient singularities and we have

$$\operatorname{lct}(X) > \frac{\dim(X)}{\dim(X)+1},$$

then X admits an orbifold Kähler–Einstein metric [1].

Suppose further that X is a Fano variety with terminal $\mathbb{Q}$-factorial singularities and $\operatorname{rk}\operatorname{Pic}(X) = 1$.

Definition 1.3. The Fano variety X is said to be *birationally superrigid* if for every linear system $\mathcal{M}$ on X without fixed components, the log pair $(X, \lambda\mathcal{M})$ has canonical singularities, where $\lambda \in \mathbb{Q}$ is such that $K_X + \lambda\mathcal{M} \equiv 0$.

Let $X_1, \dots, X_r$ be Fano varieties with at most $\mathbb{Q}$-factorial terminal singularities and $\operatorname{rk}\operatorname{Pic}(X_i) = 1$ for all $i = 1, \dots, r$. The following result is proved in [2].

Theorem 1.4. *Suppose that* X_i *is birationally superrigid and* $\operatorname{lct}(X_i) \geqslant 1$ *for all* $i = 1, \dots, r$. *Then*

$$\operatorname{Bir}(X_1 \times \dots \times X_r) = \operatorname{Aut}(X_1 \times \dots \times X_r)$$

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[1] All varieties are assumed to be projective, normal and defined over $\mathbb{C}$.

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and, for every rational dominant map $\rho\colon X_1 \times \cdots \times X_r \dashrightarrow Y$ whose generic fibre is rationally connected, there is a subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, r\}$ and a commutative diagram

$$\begin{tikzcd}
X_1 \times \cdots \times X_r \arrow[d, "\pi"'] \arrow[dr, dashed, "\rho"] & \\
X_{i_1} \times \cdots \times X_{i_k} \arrow[r, dashed, "\xi"'] & Y
\end{tikzcd}$$

where ξ is a birational map and π is the natural projection.

Example 1.5. Let X be a generic hypersurface of degree $2n \geqslant 6$ in $\mathbb{P}(1^{n+1}, n)$. Then X is birationally superrigid and $\mathrm{lct}(X) = 1$ (see [2]).

Let us show how to generalize Theorem 1.4 to the case of Fano varieties with non-biregular birational automorphisms (see [3]).

Definition 1.6. A variety X is said to be *birationally rigid* if for every non-empty linear system $\mathcal{M}$ on X without fixed components there is a birational automorphism $\xi \in \mathrm{Bir}(X)$ such that the log pair $(X, \lambda\xi(\mathcal{M}))$ has canonical singularities, where $\lambda \in \mathbb{Q}$ is such that $K_X + \lambda\xi(\mathcal{M}) \equiv 0$.

The birational rigidity of X implies that

1) there is no dominant rational map $\rho\colon X \dashrightarrow Y$ such that $\dim(Y) \geqslant 1$ and the generic fibre of ρ is rationally connected,

2) there is no birational map $\rho\colon X \dashrightarrow Y$ such that $Y \not\cong X$ has $\mathbb{Q}$-factorial terminal singularities and $\mathrm{rk}\,\mathrm{Pic}(Y) = 1$.

Definition 1.7. A subset $\Gamma \subset \mathrm{Bir}(X)$ *untwists* all maximal singularities on X if for every linear system $\mathcal{M}$ on X without fixed components there is an element $\xi \in \Gamma$ such that the log pair $(X, \lambda\xi(\mathcal{M}))$ has canonical singularities, where $\lambda \in \mathbb{Q}$ is such that $K_X + \lambda\xi(\mathcal{M}) \equiv 0$.

The existence of a subset $\Gamma \subset \mathrm{Bir}(X)$ that untwists all maximal singularities implies that the group $\mathrm{Bir}(X)$ is generated by Γ and the biregular automorphisms (see [4]).

Definition 1.8. A variety X is said to be *universally birationally rigid* if the variety $X \otimes \mathrm{Spec}(\mathbb{K})$ is birationally rigid over $\mathbb{K}$ for every finitely generated field extension $\mathbb{K}$ of $\mathbb{C}$.

We note that Definition 1.6 extends naturally to Fano varieties defined over any perfect field.

Definition 1.9. A subset $\Gamma \subset \mathrm{Bir}(X)$ *universally untwists* all maximal singularities if, for every finitely generated field extension $\mathbb{K}$ of $\mathbb{C}$, the induced subgroup

$$\Gamma \subset \mathrm{Bir}(X) \subseteq \mathrm{Bir}(X \otimes \mathrm{Spec}(\mathbb{K}))$$

untwists all maximal singularities on $X \otimes \mathrm{Spec}(\mathbb{K})$.

Definitions 1.3 and 1.9 imply that if X is birationally superrigid, then every non-empty subset of $\mathrm{Bir}(X)$ universally untwists all maximal singularities.

Remark 1.10. As noticed by Kollár, in the case when $\dim(X) \geqslant 2$, the whole group $\mathrm{Bir}(X)$ universally untwists all maximal singularities if and only if $\mathrm{Bir}(X)$ is countable.

Suppose that $X_1, \dots, X_r$ are Fano varieties with $\mathbb{Q}$-factorial terminal singularities and $\mathrm{rk}\,\mathrm{Pic}(X_i) = 1$ for all $i = 1, \dots, r$. Consider the projection

$$\pi_i\colon X_1 \times \cdots \times X_{i-1} \times X_i \times X_{i+1} \times \cdots \times X_r \longrightarrow X_1 \times \cdots \times X_{i-1} \times \widehat{X_i} \times X_{i+1} \times \cdots \times X_r$$

and write $\beth_i$ for the generic fibre of π_i in the scheme sense.

Remark 1.11. The variety $\beth_i$ is a Fano variety defined over the field of all rational functions on $X_1 \times \cdots\cdots \times X_{i-1} \times \widehat{X_i} \times X_{i+1} \times \cdots \times X_r$.

There are natural embeddings of groups

$$\prod_{i=1}^{r} \mathrm{Bir}(X_i) \subseteq \langle \mathrm{Bir}(\beth_1), \dots, \mathrm{Bir}(\beth_r) \rangle \subseteq \mathrm{Bir}(X_1 \times \cdots \times X_r),$$

and the proof of Theorem 1.4 yields the following result (see [3]).

Theorem 1.12. *Suppose that $X_1, \dots, X_r$ are universally birationally rigid and $\mathrm{lct}(X_i) \geqslant 1$ for all $i = 1, \dots, r$. Then*

$$\mathrm{Bir}(X_1 \times \cdots \times X_r) = \langle \mathrm{Bir}(\beth_1), \dots, \mathrm{Bir}(\beth_r), \mathrm{Aut}(X_1 \times \cdots \times X_r) \rangle,$$

the variety $X_1 \times \cdots \times X_r$ is non-rational and, for every dominant rational map ρ: $X_1 \times \cdots \times X_r \dashrightarrow Y$ whose generic fibre is rationally connected, there is a subset $\{i_1, \dots, i_k\} \subseteq \{1, \dots, r\}$ and a commutative diagram

$$\begin{tikzcd}
X_1 \times \cdots \times X_r \arrow[r, dashed, "\sigma"] \arrow[d, "\pi"'] & X_1 \times \cdots \times X_r \arrow[dr, dashed, "\rho"] & \\
X_{i_1} \times \cdots \times X_{i_k} \arrow[rr, dashed, "\xi"'] & & Y
\end{tikzcd}$$

where π is the natural projection and ξ, σ are birational maps.

Corollary 1.13. *Suppose that there is a subgroup $\Gamma_i \subseteq \mathrm{Bir}(X_i)$ that universally untwists all maximal singularities and we have $\mathrm{lct}(X_i) \geqslant 1$ for all $i = 1, \dots, r$. Then*

$$\mathrm{Bir}(X_1 \times \cdots \times X_r) = \left\langle \prod_{i=1}^{r} \Gamma_i, \mathrm{Aut}(X_1 \times \cdots \times X_r) \right\rangle.$$

Let X be a generic quasi-smooth well-shaped hypersurface in $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ of degree $\sum_{i=1}^{4} a_i$ with terminal singularities, where $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$. Then X is a Fano variety. For historical reasons, it is commonly referred to as a *Reid–Fletcher variety*. In [5], a finite set $\tau_1, \dots, \tau_k$ of birational involutions of X was found explicitly and the following important and complicated result was proved.

Theorem 1.14. *The variety X is birationally rigid, the sequence of groups*

$$1 \longrightarrow \langle \tau_1, \dots, \tau_k \rangle \longrightarrow \mathrm{Bir}(X) \longrightarrow \mathrm{Aut}(X) \longrightarrow 1$$

is exact and the group $\langle \tau_1, \dots, \tau_k \rangle$ universally untwists all maximal singularities.

There are 95 possibilities for the quadruple (a_1, a_2, a_3, a_4). Let $\gimel \in \{1, \dots, 95\}$ be the ordinal number of the quadruple in the standard notation (see [5]). We shall prove the following result.[2]

Theorem 1.15. *Suppose that $\gimel \notin \{1, 2, 4, 5\}$. Then* $\mathrm{lct}(X) = 1$.

In many cases one can show that the group $\mathrm{Aut}(X)$ is either trivial (see Lemma 8.3) or isomorphic to $\mathbb{Z}_2$ (see Corollary 8.2). Relations between the involutions $\tau_1, \dots, \tau_k$ are also known (see [6]). Thus one can obtain explicit applications of Theorem 1.12.

Example 1.16. Suppose that $\gimel = 41$. Using Theorems 1.12 and 1.14, one can show (see Corollary 8.2 below) that there is an exact sequence of groups

$$1 \longrightarrow \prod_{i=1}^{m} (\mathbb{Z}_2 * \mathbb{Z}_2) \longrightarrow \mathrm{Bir}\Bigl(\underbrace{X \times \dots \times X}_{m}\Bigr) \longrightarrow S_m \longrightarrow 1$$

by Theorem 1.15. Let V be a generic hypersurface of degree $2n \geqslant 6$ in $\mathbb{P}(1^{n+1}, n)$. Then

$$\mathrm{Bir}(X \times V) \cong (\mathbb{Z}_2 * \mathbb{Z}_2) \oplus \mathbb{Z}_2,$$

again by Theorems 1.12, 1.14 and 1.15 (see Example 1.5 and [7]).

It follows from [8] that $\mathrm{lct}(X) \geqslant 16/21$ for $\gimel = 1$. We shall prove the following result.

Theorem 1.17. *Suppose that $\gimel = 2$. Then* $\mathrm{lct}(X) \geqslant 7/9$.

Corollary 1.18. *Suppose that $\gimel \neq 4$ and $\gimel \neq 5$. Then X has a Kähler–Einstein metric.*

For the convenience of the reader, we organize this paper in the following way.

1) We prove Theorem 1.15 in § 2, omitting the proofs of Lemmas 2.4, 2.10, 2.11.

2) We prove the technical Lemmas 2.4, 2.10, 2.11 in §§ 3, 5, 6 respectively.

3) We explicitly describe the group $\mathrm{Bir}(X)$ for $\gimel = 9$ and $\gimel = 41$ in § 8.

4) We consider an alternative approach to the proof of Theorem 1.15 in § 9.

§ 2. Log canonical thresholds

Consider a generic quasi-smooth well-shaped hypersurface $X \subset \mathbb{P}(1, a_1, \dots, a_4)$ of degree $d = \sum_{i=1}^{4} a_i$ with terminal singularities, where $a_1 \leqslant \dots \leqslant a_4$. We write $\gimel \in \{1, \dots, 95\}$ for the ordinal number of the quadruple (a_1, a_2, a_3, a_4) according to [5]. Then $-K_X^3 \leqslant 1 \iff \gimel \geqslant 6$.

Suppose that $\gimel \notin \{1, 2, 4, 5\}$. Let D be a divisor in $|-nK_X|$, where $n \in \mathbb{N}$.

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[2] A sketch of the proof of Theorem 1.15 was given in [3].

Remark 2.1. The proof of Theorem 1.15 implies that the log pair $(X, \frac{1}{n}D)$ is not log terminal if and only if $n = 1$. However, we do not need this fact in what follows.

Suppose that the log pair $(X, \frac{1}{n}D)$ is not log canonical. To prove Theorem 1.15, we must show that this assumption leads to a contradiction.

Remark 2.2. Let V be a variety with $\mathbb{Q}$-factorial singularities and let B and B' be effective $\mathbb{Q}$-divisors on V such that (V, B) is log canonical and (V, B') is not. Then the log pair $\big(V, \frac{1}{1-\alpha}(B' - \alpha B)\big)$ is not log canonical for any $\alpha \in \mathbb{Q}$ such that $0 \leqslant \alpha < 1$ and the divisor $B' - \alpha B$ is effective.

Thus we may assume that D is an irreducible surface. It follows from [2] that $\gimel \neq 3$.

Lemma 2.3. *We have $n \neq 1$.*

Proof. Suppose that $n = 1$. Then the log pair (X, D) is log canonical at every singular point of X according to Lemma 8.12 and Proposition 8.14 of [9]. It follows that $a_1 = 1$.

Suppose that the singularities of the log pair (X, D) are not log canonical at some smooth point P of the hypersurface X. Let us derive a contradiction. We consider only the case $\gimel = 14$. The other cases are similar.

Suppose that $\gimel = 14$. Then there is a double covering $\pi\colon X \to \mathbb{P}(1,1,1,4)$ branched over a hypersurface $F \subset \mathbb{P}(1,1,1,4)$ of degree 12, which is sufficiently generic by assumption.

Put $\overline{D} = \pi(D)$ and $\overline{P} = \pi(P)$. Counting parameters, we see that

$$\mathrm{mult}_{\overline{P}}(F|_{\overline{D}}) \leqslant 2,$$

which is a contradiction because the singularities of the log pair $(\overline{D}, \frac{1}{2}F|_{\overline{D}})$ are not log canonical at $\overline{P}$ by Lemma 8.12 of [9]. The lemma is proved.

Lemma 2.4. *The log pair $(X, \frac{1}{n}D)$ is log canonical at smooth points of X.*

This lemma will be proved in § 3.

Hence there is a singular point O of X such that the log pair $(X, \frac{1}{n}D)$ is not log canonical at O. Then O is a singular point of type $\frac{1}{r}(1, a, r-a)$, where a and r are coprime positive integers with $r > 2a$. Let $\alpha\colon U \to X$ be a blow-up of O with weights $(1, a, r-a)$. Then

$$-K_U^3 = -K_X^3 - \frac{1}{r^3}E^3 = -K_X^3 - \frac{1}{ra(r-a)} = \frac{d}{a_1 a_2 a_3 a_4} - \frac{1}{ra(r-a)}, \tag{2.1}$$

where E is the exceptional divisor of α. There is a rational number μ such that

$$\overline{D} \equiv \alpha^*(D) + \mu E \equiv -nK_U + \left(\frac{n}{r} - \mu\right)E,$$

where $\overline{D}$ is the proper transform of D on U. It follows from [10] that $\mu > n/r$.

Lemma 2.5. *We have $-K_U^3 \geqslant 0$.*

Proof. Suppose that $-K_U^3 < 0$. Let C be a curve in E. By Corollary 5.4.6 of [5] there is an irreducible reduced curve $\Gamma \subset U$ such that Γ generates an extremal ray of the cone $\mathbb{NE}(U)$ different from the ray $\mathbb{R}_{\geqslant 0}C$, and we have a numerical equivalence

$$\Gamma \equiv -K_U \cdot (-bK_U + cE),$$

where $b > 0$ and $c \geqslant 0$ are integers.

Let T be a divisor in $|-K_U|$. Then $\pi(T)$ is a divisor in $|-K_X|$ and

$$D \cdot T \equiv -K_U \cdot \left(-nK_U + \left(\frac{n}{r} - \mu\right)E\right) \notin \mathbb{NE}(U)$$

because $\mu > n/r$, $b > 0$ and $c \geqslant 0$. However, the cycle $D \cdot T$ is effective since $n \neq 1$. The lemma is proved.

Taking into account the range of values of (a_1, a_2, a_3, a_4), we see that $\gimel \notin \{75, 84, 87, 93\}$.

Lemma 2.6. *We have* $-K_U^3 \neq 0$.

Proof. Suppose that $-K_U^3 = 0$ and $\gimel \neq 82$. Then $|-rK_U|$ has no base points for $r \gg 0$ and induces a morphism $\eta \colon U \to \mathbb{P}(1, a_1, a_2)$ such that the diagram

$$\begin{tikzcd}
& U \arrow[dl, "\alpha"'] \arrow[dr, "\eta"] & \\
X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1, a_1, a_2)
\end{tikzcd}$$

is commutative, where ψ is the projection. The morphism η is an elliptic fibration. Thus we have

$$\overline{D} \cdot C = -nK_U \cdot C + \left(\frac{n}{r} - \mu\right)E \cdot C = \left(\frac{n}{r} - \mu\right)E \cdot C < 0,$$

where C is the generic fibre of η, a contradiction.

Suppose that $-K_U^3 = 0$ and $\gimel = 82$. Then X is a hypersurface of degree 36 in $\mathbb{P}(1, 1, 5, 12, 18)$. Its singularities consist of points P and Q of type $\frac{1}{5}(1, 2, 3)$ and $\frac{1}{6}(1, 1, 5)$ respectively.

One can prescribe the hypersurface X by the equation

$$z^7 y + \sum_{i=0}^{6} z^i f_{36-5i}(x, y, z, t) = 0 \subset \mathbb{P}(1, 1, 5, 12, 18) \cong \operatorname{Proj}(\mathbb{C}[x, y, z, t, w]),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 5$, $\mathrm{wt}(t) = 12$, $\mathrm{wt}(w) = 18$ and f_i is a quasi-homogeneous polynomial of degree i. Then P is given by $x = z = t = w = 0$.

Suppose that $O = Q$. Then the linear system $|-rK_U|$ has no base points for $r \gg 0$, which leads to a contradiction as in the case $\gimel \neq 82$. Hence we see that $O = P$.

Let $\overline{S}$ be the proper transform on U of the surface that is cut out on X by $y = 0$. Then

$$\overline{S} \equiv \alpha^*(-K_X) - \frac{6}{5}E,$$

and the base locus of the pencil $|-K_U|$ consists of irreducible curves L and C such that L is contained in the α-exceptional divisor E and the curve $\pi(C)$ is the unique base curve of the pencil $|-K_X|$. Then $-K_U \cdot C = -1/6$ and $-K_U \cdot L > 0$. We also have $\mu \leqslant n/5$ because

$$\frac{n}{5} - \mu = (-K_U + \alpha^*(-5K_X)) \cdot \overline{S} \cdot \overline{D} \geqslant 0$$

since $\overline{D} \neq \overline{S}$ by Lemma 8.12 and Proposition 8.14 of [9], a contradiction.

Taking into account the range of (a_1, a_2, a_3, a_4), we see that

$$\daleth \notin \{11, 14, 19, 22, 28, 34, 37, 39, 49, 52, 53, 57, 59, 64,\\ 66, 70, 72, 73, 78, 80, 81, 86, 88, 89, 90, 92, 94, 95\}.$$

Lemma 2.7. *The groups* Bir(X) *and* Aut(X) *do not coincide.*

Proof. Suppose that $\mathrm{Bir}(X) = \mathrm{Aut}(X)$. Let $\overline{S}$ be a generic surface in $|-K_U|$. By Lemma 5.4.5 of [5] there is an irreducible surface $\overline{T} \subset U$ such that

1) we have $\overline{T} \sim c\overline{S} - bE$ for some integers $c \geqslant 1$ and $b \geqslant 1$,
2) the intersection $\overline{T} \cdot \overline{S}$ is a reduced irreducible curve Γ,
3) the curve Γ generates an extremal ray of the cone $\mathbb{NE}(U)$.

It is easy to construct the surface $\overline{T}$ explicitly (see [5]), and the possible values of c and b are given in [5]. The surface $\overline{T}$ is uniquely determined by the point O.

Put $T = \alpha(\overline{T})$. Then the singularities of the log pair $(X, \frac{1}{c}T)$ are log canonical by Lemma 8.12 and Proposition 8.14 of [9]. It follows that $D \neq T$.

Let $\mathcal{P}$ be the pencil generated by the effective divisors nT and cD. Then the singularities of $(X, \frac{1}{cn}\mathcal{P})$ are non-canonical, which contradicts Theorem 1.14.

It follows from [5] that

$$\daleth \notin \{11, 21, 29, 35, 50, 51, 55, 62, 63, 67, 71, 77, 82, 83, 85, 91\}.$$

Lemma 2.8. *The divisor $-K_U$ is numerically effective.*

Proof. Suppose that the anticanonical divisor $-K_U$ is not numerically effective. Then it follows from [5] that $\daleth = 47$ and O is a singular point of type $\frac{1}{5}(1, 2, 3)$. The hypersurface X can be given by the equation

$$z^4 y + \sum_{i=0}^{3} z^i f_{21-5i}(x, y, z, t) = 0 \subset \mathbb{P}(1, 1, 5, 7, 8) \cong \mathrm{Proj}(\mathbb{C}[x, y, z, t, w]),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 5$, $\mathrm{wt}(t) = 7$, $\mathrm{wt}(w) = 8$ and f_i is a generic quasi-homogeneous polynomial of degree i. Let S be the surface cut out by the equation $y = 0$ on X, and let $\overline{S}$ be the proper transform of S on U. Then

$$\overline{S} \equiv \alpha^*(-K_X) - \frac{6}{5}E,$$

but the divisor $-3K_U+\alpha^*(-5K_X)$ is numerically effective (see [11]). We also have $\mu \leqslant n/5$ because

$$\frac{n}{5}-\mu=\frac{1}{3}(-3K_U+\alpha^*(-5K_X))\cdot\overline{S}\cdot\overline{D}\geqslant 0$$

since $D\neq S$ by Lemma 8.12 and Proposition 8.14 of [9]. This contradiction proves the lemma.

Thus the divisor $-K_U$ is numerically effective and big (see Lemmas 2.5 and 2.6).

Lemma 2.9. *We have $\mu/n-1/r<1$.*

Proof. We only consider the case when $\daleth=58$ and O is a singular point of type $\frac{1}{10}(1,3,7)$ because the proof is similar in all other cases (see Lemma 6.3 below). Thus we assume that $\daleth=58$. The threefold X can be given by

$$w^2z+wf_{14}(x,y,z,t)+f_{24}(x,y,z,t)=0\subset\mathbb{P}(1,3,4,7,10)\cong\mathrm{Proj}(\mathbb{C}[x,y,z,t,w]),$$

where $\mathrm{wt}(x)=1$, $\mathrm{wt}(y)=3$, $\mathrm{wt}(z)=4$, $\mathrm{wt}(t)=7$, $\mathrm{wt}(w)=10$ and f_i is a quasi-homogeneous polynomial of degree i. Let R be the surface cut out by the equation $t=0$ on X, and let $\overline{R}$ be the proper transform of R on U. Then

$$\overline{R}\equiv\alpha^*(-4K_X)-\frac{7}{5}E$$

and $(X,\frac{1}{4}R)$ is log canonical at O according to Lemma 8.12 and Proposition 8.14 of [9]. Then $R\neq D$ and

$$0\leqslant -K_U\cdot\overline{R}\cdot\overline{D}=\frac{4}{35}n-\frac{2}{3}\mu$$

because $-K_U$ is numerically effective. Hence we have $\mu\leqslant 6n/35$.

We have shown that the log pair $(U,\frac{1}{n}\overline{D}+(\frac{\mu}{n}-\frac{1}{r})E)$ is not log canonical at some point $P\in E$ because

$$K_U+\frac{1}{n}\overline{D}\equiv\alpha^*\left(K_X+\frac{1}{n}D\right)+\left(\frac{1}{r}-\frac{\mu}{n}\right)E.$$

Lemma 2.10. *The threefold U is smooth at the point P.*

A proof of Lemma 2.10 is given in § 5.

Lemma 2.10 yields that $\mathrm{mult}_P(\overline{D})>n+n/r-\mu$. It follows from [5] that we have a dichotomy:

1) either $d=2r+a_j$ for some $j\in\{1,2,3,4\}$,

2) or $d\neq 2r+a_j$ for all $j\in\{1,2,3,4\}$ but we have $d=3r+a_j$ for some j.

Lemma 2.11. *For every $j\in\{1,2,3,4\}$ we have $d\neq 2r+a_j$.*

A proof of Lemma 2.11 is given in § 6.

Thus we have shown that $d=3r+a_j$ for some $j\in\{1,2,3,4\}$.

Remark 2.12. Let V be a threefold with isolated singularities, and let B and T be distinct irreducible effective divisors on V. We put

$$B \cdot T = \sum_{i=1}^{r} \varepsilon_i L_i + \Delta,$$

where L_i is an irreducible curve, ε_i is a non-negative integer and Δ is an effective 1-cycle whose support does not contain the curves $L_1, \ldots, L_r$. Then we have $\sum_{i=1}^{r} \varepsilon_i H \cdot L_i \leqslant B \cdot T \cdot H$ for any numerically effective divisor H on V.

It follows from Lemma 2.11 that[3] $\gimel \in \{7, 20, 23, 36, 40, 44, 61, 76\}$.

Lemma 2.13. *We have* $\gimel \neq 7$, $\gimel \neq 20$, $\gimel \neq 36$.

Proof. Suppose that $\gimel \in \{7, 20, 36\}$. Then $a_1 = 1$. The point O is a singular point of type $\frac{1}{a_2}(1, 1, a_2 - 1)$ according to Lemmas 6.4, 6.10 and 6.12 below. One can show that there is a commutative diagram

$$\begin{tikzcd}
U \arrow[dr, "\alpha"] \arrow[dd, "\sigma"'] & & & W \arrow[lll, "\gamma"'] \arrow[dd, "\eta"] \\
& X \arrow[drr, dashed, "\psi"] & & \\
V \arrow[r, hook] & \mathbb{P}(1, 1, a_3, 2a_4, 3a_4) \arrow[r, dashed, "\chi"'] & \mathbb{P}(1, 1, a_3, 2a_4) \arrow[r, dashed, "\xi"'] & \mathbb{P}(1, 1, a_3)
\end{tikzcd}$$

where ξ, χ, ψ are projections, η is an elliptic fibration, γ is a weighted blow-up of a singular point of type $\frac{1}{a_4}(1, 1, a_3)$ with weights $(1, 1, a_3)$, σ is a birational morphism that contracts l non-singular rational curves $C_1, \ldots, C_l$ to l isolated ordinary double points, $l = d(d - a_4)/a_3$, and V is a hypersurface of degree 42 in $\mathbb{P}(1, 1, 6, 14, 21)$.

Let T be the surface in $|-K_U|$ that contains the point P. Then $\mathrm{mult}_P(\overline{D}) > n + n/a_2 - \mu$.

Suppose that $P \notin \bigcup_{i=1}^{l} C_i$. Then it follows from the proof of Theorem 5.6.2 in [5] that the linear system $|-2sa_4K_U|$ contains a surface H that has multiplicity $s > 0$ at P and contains no components of the cycle $\overline{D} \cdot T$ that pass through P. Here s is a positive integer. Then

$$2sa_4\left(\frac{dn}{a_1a_2a_3a_4} - \frac{\mu}{a_2 - 1}\right) = \overline{D} \cdot T \cdot H \geqslant \mathrm{mult}_P(\overline{D})s > s\left(n + \frac{n}{a_2} - \mu\right)s,$$

which is impossible because $\mu > n/a_2$. Hence we can assume that $P \in C_1$. We put

$$\overline{D} \cdot T = mC_1 + \Delta,$$

where m is a non-negative integer and Δ is an effective cycle whose support does not contain C_1. The curve C_1 is non-singular, $\alpha^*(-K_X) \cdot C_1 = 2/a_2$ and $E \cdot C_1 = 2$.

---

[3] It follows from [5] that X has an elliptic involution $\iff$ $\gimel \in \{7, 20, 23, 36, 40, 44, 61, 76\}$.

Let $\breve{E}$ be the proper transform of E on W. Then $\breve{E} \cong \mathbb{P}(1,1,a_3/2)$ and the map

$$\eta|_{\breve{E}} \colon \mathbb{P}(1,1,a_3/2) \longrightarrow \mathbb{P}(1,1,a_3)$$

is a finite morphism of degree 2. Hence we can find a surface $R \in |-a_3K_U|$ such that R passes through the curve C_1 and contains no components of Δ that pass through P. Thus we get

$$a_3\left(\frac{dn}{a_1a_2a_3a_4} - \frac{\mu}{a_2-1}\right) = R\cdot\Delta \geqslant \operatorname{mult}_P(\Delta) > n + \frac{n}{a_2} - \mu - m,$$

whence $m > a_3n/a_4$ because $\mu > n/a_2$. Therefore we have

$$\frac{a_3n}{a_4} < m \leqslant \frac{-dnK_X\cdot\alpha(C_1)}{a_1a_2a_3a_4} = \frac{dn}{2a_1a_3a_4}$$

by Remark 2.12 because $-K_X\cdot\alpha(C_1) = 2/a_2$. It follows that $\gimel = 7$.

The fibre of the projection ψ over the point $\psi(P)$ consists of two irreducible components. One of them is the curve C_1. Let Z be the other. Then

$$C_1^2 = -2, \qquad C_1\cdot Z = 2, \qquad Z^2 = -\frac{4}{3}$$

on the surface T. We write $\Delta = \overline{m}Z + \Omega$, where $\overline{m}$ is a non-negative integer and Ω is an effective cycle whose support does not contain Z. Then

$$\frac{4n}{3} - 2\mu - \frac{5\overline{m}}{3} = (Z + C_1)\cdot\Omega > \frac{3n}{2} - \mu - m,$$

but $4\overline{m}/3 \geqslant 2m - 5n/6$ because $\Omega\cdot Z \geqslant 0$. These inequalities contradict each other because $\mu > n/2$ by [10]. The lemma is proved.

Thus we see that $\gimel \in \{23, 40, 44, 61, 76\}$ and $d = 3r + a_j$, where $r = a_3 > 2a$ and $1 \leqslant j \leqslant 2$.

The hypersurface X has a singular point Q of type $\frac{1}{\bar{r}}(1, \bar{a}, \bar{r} - \bar{a})$ such that

$$-K_X^3 = \frac{1}{ra(r-a)} + \frac{1}{\bar{r}\bar{a}(\bar{r}-\bar{a})},$$

where $\bar{r} = a_4 > 2\bar{a}$ and $\bar{a} \in \mathbb{N}$. It is known that X can be given by an equation of the form

$$x_4^2x_3 + x_4a(x_0,x_1,x_2) = x_3^3x_j + x_3^2b(x_0,x_1,x_2) + x_3c(x_0,x_1,x_2) + d(x_0,x_1,x_2) = 0$$

in $\operatorname{Proj}(\mathbb{C}[x_0,x_1,x_2,x_3,x_4])$, where $\operatorname{wt}(x_0) = 1$, $\operatorname{wt}(x_k) = a_k$ and a, b, c, d are quasi-homogeneous polynomials. We put $l = d(d-a_4)/(a_1a_2)$. It follows from [5] that there is a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\sigma"'] \arrow[dr, "\alpha"] & & & W \arrow[lll, "\gamma"'] \arrow[dd, "\eta"] \\
& X \arrow[drr, dashed, "\psi"] & & \\
V \arrow[r, hook] & \mathbb{P}(1,a_1,a_2,2a_4,3a_4) \arrow[r, dashed, "\chi"'] & \mathbb{P}(1,a_1,a_2,2a_4) \arrow[r, dashed, "\xi"'] & \mathbb{P}(1,a_1,a_2)
\end{tikzcd}$$

where ξ, χ, ψ are projections, η is an elliptic fibration, γ is a weighted blow-up with weights $(1,\bar{a},\bar{r}-\bar{a})$ of a point that dominates the point Q, and σ is a birational morphism that contracts the non-singular curves $C_1,\dots,C_l$. It is known that V is a hypersurface of degree $6a_4$ in $\mathbb{P}(1,a_1,a_2,2a_4,3a_4)$.

We note that $E\cong\mathbb{P}(1,a,r-a)$. Let L be a curve on E belonging to the linear system $|\mathcal{O}_{\mathbb{P}(1,a,r-a)}(1)|$.

Lemma 2.14. *Suppose that $P\notin L$. Then $\mu>na(r+1)/(r^2+ar)$.*

Proof. There is a curve $C\in|\mathcal{O}_{\mathbb{P}(1,a,r-a)}(a)|$ such that $P\in C$. We put

$$\overline{D}|_E=\delta C+\Upsilon\equiv r\mu L,$$

where δ is a non-negative integer and Υ is an effective divisor on E whose support does not contain the curve C. Then

$$\frac{r\mu-a\delta}{r-a}=(r\mu-a\delta)L\cdot C=C\cdot\Upsilon\geqslant\operatorname{mult}_P(\Upsilon)>n+\frac{n}{r}-\mu-\delta.$$

It follows that $\mu>na(r+1)/(r^2+ar)$ because $\delta\leqslant r\mu/a$.

Let T be a surface in $|-K_U|$. Then $-K_U\cdot T\cdot\overline{D}\geqslant 0$. It follows that $\mu\leqslant -na(r-a)K_X^3$.

Lemma 2.15. *The point P is not contained in the surface T.*

Proof. Suppose that P is contained in T. Then P is not contained in the base locus of the linear system $|-a_1K_U|$ because the base locus of $|-a_1K_U|$ contains no smooth points of E. The point P is not contained in $\bigcup_{i=1}^l C_i$ because $P\in T$.

The proof of Theorem 5.6.2 in [5] shows that there is a surface $H\in|-2sa_1a_4K_U|$ such that

$$2sa_1a_4\biggl(-nK_X^3-\frac{\mu}{a_2}\biggr)=\overline{D}\cdot H\cdot T\geqslant\operatorname{mult}_P(\overline{D})s>s\biggl(n+\frac{n}{r}-\mu\biggr)s,$$

where s is a positive integer. This is impossible because $\mu>n/r$.

It follows from Lemmas 2.14 and 2.15 that $\daleth\in\{23,44\}$ in view of the fact that $\mu\leqslant -na(r-a)K_X^3$.

Let S be a surface in $|-a_1K_U|$ that contains P. Since $\mu>n/r$, we see that $\overline{D}\neq S$.

Lemma 2.16. *The point P is contained in $\bigcup_{i=1}^l C_i$.*

Proof. Suppose that $P\notin\bigcup_{i=1}^l C_i$. Then it follows from the proof of Theorem 5.6.2 in [5] that

$$2sa_1a_4\biggl(-nK_X^3-\frac{\mu}{a_2}\biggr)=\overline{D}\cdot H\cdot S\geqslant\operatorname{mult}_P(\overline{D})s>s\biggl(n+\frac{n}{r}-\mu\biggr)s$$

for some $s\in\mathbb{N}$ and $H\in|-2sa_4K_U|$. This contradicts the inequality $\mu>n/r$.

We may assume that $P \in C_1$. Put

$$\overline{D} \cdot S = mC_1 + \Delta,$$

where m is a non-negative integer and Δ is an effective cycle whose support does not contain C_1. Then it follows from Remark 2.12 that $m \leqslant nd/(a_2d - a_2a_3)$ because $-K_X \cdot \alpha(C_1) = (d - a_3)/(a_3a_4)$.

The proof of Theorem 5.6.2 in [5] shows that there is a surface $R \in |-2sa_4K_U|$ such that

$$2sa_1a_4\left(-nK_X^3 - \frac{\mu}{a_2}\right) = R \cdot \Delta \geqslant \mathrm{mult}_P(\Delta)s > s\left(n + \frac{n}{r} - \mu - m\right),$$

where $s \in \mathbb{N}$. However, we have $m \leqslant nd/(a_2d - a_2a_3)$, whence $\gimel = 23$.

We have proved that X is a hypersurface of degree 14 in $\mathbb{P}(1, 2, 3, 4, 5)$ and O is a singular point of type $\frac{1}{4}(1, 1, 3)$. Let M be a generic surface through P in the linear system $|-3K_X|$. Then

$$S \cdot M = C_1 + Z_1,$$

where Z_1 is a curve with $-K_U \cdot Z_1 = 1/5$. We write

$$\overline{D} \cdot S = mC_1 + \overline{m}Z_1 + \Upsilon,$$

where $\overline{m}$ is a non-negative integer and Υ is an effective cycle whose support does not contain the curves C_1 or Z_1. Then $m < 7n/15$ by Remark 2.12, but $\mu > n/4$ and

$$\frac{7}{10}n - \frac{6}{3}\mu - \frac{3}{5}\overline{m} = M \cdot \Upsilon \geqslant \mathrm{mult}_P(\Upsilon) > \frac{5}{4}n - \mu - m$$

because $P \notin Z_1$. The inequalities obtained lead to a contradiction.

Thus Theorem 1.15 is completely proved.

§ 3. Non-singular points

In this section we prove Lemma 2.4. We shall use the assumptions and notation of that lemma. Let P be a smooth point of X such that the log pair $(X, \frac{1}{n}D)$ is not log canonical at P.

Lemma 3.1. *Suppose that a_4 divides d and $a_1 \neq a_2$. Then $-a_2a_3K_X^3 > 1$.*

Proof. Suppose that $-a_2a_3K_X^3 \leqslant 1$. Let L be the unique base curve of the pencil $|-a_1K_X|$, and let T be a surface in the linear system $|-K_X|$. Then $D \cdot T$ is an effective 1-cycle and $\mathrm{mult}_P(L) = 1$.

Suppose that $P \in L$. Let R be a generic surface in $|-a_1K_X|$. We write

$$D \cdot T = mL + \Delta,$$

where m is a non-negative integer and Δ is an effective cycle whose support does not contain L. Then

$$-a_1(n - a_1m)K_X^3 = D \cdot T \cdot R - mR \cdot L = R \cdot \Delta \geqslant \mathrm{mult}_P(\Delta) > n - m,$$

which is impossible because $-a_1K_X^3 \leqslant 1$. Thus we see that $P \notin L$.

Suppose that $P \in T$. Then it follows from Theorem 5.6.2 of [5] that

$$ns \geqslant -sa_1a_3nK_X^3 = D \cdot S \cdot T \geqslant \mathrm{mult}_P(D)s > ns$$

for some positive integer s and some surface $S \in |-sa_1a_3K_X|$. Hence we see that $P \notin T$.

Let G be a generic surface through P in $|-a_2K_X|$. Then $G \cdot D$ is an effective cycle. By Theorem 5.6.2 of [5] one can find an integer $s > 0$ and an effective divisor $H \in |-sa_3K_X|$ such that

$$ns \geqslant -sa_2a_3nK_X^3 = D \cdot H \cdot G \geqslant \mathrm{mult}_P(D)s > ns$$

because $-a_2a_3K_X^3 \leqslant 1$. The resulting contradiction completes the proof.

We note that $\mathrm{mult}_P(D) > n$ (see [9]).

Lemma 3.2. *Suppose that a_4 divides d and $1 = a_1 \neq a_2$. Then $-a_3K_X^3 > 1$.*

Proof. Suppose that $-a_3K_X^3 \leqslant 1$. Arguing as in the proof of Lemma 3.1, we see that P is not contained in the base locus of $|-K_X|$. Let T be the surface in $|-K_X|$ that passes through P. By Theorem 5.6.2 of [5] one can find an integer $s > 0$ and a surface $S \in |-sa_3K_X|$ such that

$$ns \geqslant -sa_3nK_X^3 = D \cdot S \cdot T \geqslant \mathrm{mult}_P(D)s > ns.$$

This is a contradiction. The lemma is proved.

Lemma 3.3. *Suppose that $a_1 \neq a_2$. Then $-a_1a_4K_X^3 > 1$.*

Proof. Assume that $-a_1a_4K_X^3 \leqslant 1$. Arguing as in the proof of Lemma 3.2, we see that $a_1 \neq 1$. Then, arguing as in the proof of Lemma 3.1, we see that P is not contained in the unique surface of the linear system $|-K_X|$.

Let S be a surface through P in $|-a_1K_X|$. We may assume that

$$\mathrm{mult}_P(S) \leqslant a_1$$

because $P \notin T$ and X is generic. Then $S \neq D$.

By Theorem 5.6.2 of [5] one can find an integer $s > 0$ and a surface $H \in |-sa_4K_X|$ such that H has a singularity of multiplicity at least s at P and contains no components of $D \cdot S$ that pass through P. We have

$$ns \geqslant -sa_1a_4nK_X^3 = D \cdot S \cdot H \geqslant \mathrm{mult}_P(D)s > ns$$

because $-a_1a_4K_X^3 \leqslant 1$. The resulting contradiction completes the proof of the lemma.

Taking into account the range of (a_1, a_2, a_3, a_4), we see that

$$\gimel \in \{6, 7, 8, 9, 10, 12, 13, 14, 16, 18, 19, 20, 22, 23, 24, 25, 32, 33, 38\}$$

by Lemmas 3.1–3.3. We now treat the remaining cases separately.

Lemma 3.4. *We have* $\gimel \neq 6$ *and* $\gimel \neq 10$.

Proof. We may assume that $\gimel = 6$ since the case $\gimel = 10$ can be treated in a similar way. It follows from [11] that X has singular points O_1 and O_2 of type $\frac{1}{2}(1,1,1)$ such that there is a commutative diagram

$$\begin{tikzcd}
 & & U \arrow[dll, "\sigma"'] \arrow[d, "\alpha"] & Y \arrow[l, "\gamma"'] \arrow[dd, "\eta"] \\
V \arrow[dr, "\omega"'] & & X \arrow[dl, dashed, "\xi"'] \arrow[dr, dashed, "\psi"] & \\
 & \mathbb{P}(1,1,1,4) \arrow[rr, dashed, "\chi"'] & & \mathbb{P}(1,1,1)
\end{tikzcd}$$

where ξ, ψ and χ are projections, α is a blow-up of O_1 with weights $(1,1,1)$, γ is a blow-up with weights $(1,1,1)$ of the point that dominates O_2, η is an elliptic fibration, ω is a double covering and σ is a birational morphism that contracts 48 irreducible curves $C_1, \ldots, C_{48}$.

The threefold U contains 48 curves $Z_1, \ldots, Z_{48}$ such that $\alpha(Z_i) \cup \alpha(C_i)$ is the fibre of the natural projection ψ over the point $\psi(C_i)$. We put $\overline{Z}_i = \alpha(Z_i)$ and $\overline{C}_i = \alpha(C_i)$. Let L be the fibre of the projection ψ that passes through the point P, and let T_1, T_2 be generic surfaces through P in the linear system $|-K_X|$.

Suppose that L is irreducible. As usual, we write

$$D \cdot T_1 = mL + \Upsilon,$$

where m is a non-negative integer and Υ is an effective cycle whose support does not contain L. Then $m \leqslant n$ (see Remark 2.12), but

$$n - m = D \cdot T_1 \cdot T_2 - mT_2 \cdot L = T_2 \cdot \Delta \geqslant \operatorname{mult}_P(\Delta) > n - m \operatorname{mult}_P(L),$$

which implies that L is singular at P. Hence there is an irreducible surface $T \in |-K_X|$ which is also singular at P. Now let S be a generic surface through P in the linear system $|-2K_X|$. Then

$$2n = D \cdot T \cdot S \geqslant \operatorname{mult}_P(D \cdot T) > 2n.$$

This contradiction shows that the curve L is reducible.

We have shown that $L = \overline{C}_i \cup \overline{Z}_i$. Write $D|_{T_1} = m_1\overline{C}_i + m_2\overline{Z}_i + \Delta$, where the m_i are non-negative integers and Δ is an effective cycle whose support does not contain $\overline{C}_i$ or $\overline{Z}_i$.

In the case when $P \in \overline{C}_i \cap \overline{Z}_i$, there is a surface $T \in |-K_X|$ such that T is singular at P. Arguing as in the previous case, we obtain a contradiction. Hence we may assume that $P \in \overline{C}_i$ and $P \notin \overline{Z}_i$.

We have equations $\overline{C}_i^2 = \overline{Z}_i^2 = -3/2$ and $\overline{C}_i \cdot \overline{Z}_i = 2$ on the surface T_1. Then

$$0 \leqslant \Delta \cdot \overline{Z}_i = \frac{1}{2}n - 2m_1 + \frac{3}{2}m_2, \qquad n - m_1 \leqslant \Delta \cdot \overline{C}_i = \frac{1}{2}n + \frac{3}{2}m_1 - 2m_2,$$

and it follows from Remark 2.12 that $m_1 + m_2 \leqslant 2n$. Hence $m_1 \leqslant n$. Therefore the log pair $\left(T_1, \overline{C}_i + \frac{1}{n}\Delta\right)$ is not log canonical at P by Theorem 7.5 of [9] since $P \notin \overline{Z}_i$. Then we have

$$\operatorname{mult}_P(\Delta|_{\overline{C}_i}) > n,$$

again by Theorem 7.5 of [9]. Thus,

$$n < \Delta \cdot \overline{C}_i = \frac{1}{2}n - m_1\overline{C}_i^{\,2} - m_2\overline{Z}_i \cdot \overline{C}_i = \frac{1}{2}n + \frac{3}{2}m_1 - 2m_2,$$

which is easily seen to contradict the inequalities obtained above.

Lemma 3.5. *We have* $\gimel \neq 12$.

Proof. Assume that $\gimel = 12$. Then X is a hypersurface of degree 10 in $\mathbb{P}(1,1,2,3,4)$ with singular points P_1, P_2, P_3, P_4 of types $\frac{1}{2}(1,1,1)$, $\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,1,2)$, $\frac{1}{4}(1,1,3)$ respectively. There is a commutative diagram

$$\begin{tikzcd}
 & & Y \arrow[dl, "\gamma_5"'] \arrow[dr, "\gamma_3"] \arrow[rrrd, "\eta"] & & & \\
 & U_{34} \arrow[dl, "\beta_4"'] \arrow[dr, "\beta_3"] & & U_{45} \arrow[dl, "\beta_5"] & & \mathbb{P}(1,1,2) \\
U_3 \arrow[dr, "\alpha_3"'] & & U_4 \arrow[dl, "\alpha_4"] & & & \\
 & X \arrow[rrrruu, dashed, "\psi"'] & & & &
\end{tikzcd}$$

where ψ is the natural projection, η is an elliptic fibration, α_3 is a weighted blow-up of P_3 with weights $(1,1,2)$, α_4 is a blow-up of P_4 with weights $(1,1,3)$, β_4 is a weighted blow-up with weights $(1,1,3)$ of the point that dominates P_4, β_3 is a weighted blow-up with weights $(1,1,2)$ of the singular point that dominates P_3, β_5 is a weighted blow-up with weights $(1,1,2)$ of the singular point of the α_4-exceptional divisor, γ_3 is a weighted blow-up with weights $(1,1,2)$ of the singular point that dominates P_3, and γ_5 is a weighted blow-up with weights $(1,1,2)$ of the singular point of the β_4-exceptional divisor.

Let L be the fibre of ψ that passes through P. Arguing as in the proof of Lemma 3.1, we see that L is not the base curve of the pencil $|-K_X|$. It follows that L does not pass through P_1 or P_2.

Since X is generic, the curve L is reduced and has at most double points outside P_4. We have $-K_X \cdot L = 5/6$, and there is a unique surface $T \in |-K_X|$ through P. The exceptional divisors of γ_5 and γ_3 are sections of the elliptic fibration η.

Assume that L is irreducible. Write

$$D \cdot T = mL + \Delta,$$

where m is a non-negative integer and Δ is an effective cycle whose support does not contain L. Let S be a generic surface through P in the linear system $|-2K_X|$. Then

$$\frac{5}{6}n - \frac{6}{3}m = S \cdot \Delta \geqslant \operatorname{mult}_P(\Delta) > n - m\operatorname{mult}_P(L) \geqslant n - 2m,$$

which implies that $m > n/2$. However, $m \leqslant n/2$ (see Remark 2.12). Thus the fibre L is reducible.

Let C be an irreducible component of L that passes through P_3, and let Z be an irreducible component of L that is different from C. Then

$$-K_X \cdot C \geqslant \frac{1}{3}, \qquad -K_X \cdot C \geqslant \frac{1}{4}, \qquad -4K_X \cdot Z \in \mathbb{N}.$$

Indeed, the curve Z does not pass through P_3 because the exceptional divisor of the birational morphism γ_3 is a section of the elliptic fibration η.

Let $\overline{C}$ be the proper transform of C on U_3. Then

$$-K_X \cdot C = \frac{1}{3} \iff -K_{U_3} \cdot \overline{C} = 0,$$

but there are only finitely many curves on U_3 whose intersection with the divisor $-K_{U_3}$ is trivial. Hence we may assume that $L = C + Z$ and $-K_X \cdot Z = 1/2$.

The hypersurface X can be given by the equation

$$w^2z + w\bigl(t^2 + tf_3(x,y,z) + f_6(x,y,z)\bigr) + tf_7(x,y,z) + f_{10}(x,y,z) = 0$$
$$\subset \operatorname{Proj}(\mathbb{C}[x,y,z,t,w]),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 2$, $\mathrm{wt}(t) = 3$, $\mathrm{wt}(w) = 4$, and f_i is a quasi-homogeneous polynomial of degree i. Let R be the surface cut out on X by the equation

$$wz + t^2 + tf_3(x,y,z) = 0,$$

and let $\breve{R}$ be the proper transform of R on the threefold U_{45}. Then

$$\breve{R} \cdot \breve{Z} = \frac{6}{5} - \frac{10}{4} E \cdot \beta_5(\breve{Z}) - \frac{6}{3} G \cdot \breve{Z} \leqslant -6K_X \cdot Z - \frac{5}{6} - 2,$$

where E and G are the exceptional divisors of α_4 and β_5 respectively and $\breve{Z}$ is the proper transform of Z on U_{45}. Hence Z is one of the finitely many curves that are contracted by the natural projection $X \dashrightarrow \mathbb{P}(1,1,2,3)$, and we have $L = C + Z$ if $-K_X \cdot Z = 1/4$. We write

$$D|_T = m_C C + m_Z Z + \Omega,$$

where m_C and m_Z are non-negative integers and Ω is an effective divisor whose support does not contain C or Z. We get a system of linear inequalities

$$\Omega \cdot C \geqslant (\operatorname{mult}_P(D) - m_Z \operatorname{mult}_P(Z) - m_C \operatorname{mult}_P(C)) \operatorname{mult}_P(C),$$
$$\Omega \cdot Z \geqslant (\operatorname{mult}_P(D) - m_Z \operatorname{mult}_P(Z) - m_C \operatorname{mult}_P(C)) \operatorname{mult}_P(Z),$$

and $-m_C K_X \cdot C - m_Z K_X \cdot Z \leqslant 5n/12$. It is easy to see that

$$Z^2 = -\frac{5}{4}, \qquad C \cdot Z = \frac{7}{4}, \qquad C^2 = -\frac{7}{12}$$

on the surface T in the case when $-K_X \cdot Z = 1/4$, and

$$Z^2 = -1, \qquad C \cdot Z = 2, \qquad C^2 = -\frac{4}{3}$$

if $-K_X \cdot Z \neq 1/4$. Simple calculations now yield a contradiction.

Lemma 3.6. *We have* $\gimel \neq 13$.

Proof. Assume that $\gimel = 13$. Then X is a hypersurface of degree 11 in $\mathbb{P}(1,1,2,3,5)$. The singularities of X consist of points P_1, P_2, P_3 of types $\frac{1}{2}(1,1,1)$, $\frac{1}{3}(1,1,2)$, $\frac{1}{5}(1,2,3)$ respectively.

Let L be the fibre of the projection $X \dashrightarrow \mathbb{P}(1,1,2)$ that contains P. The proof of Lemma 3.1 shows directly that L is not the base curve of the pencil $|-K_X|$. It follows that $P_1 \notin L$ and the curve L has at most double points outside P_3.

Arguing as in the proof of Lemma 3.5, we see that $L = C + Z$, where C and Z are irreducible curves such that $C \neq Z$ and either $-K_X \cdot C = 1/5$ or $-K_X \cdot C = 1/3$.

Suppose that $-K_X \cdot C = 1/5$. Then $-K_X \cdot Z = 8/15$ and C is one of the finitely many curves contracted by the projection $X \dashrightarrow \mathbb{P}(1,1,2,3)$. The hypersurface X is given by the equation

$$w^2y + wg(x,y,z,t) + h(x,y,z,t) = 0 \subset \mathbb{P}(1,1,2,3,5) \cong \operatorname{Proj}(\mathbb{C}[x,y,z,t,w]),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 2$, $\mathrm{wt}(t) = 3$, $\mathrm{wt}(w) = 5$, and g and h are quasi-homogeneous polynomials. Let R be the irreducible reduced surface cut out on X by the equation $y = 0$. Then R contains the curves Z and C and we have

$$C^2 = -\frac{4}{5}, \qquad C \cdot Z = \frac{6}{5}, \qquad Z^2 = -\frac{2}{15}$$

on R. (These equations make sense since the surface R is normal.) We write

$$D|_R = m_C C + m_Z Z + \Omega,$$

where m_C and m_Z are non-negative integers, and Ω is an effective divisor whose support does not contain C or Z. Thus we get

$$\frac{1}{5}n + \frac{4}{5}m_C - \frac{6}{5}m_Z = \Omega \cdot C > n - m_C - m_Z,$$
$$\frac{8}{15}n - \frac{6}{5}m_C + \frac{2}{15}m_Z = \Omega \cdot Z > n - m_C - m_Z$$

in the case when $P \in C \cap Z$. However we have $3m_C + 8m_Z \leqslant 11n/2$ by Remark 2.12. Hence $P \notin C \cap Z$ and either

$$\frac{1}{5}n + \frac{4}{5}m_C - \frac{6}{5}m_Z = \Omega \cdot C > n - m_C \quad \text{and}$$
$$\frac{8}{15}n - \frac{6}{5}m_C + \frac{2}{15}m_Z = \Omega \cdot Z \geqslant 0,$$

or $\Omega \cdot C \geqslant 0$ and $\Omega \cdot Z > n - m_Z$, which leads to a contradiction.

We have shown that $-K_X \cdot C = 1/3$ and $-K_X \cdot Z = 2/5$. Let $\alpha\colon U \to X$ be a weighted blow-up of the singular point P_2 with weights $(1,1,2)$. Then the proper transform of C on U is one of the finitely many curves on U whose intersection with $-K_U$ is trivial. Let S be the surface through P in $|-K_X|$. Then

$$C^2 = -\frac{4}{3}, \qquad C \cdot Z = 2Z^2 = -\frac{6}{5}$$

on S. Arguing as in the case when $-K_X \cdot C = 1/5$, we arrive at a contradiction. The lemma is proved.

Lemma 3.7. *We have* $\gimel \neq 14$.

Proof. Assume that $\gimel = 14$. Then X is a hypersurface of degree 12 in $\mathbb{P}(1,1,1,4,6)$. The singularities of X consist of a singular point O of type $\frac{1}{2}(1,1,1)$. Let $\psi\colon X \dashrightarrow \mathbb{P}^2$ be the natural projection, and let L be the fibre of ψ that contains P. Since $-K_X \cdot L = 1/2$, we see that L is a reduced irreducible curve.

Let T_1 and T_2 be generic surfaces through P in $|-K_X|$. We write

$$D \cdot T_1 = mL + \Delta,$$

where m is a non-negative integer and Δ is an effective cycle whose support does not contain L. Then $m \leqslant n$ by Remark 2.12. However,

$$\frac{n-m}{2} = D \cdot T_1 \cdot T_2 - mT_2 \cdot L = T_2 \cdot \Delta \geqslant \operatorname{mult}_P(\Delta) > n - m \operatorname{mult}_P(L).$$

It follows that $m > n$ if $\operatorname{mult}_P(C) = 1$. Hence the curve C is singular at P and, therefore, there is a surface $T \in |-K_X|$ which is also singular at P. We have

$$2n = D \cdot T \cdot S \geqslant \operatorname{mult}_P(D \cdot T) > 2n,$$

where S is a generic surface through P in $|-4K_X|$. This is a contradiction.

Lemma 3.8. *We have* $\gimel \neq 16$.

Proof. Assume that $\gimel = 16$. Then X is a hypersurface of degree 12 in $\mathbb{P}(1,1,2,4,5)$. The singularities of X consist of three points of type $\frac{1}{2}(1,1,1)$ and a point O of type $\frac{1}{5}(1,1,4)$.

There is a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] & Y \arrow[l, "\gamma"'] \arrow[d, "\eta"] \\
X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1,1,2)
\end{tikzcd}$$

where α is a weighted blow-up of O with weights $(1,1,4)$, β is a weighted blow-up with weights $(1,1,3)$ of the singular point of type $\frac{1}{4}(1,1,3)$, γ is a blow-up with weights $(1,1,2)$ of the singular point of type $\frac{1}{3}(1,1,2)$, η is an elliptic fibration, and ψ is the natural projection. The proof of Lemma 3.1 yields that $\psi(P)$ is a smooth point of $\mathbb{P}(1,1,2)$.

Let L be the fibre of ψ that passes through P. Then L contains no singular points of X of type $\frac{1}{2}(1,1,1)$. It follows that the curve L is reduced. Since X is generic, we see that L has at most double singular points outside O. Let T be the unique surface through P in the linear system $|-K_X|$. Then $L \subset T$.

Assume that the curve L is irreducible. Write $D \cdot T = mL + \Delta$, where m is a non-negative integer and Δ is an effective cycle whose support does not contain L. Then

$$\frac{3}{5}n - \frac{6}{5}m = S \cdot \Delta \geqslant \operatorname{mult}_P(\Delta) > n - m \operatorname{mult}_P(L) \geqslant n - 2m,$$

where S is a generic surface through P in the linear system $|-2K_X|$. We have shown that $m > n/2$. But this is impossible since $m \leqslant n/2$ by Remark 2.12. This contradiction shows that the fibre L must be reducible.

Let C be an irreducible component of L that minimizes the number $-K_X \cdot C$. Then we have $-K_X \cdot C = 1/5$ because $-K_X \cdot L = 3/5$. The hypersurface X can be given by

$$w^2z + wg(x,y,z,t) + h(x,y,z,t) = 0 \subset \mathbb{P}(1,1,2,4,5) \cong \operatorname{Proj}(\mathbb{C}[x,y,z,t,w]),$$

where $\operatorname{wt}(x) = \operatorname{wt}(y) = 1$, $\operatorname{wt}(z) = 2$, $\operatorname{wt}(t) = 4$, $\operatorname{wt}(w) = 5$, and g, h are generic quasi-homogeneous polynomials. Let R be the surface cut out by the equation $z = 0$ on X, and let $\breve{R}$ be the proper transform of R on W. Then

$$\breve{R} \equiv (\alpha \circ \beta)^*(-2K_X) - \frac{7}{5}\beta^*(E) - \frac{3}{4}G,$$

where E and G are the exceptional divisors of the birational morphisms α and β respectively.

Let $\breve{C}$ and $\overline{C}$ be the proper transforms of C on W and U respectively. Then

$$\breve{R} \cdot \breve{C} = \frac{1}{5} - \frac{7}{5}E \cdot \overline{C} - \frac{3}{4}G \cdot \breve{C} \leqslant \frac{1}{7} - \frac{7}{20} - \frac{1}{4} < 0.$$

It follows that the curve C is contained in the surface R. Since X is generic, it follows that $-K_U \cdot \overline{C} = 0$ and $E \cdot \overline{C} = 1$.

Moreover, since X is generic, we also see that the surface T is quasi-smooth and the curve L consists of two components, C and Z, where Z is an irreducible curve and $-K_X \cdot Z = 2/5$. We have

$$C^2 = -\frac{6}{5}, \qquad C \cdot Z = \frac{8}{5}, \qquad Z^2 = -\frac{4}{5}$$

on the surface T. Hence we arrive at a contradiction by repeating the proof of Lemma 3.6.

Lemma 3.9. *We have* $\gimel \neq 18$.

Proof. Assume that $\gimel = 18$. Then X is a hypersurface of degree 12 in $\mathbb{P}(1,2,2,3,5)$. The singularities of X consist of 6 points of type $\frac{1}{2}(1,1,1)$ and a point O of type $\frac{1}{5}(1,2,3)$.

It follows from [11] that there is a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & W \arrow[l, "\beta"'] \arrow[d, "\eta"] \\
X \arrow[r, dashed, "\psi"'] & \mathbb{P}(1,2,2)
\end{tikzcd}$$

where α is a weighted blow-up of O with weights $(1,2,3)$, β is a weighted blow-up with weights $(1,1,3)$ of the singular point of type $\frac{1}{3}(1,1,2)$, η is an elliptic fibration, and ψ is the natural projection.

Let C be the scheme fibre of ψ that passes through P, and let L be a reduced irreducible component of C. Then

$$-K_X \cdot C = \frac{4}{5}, \qquad -10K_X \cdot L \in \mathbb{N}.$$

But the rational number $-5K_X \cdot L$ is an integer unless the curve L passes through a singular point of type $\frac{1}{2}(1,1,1)$. Thus we see that $C = 2L$ whenever C passes through such a singular point.

Let T be the surface in $|-K_X|$, and let S and $\grave{S}$ be generic surfaces through P in $|-2K_X|$. Then S and $\grave{S}$ are irreducible. We have $S \supset L \subset \grave{S}$ but $S \neq D \neq \grave{S}$.

Assume that $L \subset T$. Then $C = 2L$ and $-K_X \cdot L = 2/5$. Since X is generic, the singularities of L are at most double points. We write $D|_T = mL + \Upsilon$, where m is a non-negative integer and Υ is an effective cycle whose support does not contain L. Then we have

$$\frac{2}{5}n - \frac{4}{5}m = S \cdot \Upsilon \geqslant \operatorname{mult}_P(\Upsilon) \geqslant \operatorname{mult}_P(D) - \operatorname{mult}_P(L) > n - 2m.$$

It follows that $m > n/2$. But $m \leqslant n/2$ by Remark 2.12, a contradiction.

We have shown that $L \not\subset T$. Assume that $C = L$. Then $\operatorname{mult}_P(L) \leqslant 2$. We write

$$D \cdot \grave{S} = \grave{m}C + \grave{\Upsilon},$$

where $\grave{m}$ is a non-negative integer and $\grave{\Upsilon}$ is an effective cycle whose support does not contain C. Then

$$\frac{4}{5}n - \frac{8}{5}\grave{m} = S \cdot \grave{\Upsilon} \geqslant \operatorname{mult}_P(\grave{\Upsilon}) > n - 2m.$$

It follows that $m > n/2$. But $m \leqslant n/2$ by Remark 2.12, a contradiction.

We have shown that $C \neq L$ but L does not pass through a point of type $\frac{1}{2}(1,1,1)$. Since the threefold X is generic, it follows that $C = L + Z$, where Z is an irreducible curve and $Z \neq L$. We write

$$D|_S = m_L L + m_Z Z + \Omega,$$

where m_L and m_Z are non-negative integers and Ω is an effective divisor on S whose support does not contain the curves L and Z. There is no loss of generality in assuming that

$$-K_X \cdot L \leqslant -K_X \cdot Z.$$

It follows that either $-K_X \cdot L = 1/5$ and $-K_X \cdot Z = 3/5$, or $-K_X \cdot L = -K_X \cdot Z = 2/5$.

Assume that $-K_X \cdot L = 2/5$. Then L and Z are smooth outside O, and

$$\frac{4}{5}n - \frac{4}{5}m_L - \frac{4}{5}m_Z = \grave{S}|_S \cdot \Omega \geqslant \operatorname{mult}_P(\Omega) > n - m_L - m_C.$$

It follows that $m_L + m_C > n$. But $m_L + m_C \leqslant n$ by Remark 2.12, a contradiction.

Thus we have $-K_X \cdot L = 1/5$. The hypersurface X can be given by

$$w^2 z + wg(x,y,z,t) + h(x,y,z,t) = 0 \subset \mathbb{P}(1,2,2,3,5) \cong \operatorname{Proj}(\mathbb{C}[x,y,z,t,w]),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = \mathrm{wt}(z) = 2$, $\mathrm{wt}(t) = 3$, $\mathrm{wt}(w) = 5$, and g, h are generic quasi-homogeneous polynomials of degree 7 and 12 respectively. Let R be the surface cut out on X by the equation $z = 0$, and let $\overline{R}$ and $\bar{L}$ be the proper transforms on U of the surface R and the curve L respectively. Then $\overline{R} \cdot L < 0$. It follows that $L \subset R \supset Z$, the curve L is contracted by the projection $X \dashrightarrow \mathbb{P}(1, 2, 2, 3)$ to a point and the intersection $L \cap Z$ contains no singular points of X different from O.

Let $\overline{Z}$ be the proper transform of Z on the threefold U, and let $\pi\colon \overline{R} \to R$ be the birational morphism induced by α. Then

$$\bar{L} + \overline{Z} = \overline{S}|_{\overline{R}} \equiv -K_U|_{\overline{R}},$$

where $\overline{S}$ is the proper transform of S on U.

Let $\overline{E}$ be the curve on $\overline{R}$ which is contracted by π to a point. Then

$$\bar{L}^2 = -1, \qquad \bar{L} \cdot \overline{Z} = \bar{L} \cdot \overline{E} = 1, \qquad \overline{Z}^2 = -\frac{1}{3}, \qquad \overline{E}^2 = -\frac{35}{6}, \qquad \overline{Z} \cdot \overline{E} = \frac{4}{3}$$

on the surface $\overline{R}$. It follows that $L^2 = -29/35$, $L \cdot Z = 43/35$, $Z^2 = -1/35$ on the surface R.

Suppose that $P \in L \cap Z$. Then $m_L + 3m_C \leqslant 5n$ by Remark 2.12. But

$$\begin{aligned} &\frac{1}{5}n + \frac{29}{35}m_L - \frac{43}{35}m_Z = \Omega \cdot L > n - m_L - m_Z, \\ &\frac{2}{5}n - \frac{43}{35}m_L + \frac{1}{35}m_Z = \Omega \cdot Z > n - m_L - m_Z, \end{aligned}$$

which leads to a contradiction. Hence either $L \ni P \notin Z$, or $Z \ni P \notin L$.

Suppose that $Z \ni P \notin L$. Then $\Omega \cdot Z > n - m_Z$ and $\Omega \cdot L \geqslant 0$, which easily yields a contradiction. Thus we see that $L \ni P \notin Z$. Then

$$\frac{1}{5}n + \frac{29}{35}m_L - \frac{43}{35}m_Z = \Omega \cdot L > n - m_L, \qquad \frac{2}{5}n - \frac{43}{35}m_L + \frac{1}{35}m_Z = \Omega \cdot Z \geqslant 0.$$

It follows that $m_L < n$. By Theorem 7.5 of [9], the log pair

$$\left(R, L + \frac{m_C}{n}C + \frac{1}{n}\Omega\right)$$

is not log canonical at P. Then $\mathrm{mult}_P(\Omega|_L) > n$ by Theorem 7.5 of [9]. Thus we see that $\Omega \cdot L > n$, which easily yields a contradiction.

Lemma 3.10. *We have* $\daleth \neq 19$.

Proof. Suppose that $\daleth = 19$. Then X is a generic hypersurface of degree 12 in $\mathbb{P}(1, 2, 3, 3, 4)$. Let T be the unique surface in the linear system $|-K_X|$, and let S be a generic surface through P in the linear system $|-6K_X|$. Then $P \notin T$ since otherwise

$$n = D \cdot S \cdot T \geqslant \mathrm{mult}_P(D) > n.$$

Let H and G be generic surfaces through P in the linear systems $|-2K_X|$ and $|-3K_X|$ respectively. Then

$$n = D \cdot H \cdot G \geqslant \mathrm{mult}_P(D) > n$$

provided that $D \neq H$. Hence we see that $D = H$, $n = 2$ and $\mathrm{mult}_P(D) \geqslant 3$. Since X is generic, an easy parameter count shows that the inequality $\mathrm{mult}_P(D) \geqslant 3$ is impossible in the case when $P \notin T$, a contradiction.

Lemma 3.11. *We have* $\gimel \neq 20$.

Proof. Suppose that $\gimel = 20$. Then X is a hypersurface of degree 13 in $\mathbb{P}(1,1,3,4,5)$. The singularities of X consist of points P_1, P_2, P_3 of types $\frac{1}{3}(1,1,2)$, $\frac{1}{4}(1,1,3)$, $\frac{1}{5}(1,1,4)$ respectively.

Arguing as in the proof of Lemma 3.2 and using Theorem 5.6.2 of [5], we see that $\mathrm{mult}_P(D) \leqslant n$ in the case when the point P is not contained in the finitely many curves contracted by the projection $X \dashrightarrow \mathbb{P}(1,1,3,4)$. Hence we may assume that P is contained in one of the curves contracted by the projection $X \dashrightarrow \mathbb{P}(1,1,3,4)$. There is a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "\gamma_3"'] \arrow[dr, "\gamma_1"] \arrow[drrr, "\omega"] & & & \\
U_1 \arrow[dr, "\alpha_1"'] & & U_3 \arrow[dl, "\alpha_3"] & & \mathbb{P}(1,1,4) \\
 & X \arrow[urrr, dashed, "\xi"'] & & &
\end{tikzcd}$$

where ξ is a projection, α_1 is a blow-up of P_1 with weights $(1,1,2)$, α_3 is a weighted blow-up of P_3 with weights $(1,1,4)$, γ_3 is a blow-up with weights $(1,1,4)$ of the singular point that dominates P_3, γ_1 is a blow-up with weights $(1,1,2)$ of the singular point that dominates P_1, and ω is an elliptic fibration.

Let Z be the fibre of ξ that contains P. Then $Z = L + C$, where L and C are irreducible curves with $-K_X \cdot L = 1/5$ and $-K_X \cdot C = 2/3$. The curves L and C are smooth at P.

Let S be a surface through P in the linear system $|-K_X|$. Then S contains L and C. One can assume that S is quasi-smooth.

We have $L^2 = -6/5$, $C^2 = 2/3$, and $L \cdot C = 2$ on the surface S. Write

$$D|_T = m_L L + m_C C + \Omega,$$

where m_L and m_C are non-negative integers and Ω is an effective 1-cycle whose support does not contain L or C. Suppose that $P \in L \cap C$. Then

$$\begin{gathered}
\frac{1}{5}n + \frac{6}{5}m_L - 2m_C = \Omega \cdot L > n - m_L - m_C, \\
\frac{2}{3}n - 2m_L - \frac{2}{3}m_C = \Omega \cdot C > n - m_L - m_C,
\end{gathered}$$

which leads to a contradiction. Hence we have shown that $P \in L$ and $P \notin C$. Then

$$\begin{gathered}
\frac{1}{5}n + \frac{6}{5}m_L - 2m_C = \Omega \cdot L > n - m_L, \\
\frac{2}{3}n - 2m_L - \frac{2}{3}m_C = \Omega \cdot C \geqslant 0.
\end{gathered}$$

This also leads to a contradiction. The lemma is proved.

Lemma 3.12. *We have* $\gimel \neq 22$.

Proof. Suppose that $\gimel = 22$. Then X is a generic hypersurface of degree 14 in $\mathbb{P}(1,2,2,3,7)$. Let T be the unique effective divisor in the linear system $|-K_X|$, and let S be a generic surface through P in the linear system $|-6K_X|$. Then $P \notin T$ since otherwise we have the contradictory inequality $n = D \cdot S \cdot T \geqslant \mathrm{mult}_P(D)$.

Let S_2 be a generic surface through P in the linear system $|-2K_X|$. It is easy to see that there is a surface $S_3 \in |-3K_X|$ that also passes through P but contains no components of the cycle $D \cdot S_2$. Thus we have

$$n = D \cdot S_2 \cdot S_3 \geqslant \mathrm{mult}_P(D) > n,$$

which contradicts the assumption. The lemma is proved.

Lemma 3.13. *We have* $\gimel \neq 23$.

Proof. Suppose that $\gimel = 23$. Then X is a hypersurface of degree 9 in $\mathbb{P}(1,2,3,4,5)$, and the natural projection $\psi\colon X \dashrightarrow \mathbb{P}(1,2,3,4)$ is a finite morphism outside 21 smooth rational curves $C_1, \dots, C_{21}$ such that $\psi(C_i)$ is a point and $-K_X \cdot C_i = 1/5$.

Arguing as in the proof of Lemma 3.1, we see that the point P is not contained in the base locus of the linear systems $|-K_X|$, $|-2K_X|$. Let R be a surface through P in the pencil $|-2K_X|$. Then the proof of Lemma 3.10 yields that $R \neq D$.

Suppose that $P \notin \bigcup_{i=1}^{21} C_i$. Then it follows from the proof of Theorem 5.6.2 in [5] that there is a surface $H \in |-4sK_X|$ that has multiplicity at least $s > 0$ at the point P and contains no components of the effective cycle $D \cdot R$, where s is a positive integer. Then

$$ns > \frac{56}{60}ns = H \cdot D \cdot R \geqslant \mathrm{mult}_P(D)s > ns,$$

a contradiction. Thus we may assume that $P \in C_1$.

Let M be a generic surface through P in $|-3K_X|$. Then $M \cdot R = C_1 + Z_1$, where Z_1 is an irreducible curve smooth at P and $-K_X \cdot Z_1 = 1/2$.

It is easy to see that $M \neq D$. We write

$$D|_M = m_1 C_1 + m_2 Z_1 + \Upsilon \equiv -nK_X|_M,$$

where m_1 and m_2 are non-negative integers and Υ is an effective cycle whose support does not contain C_1 or Z_1. The surface M is normal and smooth at P, but we have

$$C_1^2 = -\frac{8}{5}, \qquad Z_1^2 = -1, \qquad C_1 \cdot Z_1 = 2$$

on M. We may assume that $P \in Z_1$ since the case $P \notin Z_1$ is simpler. Then

$$\begin{aligned} \frac{1}{5}n + \frac{8}{5}m_1 - 2m_2 &= \Upsilon \cdot C_1 > n - m_1 - m_2, \\ \frac{1}{2}n - 2m_1 + m_2 &= \Upsilon \cdot Z_1 > n - m_1 - m_2. \end{aligned}$$

Hence we have strict inequalities $m_1 > n/2$ and $m_2 > n/2$, which contradict the inequality $m_1/5 + m_2/2 \leqslant 7n/20$ (see Remark 2.12). The lemma is proved.

Lemma 3.14. *We have* $\gimel \neq 24$.

Proof. Suppose that $\gimel = 24$. Then X is a hypersurface of degree 15 in $\mathbb{P}(1,1,2,5,7)$. The singularities of X consist of a point P_1 of type $\frac{1}{2}(1,1,1)$ and a point P_2 of type $\frac{1}{7}(1,2,5)$.

It follows from [11] that there is a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] & & Y \arrow[ll, "\gamma"'] \arrow[d, "\eta"] \\
X \arrow[rrrr, dashed, "\psi"'] & & & & \mathbb{P}(1,1,2)
\end{tikzcd}$$

where α is a weighted blow-up of P_2 with weights $(1,2,5)$, β is a weighted blow-up with weights $(1,2,3)$ of the singular point of type $\frac{1}{5}(1,2,3)$, γ is a weighted blow-up with weights $(1,1,2)$ of the singular point of type $\frac{1}{3}(1,1,2)$, and η is an elliptic fibration.

Let L be the fibre of ψ that passes through P. Arguing as in the proof of Lemma 3.1, we see that L is not the base curve of $|-K_X|$. It follows that L does not pass through P_1. The singularities of L consist of at most double points outside P_2.

Suppose that L is irreducible. Let T be a generic surface through P in the pencil $|-K_X|$. We write $D \cdot T = mL + \Delta$, where m is a non-negative integer and Δ is an effective cycle whose support does not contain L. Then

$$\frac{3}{7}n - \frac{6}{7}m = S \cdot \Delta \geqslant \operatorname{mult}_P(\Delta) > n - m \operatorname{mult}_P(L) \geqslant n - 2m,$$

where S is a generic surface through P in $|-2K_X|$. Hence $m > n/2$, which is impossible because $m \leqslant n/2$ by Remark 2.12, a contradiction. Thus the fibre L is reducible.

The divisor $-7K_X$ is a Cartier divisor in a neighbourhood of L, but $-7K_X \cdot L = 3$. Hence L consists of at most 3 components, all components of L pass through P_2, and there is a component C of L such that $-K_X \cdot C = 1/7$.

The hypersurface X can be given by the equation

$$w^2y + wg(x,y,z,t) + h(x,y,z,t) = 0 \subset \mathbb{P}(1,1,2,5,7) \cong \operatorname{Proj}(\mathbb{C}[x,y,z,t,w]),$$

where $\operatorname{wt}(x) = \operatorname{wt}(y) = 1$, $\operatorname{wt}(z) = 2$, $\operatorname{wt}(t) = 5$, $\operatorname{wt}(w) = 7$, and g, h are quasi-homogeneous polynomials. Let R be the surface cut out on X by the equation $y = 0$. Then

$$\overline{R} \equiv \alpha^*(-K_X) - \frac{8}{7}E,$$

where $\overline{R}$ is the proper transform of R on U and E is the exceptional divisor of α.

Let $\overline{C}$ be the proper transform of C on U. Then

$$\overline{R} \cdot \overline{C} = \frac{1}{7} - \frac{8}{7}E \cdot \overline{C} \leqslant \frac{1}{7} - \frac{8}{35} < 0.$$

It follows that $C \subset R$. Since X is generic, the curve C must be one of the 12 curves that satisfy $-K_U \cdot \overline{C} = 0$ and $E \cdot \overline{C} = 1$.

The surface R is normal and L consists of two components, C and Z, where Z is an irreducible curve and $-K_X \cdot Z = 2/7$. Then

$$C^2 = -\frac{23}{28}, \qquad C \cdot Z = \frac{31}{28}, \qquad Z^2 = -\frac{15}{28}$$

on the surface R. We write $D|_R = m_C C + m_Z Z + \Omega$, where m_C and m_Z are non-negative integers and Ω is an effective divisor on R whose support does not contain C or Z. Then $m_C + 2m_Z \leqslant 3n/2$ by Remark 2.12.

Suppose that $P \in C \cap Z$. Then

$$\begin{aligned} \frac{1}{7}n + \frac{23}{28}m_C - \frac{31}{28}m_Z = \Omega \cdot C > n - m_C - m_Z, \\ \frac{2}{7}n - \frac{31}{28}m_C + \frac{15}{28}m_Z = \Omega \cdot Z > n - m_C - m_Z, \end{aligned}$$

which leads to a contradiction. Hence we have either $C \ni P \notin Z$, or $Z \ni P \notin C$.

Suppose that $C \ni P \notin Z$. Then

$$\begin{aligned} \frac{1}{7}n + \frac{23}{28}m_C - \frac{31}{28}m_Z = \Omega \cdot C > n - m_C, \\ \frac{2}{7}n - \frac{31}{28}m_C + \frac{15}{28}m_Z = \Omega \cdot Z \geqslant 0, \end{aligned}$$

which leads to a contradiction. Thus we have $Z \ni P \notin C$. Then

$$\begin{aligned} \frac{1}{7}n + \frac{23}{28}m_C - \frac{31}{28}m_Z = \Omega \cdot C \geqslant 0, \\ \frac{2}{7}n - \frac{31}{28}m_C + \frac{15}{28}m_Z = \Omega \cdot Z > n - m_Z. \end{aligned}$$

It follows that $m_C > 16$. But $m_C \leqslant 3/2$ by Remark 2.12, a contradiction. The lemma is proved.

Lemma 3.15. *We have* $\gimel \neq 25$.

Proof. Suppose that $\gimel = 25$. Then X can be given by the equation

$$w^2 y + wg(x,y,z,t) + h(x,y,z,t) = 0 \subset \mathbb{P}(1,1,3,4,7) \cong \operatorname{Proj}(\mathbb{C}[x,y,z,t,w]),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 3$, $\mathrm{wt}(t) = 4$, $\mathrm{wt}(w) = 7$, and g, h are generic quasi-homogeneous polynomials of degree 8 and 15 respectively.

Let $\psi\colon X \dashrightarrow \mathbb{P}(1,1,3)$ and $\xi\colon X \dashrightarrow \mathbb{P}(1,1,3,4)$ be the natural projections. Then ξ is a finite morphism outside the smooth irreducible curves $C_1, \ldots, C_{10}$ that are cut out on X by the equations

$$y = g(x,y,z,t) = h(x,y,z,t) = 0,$$

and the normalization of the generic fibre of ψ is an elliptic curve. It follows from the proof of Lemma 3.13 that $\mathrm{mult}_P(D) \leqslant n$ in the case when $P \notin \bigcup_{i=1}^{10} C_i$.

We may assume that $P \in C_1$. The fibre of ψ over the point $\psi(C_1)$ consists of two irreducible components: let Z_1 be the one such that $Z_1 \neq C_1$. Then the curve Z_1 is smooth at P.

Let T be the surface cut out on X by the equation $y = 0$. Then $P \in C_1 \subset T$ and the surface T is normal. The intersection form of the curves C_1 and Z_1 on the surface T is given by

$$Z_1^2 = -\frac{1}{28}, \qquad C_1^2 = -\frac{11}{14}, \qquad Z_1 \cdot C_1 = \frac{17}{14}.$$

We write $D|_T = m_C C_1 + m_Z Z_1 + \Omega$, where m_C and m_Z are non-negative integers and Ω is an effective divisor on T whose support does not contain C_1 or Z_1. Then $m_C \leqslant 5n/4$ and $m_Z \leqslant 5n/11$ by Remark 2.12 because we have $-K_X \cdot C_1 = 1/7$ and $-K_X \cdot Z_1 = 11/28$ respectively.

Suppose that $P \in C_1 \cap Z_1$. Then

$$\begin{aligned} \frac{1}{7}n + \frac{11}{14}m_C - \frac{17}{14}m_Z &= \Omega \cdot C_1 > n - m_C - m_Z, \\ \frac{11}{28}n - \frac{17}{14}m_C + \frac{1}{28}m_Z &= \Omega \cdot Z_1 > n - m_C - m_Z. \end{aligned}$$

It follows that $m_Z > 5n/11$, a contradiction. Hence we have $P \in C_1$ and $P \notin Z_1$. Then

$$\begin{aligned} \frac{1}{7}n + \frac{11}{14}m_C - \frac{17}{14}m_Z &= \Omega \cdot C_1 > n - m_C, \\ \frac{11}{28}n - \frac{17}{14}m_C + \frac{1}{28}m_Z &= \Omega \cdot Z_1 \geqslant 0, \end{aligned}$$

which leads to a contradiction to the assumption. The lemma is proved.

Lemma 3.16. *We have* $\daleth \neq 32$, $\daleth \neq 33$, $\daleth \neq 38$.

Proof. Suppose that $\daleth \in \{32, 33, 38\}$. The projection $X \dashrightarrow \mathbb{P}(1, a_1, a_2, a_3)$ contracts finitely many smooth curves. Let C be one of them and let M be a generic surface containing C in the linear system $|-aK_X|$, where $a = 3$ for $\daleth \neq 33$ and $a = 2$ for $\daleth = 33$.

Let $\psi\colon X \dashrightarrow \mathbb{P}(1, a_1, a_2)$ be the natural projection, and let Z be the component of the fibre of ψ over the point $\psi(C)$ such that $Z \neq C$. Then

$$\begin{aligned} &C^2 = -\frac{10}{7}, \qquad Z^2 = -\frac{6}{7}, \qquad C \cdot Z = \frac{12}{7} \quad \text{if} \quad \daleth = 32, \\ &C^2 = -\frac{9}{7}, \qquad Z^2 = -\frac{24}{35}, \qquad C \cdot Z = \frac{12}{7} \quad \text{if} \quad \daleth = 33, \\ &C^2 = -\frac{11}{8}, \qquad Z^2 = -\frac{39}{40}, \qquad C \cdot Z = \frac{13}{8} \quad \text{if} \quad \daleth = 38 \end{aligned}$$

on the surface M. Arguing as in the proof of Lemma 3.13, we easily get a contradiction. The lemma is proved.

Thus we have shown that $\daleth \in \{7, 8, 9\}$.

Lemma 3.17. *We have* $\daleth \neq 9$.

Proof. Suppose that $\daleth = 9$. Then X is a hypersurface of degree 9 in $\mathbb{P}(1, 1, 2, 3, 3)$. The singularities of X consist of points O_1, O_2, O_3 of type $\frac{1}{3}(1, 1, 2)$ and one singular point of type $\frac{1}{2}(1, 1, 1)$.

It follows from [11] that there is a commutative diagram

$$\begin{tikzcd}
& & U \arrow[dll, "\sigma"'] \arrow[d, "\alpha"] & Y \arrow[l, "\gamma"'] \arrow[dd, "\eta"] \\
V \arrow[dr, "\omega"'] & & X \arrow[dl, dashed, "\xi"'] \arrow[dr, dashed, "\psi"] & \\
& \mathbb{P}(1,1,2,3) \arrow[rr, dashed, "\chi"'] & & \mathbb{P}(1,1,2)
\end{tikzcd}$$

where ξ, ψ and χ are projections, α is a blow-up of O_1 with weights $(1,1,3)$, γ is the composite of weighted blow-ups with weights $(1,1,3)$ of the singular points that dominate O_2 and O_3, the morphism η is an elliptic fibration, ω is a double covering and σ is a birational morphism that contracts 27 smooth rational curves $C_1,\ldots,C_{27}$.

The threefold U contains 27 irreducible curves $Z_1,\ldots,Z_{27}$ such that $\alpha(Z_i)$ is a curve and the union $\alpha(Z_i)\cup\alpha(C_i)$ is the fibre of ψ over the point $\psi(C_i)$.

We put $\overline{Z}_i=\alpha(Z_i)$ and $\overline{C}_i=\alpha(C_i)$. Then

$$-K_X\cdot\overline{Z}_i=-2K_X\cdot\overline{C}_i=\frac{2}{3},$$

but $O_1\in\overline{C}_i,\ O_2\notin\overline{C}_i,\ O_3\notin\overline{C}_i,\ O_1\notin\overline{Z}_i,\ O_2\in\overline{Z}_i$ and $O_3\in\overline{Z}_i$.

It follows from the proof of Lemma 3.1 that P is not contained in the base curve of $|-K_X|$.

Let L be the fibre of ψ that contains P. Since X is generic, it follows that L is reduced and its singularities consist of finitely many double points. We easily see that $L=\alpha(Z_i)\cup\alpha(C_i)$ for some i if L is reducible.

Suppose that L is irreducible. Let T be the unique surface through P in the linear system $|-K_X|$. We write $D\cdot T=mL+\Delta$, where m is a non-negative integer and Δ is an effective cycle whose support does not contain L. Then $m\leqslant n/2$.

We may assume that L is singular at P since otherwise we easily get a contradiction. Suppose that T is smooth at P. Then

$$n-2m=\Delta\cdot L>n-2m,$$

which is a contradiction. Thus the surface T is singular at P and

$$n-2m=\Delta\cdot S\geqslant\operatorname{mult}_P(D)\operatorname{mult}_P(T)-mS\cdot L>2n-2m,$$

where S is a generic surface through P in $|-2K_X|$, a contradiction.

Thus the curve L is reducible. We may assume that $L=\overline{C}_1\cup\overline{Z}_1$ and the surface T is quasi-smooth. Then $\overline{C}_1^2=-4/3$, $\overline{Z}_1^2=-2/3$ and $\overline{C}_1\cdot\overline{Z}_1=2$ on the surface T, but

$$D|_T=m_1\overline{C}_1+m_2\overline{Z}_1+\Upsilon\equiv-nK_X|_T,$$

where m_1 and m_2 are non-negative integers and Υ is an effective divisor whose support does not contain the curves $\overline{C}_1$ or $\overline{Z}_1$.

Suppose that $P \in \overline{Z}_1 \cap \overline{C}_1$. Then

$$\frac{1}{3}n+\frac{4}{3}m_1-2m_2 = \Upsilon\cdot\overline{C}_1 > n-m_1-m_2, \qquad \frac{2}{3}n-2m_1+\frac{2}{3}m_2 = \Upsilon\cdot\overline{Z}_1 > n-m_1-m_2.$$

It follows that $m_1 > n/2$ and $m_2 > n/2$, but $m_1/3 + 2m_2/3 \leqslant n/2$ by Remark 2.12, a contradiction.

We may assume that $\overline{C}_1 \ni P \notin \overline{Z}_1$ because the case $\overline{Z}_1 \ni P \notin \overline{C}_1$ is simpler. Then

$$\frac{1}{3}n + \frac{4}{3}m_1 - 2m_2 = \Upsilon \cdot \overline{C}_1 > n - m_1, \qquad \frac{2}{3}n - 2m_1 + \frac{2}{3}m_2 = \Upsilon \cdot \overline{Z}_1 \geqslant 0,$$

which gives $m_1 < n$ because $m_1/3 + 2m_2/3 \leqslant n/2$. It follows from the proof of Lemma 3.9 that $\mathrm{mult}_P(\Upsilon \cdot \overline{C}_1) > n$, which implies that $n/3 + 4m_1/3 - 2m_2 > n$. The resulting inequalities are incompatible.

Lemma 3.18. *We have* $\gimel \neq 8$.

Proof. Suppose that $\gimel = 8$. Then X is a hypersurface of degree 9 in $\mathbb{P}(1,1,1,3,4)$. Its singularities consist of one singular point O of type $\frac{1}{4}(1,1,3)$. There is a commutative diagram

$$\begin{tikzcd}
 & & U \arrow[dll, "\sigma"'] \arrow[d, "\alpha"] & W \arrow[l, "\beta"'] & \\
Y \arrow[dr, "\omega"'] & & X \arrow[dl, dashed, "\xi"'] \arrow[dr, dashed, "\psi"] & & Z \arrow[ul, "\gamma"'] \arrow[dl, "\eta"] \\
 & \mathbb{P}(1,1,1,3) \arrow[rr, dashed, "\chi"'] & & \mathbb{P}^2 &
\end{tikzcd}$$

where ξ, ψ and χ are projections, α is a weighted blow-up of O with weights $(1,1,3)$, β is a weighted blow-up with weights $(1,1,2)$ of the singular point of type $\frac{1}{3}(1,1,2)$, γ is a blow-up with weights $(1,1,1)$ of the singular point of type $\frac{1}{2}(1,1,1)$, η is an elliptic fibration, σ is a birational morphism that contracts smooth rational curves $\overline{C}_1, \dots, \overline{C}_{15}$, and ω is a double covering.

We put $C_i = \alpha(\overline{C}_i)$. Then $-K_X \cdot C_i = 1/4$.

Let L be the fibre of ψ that passes through P, and let S be a generic surface through P in the linear system $|-K_X|$. Then L is reduced and $L \subset S$.

Suppose that the curve L is irreducible. Arguing as in the proof of Lemma 3.5, we see that L must be singular at P. Hence some surface $T \in |-K_X|$ is singular at P. We write $D \cdot T = mL + \Delta$, where m is a non-negative integer and Δ is an effective cycle whose support does not contain L. Then

$$\frac{3}{4}n - \frac{3}{4}m = S \cdot \Delta \geqslant \mathrm{mult}_P(\Delta) > 2n - m\,\mathrm{mult}_P(L) = 2n - 2m.$$

It follows that $m > n$. But $m \leqslant n$ by Remark 2.12, a contradiction.

Thus the fibre L is reducible. Arguing as in the proof of Lemma 3.9, we see that $L = C_i + Z_i$, where Z_i is an irreducible curve and $-K_X \cdot Z_i = 1/2$. The hypersurface X can be given by the equation

$$w^2z + f_5(x, y, z, t)w + f_9(x, y, z, t) = 0 \subset \mathbb{P}(1, 1, 1, 3, 4) \cong \operatorname{Proj}(\mathbb{C}[x, y, z, t, w]),$$

where $\operatorname{wt}(x) = \operatorname{wt}(y) = \operatorname{wt}(z) = 1$, $\operatorname{wt}(t) = 3$, $\operatorname{wt}(w) = 4$, and f_i is a quasi-homogeneous polynomial of degree i. Let R be the surface cut out on X by the equation $z = 0$. Then

$$C_i^2 = -\frac{17}{20}, \qquad Z_i^2 = -\frac{3}{5}, \qquad C_i \cdot Z_i = \frac{11}{10}$$

on the surface R. We write $D|_R = m_C C_i + m_Z Z_i + \Upsilon$, where m_C and m_Z are non-negative integers and Υ is an effective cycle whose support does not contain C_i or Z_i. Then

$$\frac{1}{4}n + \frac{17}{20}m_C - \frac{11}{10}m_Z = \Upsilon \cdot C_i \geqslant 0, \qquad \frac{1}{2}n - \frac{11}{10}m_C + \frac{3}{5}m_Z = \Upsilon \cdot Z_i \geqslant 0.$$

It follows that $m_C \leqslant n$ and $m_Z \leqslant n$ because $m_C + 2m_Z \leqslant 3n$ by Remark 2.12.

Suppose that $P \in \overline{Z}_1 \cap \overline{C}_1$. Arguing as in the proofs of Lemmas 3.9 and 3.17, we get

$$\frac{1}{4}n + \frac{17}{20}m_C - \frac{11}{10}m_Z = \Upsilon \cdot C_i > n - m_Z,$$
$$\frac{1}{2}n - \frac{11}{10}m_C + \frac{3}{5}m_Z = \Upsilon \cdot Z_i > n - m_C,$$

which contradicts the inequality $m_C + 2m_Z \leqslant 3n$. Now the proofs of Lemmas 3.9 and 3.17 show that either

$$\frac{1}{4}n + \frac{17}{20}m_C - \frac{11}{10}m_Z = \Upsilon \cdot C_i > n, \qquad \frac{1}{2}n - \frac{11}{10}m_C + \frac{3}{5}m_Z = \Upsilon \cdot Z_i \geqslant 0,$$

or we have a system of linear inequalities

$$\frac{1}{4}n + \frac{17}{20}m_C - \frac{11}{10}m_Z \geqslant 0, \qquad \frac{1}{2}n - \frac{11}{10}m_C + \frac{3}{5}m_Z > n.$$

In both cases we easily derive a contradiction. The lemma is proved.

Lemma 3.19. *We have* $\gimel \neq 7$.

Proof. Suppose that $\gimel = 7$. Then X is a hypersurface of degree 8 in $\mathbb{P}(1, 1, 2, 2, 3)$. The singularities of X consist of a singular point Q of type $\frac{1}{3}(1, 1, 2)$ and 4 singular points of type $\frac{1}{2}(1, 1, 1)$.

Let $\psi\colon X \dashrightarrow \mathbb{P}(1, 1, 2)$ be the natural projection. Then there is a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & Y \arrow[l, "\beta"'] \arrow[dr, "\eta"] & \\
X \arrow[rr, dashed, "\psi"'] & & \mathbb{P}(1, 1, 2)
\end{tikzcd}$$

where α is a weighted blow-up of Q with weights $(1,1,2)$, β is a blow-up with weights $(1,1,1)$ of the singular point that dominates Q, and η is an elliptic fibration.

Let C be the fibre of ψ that passes through P. Arguing as in the proof of Lemma 3.1, we see that C is not the base curve of the pencil $|-K_X|$. It follows that $|-K_X|$ contains a unique surface S that passes through P, and the curve C is reduced and contains no singular points of type $\frac{1}{2}(1,1,1)$.

Suppose that C is irreducible. Then the singularities of C consist of finitely many double points outside Q. Arguing as in the proof of Lemma 3.5, we see that C must be singular at P. Thus either the surface S is singular at P, or there is an irreducible surface in $|-2K_X|$ which is singular at P. Arguing as in the proof of Lemma 3.18, we obtain a contradiction. Hence C is reducible.

Arguing as in the proof of Lemma 3.5, we see that $C=L+Z$, where L and Z are irreducible curves with $L\neq Z$ and either

$$-K_X\cdot L=-K_X\cdot Z=\frac{2}{3},$$

or $-K_X\cdot L=1/3$ and $-K_X\cdot Z=1$. The proof of Lemma 3.9 shows that L is one of the finitely many curves contracted by the projection $X\dashrightarrow\mathbb{P}(1,1,2,2)$. Then

$$L^2=-\frac{4}{3},\qquad Z^2=0,\qquad Z\cdot L=2$$

on the surface S. Arguing as in the proof of Lemma 3.18, we obtain a contradiction. The lemma is proved.

This completes the proof of Lemma 2.4.

§ 4. Non-superrigid threefolds

We use the notation and assumptions of Lemma 2.10. Then X is not birationally superrigid by Lemma 2.7. Let us show that $\daleth\notin\{27,30,41,68\}$.

Lemma 4.1. *We have* $\daleth\neq 30$ *and* $\daleth\neq 41$.

Proof. We may assume that $\daleth=41$ because the proof of the inequality $\daleth\neq 30$ is similar. Using the notation of § 8, we can assume that $O=O_1$ by Lemmas 2.4 and 2.6.

Let G be the surface on X cut out by the equation $w=0$, and let $\overline{G}$ be the proper transform of G on U. Suppose that $\mu>3n/10$. Then $D\neq G$ and

$$\begin{aligned}0\leqslant -K_U\cdot\overline{G}\cdot\overline{D}&=\left(\alpha^*(-K_X)-\frac{1}{5}E\right)\cdot(\alpha^*(-10K_X)-3E)\cdot(\alpha^*(-nK_X)-\mu)\\&=n-\frac{15\mu}{4}\,.\end{aligned}$$

It follows that $\mu\leqslant 4n/15<3n/10$.

Suppose that P is a smooth point of U. Then

$$1<\operatorname{mult}_P\left(\frac{1}{n}\overline{D}+\left(\frac{\mu}{n}-\frac{1}{5}\right)E\right)=\frac{\operatorname{mult}_P(\overline{D})}{n}+\frac{\mu}{n}-\frac{1}{5}\,.$$

It follows that $\mathrm{mult}_P(\overline{D}) > 6n/5 - \mu$. Let S be the unique surface through P in the linear system $|-K_U|$. Then $\overline{D} \neq S$.

Suppose that $P \notin \bigcup_{i=1}^{75} C_i$. Let H be a generic surface through P in $|-10K_U|$. Then H contains no components of the cycle $\overline{D} \cdot S$. It follows that

$$n - \frac{5}{2}\mu = \overline{D} \cdot S \cdot H > \frac{6}{5}n - \mu,$$

which is a contradiction. Thus there is a curve C_i such that $P \in C_i$.

The fibre of the rational map $\psi \circ \alpha$ over the point $\psi(P)$ consists of the curve C_i and another irreducible curve $\overline{C}_i$ such that $-K_U \cdot \overline{C}_i = 1/5$ and $E \cdot \overline{C}_i = 0$. We write

$$\overline{D} \cdot S = mC_i + \overline{m}\overline{C}_i + \Delta,$$

where m and $\overline{m}$ are non-negative integers and Δ is an effective cycle whose support does not contain C_i or $\overline{C}_i$. Let R be a generic surface through P in the linear system $|-4K_U|$. Then

$$\frac{2}{5}n - \mu + \frac{4}{5}\overline{m} = R \cdot \Delta > \frac{6}{5}n - \mu - m.$$

It follows that $m - 4\overline{m}/5 > 4n/5$. But $m + \overline{m} \leqslant n/2$ by Remark 2.12, a contradiction.

Thus the threefold U is singular at P. Let $\iota\colon \breve{U} \to U$ be a weighted blow-up of P with weights $(1,1,3)$. Then

$$\breve{D} \equiv \iota^*(\overline{D}) - \nu F,$$

where ν is a positive rational number, F is the exceptional divisor of the birational morphism ι, and $\breve{D}$ is the proper transform of D on $\breve{U}$.

Let $\breve{E}$ be the proper transform of E on $\breve{U}$. Then

$$\breve{D} + \left(\mu - \frac{1}{5}n\right)\breve{E} \equiv \iota^*\left(\overline{D} + \left(\mu - \frac{1}{5}n\right)E\right) - \left(\nu + \frac{3}{4}\mu - \frac{3}{20}n\right)F.$$

It follows that $\nu > 2n/5 - 3\mu/4$ because of [10].

Let $\breve{T}$ and $\breve{T}'$ be generic surfaces in $|-K_{\breve{U}}|$. Then $\breve{T} \cdot \breve{T}' = \breve{L}$, where $\breve{L}$ is an irreducible curve. We write $\breve{D} \cdot \breve{T} = \varepsilon \breve{L} + \Upsilon$, where ε is a non-negative integer and Υ is an effective cycle whose support does not contain $\breve{L}$. Then

$$0 \leqslant \breve{T}' \cdot \Upsilon = \breve{T}' \cdot (\breve{D} \cdot \breve{T} - \varepsilon \breve{L}) = \frac{1}{10}n - \frac{1}{4}\mu - \frac{1}{3}\nu + \frac{\varepsilon}{30}.$$

It follows that $\nu \leqslant 3n/10 - 3\mu/4 + \varepsilon/10$. But $\nu > 2n/5 - 3\mu/4$. Then

$$\frac{3}{10}n - \frac{3}{4}\mu + \frac{\varepsilon}{10} > \frac{2}{5}n - \frac{3}{4}\mu,$$

which is impossible because $\varepsilon \leqslant n$ by Remark 2.12. The lemma is proved.

Lemma 4.2. *We have* $\gimel \neq 27$ *and* $\gimel \neq 68$.

Proof. We may assume that $\gimel = 68$ because the proof of $\gimel \neq 27$ is similar. Then X is a hypersurface of degree 28 in $\mathbb{P}(1,3,4,7,14)$. It contains singular points O_1 and O_2 of type $\frac{1}{7}(1,3,4)$.

We may assume that $O = O_1$ (see Lemmas 2.4, 2.5). Then X can be given by the equation

$$t^2w + tf_{21}(x,y,z,w) + f_{28}(x,y,z,w) = 0 \subset \mathbb{P}(1,3,4,7,14) \cong \operatorname{Proj}(\mathbb{C}[x,y,z,t,w]),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = 3$, $\mathrm{wt}(z) = 4$, $\mathrm{wt}(t) = 7$, $\mathrm{wt}(w) = 14$ and f_i is a quasi-homogeneous polynomial of degree i. The point O_1 is given by $x = y = z = w = 0$. There is a commutative diagram

$$\begin{tikzcd}
 & & U \arrow[dll, "\sigma"'] \arrow[d, "\alpha"] & & Y \arrow[ll, "\gamma"'] \arrow[dd, "\eta"] \\
V \arrow[dr, "\omega"'] & & X \arrow[dl, dashed, "\xi"'] \arrow[drr, dashed, "\psi"] & & \\
 & \mathbb{P}(1,3,4,14) \arrow[rrr, dashed, "\chi"'] & & & \mathbb{P}(1,3,4)
\end{tikzcd}$$

where ξ, ψ and χ are projections, γ is a weighted blow-up with weights $(1,3,4)$ of the point that dominates O_2, the morphism η is an elliptic fibration, σ is a birational morphism that contracts 49 curves $C_1,\dots,C_{49}$, and ω is a double covering.

Exceptional divisors of the birational morphism $\alpha \circ \gamma$ are sections of η, and U contains 49 smooth irreducible curves $Z_1,\dots,Z_{49}$ such that $\alpha(Z_i)$ is a curve and $\alpha(Z_i) \cup \alpha(C_i)$ is the fibre of the projection ψ over the point $\psi(C_i)$. It follows that $-K_U \cdot Z_i = 1/7$.

Suppose that $\mu > 3n/14$. Let M be the surface on X cut out by the equation $w = 0$, and let $\overline{M}$ be the proper transform of M on U. Then

$$\overline{M} \equiv \alpha^*(-14K_X) - 3E.$$

It follows that $D \neq M$. The divisor $-K_U$ is numerically effective and big. Thus we have

$$0 \leqslant -K_U \cdot \overline{M} \cdot \overline{D} = \left(\alpha^*(-K_X) - \frac{1}{7}E\right) \cdot (\alpha^*(-14K_X) - 3E) \cdot (\alpha^*(-nK_X) - \mu)$$
$$= \frac{1}{42}n - \frac{1}{8}\mu.$$

It follows that $\mu \leqslant 4n/21 < 3n/14$. So the inequalities $3n/14 \geqslant \mu > n/7$ hold.

Suppose that the threefold U is smooth at P. Then

$$\mathrm{mult}_P(\overline{D}) > \frac{8}{7}n - \mu$$

and there is a surface $S \in |-3K_U|$ that contains P. Let T be the unique surface in the linear system $|-K_U|$.

Suppose that $P \in T$. Then $P \notin \bigcup_{i=1}^{49} C_i$. In particular, one can show that there is a surface $H \in |-84K_U|$ that contains P and does not contain components of the effective cycle $\overline{D} \cdot T$. Hence we have

$$2n - 7\mu = \overline{D} \cdot T \cdot H > \frac{8}{7}n - \mu.$$

It follows that $\mu < n/7$. But $\mu > n/7$ by [10], a contradiction.

This proves that P is not contained in T. Let L be the unique curve on the surface $E \cong \mathbb{P}(1,3,4)$ which is contained in the linear system $|\mathcal{O}_{\mathbb{P}(1,3,4)}(1)|$. Then $P \notin L = T \cdot E$. It follows that there is a unique smooth irreducible curve $C \subset E$ through P in the linear system $|\mathcal{O}_{\mathbb{P}(1,3,4)}(3)|$. We write

$$\overline{D}|_E = \varepsilon C + \Upsilon \equiv 7\mu L,$$

where ε is a non-negative integer and Υ is an effective cycle on E whose support does not contain C. Then

$$\frac{7\mu - 3\varepsilon}{4} = (7\mu - 3\varepsilon)L \cdot C = C \cdot \Upsilon \geqslant \operatorname{mult}_P(\Upsilon) > \frac{8}{7}n - \mu - \varepsilon.$$

It follows that $11\mu + \varepsilon > 32n/7$. But $\varepsilon \leqslant 7\mu/3$ because $\Upsilon \equiv (7\mu - 3\varepsilon)L$. We have

$$\frac{40}{3}\mu \geqslant 11\mu + \varepsilon > \frac{32}{7}n.$$

It follows that $\mu > 12n/35$. This is a contradiction because $\mu \leqslant 3n/14$.

Suppose that P is a singular point of type $\frac{1}{4}(1,1,3)$. Let $\iota\colon \breve{U} \to U$ be the weighted blow-up of the point P with weights $(1,1,3)$. Then

$$\breve{D} \equiv \iota^*(\overline{D}) - \nu F,$$

where ν is a positive rational number, F is the exceptional divisor of the birational morphism ι and $\breve{D}$ is the proper transform of D on $\breve{U}$.

Let $\breve{E}$ be the proper transform of E on $\breve{U}$. Then

$$\breve{D} + \left(\mu - \frac{1}{7}\right)\breve{E} \equiv \iota^*\left(\overline{D} + \left(\mu - \frac{1}{7}n\right)E\right) - \left(\nu + \frac{1}{4}\mu - \frac{1}{28}n\right)F.$$

It follows that $\nu > 2n/7 - \mu/4$ according to [10].

Let $\breve{T}$ be the proper transform of T on $\breve{U}$, and let $\breve{H}$ be a generic surface in the linear system $|-3K_{\breve{U}}|$. We write $\breve{D} \cdot \breve{T} = \varepsilon\breve{L} + \Phi$, where ε is a non-negative integer, $\breve{L}$ is the base curve of the pencil $|-3K_{\breve{U}}|$, and Φ is an effective cycle whose support does not contain the curve $\breve{L}$. Then

$$0 \leqslant \breve{H} \cdot \Phi = \breve{H} \cdot (\breve{D} \cdot \breve{T} - \varepsilon\breve{L}) = \frac{1}{14}n - \frac{1}{4}\mu - \nu + \frac{9}{14}\varepsilon,$$

but $\nu > 2n/7 - \mu/4$, which implies that $\varepsilon > n/3$. The last inequality is impossible since $\varepsilon \leqslant n/3$ by Remark 2.12.

Thus P is a singular point of type $\frac{1}{3}(1,1,2)$. Let $\upsilon\colon \grave{U}\to U$ be the weighted blow-up of P with weights $(1,1,2)$. Then

$$\grave{D}\equiv \upsilon^*(\overline{D})-\theta G,$$

where θ is a rational number, G is the exceptional divisor of the birational morphism υ, and $\grave{D}$ is the proper transform of the surface D on the threefold $\grave{U}$.

Let $\grave{E}$ be the proper transform of E on $\grave{U}$. Then

$$\grave{D}+\biggl(\mu-\frac{1}{7}n\biggr)\grave{E}\equiv \upsilon^*\biggl(\overline{D}+\biggl(\mu-\frac{1}{7}n\biggr)E\biggr)-\biggl(\theta+\frac{2}{3}\mu-\frac{2}{21}n\biggr)G.$$

It follows that $\theta>3n/7-2\mu/3$ according to [10].

Let S be a generic surface in $|-4K_U|$. Then $T\cdot S=L$, where L is an irreducible curve such that $\alpha(L)$ is the base curve of the pencil $|-4K_X|$. We write $\overline{D}\cdot T=\varepsilon L+\Psi$, where ε is a non-negative integer and Ψ is an effective cycle whose support does not contain L.

Let $\grave{T}$ and $\grave{S}$ be the proper transforms on $\grave{U}$ of the surfaces T and S respectively. Then

$$\grave{T}\equiv \upsilon^*(-K_U)-\frac{1}{3}G,\qquad \grave{S}\equiv \upsilon^*(-4K_U)-\frac{1}{3}G,$$

but $\grave{T}\cdot\grave{S}=\grave{L}$, where $\grave{L}$ is the proper transform of the curve L. Write $\grave{D}\cdot\grave{T}=\varepsilon\grave{L}+\Xi$ for some effective cycle Ξ whose support does not contain the curve $\grave{L}$. Then

$$0\leqslant \grave{S}\cdot\Xi=\grave{S}\cdot(\grave{D}\cdot\grave{T}-\varepsilon\grave{L})=\frac{2}{21}n-\frac{1}{3}\mu-\frac{1}{2}\theta-\frac{1}{42}\varepsilon$$

because $\grave{S}\cdot\grave{L}=1/42$. On the other hand, $\theta>3n/7-2\mu/3$. Therefore we have

$$0\leqslant\frac{1}{42}\varepsilon\leqslant\frac{2}{21}n-\frac{1}{3}\mu-\frac{1}{2}\theta<\frac{2}{21}n-\frac{1}{3}\mu-\frac{1}{2}\biggl(\frac{3}{7}n-\frac{2}{3}\mu\biggr)=-\frac{5}{42}n<0.$$

This is a contradiction. The lemma is proved.

We note that the approach used to prove Lemmas 4.1 and 4.2 may also be applied to prove Lemmas 2.10 and 2.11.

§ 5. Singular points

In this section we prove Lemma 2.10. We shall use the hypotheses and notation of that lemma. Suppose that P is a singular point of U. Let us derive a contradiction.

The point P is a singular point of type $\frac{1}{\bar{r}}(1,\bar{a},\bar{r}-\bar{a})$, where $\bar{a}$ and $\bar{r}$ are coprime positive integers with $\bar{r}>2\bar{a}$. Let $\beta\colon W\to U$ be the blow-up of P with weights $(1,\bar{a},\bar{r}-\bar{a})$. Then

$$-K_W^3=-K_X^3-\frac{1}{ra(r-a)}-\frac{1}{\bar{r}\bar{a}(\bar{r}-\bar{a})}=\frac{\sum_{i=1}^4 a_i}{a_1a_2a_3a_4}-K_X^3-\frac{1}{ra(r-a)}-\frac{1}{\bar{r}\bar{a}(\bar{r}-\bar{a})}\,.$$

Let $\breve{D}$ be the proper transform of D on W. There is a rational number ν such that

$$\breve{D}\equiv(\alpha\circ\beta)^*(-nK_X)-\mu\beta^*(E)-\nu G,$$

where G is the β-exceptional divisor. Then

$$K_W + \frac{1}{n}\breve{D} + \left(\frac{\mu}{n} - \frac{1}{r}\right)\breve{E} \equiv \beta^*\left(K_U + \frac{1}{n}\overline{D} + \left(\frac{\mu}{n} - \frac{1}{r}\right)E\right) - \varepsilon G \equiv -\varepsilon G,$$

where $\breve{E}$ is the proper transform of E on W and ε is a rational number. Then $\varepsilon > 0$ because of [10].

Lemma 5.1. *We have* $-K_W^3 \neq 0$.

Proof. Suppose that $-K_W^3 = 0$. It follows from [11] that the linear system $|-rK_W|$ is free and induces an elliptic fibration $\eta\colon W \to Y$ for $r \gg 0$. Then

$$0 \leqslant \breve{D} \cdot C = -\varepsilon G \cdot C < 0,$$

where C is a generic fibre of the elliptic fibration η. This contradiction proves the lemma.

Thus it follows from [11] that either $-K_W^3 < 0$ or the anticanonical divisor $-K_W$ is numerically effective and big.

Lemma 5.2. *Suppose that* $-K_W^3 < 0$. *Then* $-K_W$ *is not big.*

Proof. Suppose that $-K_W$ is big. Then it follows from [11] that we have the following alternative:

1) either $\gimel = 25$ and O is a singular point of type $\frac{1}{7}(1,3,4)$,
2) or $\gimel = 43$ and O is a singular point of type $\frac{1}{9}(1,4,5)$.

Suppose that $\gimel = 43$. Then the divisor $-K_W - 4\beta^*(K_U)$ is numerically effective (see [11]) and there is a surface H in the linear system $|-2K_X|$ such that

$$\breve{H} \equiv (\alpha \circ \beta)^*(-2K_X) - \frac{11}{9}\beta^*(E) - \frac{3}{2}G,$$

where $\breve{H}$ is the proper transform of H on W. Hence

$$0 \leqslant \breve{H} \cdot \breve{D} \cdot (-K_W - 4\beta^*(K_U)) = \frac{5}{9}n - \frac{11}{4}\mu - \nu.$$

This is a contradiction because $\nu - n/3 + 3\mu/4 = n\varepsilon > 0$ and $\mu > n/9$.

Thus we see that $\gimel = 25$. It follows from [11] that the divisor $-K_W - 3\gamma^*(K_U)$ is numerically effective and there is a surface $R \subset W$ such that

$$R \equiv (\alpha \circ \beta)^*(-K_X) - \frac{8}{7}\beta^*(E) - \frac{2}{3}G$$

and $\nu + 2\mu/3 - 3n/7 = n\varepsilon > 0$. Then

$$0 \leqslant R \cdot \breve{D} \cdot (-K_W - 3\beta^*(K_U)) = \frac{5}{7}n - \frac{8}{3}\mu - \nu.$$

This is a contradiction because $\mu > n/7$. The lemma is proved.

Let T be a surface in $|-K_X|$, and let $\mathcal{P}$ be the pencil generated by the divisors nT and D. Then

$$\mathcal{B} \equiv -nK_W \equiv (\alpha \circ \beta)^*(-nK_X) - \frac{n}{r}\beta^*(E) - \frac{n}{\bar{r}}G, \tag{5.1}$$

where $\mathcal{B}$ is the proper transform of the pencil $\mathcal{P}$ on the threefold W.

Lemma 5.3. *The divisor $-K_W$ is numerically effective and big.*

Proof. Suppose that $-K_W$ is not numerically effective and big. Then $-K_W^3 < 0$ and $-K_W$ is not big by Lemma 5.2. It follows from [12] that the equivalence (5.1) almost uniquely determines[4] the pencil $\mathcal{P}$.

Suppose that $\daleth \in \{45, 48, 58, 69, 74, 79\}$. Then O is of type $\frac{1}{a_4}(1, a_1, a_3)$ and X can be given by

$$w^2z + wf(x, y, z, t) + g(x, y, z, t) = 0 \subset \mathbb{P}(1, a_1, a_2, a_3, a_4) \cong \mathrm{Proj}(\mathbb{C}[x, y, z, t, w]),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(t) = a_3$, $\mathrm{wt}(w) = a_4$, and f, g are quasi-homogeneous polynomials. Let S be the surface on X cut out by the equation $z = 0$, and let $\mathcal{M}$ be the pencil generated by the divisors a_2T and S. It follows from [12] that $\mathcal{P} = \mathcal{M}$ or $\mathcal{P} = |-a_1K_X|$.

Suppose that $\mathcal{P} = |-a_1K_X|$. Then $\mu = n/a_1$, which is impossible because $\mu > n/a_4$.

Thus we see that $\mathcal{P} = \mathcal{M}$. Let M be a divisor in $\mathcal{M}$, and let $\overline{M}$ be the proper transform of M on U. If $M \neq S$, then the following numerical equivalence holds:

$$\overline{M} \equiv \alpha^*(M) - \frac{a_3}{a_4}E.$$

The inequality $\mu > n/a_4$ implies that $D = S$. On the other hand, since X is generic, we see from Lemma 8.12 and Proposition 8.14 of [9] that the log pair $(X, \frac{1}{a_2}S)$ has log canonical singularities at O, a contradiction.

Thus we see that $\daleth \notin \{45, 48, 58, 69, 74, 79\}$. Suppose that $\daleth \neq 76$. Then O is a singular point of type $\frac{1}{a_4}(1, a_1, a_3)$ and X can be given by

$$w^2z + wf(x, y, z, t) + g(x, y, z, t) = 0 \subset \mathbb{P}(1, a_1, a_2, a_3, a_4) \cong \mathrm{Proj}(\mathbb{C}[x, y, z, t, w]),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(t) = a_3$, $\mathrm{wt}(w) = a_4$ and f, g are quasi-homogeneous polynomials. Here O is given by the equations $x = y = z = t = 0$.

Let S be the surface on X cut out by the equation $z = 0$, and let $\mathcal{M}$ be the sheaf generated by the divisors a_2T and S. It follows from [12] and the numerical equivalence (5.1) that either $\mathcal{P} = |-a_1K_X|$ or $\mathcal{P} = \mathcal{M}$. But we have $n \neq a_1$ because $\mu > n/a_4$. Thus we see that $\mathcal{P} = \mathcal{M}$ and $n = a_2$.

Let M be any divisor in the pencil $\mathcal{M}$, and let $\overline{M}$ be the proper transform of M on U. If $M \neq S$, then

$$\overline{M} \equiv \alpha^*(M) - \frac{a_3}{a_4}E.$$

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[4] For example, by [12], (5.1) implies that $n = 1$ if $a_1 = 1$.

But $\mu > n/a_4$. We see that $D = S$, but the log pair $(X, \frac{1}{a_2}S)$ has log canonical singularities at O according to Lemma 8.12 and Proposition 8.14 of [9] since X is generic by hypothesis. This is a contradiction.

Thus we see that $\daleth = 76$. Arguing as in the previous case, we easily get a contradiction. The lemma is proved.

We have proved that $\daleth \in \{8, 12, 13, 16, 20, 24, 25, 26, 31, 33, 36, 38, 46, 47, 48, 54, 56, 58, 65, 74, 79\}$.

Lemma 5.4. *The case* $\daleth \notin \{12, 13, 20, 25, 31, 33, 38, 58\}$ *is impossible.*

Proof. Suppose that $\daleth \notin \{12, 13, 20, 25, 31, 33, 38, 58\}$. Then

$$r = a_4, \quad r - a = a_3, \quad \bar{r} = r - a, \quad \bar{a} = a, \quad n\varepsilon = \nu - \frac{1}{\bar{r}}(\bar{r} - \bar{a})\left(\frac{n}{r} - \mu\right) - \frac{n}{\bar{r}}.$$

Suppose that $\daleth \neq 24$. Then X can be given by the equation

$$w^2 z + w f(x, y, z, t) + g(x, y, z, t) = 0 \subset \mathbb{P}(1, a_1, a_2, a_3, a_4) \cong \mathrm{Proj}(\mathbb{C}[x, y, z, t, w]),$$

where $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = a$, $\mathrm{wt}(z) = d - 2a_4$, $\mathrm{wt}(t) = a_3$, $\mathrm{wt}(w) = a_4$, the point O is given by the equations $x = y = z = t = 0$, and f, g are quasi-homogeneous polynomials. Then

$$\breve{R} \equiv (\alpha \circ \beta)^*(-a_2 K_X) - \frac{d - r}{r}\beta^*(E) - \frac{\bar{r} - \bar{a}}{\bar{r}}G,$$

where $\breve{R}$ is the proper transform on W of the surface cut out by the equation $z = 0$ on X. Then $\breve{D} \neq \breve{R}$ and

$$\frac{n\sum_{i=1}^{4} a_i}{a_1 a_3 a_4} - \frac{\mu(d - r)}{a(r - a)} - \frac{\nu(\bar{r} - \bar{a})}{\bar{a}(\bar{r} - \bar{a})} = -K_W \cdot \breve{D} \cdot \breve{R} \geqslant 0.$$

It follows that $\mu < n/r$ because $\varepsilon > 0$, a contradiction.

Thus $\daleth = 24$. We use the notation in the proof of Lemma 3.14. Then

$$\breve{R} \equiv (\alpha \circ \beta)^*(-K_X) - \frac{8}{7}\beta^*(E) - \frac{3}{5}G,$$

where $\breve{R}$ is the proper transform of the surface R on the threefold W. We have

$$\frac{3}{14}n - \frac{8}{10}\mu - \frac{1}{2}\nu = -K_W \cdot \breve{D} \cdot \breve{R} \geqslant 0,$$

but $n\varepsilon = \nu + 3\mu/5 - 2n/7 > 0$ and $\mu < n/7$. This leads to a contradiction. The lemma is proved.

The divisor $-K_W$ is numerically effective and big and we have $\daleth \in \{12, 13, 20, 25, 31, 33, 38, 58\}$. Then

$$r = a_4, \qquad r - a = a_3, \qquad \bar{a} = a_1, \qquad \bar{r} - \bar{a} = a_2, \qquad a_2 \neq a_3,$$
$$n\varepsilon = \nu + \frac{r - 2a}{r - a}\mu - \frac{2}{r}n$$

according to [11], and W has a singular point $\overline{P} \neq P$ of type $\frac{1}{\bar{r}}(1, \bar{a}, \bar{r} - \bar{a})$ such that the diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & & W \arrow[ll, "\beta"'] & & V \arrow[ll, "\gamma"'] \arrow[d, "\eta"] \\
X \arrow[rrrr, dashed, "\psi"'] & & & & \mathbb{P}(1, a_1, a_2)
\end{tikzcd}$$

is commutative, where ψ is a projection, γ is a blow-up of $\overline{P}$ with weights $(1, \bar{a}, \bar{r} - \bar{a})$, and η is an elliptic fibration. Let F be the exceptional divisor of γ, and let $\overline{G}$ be the proper transform of the surface G on the threefold V. Then F and $\overline{G}$ are sections of η and $G \not\ni \overline{P} \notin \breve{E}$.

Lemma 5.5. *We have* $\varepsilon < 1$.

Proof. We may assume that $\gimel = 25$ since the proof is similar in the other cases. If $\gimel = 25$, then

$$0 \leqslant -K_W \cdot \breve{D} \cdot \breve{E} = \left(\beta^*(E) - \frac{1}{4}G\right) \cdot \left(-\frac{1}{7}\beta^*(E) - \frac{1}{4}G\right) \cdot (-\mu\beta^*(E) - \nu G) = \frac{7}{12}\mu - \frac{1}{3}\nu,$$

where O and P are singular points of types $\frac{1}{7}(1, 3, 4)$ and $\frac{1}{4}(1, 1, 3)$ respectively. Then

$$n\varepsilon = \nu + \frac{1}{4}\mu - \frac{2}{7}n \leqslant 2\mu - \frac{2}{7}n \leqslant \frac{1}{4}n$$

because the proof of Lemma 2.9 yields that $\mu \leqslant 15n/56$. The lemma is proved.

Thus the singularities of the log pair $\bigl(W, \frac{1}{n}\breve{D} + \bigl(\frac{\mu}{n} - \frac{1}{r}\bigr)\breve{E} + \varepsilon G\bigr)$ are not log canonical at some point $Q \in G$.

Lemma 5.6. *The threefold W is smooth at the point Q.*

Proof. Suppose that W is singular at Q. Then Q is a singular point of type $\frac{1}{\breve{r}}(1, 1, \breve{r} - 1)$, where either $\breve{r} = \bar{r} - \bar{a}$ or $\breve{r} = \bar{a} \neq 1$. Let $\omega \colon \breve{W} \to W$ be a weighted blow-up of Q with weights $(1, 1, \breve{r} - 1)$, and let $\mathcal{H}$ be the proper transform of the pencil $\mathcal{P}$ on the threefold $\breve{W}$. Then $\mathcal{H} \equiv -nK_{\breve{W}}$ by [10].

It follows from [12] and the equivalence $\mathcal{H} \equiv -nK_{\breve{W}}$ that $n = r\mu = a_1$. But we saw earlier that $\mu > n/r$, a contradiction. The lemma is proved.

The log pair $\bigl(W, \frac{1}{n}\breve{D} + \bigl(\frac{\mu}{n} - \frac{1}{r}\bigr)\breve{E} + \varepsilon G\bigr)$ is not canonical at Q. It follows that

$$\mathrm{mult}_Q(\breve{D}) > \begin{cases} n + n/r - \mu - n\varepsilon, & \text{if } Q \in \breve{E}, \\ n - n\varepsilon, & \text{if } Q \notin \breve{E}. \end{cases}$$

Lemma 5.7. *There is a surface $T \in |-K_W|$ such that $Q \in T$.*

Proof. If $a_1 = 1$, then the existence of a surface $T \in |-K_W|$ passing through Q is obvious. Hence we may assume that $a_1 \neq 1$. Then $\gimel \in \{33, 38, 58\}$.

Suppose that $\daleth = 38$. Then there is a unique surface $T \in |-K_W|$. Suppose that Q is not contained in T. Arguing as in the proof of Lemma 2.14, we see that $\mathrm{mult}_Q(\breve{D}) \leqslant (a_1+a_2)\nu/a_1$. Then

$$\nu\frac{a_1+a_2}{a_1} > n - \left(\mu - \frac{1}{7}n\right) - \left(\nu + \frac{3}{5}\mu - \frac{2}{7}n\right),$$

but $\mathrm{mult}_Q(\breve{D}) > n + n/r - \mu - n\varepsilon$. Hence $\mu > 55n/56 - 5\nu/2$. But the inequality $-K_W \cdot \breve{D} \geqslant 0$ and the proof of Lemma 2.9 yield that $\nu \leqslant 10\mu/7$ and $\mu \leqslant 9n/40$ respectively.

The hypersurface X can be given by the equation

$$w^2y + w\big(t^2 + tf_5(x,y,z) + f_{10}(x,y,z)\big) + tf_{13}(x,y,z) + f_{18}(x,y,z) = 0 \\ \subset \mathrm{Proj}(\mathbb{C}[x,y,z,t,w]),$$

where $\mathrm{wt}(x)=1$, $\mathrm{wt}(y)=2$, $\mathrm{wt}(z)=3$, $\mathrm{wt}(t)=5$, $\mathrm{wt}(w)=8$ and $f_i(x,y,z)$ is a quasi-homogeneous polynomial of degree i. Let $\breve{S}$ be the proper transform on W of the surface cut out on X by the equation $wy + \big(t^2 + tf_5(x,y,z) + f_{10}(x,y,z)\big) = 0$. Then

$$\breve{S} \equiv (\alpha\circ\beta)^*(-10K_X) - \frac{18}{8}\beta^*(E) - \frac{13}{5}G,$$

but $\breve{S} \neq \breve{D}$. Since the divisor $-K_W$ is numerically effective, we see that

$$0 \leqslant -K_W \cdot \breve{D} \cdot \breve{S} = \frac{3}{4}n - \frac{6}{5}\mu - \frac{13}{6}\nu,$$

but $\nu \leqslant 8\mu/5$. It follows that $\nu \leqslant 9n/35$. We now easily obtain a contradiction. This proves that $\daleth \neq 38$.

Suppose that $\daleth = 33$. Then X is a hypersurface of degree 17 in $\mathbb{P}(1,2,3,5,7)$, O is a singular point of type $\frac{1}{7}(1,2,5)$, and P is a singular point of type $\frac{1}{5}(1,2,3)$.

The proofs of Lemmas 5.5, 2.9 yield that $\nu \leqslant 7\mu/5$ and $\mu \leqslant 17/70$. Arguing as in the proof of Lemma 2.14, we see that $\mathrm{mult}_Q(\breve{D}) \leqslant 5\nu/2$. It follows that

$$\frac{5}{2}\nu > n - \left(\mu - \frac{1}{7}n\right) - \left(\nu + \frac{3}{5}\mu - \frac{2}{7}n\right),$$

whence $7\nu/2 + 8\mu/5 > 10n/7$. The threefold X can be given by the equation

$$w^2z + w\big(t^2 + tf_5(x,y,z) + f_{10}(x,y,z)\big) + tf_{12}(x,y,z) + f_{17}(x,y,z) = 0 \\ \subset \mathrm{Proj}(\mathbb{C}[x,y,z,t,w]),$$

where $\mathrm{wt}(x)=1$, $\mathrm{wt}(y)=2$, $\mathrm{wt}(z)=3$, $\mathrm{wt}(t)=5$, $\mathrm{wt}(w)=7$, and f_i is a quasi-homogeneous polynomial of degree i. Let S be the surface cut out on X by the equation

$$wz + \big(t^2 + tf_5(x,y,z) + f_{10}(x,y,z)\big) = 0,$$

and let $\breve{S}$ be the proper transform of S on W. Then

$$\breve{S} \equiv (\alpha\circ\beta)^*(-10K_X) - \frac{17}{8}\beta^*(E) - \frac{12}{5}G,$$

but the singularities of the log pair $(X, \frac{1}{10}S)$ are log canonical by Lemma 8.12 and Proposition 8.14 of [9]. Thus we see that $S \neq D$.

The divisor $-K_W$ is numerically effective. Hence

$$0 \leqslant -K_W \cdot \breve{D} \cdot \breve{S} = \frac{17}{21}n - \frac{17}{10}\mu - 2\nu,$$

but $\nu \leqslant 7\mu/5$. Then $\nu \leqslant 34n/135$ contrary to the inequalities $\mu \leqslant 17n/70$ and $7\nu/2 + 8\mu/5 > 10n/7$.

Thus we have $\gimel = 58$. Then X is a hypersurface of degree 24 in $\mathbb{P}(1,3,4,7,10)$, O is a singular point of type $\frac{1}{10}(1,3,7)$, and P is a singular point of type $\frac{1}{7}(1,3,4)$.

The proofs of Lemmas 5.5 and 2.9 yield that $\nu \leqslant 10\mu/7$ and $\mu \leqslant 6/35$. Arguing as in the proof of Lemma 2.14, we see that

$$\frac{7}{3}\nu \geqslant \operatorname{mult}_Q(\breve{D}) > n - \biggl(\mu - \frac{1}{10}n\biggr) - \biggl(\nu + \frac{4}{7}\mu - \frac{1}{5}n\biggr)$$

since $n\varepsilon = \nu + 4\mu/7 - n/5$. It follows that $10\nu/3 + 11\mu/7 > 13n/10$. Then

$$\frac{39}{190}n > \frac{6}{35}n \geqslant \mu > \frac{39}{190}n$$

because $\nu \leqslant 10\mu/7$. The resulting contradiction completes the proof of the lemma.

It follows from [11] that $|-rK_W|$ has no base points for $r \gg 0$ and induces a birational morphism $\omega \colon W \to \overline{W}$ such that $\overline{W}$ is a hypersurface of degree $6a_3$ with only canonical singularities in $\mathbb{P}(1, a_1, a_2, 2a_3, 3a_3)$.

Lemma 5.8. *The morphism ω is not an isomorphism in a neighbourhood of Q.*

Proof. Suppose that ω is an isomorphism in a neighbourhood of Q. Then it follows from the proof of Theorem 5.6.2 in [5] that there is a divisor $R \in |-2sa_1a_3K_W|$ such that $\operatorname{mult}_Q(R) \geqslant s$ for some positive integer s, but the set $\operatorname{Supp}(R)$ contains no components through Q of the cycle $\breve{D} \cdot S$. Then

$$\begin{aligned} 2sa_1a_3\biggl(\frac{n\sum_{i=1}^{4} a_i}{a_1a_2a_3a_4} - \frac{\mu}{a_1a_3} - \frac{\nu}{a_1a_2}\biggr) &= R \cdot \breve{D} \cdot T \\ &\geqslant \operatorname{mult}_Q(\breve{D} \cdot T)s > \biggl(n - \nu - \mu\frac{a_3 - a_1}{a_3} + \frac{2n}{a_4}\biggr)s \end{aligned}$$

because $Q \notin \breve{E}$. For all possible values of $\gimel$ we easily see that this inequality cannot hold because $n\varepsilon = \nu + (a_3 - a_1)\mu/a_3 - 2n/a_4 > 0$. The lemma is proved.

Thus there is a unique curve $C \subset W$ that contains Q and satisfies

$$-K_W \cdot C = 0, \qquad \beta^*(-K_U) \cdot C = \frac{1}{a_4}, \qquad C \cdot G = 1.$$

It follows that $\gimel \notin \{33, 38, 58\}$ by Lemma 5.7. Hence we have $\gimel \in \{12, 13, 20, 25, 31\}$.

We write $\breve{D}\cdot T = mC+\Omega$, where m is a non-negative integer and Ω is an effective 1-cycle whose support does not contain C. Then it follows from Remark 2.12 that

$$m \leqslant \frac{5}{4}n-\mu, \qquad m \leqslant \frac{11}{15}n-\frac{1}{2}\mu, \qquad m \leqslant \frac{13}{15}n-\mu,$$
$$m \leqslant \frac{5}{7}n-\frac{1}{3}\mu, \qquad m \leqslant \frac{2}{3}n-\mu$$

in the cases when $\gimel = 12, 13, 20, 25, 31$ respectively. We recall that $\overline{G}$ is a section of the fibration η.

Let $\mathcal{H}$ be the pencil consisting of all surfaces through Q in the linear system $|-a_2K_W|$, and let H be a generic surface in $\mathcal{H}$. Then C is the only curve in the base locus of $\mathcal{H}$ that passes through Q. Hence,

$$a_2\left(\frac{n\sum_{i=1}^{4}a_i}{a_1a_2a_3a_4}-\frac{\mu}{a_1a_3}-\frac{\nu}{a_1a_2}\right) = H\cdot\Omega \geqslant \mathrm{mult}_Q(\Omega) > n-\nu-\mu\frac{a_3-a_1}{a_3}+\frac{2n}{a_4}-m.$$

It follows that either $\gimel = 12$ or $\gimel = 13$.

Lemma 5.9. *We have* $\gimel \neq 12$.

Proof. Suppose that $\gimel = 12$. Let R be a generic surface through Q in the linear system $|-2K_W|$. Then

$$R|_T = C+L+Z,$$

where $L = G|_T$, Z is a reduced curve and $P \notin \beta(Z)$.

Suppose that Z is irreducible. Then we have

$$Z^2 = -\frac{4}{3}, \qquad C^2 = -2, \qquad L^2 = -\frac{3}{2}$$

on the surface T. As usual, we write

$$\breve{D}|_T = m_CC+m_LL+m_ZZ+\Upsilon,$$

where m_C, m_L and m_Z are non-negative integers and Υ is an effective cycle whose support does not contain the curves C, L or Z.

Suppose that $Q \notin \breve{E}$. Then $m_C > 2n/3 - m_Z/3$ because

$$\frac{5}{6}n-\frac{2}{3}\mu-\nu = R\cdot\breve{D}\cdot T = m_L+\frac{1}{3}m_Z+R\cdot\Upsilon > m_L+\frac{1}{3}m_Z+\frac{3}{2}n-\nu-\frac{2}{3}\mu-m_L-m_C,$$

but $4m_Z/3 \geqslant 2m_C - n/3$ because $\Upsilon\cdot Z \geqslant 0$. Therefore we have

$$m_C > \frac{2}{3}n+\frac{1}{3}m_Z \geqslant \frac{7}{12}n+\frac{1}{2}m_C,$$

whence $m_C > 7n/6$. But $m_C \leqslant 5n/6$ by Remark 2.12 since $-K_X\cdot\alpha\circ\beta(C) = 5/6$.

This proves that $Q \in \breve{E}$. Then $C \subset \breve{E}$ and $\beta(C) \in |\mathcal{O}_{\mathbb{P}(1,1,3)}(1)|$. But

$$\frac{5}{6}n-\frac{2}{3}\mu-\nu = R\cdot\breve{D}\cdot T = m_L+\frac{1}{3}m_Z+R\cdot\Upsilon > m_L+\frac{1}{3}m_Z+\frac{7}{4}n-\nu-\frac{5}{3}\mu-m_L-m_C.$$

It follows that $m_C > 11n/12 - \mu + m_Z/3$. We have $-K_X \cdot \alpha \circ \beta(Z) = 5/6$ and $Z \cdot \breve{E} = 2$. But

$$\frac{4}{3}m_Z \geqslant 2m_C + 2\mu - \frac{5}{6}n$$

because $Z \cdot \Upsilon \geqslant 0$. Then $m_Z > 3n/2$. But $m_Z \leqslant n/2$ by Remark 2.12, a contradiction.

Therefore the curve Z is reducible. Then $Q \in \breve{E}$ and $Z = \acute{Z} + \grave{Z}$, where $\acute{Z}$ and $\grave{Z}$ are irreducible curves such that

$$G \cdot \acute{Z} = G \cdot \grave{Z} = -K_U \cdot \beta(\grave{Z}) = 0$$

and $-K_X \cdot \alpha \circ \beta(\acute{Z}) = 7/12$. It is easy to calculate that

$$\acute{Z}^2 = -\frac{4}{3}, \qquad \grave{Z}^2 = C^2 = -2, \qquad L^2 = -\frac{3}{2},$$
$$L \cdot C = \acute{Z} \cdot C = \acute{Z} \cdot \grave{Z} = \grave{Z} \cdot C = 1, \qquad L \cdot \grave{Z} = L \cdot \acute{Z} = 0$$

on the surface T. As in the previous case, we write

$$\breve{D}|_T = \overline{m}_C C + \overline{m}_L L + \overline{m}_Z \acute{Z} + \Phi,$$

where $\overline{m}_C$, $\overline{m}_L$, $\overline{m}_Z$ are non-negative integers and Φ is an effective cycle whose support does not contain C, L or $\acute{Z}$. Then

$$R|_T \cdot \Phi \geqslant \operatorname{mult}_Q(\Phi) > \frac{7}{4}n - \nu - \frac{5}{3}\mu - m_L - m_C$$

and $\Phi \cdot \acute{Z} \geqslant 0$. Clearly, $\beta^*(-K_U)|_T \cdot \Phi \geqslant 0$. Hence we see that

$$\overline{m}_C > \frac{11}{12}n - \mu + \frac{1}{3}\overline{m}_Z, \qquad \frac{4}{3}\overline{m}_Z \geqslant \overline{m}_C + \mu - \frac{5}{6}n, \qquad \overline{m}_C + \mu \leqslant \frac{5}{4} - \overline{m}_Z.$$

But these linear inequalities are incompatible. The resulting contradiction completes the proof of the lemma.

Lemma 5.10. *We have* $\gimel \neq 13$.

Proof. Suppose that $\gimel = 13$. Then $C \subset \breve{E}$ because otherwise

$$2\biggl(\frac{11}{30}n - \frac{1}{6}\mu - \frac{1}{2}\nu\biggr) = H \cdot \Omega \geqslant \operatorname{mult}_Q(\Omega) > \frac{7}{5}n - \nu - \frac{1}{3}\mu - m$$

and, therefore, $m > 2n/3$ contrary to the inequalities $m \leqslant 11n/15 - \mu/2$ and $\mu > n/5$. We write

$$\overline{D}|_{\breve{E}} = \overline{m}C + \Upsilon,$$

where $\overline{m}$ is a non-negative integer and Υ is an effective cycle whose support does not contain C. Then $\overline{m} \leqslant 5\mu/2$ because $\beta(C) \in |\mathcal{O}_{\mathbb{P}(1,2,3)}(2)|$ and the curve C is reduced, where $E \cong \mathbb{P}(1,2,3)$. Hence we have $\overline{m} \leqslant 11n/12$ because $\mu \leqslant 11n/30$.

Clearly, the log pair $(W, \frac{1}{n}\breve{D} + \breve{E} + \varepsilon G)$ is not log canonical at Q. Hence the log pair

$$\biggl(\breve{E}, C + \frac{\nu + \mu/3 - 2n/5}{n}G\bigg|_{\breve{E}} + \frac{1}{n}\Upsilon\biggr)$$

is not log canonical at Q by Theorem 7.5 of [9]. Then

$$\frac{5}{3}\mu-\nu=(\overline{m}C+\Upsilon)\cdot C=\Upsilon\cdot C\geqslant \operatorname{mult}_Q(\Upsilon|_C)>\frac{7}{5}n-\nu-\frac{1}{3}\mu$$

by Theorem 7.5 of [9]. It follows that $\mu>7n/10$, but $\mu\leqslant 11n/30$, a contradiction. The lemma is proved.

This completes the proof of Lemma 2.10.

§ 6. Quadratic involutions

In this section we prove Lemma 2.11. We shall use the hypotheses and notation of that lemma. Suppose that $d=2r+a_j$. To prove Lemma 2.11, we must derive a contradiction.

Lemma 6.1. *We have* $\gimel\neq 9$ *and* $\gimel\neq 17$.

Proof. We may assume that $\gimel=9$ since the proof of $\gimel\neq 17$ is similar. We use the notation from the proof of Lemma 3.17 and identify the point O with O_1.

Suppose that $\mu>2n/3$. Let G be the surface cut out by the equation $w=0$ on X, and let $\overline{G}$ be the proper transform of G on U. Then

$$0\leqslant -K_U\cdot\overline{G}\cdot\overline{D}=\biggl(\alpha^*(-K_X)-\frac{1}{3}E\biggr)\cdot(\alpha^*(-3K_X)-2E)\cdot(\alpha^*(-nK_X)-\mu)=\frac{3}{2}n-3\mu.$$

It follows that $\mu\leqslant n/2<2n/3$, a contradiction. Thus $\mu\leqslant 2n/3$.

Suppose that $P\in C_i$. Let S be a surface through P in the linear system $|-K_U|$. We write

$$\overline{D}\cdot S=mC_i+\overline{m}Z_i+\Delta,$$

where m and $\overline{m}$ are non-negative integers and Δ is an effective cycle whose support does not contain the curves C_i or Z_i. Let R be a generic surface through P in the linear system $|-2K_U|$. Then we have

$$n-\mu-\frac{2}{3}\overline{m}=R\cdot\Delta>\frac{4}{3}n-\mu-m,$$

whence $m-4\overline{m}/3>n/3$. Put $H=\alpha(S)$. Then

$$\overline{C}_i^{\,2}=-\frac{4}{3}\,,\qquad \overline{Z}_i^{\,2}=-\frac{2}{3}\,,\qquad \overline{C}_i\cdot\overline{Z}_i=2$$

on the surface H. On the other hand, if we write

$$D|_H=m\overline{C}_i+\overline{m}\overline{Z}_i+\Omega\equiv -nK_X|_H,$$

where Ω is an effective divisor whose support does not contain the curves $\overline{C}_i$ or $\overline{Z}_i$, then

$$0\leqslant\Omega\cdot\overline{Z}=\frac{2}{3}n-m\overline{C}_i\cdot\overline{Z}_i-\overline{m}\overline{Z}_i^{\,2}=\frac{2}{3}n-2m+\frac{2}{3}\overline{m}.$$

This contradicts the inequality $m-4\overline{m}/3>n/3$. We see that $P\notin\bigcup_{i=1}^{27}C_i$.

Let L be the fibre of the rational map $\psi\circ\alpha$ over the point $\psi\circ\alpha(P)$, and let S be a surface through P in the linear system $|-K_U|$. We write

$$\overline{D}\cdot S=\grave{m}L+\Upsilon,$$

where $\grave{m}$ is a non-negative integer and Υ is an effective cycle whose support does not contain the curve L. Let R be a generic surface through P in $|-2K_U|$. Then

$$n-\mu-\frac{2}{3}\grave{m}=R\cdot\Delta>\frac{4}{3}n-\mu-\grave{m}$$

because the curve L is smooth at P. Hence $\grave{m}>n$. But Remark 2.12 yields that $\grave{m}\leqslant n/2$, a contradiction. The lemma is proved.

Lemma 6.2. *We have* $\daleth\neq 6$ *and* $\daleth\neq 15$.

Proof. We may assume that $\daleth=15$ since the proof of $\daleth\neq 6$ is similar. Then O is a singular point of type $\frac{1}{3}(1,1,2)$ and X is a hypersurface of degree 12 in $\mathbb{P}(1,1,2,3,6)$. We have a commutative diagram

$$\begin{tikzcd}
 & & U \arrow[dll, "\sigma"'] \arrow[d, "\alpha"] & Y \arrow[l, "\gamma"'] \arrow[ddr, "\eta"] & \\
V \arrow[dr, "\omega"'] & & X \arrow[dl, dashed, "\xi"'] \arrow[drr, dashed, "\psi"] & & \\
 & \mathbb{P}(1,1,2,6) \arrow[rrr, dashed, "\chi"'] & & & \mathbb{P}(1,1,2)
\end{tikzcd}$$

where ξ, ψ and χ are projections, γ is a weighted blow-up with weights $(1,1,2)$ of the singular point of type $\frac{1}{3}(1,1,2)$, σ is a birational morphism that contracts rational curves $C_1,\ldots,C_{54}$, η is an elliptic fibration, and ω is a double covering.

Let S be the unique surface through the non-singular point P in the linear system $|-K_U|$. Then $\mathrm{mult}_P(\overline{D}\cdot S)>\frac{4}{3}n-\mu$.

Suppose that $P\notin\bigcup_{i=1}^{54}C_i$. Let H be a generic surface through P in the linear system $|-6K_U|$. Then H contains no components of the effective cycle $\overline{D}\cdot S$. Hence,

$$2n-3\mu=\overline{D}\cdot S\cdot H>\frac{4}{3}n-\mu.$$

It follows that $\mu<n/3$. Hence there is a curve C_i such that $P\in C_i$.

The fibre of the rational map $\psi\circ\alpha$ over the point $\psi(P)$ consists of the curve C_i and an irreducible curve $\overline{C}_i$ such that $-K_U\cdot\overline{C}_i=1/3$ and $E\cdot\overline{C}_i=0$. We write

$$\overline{D}\cdot S=mC_i+\overline{m}\overline{C}_i+\Delta,$$

where m and $\overline{m}$ are non-negative integers and Δ is an effective cycle whose support does not contain the curves C_i or $\overline{C}_i$. Let R be a generic surface through P in the linear system $|-2K_U|$. Then

$$\frac{2}{3}n-\mu-\frac{2}{3}\overline{m}=R\cdot\Delta>\frac{4}{3}n-\mu-m.$$

It follows that $m-2\overline{m}/3>2n/3$.

We put $\breve{S} = \alpha(S)$, $Z = \alpha(C_i)$ and $\overline{Z} = \alpha(\overline{Z}_i)$. Then $\breve{S}$ is a generic surface of degree 12 in $\mathbb{P}(1,3,4,8)$, and the curves Z and $\overline{Z}$ are contained in $\breve{S}$. We have

$$Z^2 = \overline{Z}^2 = -\frac{4}{3}, \qquad Z \cdot \overline{Z} = 2$$

on the surface $\breve{S}$. Write $D|_{\breve{S}} = mZ + \overline{m}\overline{Z} + \Omega$, where Ω is an effective divisor on $\breve{S}$ whose support does not contain Z or $\overline{Z}$. Then

$$0 \leqslant \Omega \cdot \overline{Z} = \frac{1}{3}n - mZ \cdot \overline{Z} - \overline{m}\overline{Z}^2 = \frac{1}{3}n - 2m + \frac{4}{3}\overline{m},$$

but $m > 2n/3 + 2\overline{m}/3$. The resulting inequalities are incompatible. The lemma is proved.

It follows from the equation $d = 2r + a_j$ that the threefold X can be given by

$$x_i^2 x_j + x_i f(x_0, x_1, x_2, x_3, x_4) + g(x_0, x_1, x_2, x_3, x_4) = 0 \subset \operatorname{Proj}(\mathbb{C}[x_0, x_1, x_2, x_3, x_4]),$$

where $i \neq j$, $a_i = r$, $\mathrm{wt}(x_0) = 1$, $\mathrm{wt}(x_k) = a_k$, and f, g are general quasi-homogeneous polynomials that do not depend on x_i. We put $\bar{a}_3 = a_{3+4-i}$, $\bar{a}_4 = a_i a_j$, $\bar{d} = 2\bar{a}_4$. Then there is a commutative diagram

$$\begin{tikzcd}
U \arrow[rr, "\alpha"] \arrow[d, "\sigma"'] & & X \arrow[d, dashed, "\xi"] \\
V \arrow[r, hook] & \mathbb{P}(1, a_1, a_2, \bar{a}_3, \bar{a}_4) \arrow[r, dashed, "\chi"'] & \mathbb{P}(1, a_1, a_2, \bar{a}_3)
\end{tikzcd}$$

where ξ and χ are projections and σ is a birational morphism that contracts rational curves $C_1, \ldots, C_l$, where $l = a_i a_j (d - a_i) \sum_{i=1}^{4} a_i$ and V is a hypersurface of degree $\bar{d}$ in $\mathbb{P}(1, a_1, a_2, \bar{a}_3, \bar{a}_4)$ with terminal singularities. Then $-K_X \cdot \alpha(C_k) = 1/a_i$.

Let M be the surface cut out on X by the equation $x_j = 0$, and let $\overline{M}$ be the proper transform of M on the threefold U. Since X is generic, it follows from Lemma 8.12 and Proposition 8.14 of [9] that $M \neq D$.

Lemma 6.3. *We have* $\mu \leqslant -a_j n K_X^3 (r-a)a/(d-r) \leqslant n(d-r)/(r a_j)$.

Proof. The inequality $\mu \leqslant -a_j n K_X^3 (r-a)a/(d-r)$ is trivial: we have

$$0 \leqslant -K_U \cdot \overline{M} \cdot \overline{D} = -a_j n K_X^3 - \frac{\mu(d-r)}{a(r-a)}$$

since the divisor $-K_U$ is numerically effective. It remains to show that $-a_j n K_X^3 \times (r-a)a/(d-r) \leqslant n(d-r)/(r a_j)$.

Suppose that $-a_j n K_X^3 (r-a)a/(d-r) > n(d-r)/(r a_j)$. Then

$$\frac{d-r}{r a_j} < -a_j K_X^3 \frac{(r-a)a}{d-r} = \frac{d a_j (r-a)a}{(d-r) a_1 a_2 a_3 a_4},$$

but $a_1 a_2 a_3 a_4 \geqslant a_j r (r-a) a$. Thus we have $(d-r)^2 < d(d-2r)$, a contradiction. The lemma is proved.

We note that $E \cong \mathbb{P}(1, a, r-a)$ and the linear system $|\mathcal{O}_{\mathbb{P}(1,a,r-a)}(1)|$ consists of a single curve when $a \neq 1$. Taking into account the possible values of (a_1, a_2, a_3, a_4), we see that

$$a = 1 \Rightarrow \gimel \in \{7, 8, 12, 13, 16, 20, 25, 26, 30, 36, 31, 41, 47, 54\}$$

by Lemmas 6.1, 6.2.

Lemma 6.4. *We have* $\gimel \neq 7$.

Proof. Suppose that $\gimel = 7$. Then X is a hypersurface of degree 8 in $\mathbb{P}(1, 1, 2, 2, 3)$ and O is a singular point of type $\frac{1}{3}(1, 1, 2)$. Let S be the unique surface through P in the linear system $|-K_U|$. Then S is smooth at P.

The singularities of U consist of singular points P_0, P_1, P_2, P_3 and P_4 of type $\frac{1}{2}(1, 1, 1)$ such that P_0 is a singular point of E. There is a commutative diagram

$$\begin{tikzcd}
U \arrow[d, "\alpha"'] & & Y_i \arrow[ll, "\beta_i"'] \arrow[dr, "\eta_i"] & \\
X \arrow[rrr, dashed, "\xi_i"'] & & & \mathbb{P}(1,1,2)
\end{tikzcd}$$

where ξ_i is a projection, β_i is a blow-up of P_i with weights $(1, 1, 1)$, and η_i is an elliptic fibration.

Suppose that $P \notin \bigcup_{i=1}^{l} C_i$. We easily see that the proper transform of the surface E on the variety Y_i is a section of the elliptic fibration η_i if $i \neq 0$. Hence there is a surface $H \in |-2K_U|$ such that

$$2\left(\frac{2}{3}n - \frac{1}{2}\mu\right) = \overline{D} \cdot H \cdot S \geqslant \operatorname{mult}_P(\overline{D}) > \frac{4}{3}n - \mu.$$

This is a contradiction. Thus, we may assume that $P \in C_1$.

Clearly, $-K_X \cdot \alpha(C_1) = 1/3$. Then

$$\overline{M}|_S = C_1 + Z_1 \equiv (-2K_U - E)|_S,$$

where Z_1 is an irreducible curve and $-K_X \cdot \alpha(Z_1) = 1$. We put $L = E|_S$. Then

$$C_1^2 = -2, \qquad Z_1^2 = L^2 = -\frac{3}{2}, \qquad C_1 \cdot Z_1 = L \cdot C_1 = 1, \qquad L \cdot Z_1 = \frac{3}{2}$$

on the surface S. Write $\overline{D}|_S = m_C C_1 + m_Z Z_1 + m_L L + \Omega$, where m_C, m_Z and m_L are non-negative integers and Ω is an effective cycle on S whose support does not contain the curves C_1, Z_1 or L. Then

$$n - \frac{3}{2}\mu + \frac{3}{2}m_Z - \frac{3}{2}m_L - m_C = Z_1 \cdot \Omega \geqslant 0.$$

It follows that $3m_Z/2 \geqslant 3(\mu + m_L)/2 + m_C - n$. We similarly see that

$$\frac{3}{2}\mu - \frac{3}{2}m_Z + \frac{3}{2}m_L - m_C = L \cdot \Omega \geqslant 0.$$

It follows that $3(\mu+m_L)/2 \geqslant 3m_Z/2+m_C$. We also have

$$\frac{4}{3}n-\mu-m_L-m_Z=(L+C_1+Z_1)\cdot\Omega\geqslant \mathrm{mult}_P(\Omega)>\frac{4}{3}n-\mu-m_L-m_C,$$

whence $m_C>m_Z$ and $4n/3\geqslant \mu+m_L+m_Z$. This proves that $m_Z\leqslant n/2$ and $m_C\leqslant n/2$.

By Theorem 7.5 of [9], the log pair

$$\left(S, C_1+\frac{\overline{m}_L+\mu-n/3}{n}L+\frac{m_Z}{n}Z+\frac{1}{n}\Omega\right)$$

is not log canonical at P because $m_C\leqslant n$. It follows that

$$C_1\cdot\Omega\geqslant \mathrm{mult}_P(\Omega|_{C_1})>n-m_L-\mu+\frac{1}{3}n$$

by Theorem 7.5 of [9]. Hence $m_C>m_Z/2+n/2$. But we already know that $m_C\leqslant n/2$, a contradiction. The lemma is proved.

Lemma 6.5. *We have* $\gimel\neq 8$.

Proof. Suppose that $\gimel=8$. We use the notation from the proof of Lemma 3.18. Let $\overline{R}$ be the proper transform of the surface R on the threefold U, and let S be a generic surface through P in $|-K_U|$. Then $\mathrm{mult}_P(S)=1$.

Suppose that $P\notin\overline{R}$. Then $\overline{R}|_S$ is an irreducible curve. We denote it by Z. Write $\overline{D}|_S=m_FF+m_ZZ+\Upsilon$, where m_F and m_Z are non-negative integers, F is a smooth irreducible curve with $E|_S=F$, and Υ is an effective divisor on S whose support does not contain the curves F or Z. Then

$$F^2=Z^2=-\frac{4}{3},\qquad Z\cdot F=\frac{5}{3}$$

on the surface S. Hence $4m_Z/3\geqslant 5(\mu+m_F)/4-3n/4$ because $\Upsilon\cdot Z\geqslant 0$. We have

$$\frac{4}{3}\mu+\frac{4}{3}m_F-\frac{5}{3}m_Z=\Upsilon\cdot F>\frac{5}{4}n-\mu-m_F,$$

whence $7(\mu+m_F)/3>5n/4+5m_Z/3$. Therefore we see that

$$\frac{4}{3}m_Z>\frac{5}{7}\left(\frac{5}{4}n+\frac{5}{3}m_Z\right)-\frac{3}{4}n.$$

It follows that $m_Z>n$. But Remark 2.12 implies that $m_Z\leqslant n$ because $\alpha^*(-K_X)\cdot Z=3/4$, a contradiction.

Thus $P\in\overline{R}$. Suppose that $P\notin\bigcup_{i=1}^{l}C_i$. There is a surface $H\in|-3K_U|$ such that

$$3\left(\frac{3}{4}n-\frac{5}{3}\mu\right)=\overline{D}\cdot H\cdot\overline{R}\geqslant \mathrm{mult}_P(\overline{D})>\frac{5}{4}n-\mu.$$

It follows that $\mu<n/4$. But $\mu>n/4$ by [10], a contradiction. Thus, we may assume that $P\in C_1$.

Put $B = E|_{\overline{R}}$ and $C_1 + Z_1 = S|_{\overline{R}}$, where B and Z_1 are irreducible curves. Write

$$\overline{D}|_{\overline{R}} = \overline{m}_C C_1 + \overline{m}_Z Z_1 + \overline{m}_B B + \Omega,$$

where $\overline{m}_C$, $\overline{m}_Z$ and $\overline{m}_B$ are non-negative integers and Ω is an effective divisor (on the surface $\overline{R}$) whose support does not contain the curves C_1, Z_1 or B. We easily see that

$$C_1^2 = -1, \qquad Z_1^2 = -\frac{2}{3}, \qquad B^2 = -\frac{20}{3}, \qquad B \cdot Z_1 = \frac{2}{3}, \qquad B \cdot C_1 = Z_1 \cdot C_1 = 1$$

on the surface $\overline{R}$. Then $2\overline{m}_Z/3 \geqslant 2(\overline{m}_B + \mu) + \overline{m}_C - n/2$ because $\Omega \cdot Z \geqslant 0$, but

$$\frac{3}{4}n - \frac{5}{3}\mu - \frac{1}{3}\overline{m}_Z - \frac{1}{3}\overline{m}_B = (C_1 + Z_1) \cdot \Omega \geqslant \operatorname{mult}_P(\Omega) > \frac{5}{4}n - \mu - \overline{m}_B - \overline{m}_C$$

since the curve Z_1 does not pass through P. Therefore we have

$$\frac{2}{3}\overline{m}_Z \geqslant 2(\overline{m}_B + \mu) + \overline{m}_C - \frac{1}{2}n > \frac{1}{3}\overline{m}_Z + \frac{4}{3}\mu,$$

whence $\overline{m}_Z > 4\mu > n$. On the other hand, we have

$$\frac{9}{4}n - 5\mu - \overline{m}_Z - \overline{m}_B = 3(C_1 + Z_1) \cdot \Omega \geqslant \operatorname{mult}_P(\Omega) > \frac{5}{4}n - \mu - \overline{m}_B - \overline{m}_C,$$

whence $\overline{m}_C > \overline{m}_Z + 4\mu - n > n$. But Remark 2.12 implies that $\overline{m}_C + 2\overline{m}_Z \leqslant 3n$ because $\alpha^*(-K_X) \cdot Z_1 = 1/2$ and $\alpha^*(-K_X) \cdot C_1 = 1/4$, a contradiction. The lemma is proved.

Lemma 6.6. *We have* $\daleth \neq 12$.

Proof. Suppose that $\daleth = 12$. Then X is a hypersurface of degree 10 in $\mathbb{P}(1,1,2,3,4)$, and O is either a singular point of type $\frac{1}{3}(1,1,2)$ or a singular point of type $\frac{1}{4}(1,1,3)$.

Let S be a surface through P in the pencil $|-K_U|$. Then S is smooth at P.

Suppose that $P \notin \bigcup_{i=1}^{l} C_i$. Then we have $\mu \leqslant n/r$ because the proof of Theorem 5.6.2 in [5] yields that there is a surface $H \in |-s(7-r)K_U|$ such that

$$s(7-a)\left(\frac{5}{12}n - \frac{\mu}{r-a}\right) = \overline{D} \cdot H \cdot S \geqslant \operatorname{mult}_P(\overline{D})s > \left(n + \frac{n}{r} - \mu\right)s,$$

where s is a positive integer. But we have $\mu > n/r$ by [10], a contradiction.

We may assume that $P \in C_1$. Let R be a generic hypersurface through P in the linear system $|-2K_U|$. Then $R \cdot S = C_1 + Z + L$, where Z and L are irreducible curves such that $L \subset E$ and $Z \neq C_1$. We write

$$\overline{D}|_S = m_C C_1 + m_Z Z + m_L L + \Upsilon,$$

where m_C, m_Z and m_L are non-negative integers and Υ is an effective divisor (on the surface S) whose support does not contain the curves C_1, Z or L.

Suppose that $r = 3$. Then $C_1^2 = -2$, $Z^2 = -1$ and $C_1 \cdot Z = 2$ on the surface S. Therefore we have

$$\frac{5}{6}n - \mu - m_Z = R \cdot (\Upsilon + m_L L) > \frac{4}{3}n - \mu - m_C$$

because $R \cdot L > 0$. Thus we have $m_C > n/2 + m_Z$. But

$$\frac{1}{2}n - 2m_C + m_Z = (\Omega + m_L L) \cdot Z \geqslant 0.$$

This leads to a contradiction because $m_C > n/2 + m_Z$.

Thus we see that $r = 4$. Then $Z^2 = L^2 = -4/3$, $C_1^2 = -2$ and $L \cdot C_1 = Z \cdot C_1 = Z \cdot L = 1$, but

$$\frac{5}{6}n - \mu = R \cdot \overline{D} \cdot S = \frac{2}{3}m_L + \frac{2}{3}m_Z + R \cdot \Upsilon > \frac{2}{3}m_L + \frac{2}{3}m_Z + \frac{5}{4}n - \mu - m_C - m_L.$$

It follows that $m_C > 5n/12 + 2m_Z/3 - (\mu + m_L)/3$. But

$$\frac{4}{3}\mu = L \cdot (m_C C_1 + m_Z Z + m_L L + \Upsilon) > -\frac{4}{3}m_L + m_Z + m_C + \frac{5}{4}n - \mu - m_L - m_C.$$

Therefore $7(\mu + m_L)/3 > 5n/4 + m_Z$. We have $\mu + m_L \leqslant \frac{5}{4}n - m_Z$ and $\frac{4}{3}m_Z \geqslant (\mu + m_L) + m_C - \frac{7}{12}n$ because $-K_U \cdot \Upsilon \geqslant 0$ and $\Upsilon \cdot Z \geqslant 0$ respectively. So we have $m_Z > n/2$ because

$$\frac{4}{3}m_Z > (\mu + m_L) - \frac{1}{6}n + \frac{2}{3}m_Z - \frac{1}{3}(\mu + m_L) > \frac{2}{7}\left(\frac{5}{4}n + m_Z\right) - \frac{1}{6}n + \frac{2}{3}m_Z.$$

But it follows from the proof of Theorem 5.6.2 in [5] that

$$3s\left(\frac{5}{12}n - \frac{1}{3}\mu - m_L - m_Z\right) = M \cdot \Upsilon \geqslant \mathrm{mult}_P(\overline{D})s > \left(\frac{5}{4}n - \mu - m_L - m_C\right)s$$

for some integer $s > 0$ and some surface $M \in |-3sK_U|$. We have $m_C > m_Z$ and

$$\frac{4}{3}m_Z > (\mu + m_L) + m_Z - \frac{7}{12}n > \frac{3}{7}\left(\frac{5}{4}n + m_Z\right) + m_Z - \frac{7}{12}n,$$

whence $m_Z < n/2$. This contradicts the previous inequality $m_Z > n/2$. The lemma is proved.

Lemma 6.7. *Suppose that* $\gimel = 13$. *Then* O *is a singular point of type* $\frac{1}{5}(1,2,3)$.

Proof. Suppose that O is not a singular point of type $\frac{1}{5}(1,2,3)$. Then O is a singular point of type $\frac{1}{3}(1,1,2)$. Let S be a surface through P in $|-K_U|$. Then $\overline{D} \neq S$ by Lemma 2.3.

Suppose that the birational morphism σ is an isomorphism in a neighbourhood of P. Then it follows from the proof of Theorem 5.6.2 in [5] that one can find an integer $s > 0$ and a surface $H \in |-5sK_U|$ such that

$$5s\left(\frac{11}{30}n - \frac{1}{2}\mu\right) = \overline{D} \cdot H \cdot S \geqslant \mathrm{mult}_P(\overline{D})s > \left(\frac{4}{3}n - \mu\right)s.$$

It follows that $\mu < n/3$. But this is impossible since $\mu > n/3$ by [10].

We see that σ contracts some irreducible curve passing through P. This curve is actually unique. We denote it by L. Then $P \in L$ and $-K_U \cdot L = 0$. The curve L is smooth and rational.

Let H be a generic surface through P in the linear system $|-2K_U|$. There is an irreducible curve $C \subset U$ such that $H|_S = L + C$. Then

$$L^2 = -2, \qquad L \cdot C = 2, \qquad C^2 = -\frac{6}{5}$$

on the surface S. We write $D|_S = m_L L + m_C C + \Omega$, where m_L and m_C are non-negative integers and Ω is an effective divisor whose support does not contain L or Z. Then $P \notin C$ and

$$\frac{1}{3}n - \mu - 2m_C + 2m_L = \Omega \cdot L > \frac{4}{3}n - mu - m_L,$$

but $2n/5 - 2m_L + 6m_C/5 = \Omega \cdot C \geqslant 0$. This easily leads to a contradiction. The lemma is proved.

Lemma 6.8. *We have* $\daleth \neq 16$.

Proof. Suppose that $\daleth = 16$. Then X is a hypersurface of degree 12 in $\mathbb{P}(1,1,2,4,5)$, and O is a singular point of type $\frac{1}{5}(1,1,4)$. Let S be the unique surface through P in the pencil $|-K_U|$. Then S is smooth at P.

Suppose that $P \notin \bigcup_{i=1}^{l} C_i$. Then there is a surface $H \in |-4K_U|$ that passes through P and contains no components of the effective cycle $\overline{D} \cdot S$. Then

$$4\biggl(\frac{3}{10}n - \frac{1}{4}\mu\biggr) = \overline{D} \cdot H \cdot S \geqslant \operatorname{mult}_P(\overline{D}) > \frac{6}{5}n - \mu.$$

This leads to a contradiction. Thus we may assume that $P \in C_1$. Then $C_1 \subset S$.

Let $\psi\colon X \dashrightarrow \mathbb{P}(1,1,2)$ be the natural projection, and let F be the curve cut out on the surface S by the divisor E. Then the fibre of the rational map $\psi \circ \alpha$ over the point $\psi \circ \alpha(P)$ consists of the curves F and C_1 and another irreducible curve Z such that

$$Z^2 = F^2 = -\frac{5}{4}, \qquad C_1^2 = -2, \qquad Z \cdot C_1 = F \cdot C_1 = 1, \qquad Z \cdot F = \frac{3}{4}$$

on the surface S. It is easy to see that $P = F \cap C_1$. We write

$$\overline{D}|_S = m_C C_1 + m_F F + m_Z Z + \Omega,$$

where m_C, m_F and m_Z are non-negative integers and Ω is an effective cycle whose support does not contain the curves C_1, F or Z. Then the linear system $|-4K_U|$ contains a surface M that passes through P and contains no components of Ω. We have

$$4\biggl(\frac{3}{10}n - \frac{1}{4}\mu - \frac{1}{4}m_Z - \frac{1}{4}m_F\biggr) = M|_S \cdot \Omega \geqslant \operatorname{mult}_P(\Omega) > \frac{6}{5}n - \mu - m_F - m_C.$$

It follows that $m_C > m_Z$. On the other hand, we have

$$\frac{3}{10}n - \frac{1}{4}\mu = -K_U|_S \cdot (m_C C_1 + m_F F + m_Z Z + \Omega)$$
$$\geqslant -K_U|_S \cdot (m_C C_1 + m_F F + m_Z Z) = \frac{m_F + m_Z}{4}\,.$$

It follows that $m_F + m_Z \leqslant 6n/5 - \mu$. We similarly have

$$\frac{5}{4}\mu + \frac{5}{4}m_F - \frac{3}{4}m_Z - m_C = \Omega \cdot F > \frac{6}{5}n - \mu - m_C - m_F,$$

whence $9(\mu + m_F)/4 > 6n/5 + 3m_Z/4$. Since $\Omega \cdot Z \geqslant 0$, it follows that

$$\frac{5}{4}m_Z \geqslant \frac{3}{4}\mu + m_C + \frac{3}{4}m_F - \frac{2}{5}n.$$

But $m_Z \leqslant 3n/4$ by Remark 2.12 because $-K_X \cdot \alpha(Z) = 2/5$. Thus we have

$$\frac{9}{4}\left(\frac{6}{5}n - \mu\right) \geqslant \frac{9}{4}(\mu + m_F) > \frac{6}{5}n + \frac{3}{4}m_Z,$$
$$\frac{15}{16}n \geqslant \frac{5}{4}m_Z \geqslant \frac{3}{4}\mu + m_C + \frac{3}{4}m_F - \frac{2}{5}n > \frac{3}{4}\mu + m_Z + \frac{3}{4}m_F - \frac{2}{5}n.$$

But these linear inequalities are easily seen to be incompatible, a contradiction. The lemma is proved.

Lemma 6.9. *We have* $\gimel \neq 25$.

Proof. Suppose that $\gimel = 25$. Then X is a hypersurface of degree 15 in $\mathbb{P}(1,1,3,4,7)$, and O is either a singular point of type $\frac{1}{4}(1,1,3)$ or a singular point of type $\frac{1}{7}(1,3,4)$.

Suppose that O is of type $\frac{1}{4}(1,1,3)$. Let S be the unique surface through P in the pencil $|-K_U|$. Then $\overline{D} \neq S$.

Suppose that the birational morphism σ is an isomorphism in a neighbourhood of P. Then the proof of Theorem 5.6.2 in [5] shows the existence of an integer $s > 0$ and a surface $H \in |-7sK_U|$ that has multiplicity at least s at P and contains no components through P of the cycle $\overline{D} \cdot S$. Then

$$7s\left(\frac{5}{28}n - \frac{1}{3}\mu\right) = \overline{D} \cdot H \cdot S \geqslant \mathrm{mult}_P(\overline{D})s > \left(\frac{5}{4}n - \mu\right)s.$$

This contradicts the inequality $\mu > n/4$ which follows from [10].

Thus σ is not an isomorphism in a neighbourhood of P. Then there is a unique irreducible curve $L \subset U$ such that $P \in L$ and $-K_U \cdot L = 0$. The curve L is smooth and rational.

Let H be a generic surface through P in the linear system $|-3K_U|$. Then $H|_S = L + C$, where C is an irreducible curve such that

$$L^2 = -2, \qquad L \cdot C = 2, \qquad C^2 = -\frac{8}{7}$$

on the surface S. We write $D|_S = m_L L + m_C C + \Omega$, where m_L and m_C are non-negative integers and Ω is an effective divisor whose support does not contain L or Z. Then $P \notin C$ and

$$\frac{1}{4}n - \mu - 2m_C + 2m_L = \Omega \cdot L > \frac{5}{4}n - mu - m_L,$$
$$\frac{2}{7}n - 2m_L + \frac{8}{7}m_C = \Omega \cdot C \geqslant 0.$$

This easily leads to a contradiction.

Hence the point P is a singular point of type $\frac{1}{7}(1,3,4)$. Let T be a generic surface in the pencil $|-K_U|$.

Suppose that $P \in T$. The base locus of the pencil $|-K_U|$ consists of irreducible curves $\overline{C}$ and $\bar{L}$ such that $\overline{C} = E|_T$ and $\bar{L} = \overline{M}|_T$. Then

$$\overline{C}^2 = \bar{L}^2 = -\frac{7}{12}, \qquad \bar{L} \cdot \overline{C} = \frac{8}{12}$$

on the surface S, but $\alpha^*(-K_X) \cdot \bar{L} = -K_X^3$. We write $\overline{D}|_T = \overline{m}_L \bar{L} + \overline{m}_C \overline{C} + \Upsilon$, where $\overline{m}_L$ and $\overline{m}_C$ are non-negative integers and Υ is an effective divisor (on the surface T) whose support does not contain the curves $\bar{L}$ or $\overline{C}$. The inequalities $\Upsilon \cdot \bar{L} \geqslant 0$ and $\Upsilon \cdot \overline{C} \geqslant \mathrm{mult}_P(\Upsilon)$ imply that

$$\frac{19}{12}(\mu + \overline{m}_C) > \frac{8}{7}n + \frac{2}{3}\overline{m}_L, \qquad \frac{2}{3}(\mu + \overline{m}_C) \leqslant \frac{5}{28}n + \frac{7}{12}\overline{m}_L,$$

whence $\overline{m}_L > n$. The last inequality contradicts Remark 2.12.

Thus P is not contained in T. Then $\mu \geqslant 24n/70$ by Lemma 2.14. But we have $\mu \leqslant 15n/56$ by Lemma 6.3, a contradiction. The lemma is proved.

Lemma 6.10. *We have* $\gimel \neq 26$ *and* $\gimel \neq 36$.

Proof. We may assume that $\gimel = 36$ because the proof of $\gimel \neq 26$ is almost the same. Then X is a hypersurface of degree 18 in $\mathbb{P}(1,1,4,6,7)$, and O is a singular point of type $\frac{1}{7}(1,1,6)$. There is a unique surface through P in the pencil $|-K_U|$. We denote this surface by S.

Suppose that $P \notin \bigcup_{i=1}^{l} C_i$. It follows from the proof of Theorem 5.6.2 in [5] that one can find an integer $s > 0$ and a surface $H \in |-6sK_U|$ such that $\mathrm{mult}_P(H) \geqslant s$ and H contains no components through P of the cycle $\overline{D} \cdot S$. Then

$$6s\left(\frac{3}{28}n - \frac{1}{6}\mu\right) = \overline{D} \cdot H \cdot S \geqslant \mathrm{mult}_P(\overline{D})s > \left(\frac{8}{7}n - \mu\right)s.$$

This is a contradiction. Hence $P \in \bigcup_{i=1}^{l} C_i$. We may assume that $P \in C_1$.

Put $L = C_1$ and $C = E|_S$. Then $\overline{M}|_S = L + Z$, where Z is an irreducible curve. Let us find the intersection form of the curves C, L and Z on the surface S. This is easy. We have

$$Z^2 = C^2 = -\frac{7}{6}, \qquad L^2 = -2, \qquad Z \cdot L = C \cdot L = 1, \qquad Z \cdot C = \frac{5}{6}.$$

The point P is the intersection point of the curves L and C. We put

$$\overline{D}|_S = m_L L + m_C C + m_Z Z + \Omega,$$

where m_L, m_C and m_Z are non-negative integers and Ω is an effective divisor whose support does not contain the curves L, C or Z. It follows from the proof of Theorem 5.6.2 in [5] that one can find an integer $s > 0$ and a surface $H \in |-6sK_U|$ such that $\mathrm{mult}_P(H) \geqslant s$ and H contains no components through P of the support of the effective cycle Ω. Then

$$6s\left(\frac{3}{28}n - \frac{1}{6}\mu - \frac{1}{6}m_C - \frac{1}{6}m_Z\right) = H|_S \cdot \Omega \geqslant \mathrm{mult}_P(\Omega)s > \left(\frac{8}{7}n - \mu - m_L - m_C\right)s.$$

It follows that $m_L > n/2 + m_Z$. But $m_L \leqslant 3n/4$ by Remark 2.12. We have

$$\begin{aligned}\frac{3}{28}n - \frac{1}{6}\mu &= -K_U|_S \cdot (m_L L + m_C C + m_Z Z + \Omega)\\ &\geqslant -K_U|_S \cdot (m_L L + m_C C + m_Z Z) = \frac{m_C + m_Z}{6}.\end{aligned}$$

Hence $m_C + m_Z \leqslant 9n/14 - \mu$. We similarly have

$$\frac{7}{6}\mu + \frac{7}{6}m_C - \frac{5}{6}m_Z - m_L = \Omega \cdot C > \frac{8}{7}n - \mu - m_L - m_C.$$

It follows that $13(\mu + m_C)/6 > 8n/7 + 5m_Z/6$. The inequality $\Omega \cdot Z \geqslant 0$ implies that

$$\frac{2}{7}n - \frac{5}{6}\mu - m_L - \frac{5}{6}m_C + \frac{7}{6}m_Z \geqslant 0.$$

Hence we have $7m_Z/6 \geqslant 5\mu/6 + m_L + 5m_C/6 - 2n/7$. But $m_Z \leqslant 3n/8$ by Remark 2.12.

It follows from Lemma 6.3 that $18n/77 \geqslant \mu > n/7$. The resulting system of linear inequalities

$$\frac{13}{6}(\mu + m_C) > \frac{8}{7}n + \frac{5}{6}m_Z,$$

$$\frac{21}{48}n \geqslant \frac{7}{6}m_Z \geqslant \frac{5}{6}\mu + m_L + \frac{5}{6}m_C - \frac{2}{7}n,$$

$$m_C + m_Z \leqslant \frac{9}{14}n - \mu, \qquad \frac{3}{4}n \geqslant m_L > \frac{1}{2}n + m_Z, \qquad \frac{18}{77}n \geqslant \mu > \frac{1}{7}n$$

is easily seen to be incompatible, a contradiction. The lemma is proved.

Lemma 6.11. *We have* $\gimel \neq 31$.

Proof. Suppose that $\gimel = 31$. Then X is a hypersurface of degree 16 in $\mathbb{P}(1,1,4,5,6)$, and O is either a singular point of type $\frac{1}{5}(1,1,4)$ or a singular point of type $\frac{1}{6}(1,1,5)$.

Let S be a surface through P in the pencil $|-K_U|$. Then $\overline{D} \neq S$, and the argument used to prove Lemma 6.10 yields that $P \in C_1 \subset S$.

Suppose that O is a singular point of type $\frac{1}{6}(1,1,5)$. Arguing as in the proof of Lemma 6.10, we arrive at a contradiction. Hence O is a singular point of type $\frac{1}{5}(1,1,4)$.

Let $\omega\colon U \dashrightarrow \mathbb{P}(1,1,4)$ be the composite of the weighted blow-up α and the natural projection, and let L be the component of the fibre of ω over the point $\omega(C_1)$ such that $L \neq C_1$. We write

$$\overline{D}|_S = mC_1 + m'L + \Upsilon,$$

where m and m' are non-negative integers and Υ is an effective divisor whose support does not contain the curves L or C_1. The curve L is smooth and $P \notin L$. Using the inequalities $\Upsilon \cdot L \geqslant 0$ and $\Upsilon \cdot C_1 \geqslant \operatorname{mult}_P(\Upsilon)$, we easily obtain a contradiction because we have $L^2 = -2/3$, $C_1^2 = -2$, and $L \cdot C_1 = 2$ on the surface S. The lemma is proved.

Lemma 6.12. *We have* $\daleth \neq 20$.

Proof. Suppose that $\daleth = 20$. Arguing as in the proof of Lemma 6.10, we see that O must be a singular point of type $\frac{1}{4}(1,1,3)$. Then we easily get a contradiction as in the proof of Lemma 6.11. The lemma is proved.

Lemma 6.13. *We have* $\daleth \neq 47$ *and* $\daleth \neq 54$.

Proof. We may assume that $\daleth = 47$ because the proof that $\daleth \neq 54$ is similar. Then X is a hypersurface of degree 21 in $\mathbb{P}(1,1,5,7,8)$ and it follows from Lemma 2.8 that O is a singular point of type $\frac{1}{8}(1,1,7)$.

Let T be the unique surface through P in the pencil $|-K_U|$. Then $\overline{D} \neq T$.

Suppose that $P \notin \bigcup_{i=1}^{l} C_i$. Then it follows from the proof of Theorem 5.6.2 in [5] that one can find an integer $s > 0$ and a surface $H \in |-7sK_U|$ such that the multiplicity of H at P is greater than or equal to s and H contains no components through P of the effective cycle $\overline{D} \cdot T$. We have

$$s\biggl(\frac{21}{40}n - \mu\biggr) = \overline{D} \cdot H \cdot T \geqslant \operatorname{mult}_P(\overline{D})s > s\biggl(\frac{9}{8}n - \mu\biggr)s.$$

This contradicts the inequality $\mu > n/8$. Thus we may assume that $P \in C_1$.

We write $\overline{D} \cdot T = mC_1 + \Delta$, where m is a non-negative integer and Δ is an effective cycle whose support does not contain C_1. It follows from the proof of Theorem 5.6.2 in [5] that one can find an integer $s > 0$ and a surface $R \in |-7sK_U|$ such that $\operatorname{mult}_P(R) \geqslant s$ and R contains no components through P of the effective cycle Δ. Then

$$s\biggl(\frac{21}{40}n - \mu\biggr) = R \cdot \Delta \geqslant \operatorname{mult}_P(\Delta)s > s\biggl(\frac{9}{8}n - \mu - m\biggr).$$

It follows that $m > 3n/5$. But we have $m \leqslant 3n/5$ by Remark 2.12 because $\alpha^*(-K_X) \cdot C_1 = 1/8$, a contradiction. The lemma is proved.

Lemmas 6.1, 6.2, 6.4–6.6, 6.8–6.13 yield that

$$\daleth \in \{13, 18, 23, 24, 32, 38, 40, 42, 43, 44, 45, 46, 48, 56, 58, 60, 61, 65, 69, 74, 76, 79\}$$

and $a \neq 1$. Let T be a generic surface in $|-K_U|$. Then $T|_E \in |\mathcal{O}_{\mathbb{P}(1,a,r-a)}(1)|$.

Lemma 6.14. *The point P is contained in the surface T.*

Proof. It follows from Lemmas 2.14 and 6.3 that $P \in T$ if $\daleth \notin \{13, 24\}$. Therefore we may assume that $\daleth \in \{13, 24\}$ and $P \notin T$. Let us derive a contradiction.

Let L be the unique curve in the linear system $|\mathcal{O}_{\mathbb{P}(1,a,r-a)}(1)|$. Then $P \notin L$ because $P \notin T$. Hence there is a unique smooth irreducible curve C through P in the linear system $|\mathcal{O}_{\mathbb{P}(1,a,r-a)}(a)|$. We write

$$\overline{D}|_E = \delta C + \Upsilon \equiv r\mu L,$$

where δ is a non-negative integer and Υ is an effective divisor (on the surface E) whose support does not contain C. Arguing as in the proof of Lemma 2.14, we see that $\delta \leqslant r\mu/a$. Hence $\delta < n$ by Lemma 6.3.

The log pair $(E, \frac{1}{n}\overline{D}|_E)$ is not log canonical at P (see Theorem 7.5 in [9]). Since $\delta < n$, it follows that the log pair $(E, C + \frac{1}{n}\Upsilon)$ is not log canonical at P. Again using Theorem 7.5 of [9], we see that

$$\frac{r\mu}{r-a} \geqslant \frac{r\mu - a\delta}{r-a} = C \cdot \Upsilon \geqslant \mathrm{mult}_P(\Upsilon|_C) > n.$$

It follows that $\mu \geqslant n(r-a)/r$. This inequality is impossible by Lemma 6.3. The lemma is proved.

It follows from easy calculations (see the proof of Theorem 5.6.2 in [5]) that

$$T \cap E \cap \bigcup_{i=1}^{l} C_i \neq \varnothing \iff \daleth \in \{43, 46, 69, 74, 76, 79\}.$$

Lemma 6.15. *The case $\daleth \notin \{13, 24, 32, 43, 46\}$ is impossible.*

Proof. Suppose that $\daleth \notin \{13, 24, 32, 43, 46, 56\}$. It follows from the proof of Theorem 5.6.2 in [5] that one can find an integer $s > 0$ and a surface $H \in |-sa_1\bar{a}_3K_U|$ such that $\mathrm{mult}_P(H) \geqslant s$ and H contains no components through P of the cycle $\overline{D} \cdot T$ except possibly for one of the curves $C_1, \ldots, C_l$.

It is easy to see that $\daleth \in \{69, 74, 76, 79\}$ and $P \in \bigcup_{i=1}^{l} C_i$ since otherwise we obtain a contradiction from the inequalities

$$sa_1\bar{a}_3\left(-nK_X^3 - \frac{\mu}{a(r-a)}\right) = \overline{D} \cdot H \cdot T \geqslant \mathrm{mult}_P(\overline{D})s > \left(n + \frac{n}{r} - \mu\right)s.$$

We may assume that $P \in C_1$. Write $\overline{D} \cdot T = mC_1 + \Delta$, where m is a non-negative integer and Δ is an effective cycle whose support does not contain C_1. Then

$$sa_1\bar{a}_3\left(-nK_X^3 - \frac{\mu}{a(r-a)}\right) = H \cdot \Delta \geqslant \mathrm{mult}_P(\Delta)s > \left(n + \frac{n}{r} - \mu - m\right)s.$$

This is impossible because $m \leqslant -a_i nK_X^3$ by Remark 2.12.

This shows that $\daleth = 56$. As in the previous case, one can find an integer $s > 0$ and a surface H in the linear system $|-24sK_U|$ such that

$$24s\left(\frac{1}{22}n - \frac{1}{24}\mu\right) = \overline{D} \cdot H \cdot T \geqslant \mathrm{mult}_P(\overline{D})s > \left(\frac{12}{11}n - \mu\right)s.$$

This contradicts the inequality $\mu > n/r$. The lemma is proved.

Thus $\gimel \in \{13, 24, 32, 43, 46\}$. We successively treat these cases.

Lemma 6.16. *We have* $\gimel \neq 13$.

Proof. Suppose that $\gimel = 13$. Then the point O is a singular point of type $\frac{1}{5}(1,2,3)$ by Lemma 6.7, the base locus of the pencil $|-K_U|$ consists of curves $\overline{C}$ and $\bar{L}$ with $\overline{C} = E|_T$, and $\alpha(\bar{L})$ is the base curve of the pencil $|-K_X|$. The curves $\overline{C}$ and $\bar{L}$ are irreducible and we have

$$\overline{C}^2 = \bar{L}^2 = -\frac{5}{6}, \qquad \bar{L} \cdot \overline{C} = 1$$

on the surface T. We write $\overline{D}|_T = \overline{m}_L \bar{L} + \overline{m}_C \overline{C} + \Upsilon$, where $\overline{m}_L$ and $\overline{m}_C$ are non-negative integers and Υ is an effective divisor whose support does not contain $\bar{L}$ or $\overline{C}$. Then

$$\frac{11}{5}n - \frac{11}{6}\mu = (6L + 5C) \cdot (\overline{m}_L \bar{L} + \overline{m}_C \overline{C} + \Upsilon) = \frac{11}{6}\overline{m}_C + (6L + 5C) \cdot \Upsilon \geqslant \frac{11}{6}\overline{m}_C.$$

It follows that $\overline{m}_C \leqslant 6n/5 - \mu$. Thus we have $\overline{m}_C < n$ because $\mu > n/5$.

Suppose that $P \notin \bar{L}$. Then it follows from Theorem 7.5 of [9] that the log pair

$$\left(S, \overline{C} + \frac{\overline{m}_L}{n} L + \frac{1}{n}\Upsilon\right)$$

is not log canonical at P because $\overline{m}_C + \mu - n/5 \leqslant n$. Hence we have $\mathrm{mult}_P(\Upsilon|_{\overline{C}}) > n$ by Theorem 7.5 of [9]. Therefore,

$$\frac{5}{6}\mu + \frac{5}{6}\overline{m}_C \geqslant \frac{5}{6}\mu - \overline{m}_L + \frac{5}{6}\overline{m}_C = \Upsilon \cdot \overline{C} > n.$$

This is impossible because $\overline{m}_C \leqslant 6/5 - \mu$. Thus we see that $P = \bar{L} \cap \overline{C}$.

Write $\overline{D}|_{\overline{M}} = m\bar{L} + \Omega$, where m is a non-negative integer and Ω is an effective divisor whose support does not contain $\bar{L}$. Then $L^2 = 1/6$ on the surface $\overline{M}$. But we have $m \leqslant n$ by Remark 2.12 because $\alpha^*(-K_X) \cdot \bar{L} = 11/30$.

Arguing as in the case when $P \notin \bar{L}$, we see that $\mathrm{mult}_P(\Omega|_{\bar{L}}) > n$. We have

$$\frac{11}{30}n - \mu = \overline{D} \cdot \bar{L} = \frac{1}{6}m + \Omega \cdot \bar{L} > \frac{1}{6}m + n.$$

It follows that $m < 0$, a contradiction. The lemma is proved.

Lemma 6.17. *We have* $\gimel \neq 24$.

Proof. Suppose that $\gimel = 24$. The base locus of $|-K_U|$ consists of irreducible curves L and C such that $\alpha(C)$ is the base curve of $|-K_X|$, and the curve L is contained in the surface $E \cong \mathbb{P}(1,2,5)$ and is the unique curve in the linear system $|\mathcal{O}_{\mathbb{P}(1,2,5)}(1)|$.

Let $\overline{S}$ be a generic surface in the pencil $|-K_U|$. Then $P \in L$ by Lemma 6.14. But $P \notin C$ because the intersection $L \cap C$ consists of a singular point of U of type $\frac{1}{5}(1,2,3)$.

The surface $\overline{S}$ is normal. We easily see that

$$L^2 = C^2 = -\frac{7}{10}, \qquad L \cdot C = \frac{4}{5}$$

on the surface $\overline{S}$. We write $\overline{D}|_{\overline{S}} = m_L L + m_C C + \Delta$, where m_L and m_C are non-negative integers and Δ is an effective divisor (on the surface $\overline{S}$) whose support does not contain L or C. Then

$$\frac{3}{14}n - \frac{4}{5}\mu - \frac{4}{5}m_L + \frac{7}{10}m_C = \Delta \cdot C \geqslant 0.$$

But $m_C \leqslant n$ by Remark 2.12 because $\alpha^*(-K_X) \cdot C = 3/14$. Thus $\mu + m_L \leqslant 8n/7$.

The surface $\overline{S}$ is smooth at P. By Theorem 7.5 of [9] the log pair

$$\left(\overline{S}, \frac{m_L + \mu - n/7}{n}L + \frac{m_C}{n}C + \frac{1}{n}\Delta\right)$$

is not log canonical at P, but $\mu + m_L - n/7 \leqslant n$. It follows that the log pair

$$\left(\overline{S}, L + \frac{m_C}{n}C + \frac{1}{n}\Delta\right)$$

is not log canonical at P. Using Theorem 7.5 of [9], we see that

$$\frac{7}{10}\mu + \frac{7}{10}m_L = \Delta \cdot L \geqslant \mathrm{mult}_P(\Delta|_L) > n.$$

It follows that $\mu + m_L > 10n/7$. But $\mu + m_L \leqslant 8n/7$, a contradiction. The lemma is proved.

Lemma 6.18. *We have* $\daleth \neq 32$.

Proof. Suppose that $\daleth = 32$. Then X is a hypersurface of degree 16 in $\mathbb{P}(1,2,3,4,7)$, the point O is a singular point of type $\frac{1}{7}(1,3,4)$, the base locus of the pencil $|-2K_U|$ consists of non-singular curves L and C with $L = T \cdot E$, and $\alpha(C)$ is the base curve of $|-2K_X|$.

We write $\overline{D} \cdot T = m_1 L + m_2 C + \Delta$, where m_1 and m_2 are non-negative integers and Δ is an effective cycle whose support does not contain the curves L or C. Then

$$-K_U \cdot L = -K_U \cdot C = \frac{1}{12}.$$

It follows that $m_1 + m_2 + \mu \leqslant 8n/7$ by Remark 2.12.

It is easy to see that the intersection $C \cap L$ consists of a singular point of U. Hence the curve C does not contain P. Let S be a generic surface in $|-2K_U|$. Then

$$\frac{4}{21}n - \frac{1}{6}\mu - \frac{1}{6}(m_1 + m_2) = S \cdot \Delta \geqslant \mathrm{mult}_P(\Delta) > \frac{8}{7}n - \mu - m_1.$$

It follows that $40n/7 \geqslant 40n/7 - 6m_2 \geqslant 5n(m_1 + m_2 + \mu) - 6m_2 > 40n/7$, a contradiction. The lemma is proved.

Lemma 6.19. *We have* $\daleth \neq 43$.

Proof. Suppose that $\daleth = 43$. Then X is a hypersurface of degree 20 in $\mathbb{P}(1,2,3,5,9)$, the point O is a singular point of type $\frac{1}{9}(1,4,5)$ and $j = 1$. The base locus of $|-2K_U|$ consists of irreducible curves C and L, where $L = T \cdot E$ and C is the only one of the curves $C_1, \ldots, C_l$ that intersects L.

Suppose that $P \notin C$. Then it follows from the proof of Theorem 5.6.2 in [5] that one can find an integer $s > 0$ and a surface H in the linear system $|-20sK_U|$ such that the multiplicity of H at P is greater than or equal to $s > 0$ and H contains no components through P of the effective cycle $\overline{D} \cdot T$. Hence,

$$20s\left(\frac{1}{18}n - \frac{1}{20}\mu\right) = \overline{D} \cdot H \cdot T \geqslant \mathrm{mult}_P(\overline{D})s > \left(\frac{10}{9}n - \mu\right)s.$$

It follows that $\mu < n/9$. This contradicts the inequality $\mu > n/10$.

We see that $P \in C$. Then $\overline{M}$ contains C and L. We write

$$\overline{D}|_{\overline{M}} = m_1 L + m_2 C + \Delta,$$

where m_1 and m_2 are non-negative integers and Δ is an effective cycle whose support does not contain the curves L or C. It follows from Remark 2.12 that $m_2 \leqslant n$ because $\alpha^*(-K_X) \cdot C = 1/9$.

The surface $\overline{M}$ is smooth at P. Hence, by Theorem 7.5 of [9], the log pair

$$\left(\overline{M}, \frac{1}{n}\overline{D}|_{\overline{M}} + \left(\frac{\mu}{n} - \frac{1}{9}\right)E|_{\overline{M}}\right)$$

is not log canonical at P. We have $E|_{\overline{M}} = L + Z$, where Z is an irreducible curve that does not pass through P. Thus the log pair

$$\left(\overline{M}, \left(\frac{m_1}{n} + \frac{\mu}{n} - \frac{1}{9}\right)L + C + \frac{1}{n}\Delta\right)$$

is not log canonical at P. Using Theorem 7.5 of [9], we see that

$$\frac{1}{9}n - \mu - m_1 + m_2 = \Delta \cdot C \geqslant \mathrm{mult}_P(\Delta|_C) > n - m_1 - \mu + \frac{1}{9}n$$

since $C^2 = -1$ and $C \cdot L = 1$ on the surface $\overline{M}$. It follows that $m_2 > n$, a contradiction. The lemma is proved.

Lemma 6.20. *We have* $\gimel \neq 46$.

Proof. Suppose that $\gimel = 46$. Then X is a hypersurface of degree 21 in $\mathbb{P}(1, 1, 3, 7, 10)$, the point O is a singular point of type $\frac{1}{10}(1, 3, 7)$, and the base locus of $|-K_U|$ consists of irreducible smooth rational curves C and L such that $\alpha(C)$ is the unique base curve of $|-K_X|$ and L is contained in the surface E.

The curve C is the only one of $C_1, \dots, C_l$ that is contained in the surface T. The surface $\overline{M}$ contains C and L, whence $P \in \overline{M}$.

Suppose that $P \notin C$. Then it follows from the proof of Theorem 5.6.2 of [5] that one can find an integer $s > 0$ and a surface H in the linear system $|-21sK_U|$ such that the multiplicity of H at P is greater than or equal to s and H contains no components through P of the cycle $\overline{D} \cdot \overline{M}$. Then

$$21s\left(\frac{1}{10}n - \frac{11}{21}\mu\right) = \overline{D} \cdot H \cdot \overline{S} \geqslant \mathrm{mult}_P(\overline{D})s > \left(\frac{11}{10}n - \mu\right)s.$$

It follows that $\mu < n/10$. But $\mu > n/10$ by [10], a contradiction.

Thus $P = C \cap L$. We write $\overline{D}|_{\overline{M}} = m_1 L + m_2 C + \Delta$, where m_1 and m_2 are non-negative integers and Δ is an effective divisor (on the surface $\overline{M}$) whose support does not contain L or C. Then $m_2 \leqslant n$ by Remark 2.12.

The surface $\overline{M}$ is smooth at P. It follows from the proof of Theorem 7.5 in [9] that the log pair

$$\left(\overline{M}, \left(\frac{m_1}{n} + \frac{\mu}{n} - \frac{1}{10}\right)L + C + \frac{1}{n}\Delta\right)$$

is not log canonical at P since $E|_{\overline{M}} = L + Z$, where Z is an irreducible curve that does not pass through P. It now follows from Theorem 7.5 of [9] that

$$\frac{1}{10}n - \mu - m_1 + m_2 = \Delta \cdot C \geqslant \operatorname{mult}_P(\Delta|_C) > n - m_1 - \mu + \frac{1}{10}n$$

because $C^2 = -1$ and $C \cdot L = 1$. Thus $m_2 > n$. This contradicts the inequality $m_2 \leqslant n$ which follows from Remark 2.12. The lemma is proved.

This completes the proof of Lemma 2.11.

§ 7. The hypersurface of degree 5/2

In this section we prove Theorem 1.17. Let X be a generic surface of degree 5 in $\mathbb{P}(1,1,1,1,2)$. The singularities of X consist of a singular point O of type $\frac{1}{2}(1,1,1)$, and X can be given by the equation

$$xw^2 + f_3(x,y,z,t)w + f_5(x,y,z,t) = 0 \subset \mathbb{P}(1,1,1,1,2) \cong \operatorname{Proj}(\mathbb{C}[x,y,z,t,w]),$$

where $\operatorname{wt}(x) = \operatorname{wt}(y) = \operatorname{wt}(z) = \operatorname{wt}(t) = 1$ and $\operatorname{wt}(w) = 2$, $f_i(x,y,z,t)$ is a homogeneous polynomial of degree $i \geqslant 1$, and the point O is given by $x = y = z = t = 0$.

Let $\psi\colon X \dashrightarrow \mathbb{P}^3$ be the natural projection. Then there is a commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "\pi"'] \arrow[dr, "\gamma"] & & W_i \arrow[ll, "\alpha_i"'] \arrow[dr, "\omega_i"] & \\
X \arrow[dr, dashed, "\psi"'] & & Z \arrow[dl, "\eta"] & & U_i \arrow[ll, "\beta_i"'] \arrow[dl, "\xi_i"] \\
 & \mathbb{P}^3 \arrow[rr, dashed, "\chi_i"'] & & \mathbb{P}^2 &
\end{tikzcd} \tag{7.1}$$

where π is a blow-up of O with weights $(1,1,1)$, γ is a birational morphism that contracts 15 smooth rational curves $C_1, \ldots, C_{15}$ to 15 ordinary double points $P_1, \ldots, P_{15}$ of the variety Z, η is a double covering branched over the irreducible surface $R \subset \mathbb{P}^3$ of degree 6 that is given by an equation of the form

$$f_3^2(x,y,z,t) = 4xf_5(x,y,z,t) \subset \mathbb{P}^3 \cong \operatorname{Proj}(\mathbb{C}[x,y,z,t])$$

and has 15 ordinary double points $\eta(P_1), \ldots, \eta(P_{15})$, the morphism α_i is a blow-up of the smooth curve C_i, the morphism β_i is a blow-up of the point P_i, w_i is a birational morphism, χ_i is the projection from the point $\eta(P_i)$, and ξ_i is an elliptic fibration.

Put $\lambda = 7/9$. Let D be any effective $\mathbb{Q}$-divisor on X such that $D \equiv -K_X$. We claim that the log pair $(X, \lambda D)$ is log canonical, which implies that $\mathrm{lct}(X) \geqslant 7/9$.

Suppose that the log pair $(X, \lambda D)$ is not log canonical. Let $\mathcal{L}(X, \lambda D)$ be its subscheme of log canonical singularities (see [4]), and let $\mathcal{J}(\lambda D)$ be the corresponding ideal sheaf. Then

$$H^1(X, \mathcal{J}(\lambda D)) = 0 \tag{7.2}$$

by the Nadel vanishing theorem (see Theorem 2.16 in [9]).

Lemma 7.1. *Let T be a divisor in $|-K_X|$. Then the log pair (X, T) is log canonical.*

Proof. Let T be the surface cut out by the equation $x = 0$ on X. Since the hypersurface X is generic, Lemma 8.12 and Proposition 8.14 of [9] imply that the singularities of the log pair (X, T) are log canonical.

Hence we may assume that T is not cut out by $x = 0$. Then the diagram (7.1) and Lemma 8.12 of [9] yield that T is normal and the log pair (X, T) is log canonical at O.

Let P be a point of T different from the point O. If $\psi(P) \notin R$, then T is smooth at P. On the other hand, an easy parameter count yields that the singularity of T at P is at most $\mathbb{A}_n$ if $P \in \bigcup_{i=1}^{15} \pi(C_i)$.

We may assume that ψ is a double covering in a neighbourhood of P, the log pair (X, T) is not log canonical at P and $\psi(P) \in R$.

Put $\overline{T} = \psi(T)$ and $\overline{P} = \psi(P)$. It follows from Lemma 8.12 of [9] that the log pair $(\mathbb{P}^3, \overline{T} + \frac{1}{2}R)$ is not log canonical at $\overline{P}$. Then Theorem 7.5 of [9] yields that the log pair $(\overline{T}, \frac{1}{2}R|_{\overline{T}})$ is not log canonical at $\overline{P}$. Hence,

$$\mathrm{mult}_{\overline{P}}(R|_{\overline{T}}) \geqslant 3, \tag{7.3}$$

where $\overline{T}$ is a plane in $\mathbb{P}^3$. However, it follows from a parameter count that inequality (7.3) never holds for generic polynomials f_3 and f_5. The lemma is proved.

It follows from Remark 2.2 that to complete the proof of Theorem 1.17 we may assume that the support of the divisor D contains no surfaces of the linear system $|-K_X|$.

Lemma 7.2. *The scheme $\mathcal{L}(X, \lambda D)$ is zero-dimensional.*

Proof. Clearly, the scheme $\mathcal{L}(X, \lambda D)$ contains no two-dimensional components since $\lambda < 1$ and the divisor class group of X is generated by the divisor $-K_X$. Suppose that the scheme $\mathcal{L}(X, \lambda D)$ is not zero-dimensional.

There is a curve $C \subset X$ such that the singularities of the log pair $(X, \lambda D)$ are not log canonical at a generic point of C. In particular, we have

$$\mathrm{mult}_C(D) > \frac{1}{\lambda} = \frac{9}{7}$$

and we may assume that the curve C is irreducible.

Suppose that $\psi(C)$ is a curve and the intersection $R \cap \psi(C)$ contains some smooth point Q of the ramification surface R. Let $\overline{Q}$ be the point of X such that $\psi(\overline{Q}) = Q$. Then there is a surface $H' \in |-K_X|$ with a singularity at $\overline{Q}$. Then

$$\frac{5}{2} = H \cdot H' \cdot D \geqslant \mathrm{mult}_{\overline{Q}}(T)\,\mathrm{mult}_C(D) > \frac{2}{\lambda} = \frac{18}{7} > \frac{5}{2},$$

where H is a generic surface through $\overline{Q}$ in $|-K_X|$.

Hence either $\psi(C)$ is a point or $R \cap \psi(C) \subseteq \mathrm{Sing}(R)$.

Suppose that C is not contracted by ψ and the curve $\psi(C)$ is not a line. Let Q_1 and Q_2 be generic points on the curve C. Then

$$\frac{5}{2} = H_1 \cdot H_2 \cdot D \geqslant 2\,\mathrm{mult}_C(D) > \frac{2}{\lambda} = \frac{18}{7} > \frac{5}{2},$$

where H_i is a generic surface through Q_i in $|-K_X|$, a contradiction.

We easily see that no line in $\mathbb{P}^3$ can intersect the surface R only at singular points of R. It follows that C is one of the curves $\pi(C_1), \dots, \pi(C_{15})$.

We put $\overline{C}_i = \pi(C_i)$ and assume that $C = \overline{C}_1$. Let $\overline{T}$ be the surface cut out by the equation $x = 0$ on X, and let T be the proper transform of the surface $\overline{T}$ on Y. The surface $\overline{T}$ contains all the curves $\overline{C}_1, \dots, \overline{C}_{15}$, the surface T is smooth and the morphism γ induces a birational morphism

$$\gamma|_T \colon T \longrightarrow \gamma(T) \cong \mathbb{P}^2,$$

which contracts the curves $C_1, \dots, C_{15}$ to the points $P_1, \dots, P_{15}$ respectively.

We put $\grave{T} = \gamma(T)$ and $\breve{T} = \nu(\grave{T})$. Then $\breve{T}$ is a plane in $\mathbb{P}^3$.

Let L_j be the proper transform on the surface T of the line through the points $\nu(P_1)$ and $\nu(P_j)$ in $\breve{T}$, where $j \neq 1$. Then

$$C_1 \cdot L_j = C_j \cdot L_j = 1, \qquad C_i^2 = L_j^2 = -1, \qquad L_j \cdot C_k = L_i \cdot L_j = C_i \cdot C_k = 0$$

on the surface T, where $i \neq j \neq k$ and $j \neq 1$. Let E be the curve contracted by the birational morphism $\pi|_T$ to the point O. Then

$$E \cdot C_i = E \cdot L_j = 1, \qquad E^2 = -6$$

on the surface T. We put $\bar{L}_j = \pi(L_j)$. Then we have

$$\overline{C}_1 \cdot \bar{L}_j = \overline{C}_j \cdot \bar{L}_j = \frac{7}{6}, \qquad \overline{C}_i^2 = \bar{L}_j^2 = -\frac{5}{6}, \qquad \overline{C}_i \cdot \overline{C}_k = \overline{C}_k \cdot \bar{L}_j = \bar{L}_i \cdot \bar{L}_j = \frac{1}{6}$$

on the surface $\overline{T}$, where $i \neq j \neq k$ and $j \neq 1$. We now write

$$D|_{\overline{T}} = \overline{m}\,\overline{C}_1 + \sum_{i=2}^{15} \varepsilon_i \bar{L}_i + \Delta \equiv -K_X|_{\overline{T}},$$

where $\overline{m}$ and ε_i are non-negative rational numbers and Δ is an effective $\mathbb{Q}$-divisor (on the surface $\overline{T}$) whose support does not contain the curves $C_1, L_2, \dots, L_{15}$. Then $\overline{m} > 1/\lambda$ and

$$\frac{3}{2} = D|_{\overline{T}} \cdot \bar{L}_k = \overline{m}\,\overline{C}_1 \cdot \bar{L}_k + \sum_{i=2}^{15} \varepsilon_i \bar{L}_i \cdot \bar{L}_k + \Delta \cdot \bar{L}_k \geqslant \overline{m}\,\overline{C}_1 \cdot \bar{L}_k + \sum_{i=2}^{15} \varepsilon_i \bar{L}_i \cdot \bar{L}_k$$
$$= \frac{7\overline{m}}{6} - \varepsilon_k + \frac{\sum_{i=2}^{15} \varepsilon_i}{6},$$

where $k \neq 1$. Summing the last inequality over k, we get

$$21 \geqslant \frac{49\overline{m}}{3} + \frac{4}{3}\sum_{i=2}^{15}\varepsilon_i \geqslant \frac{49\overline{m}}{3}\,.$$

It follows that $\overline{m} \leqslant 9/7$. This contradicts the inequality $\overline{m} > 1/\lambda = 9/7$. The lemma is proved.

Equation (7.2) implies that there is a point $P \in X$ such that the log pair $(X, \lambda D)$ has log canonical singularities outside P. Let E be the exceptional divisor of the weighted blow-up π, and let $\overline{D}$ be the proper transform of the divisor D on the variety Y. Then

$$\overline{D} \equiv \pi^*(D) - mE,$$

where m is a non-negative rational number.

Lemma 7.3. *The point P is the point O.*

Proof. Suppose that $P \notin \bigcup_{i=1}^{15}\pi(C_i)$. Then it follows from Proposition 3 in [2] that there is a surface $T \in |-K_X|$ such that

$$\frac{5}{2} = D \cdot H \cdot T \geqslant \operatorname{mult}_P(D \cdot T) > \frac{2}{\lambda} = \frac{18}{7} > \frac{5}{2}\,,$$

where H is a generic surface through P in $|-K_X|$, a contradiction.

Assume that $P \neq O$ and $P \in \pi(C_1)$. Restricting the divisor D to the surface E, we see that $2m \geqslant \operatorname{mult}_{C_1}(\overline{D})$. It follows from the proof of Lemma 7.2 that $\operatorname{mult}_{C_1}(\overline{D}) \leqslant 9/7$.

Let $\breve{D}$ be the proper transform of the divisor D on the variety W_1. Then

$$\breve{D} \equiv (\pi \circ \alpha_1)^*(D) - m\breve{E} - \operatorname{mult}_{C_1}(\overline{D})G_1,$$

where G_1 is the exceptional divisor of the birational morphism α_1 and $\breve{E}$ is the proper transform of the divisor E on the variety W_1. The assumption $P \neq O$ and the equivalence

$$K_{W_1} + \lambda\breve{D} + \left(\lambda m - \frac{1}{2}\right)\breve{E} + (\lambda \operatorname{mult}_{C_1}(\overline{D}) - 1)G_1 \equiv (\pi \circ \alpha_1)^*(K_X + \lambda D)$$

imply that the log pair $(W_1, \lambda\breve{D})$ is not log canonical since $\operatorname{mult}_{C_1}(\overline{D}) \leqslant 9/7$. Then the log pair $(G_1, \lambda\breve{D})$ is not log canonical (Theorem 7.5 of [9]).

We have $G_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let L be the fibre of the morphism $\pi \circ \alpha_1$ over the point P. The curve L is contained in the surface G_1. Let $\acute{L}$ be the curve on G_1 such that $L \cdot \acute{L} = 0$ and $\acute{L}^2 = 0$. Then

$$\lambda\breve{D}|_{G_1} \equiv \left(\frac{\lambda}{2} - \lambda m + \lambda \operatorname{mult}_{C_1}(\overline{D})\right)L + \lambda \operatorname{mult}_{C_1}(\overline{D})\acute{L}.$$

It follows (see Lemma 1.7.9 in [4]) that the log pair $(G_1, \lambda\breve{D})$ is log canonical if

$$\frac{\lambda}{2} - \lambda m + \lambda \operatorname{mult}_{C_1}(\overline{D}) \leqslant 1.$$

But the log pair $(G_1, \lambda\breve{D})$ is not log canonical. Hence $\mathrm{mult}_{C_1}(\overline{D}) > 11/14 + m$. It follows that $9/7 \geqslant \mathrm{mult}_{C_1}(\overline{D}) > 11/7$ since $2m \geqslant \mathrm{mult}_{C_1}(\overline{D})$, a contradiction. The lemma is proved.

Let T be the surface cut out by the equation $x = 0$ on X. Then

$$\overline{T} \equiv \pi^*(T) - \frac{3}{2}E,$$

where $\overline{T}$ is the proper transform of the divisor T on Y. Hence, $5/2 - 3m = \overline{D} \cdot \overline{T} \cdot \overline{H} \geqslant 0$, where $\overline{H}$ is a generic surface in the linear system $|-K_Y|$.

Corollary 7.4. *We have* $m \leqslant 5/6$.

It follows from [10] or [9] that $m > 9/14$. Hence the equivalence

$$K_Y + \lambda\overline{D} + \left(\lambda m - \frac{1}{2}\right)E \equiv \pi^*(K_X + \lambda D)$$

implies the existence of a point $\overline{Q} \in E \cong \mathbb{P}^2$ such that the log pair $(Y, \lambda\overline{D} + (\lambda m - 1/2)E)$ is not log canonical at $\overline{Q}$. In particular, we have

$$\mathrm{mult}_{\overline{Q}}(\overline{D}) > \frac{3}{2\lambda} - m$$

because the divisor $\lambda\overline{D} + (\lambda m - 1/2)E$ is effective.

Lemma 7.5. *If* $\overline{Q} \in \overline{T}$, *then* $\overline{Q} \in \bigcup_{i=1}^{15} C_i$.

Proof. Suppose that $\overline{Q} \in \overline{T}$ and $\overline{Q} \notin \bigcup_{i=1}^{15} C_i$. Let $\overline{H}$ be a generic surface through $\overline{Q}$ in the linear system $|-K_Y|$. Then $\overline{H}$ contains no components of the effective cycle $\overline{D} \cdot \overline{T}$. Thus we see that

$$\frac{5}{2} - 3m = \overline{D} \cdot \overline{T} \cdot \overline{H} \geqslant \mathrm{mult}_{\overline{Q}}(\overline{D}) > \frac{3}{2\lambda} - m.$$

It follows that $m < 1/2$, a contradiction. The lemma is proved.

Lemma 7.6. *Let* $\iota\colon S \to \mathbb{P}^2$ *be a double covering branched over a smooth curve* $Z \subset \mathbb{P}^2$ *of degree* 6, *and let* B *be an effective* $\mathbb{Q}$*-divisor on* S *such that*

1) *we have* $B \equiv \iota^*(\mathcal{O}_{\mathbb{P}^2}(1))$,

2) *the log pair* (S, B) *is log canonical in a punctured neighbourhood of a point* $\Theta \in S$ *such that* $\iota(\Theta) \notin Z$.

Then the log pair (S, B) *is log canonical at* Θ.

Proof. Suppose that the log pair (S, B) is not log canonical at Θ. Taking the intersection of the divisor B with the proper transform of a generic line through the point $\iota(\Theta)$ in the plane $\mathbb{P}^2$, we see that $\mathrm{mult}_\Theta(B) \leqslant 2$.

Let $\upsilon\colon \overline{S} \to S$ be a blow-up of the point Θ. Then

$$K_{\overline{S}} + \overline{B} + (\mathrm{mult}_\Theta(B) - 1)F \equiv \upsilon^*(K_S + B),$$

where F is the exceptional curve of the blow-up υ and $\overline{B}$ is the proper transform of the divisor B on the surface $\overline{S}$. Then there is a point $\overline{\Theta} \in F$ such that the singularities of the log pair $(\overline{S}, \overline{B} + (\mathrm{mult}_{\Theta}(B) - 1)F)$ are not log canonical at $\overline{\Theta}$. Hence $\mathrm{mult}_{\overline{\Theta}}(\overline{B}) > 2 - \mathrm{mult}_{\Theta}(B)$.

Let L be a generic surface in the linear system $\left|\iota^*(\mathcal{O}_{\mathbb{P}^2}(1))\right|$ such that the proper transform $\bar{L}$ of L on the surface $\overline{S}$ passes through the point $\overline{\Theta}$. Then L consists of at most two components and one of them contains the point Θ, but L is smooth at Θ.

By Remark 2.2 we may assume that the support of the divisor B does not contain at least one component of L. Therefore, if L is irreducible, then

$$2 - \mathrm{mult}_{\Theta}(B) = \overline{B} \cdot \bar{L} \geqslant \mathrm{mult}_{\overline{\Theta}}(\overline{B}) > 2 - \mathrm{mult}_{\Theta}(B),$$

which is a contradiction. Hence we see that L is reducible.

Let L_1 and L_2 be the components of L such that $L_2 \not\ni \Theta \in L_1$, and let $\bar{L}_1$ be the proper transform of L_1 on the surface $\overline{S}$. Then we have

$$1 - \mathrm{mult}_{\Theta}(B) = \overline{B} \cdot \bar{L}_1 \geqslant \mathrm{mult}_{\overline{\Theta}}(\overline{B}) > 2 - \mathrm{mult}_{\Theta}(B) > 1 - \mathrm{mult}_{\Theta}(B)$$

in the case when the support of B does not contain L_1. Thus we see that the support of B contains L_1 but not L_2. We write

$$B = \varepsilon L_1 + \Omega,$$

where ε is a positive rational number and Ω is an effective $\mathbb{Q}$-divisor (on the surface S) whose support does not contain L_1. Then

$$1 = B \cdot L_2 = 3\varepsilon + \Omega \cdot L_2.$$

It follows that $\varepsilon \leqslant 1/3$.

Let $\overline{\Omega}$ be the proper transform of the divisor Ω on the surface $\overline{S}$. Then the log pair $(\overline{S}, \varepsilon\bar{L}_1 + \overline{\Omega} + (\mathrm{mult}_{\Theta}(\Omega) + \varepsilon - 1)F)$ is not log canonical at the point $\overline{\Theta}$. By Theorem 7.5 of [9], the log pair $(\bar{L}_1, \overline{\Omega}|_{\bar{L}_1} + (\mathrm{mult}_{\Theta}(\Omega) + \varepsilon - 1)F|_{\bar{L}_1})$ is not log canonical at $\overline{\Theta}$. This is equivalent to the inequality

$$\mathrm{mult}_{\overline{\Theta}}(\overline{\Omega}|_{\bar{L}_1}) > 2 - \mathrm{mult}_{\Theta}(\Omega) - \varepsilon,$$

whence $\overline{\Omega} \cdot \bar{L}_1 > 2 - \mathrm{mult}_{\Theta}(\Omega) - \varepsilon$. We have

$$1 - \mathrm{mult}_{\Theta}(\Omega) + 2\varepsilon = \overline{\Omega} \cdot \bar{L}_1 > 2 - \mathrm{mult}_{\Theta}(\Omega) - \varepsilon.$$

It follows that $\varepsilon > 1/3$. But we have seen that $\varepsilon \leqslant 1/3$. The lemma is proved.

Lemma 7.6 yields the following result.

Lemma 7.7. *The point $\overline{Q}$ belongs to the set $\bigcup_{i=1}^{15} C_i$.*

Proof. Suppose that $\overline{Q} \notin \bigcup_{i=1}^{15} C_i$. Let $\overline{H}$ be a generic surface through Q in the linear system $|-K_Y|$. Then the log pair $(\overline{H}, \lambda\overline{D}|_{\overline{H}} + (\lambda m - 1/2)E|_{\overline{H}})$ is not log canonical at $\overline{Q}$ (see Theorem 7.5 in [9]). This contradicts Lemma 7.6 since $\overline{Q} \notin \overline{T}$ by Lemma 7.5. The lemma is proved.

We may assume that $\overline{Q} \in C_1$. Let S be a generic surface through $\overline{Q}$ in the linear system $|-K_Y|$. We put $\Theta = \overline{Q}$, $L_1 = E|_S$, $C = C_1$ and $B = \overline{D}|_S + (m - 1/2)L_1$. The proof of Lemma 7.2 yields that $\mathrm{mult}_C(D) \leqslant 1/\lambda$. Hence the following assertions hold.

1) We have $B \equiv \iota^*(\mathcal{O}_{\mathbb{P}^2}(1))$, where $\iota = \eta \circ \gamma|_S \colon S \to \mathbb{P}^2$.

2) The log pair $(S, \lambda B)$ is log canonical in a punctured neighbourhood of the point $\Theta \in S$.

Lemma 7.8. *The log pair $(S, \lambda B)$ is log canonical at the point Θ.*

Proof. Suppose that the log pair $(S, \lambda B)$ is not log canonical at Θ. Let L_2 be an irreducible curve such that $\iota(L_2) = \iota(L_1)$ and $L_1 \neq L_2$. Then

$$L_1 + L_2 + C \equiv \iota^*(\mathcal{O}_{\mathbb{P}^2}(1))$$

and the log pair $(S, L_1 + L_2 + C)$ is log canonical.

We may assume (see Remark 2.2) that the support of the divisor B does not contain one of the curves L_1, L_2, C. Taking the intersection of B with these curves, we see that the support of B does not contain L_2. We write

$$B = \varepsilon C + \overline{m} L_1 + \Omega,$$

where ε and $\overline{m}$ are non-negative rational numbers and Ω is an effective $\mathbb{Q}$-divisor (on the surface S) whose support does not contain the curves L_1, L_2, C. Then

$$1 = B \cdot L_2 = 2\overline{m} + \varepsilon + \Omega \cdot L_2 \geqslant 2\overline{m} + \varepsilon.$$

It follows that $2\overline{m} + \varepsilon \leqslant 1$. By Theorem 7.5 in [9], the log pairs

$$(C, \lambda \overline{m} L_1|_C + \lambda \Omega|_C), \qquad (L_1, \lambda \varepsilon C|_{L_1} + \lambda \Omega|_{L_1})$$

are not log canonical at Θ. Hence we have $2\varepsilon > 1/\lambda$ and $2\overline{m} > 1/\lambda - 1$ respectively.

Let $\upsilon \colon \overline{S} \to S$ be a blow-up of the point Θ. Then

$$K_{\overline{S}} + \lambda\varepsilon\overline{C} + \lambda\overline{m}\bar{L}_1 + \lambda\overline{\Omega} + (\lambda\varepsilon + \lambda\overline{m} + \mathrm{mult}_\Theta(\Omega) - 1)F \equiv \upsilon^*(K_S + \lambda\varepsilon C + \lambda\overline{m}L_1 + \lambda\Omega),$$

where F is the exceptional curve of the birational morphism υ and $\overline{C}$, $\bar{L}_1$, $\overline{\Omega}$ are the proper transforms of the divisors C, L_1, Ω respectively. Then it follows that $\lambda\varepsilon + \lambda\overline{m} + \mathrm{mult}_\Theta(\Omega) - 1 < 1$ because

$$0 = B \cdot C = -2\varepsilon + \overline{m} + C \cdot \Omega \geqslant -2\varepsilon + \overline{m} + \mathrm{mult}_\Theta(\Omega)$$

and $2\overline{m} + \varepsilon \leqslant 1$. Hence there is a point $\overline{\Theta} \in F$ such that the log pair

$$(\overline{S}, \lambda\varepsilon\overline{C} + \lambda\overline{m}\bar{L}_1 + \lambda\overline{\Omega} + (\lambda\varepsilon + \lambda\overline{m} + \mathrm{mult}_\Theta(\Omega) - 1)F)$$

is not log canonical at $\overline{\Theta}$.

Suppose that $\overline{\Theta} \notin \overline{C} \cup \bar{L}_1$. Then $\mathrm{mult}_\Theta(\Omega) = F \cdot \overline{\Omega} > 1/\lambda = 9/7$ because the log pair $(F, \lambda\overline{\Omega}|_F)$ is not log canonical at $\overline{\Theta}$ by Theorem 7.5 of [9]. We have

$$0 = C \cdot B = \overline{m} - 2\varepsilon + \Omega \cdot C > \overline{m} - 2\varepsilon + \frac{1}{\lambda},$$

$$1 = L_1 \cdot B = -2\overline{m} + \varepsilon + \Omega \cdot L_1 > -2\overline{m} + \varepsilon + \frac{1}{\lambda}.$$

This contradicts the inequality $2\overline{m} + \varepsilon \leqslant 1$.

Suppose that $\overline{\Theta} \in \bar{L}_1$. Then

$$1 - \mathrm{mult}_{\Theta}(\Omega) - \varepsilon + 2\overline{m} = \bar{L}_1 \cdot \overline{\Omega} > \frac{2}{\lambda} - \mathrm{mult}_{\Theta}(\Omega) - \varepsilon - \overline{m}$$

because the singularities of the log pair $(\bar{L}_1, \lambda\overline{\Omega}|_{\bar{L}_1} + (\lambda\varepsilon + \lambda\overline{m} + \mathrm{mult}_{\Theta}(\Omega) - 1)F|_{\bar{L}_1})$ are not log canonical at the point $\overline{\Theta}$ by Theorem 7.5 of [9]. Hence $\overline{m} > 11/21$. This contradicts the inequality $2\overline{m} + \varepsilon \leqslant 1$.

Hence we see that $\overline{\Theta} \in \overline{C}$. Then

$$-\mathrm{mult}_{\Theta}(\Omega) + 2\varepsilon - \overline{m} = \overline{C} \cdot \overline{\Omega} > \frac{2}{\lambda} - \mathrm{mult}_{\Theta}(\Omega) - \varepsilon - \overline{m}$$

because the singularities of the log pair $(\overline{C}, \lambda\overline{\Omega}|_{\overline{C}} + (\lambda\varepsilon + \lambda\overline{m} + \mathrm{mult}_{\Theta}(\Omega) - 1)F|_{\overline{C}})$ are not log canonical at the point $\overline{\Theta}$ by Theorem 7.5 in [9]. Hence $\varepsilon > 6/7$. This contradicts the inequalities $2\overline{m} > 1/\lambda - 1 = 2/7$ and $2\overline{m} + \varepsilon \leqslant 1$. The lemma is proved.

By Theorem 7.5 of [9], the log pair $(S, \lambda B)$ is not log canonical at Θ. This contradicts Lemma 7.8. Theorem 1.17 is proved.

§ 8. Birational automorphisms

Let X be a generic hypersurface of degree 20 in $\mathbb{P}(1,1,4,5,10)$. Then the singularities of X consist of a singular point Q of type $\frac{1}{2}(1,1,1)$ and singular points O_1 and O_2 of type $\frac{1}{5}(1,1,4)$.

Let $\alpha\colon U \to X$ be a weighted blow-up of the point O_1 with weights $(1,1,4)$. Then there is a commutative diagram

$$\begin{tikzcd}
 & & U \arrow[dll, "\sigma"'] \arrow[d, "\alpha"] & & Y \arrow[ll, "\gamma"'] \arrow[dd, "\eta"] \\
V \arrow[dr, "\omega"'] & & X \arrow[dl, dashed, "\xi"'] \arrow[drr, dashed, "\psi"] & & \\
 & \mathbb{P}(1,1,4,10) \arrow[rrr, dashed, "\chi"'] & & & \mathbb{P}(1,1,4)
\end{tikzcd}$$

where ξ, ψ and χ are projections, γ is a weighted blow-up with weights $(1,1,4)$ of the singular point that dominates O_2, η is an elliptic fibration, σ is a birational morphism that contracts 75 rational curves $C_1, \dots, C_{75}$, and ω is a double covering.

Let τ be the involution of X that is induced by the projection $X \to \mathbb{P}(1,1,4,5)$.

Lemma 8.1. *The group* $\mathrm{Aut}(X)$ *is generated by the involution* τ.

Proof. We put $P_i = \sigma(C_i)$. Then X can be given by the equation

$$t^2w + tg(x,y,z,w) + h(x,y,z,w) = 0 \subset \mathbb{P}(1,1,4,5,10) \cong \mathrm{Proj}(\mathbb{C}[x,y,z,t,w]),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 4$, $\mathrm{wt}(t) = 5$, $\mathrm{wt}(w) = 10$, and g, h are quasihomogeneous polynomials of degree 15 and 20 respectively. The point O_1 is given by $x = y = z = w = 0$.

Let υ be any biregular automorphism of X such that $\upsilon \neq \tau$. Twisting υ by τ if necessary, we may assume that $\upsilon(O_1) = O_1$. We claim that then υ is the identity map.

Indeed, the automorphism υ induces a biregular automorphism $\grave{\upsilon}$ of the threefold U such that the divisor E is $\grave{\upsilon}$-invariant. Put $\grave{P}_i = C_i \cap E$. Then the points $\grave{P}_1, \ldots, \grave{P}_{75}$ can be given by

$$\grave{g}(x, y, z) = \grave{h}(x, y, z) = 0 \subset \mathbb{P}(1, 1, 4) \cong \operatorname{Proj}(\mathbb{C}[x, y, z]),$$

where $\operatorname{wt}(x) = \operatorname{wt}(y) = 1$, $\operatorname{wt}(z) = 4$ and $\grave{g}$, $\grave{h}$ are generic quasi-homogeneous polynomials of degree 15 and 20 respectively. The set $\{\grave{P}_1, \ldots, \grave{P}_{75}\}$ is $\grave{\upsilon}|_E$-invariant. The proper transform on U of the surface that is cut out on X by the equation $w = 0$ is $\grave{\upsilon}$-invariant. It follows that $\grave{\upsilon}|_E$ is the identity map.

We easily see that υ induces a biregular automorphism $\bar{\upsilon}$ of Y such that the fibration η is invariant with respect to $\bar{\upsilon}$. Let $\overline{E}$ and $\overline{F}$ be exceptional divisors of the birational morphism $\alpha \circ \gamma$ such that $\gamma(\overline{E}) = E$ and $\alpha \circ \gamma(\overline{F}) = O_2$. Then the surfaces $\overline{E}$ and $\overline{F}$ are $\bar{\upsilon}$-invariant sections of η and the restrictions $\bar{\upsilon}|_{\overline{E}}$ and $\bar{\upsilon}|_{\overline{F}}$ are the identity maps.

As shown in [6], the sections $\overline{E}$ and $\overline{F}$ induce $\mathbb{Z}$-linearly independent points in the Picard group of a generic fibre of η. Hence the induced automorphism $\bar{\upsilon}$ acts trivially on the generic fibre of η. It follows that $\bar{\upsilon}$ is the identity map. Thus the automorphism υ is the identity map. The lemma is proved.

The following result is a corollary of [6] and Lemma 8.1.

Corollary 8.2. *There is an exact sequence of groups*

$$1 \longrightarrow \mathbb{Z}_2 * \mathbb{Z}_2 \longrightarrow \operatorname{Bir}(X) \longrightarrow \mathbb{Z}_2 \longrightarrow 1,$$

whence $\operatorname{Bir}(X) \cong \mathbb{Z}_2 * \mathbb{Z}_2$.

The following result can similarly be deduced from [6].

Lemma 8.3. *Let* V *be a generic hypersurface of degree* 9 *in* $\mathbb{P}(1, 1, 2, 3, 3)$. *Then*

$$\operatorname{Bir}(V) \cong \langle a, b, c \mid a^2 = b^2 = c^2 = (abc)^2 = 1 \rangle.$$

Proof. The singularities of the threefold V consist of points O_1, O_2, O_3 of type $\frac{1}{3}(1, 1, 2)$ and a singular point O of type $\frac{1}{2}(1, 1, 1)$. Let υ be a biregular automorphism of V. We claim that υ is the identity map (see [6]).

Indeed, let Z be the base curve of the pencil $|-K_V|$. Then Z is a smooth υ-invariant rational curve that contains the points O_1, O_2, O_3 and O. It follows that $\upsilon(O_i) = O_i$ because

$$\upsilon(\{O_1, O_2, O_3\}) = \{O_1, O_2, O_3\}$$

and $\upsilon(O) = O$. Arguing as in the proof of Lemma 8.1, we see that υ is the identity map. The lemma is proved.

It would be interesting to prove analogues of Corollary 8.2 and Lemma 8.3 for all birationally rigid Fano threefolds that satisfy the hypotheses of Theorem 1.14.

§ 9. Kollár's method

In this section we consider an alternative approach to the proof of Theorem 1.15 due to J. Kollár. We use the hypotheses and notation of Theorem 1.14.

The hypersurface X can be given by the equation

$$f(x,y,z,t,w)=0\subset\mathbb{P}(1,a_1,a_2,a_3,a_4)\cong\operatorname{Proj}(\mathbb{C}[x,y,z,t,w]),$$

where $\operatorname{wt}(x)=1$, $\operatorname{wt}(y)=a_1$, $\operatorname{wt}(z)=a_2$, $\operatorname{wt}(t)=a_3$, $\operatorname{wt}(w)=a_3$, and $f(x,y,z,t,w)$ is a generic quasi-homogeneous polynomial of degree $d=\sum_{i=1}^4 a_i$. We introduce the following notation: S is a generic surface in the linear system $|-K_X|$, Z is the curve that is cut out on S by the equations $x=y=0$, and L is the curve in $\mathbb{P}(1,a_1,a_2,a_3,a_4)$ given by the equations $x=y=z=0$.

Proposition 9.1. *We have* $\operatorname{lct}(X)=1$ *if one of the following conditions holds*:

1) $d\leqslant a_1a_2$,
2) $d\leqslant a_1a_3$ *and the curve* L *is not contained in* X,
3) $d\leqslant a_2a_3$, *the curve* Z *is irreducible and the log pair* $(S,\frac{1}{a_1}Z)$ *is log canonical.*

Proof. Let D be an arbitrary effective $\mathbb{Q}$-divisor on X such that the numerical equivalence $D\equiv -K_X$ holds. It follows from Lemma 2.4 that the log pair (X,D) is log canonical outside the singular points of X, but S contains all singular points of X.

Suppose that the log pair (X,D) is not log canonical. Then Theorem 7.5 of [9] yields that the log pair $(S,D|_S)$ is not log canonical. This contradicts Corollary 12 of [13] if either 1) or 2) holds.

To complete the proof, we assume that $d\leqslant a_2a_3$, the curve Z is irreducible and the log pair $(S,\frac{1}{a_1}Z)$ has log canonical singularities. We may assume that the support of the divisor $D|_S$ does not contain the curve Z (see Remark 2.2). Then the proof of Proposition 11 in [13] yields that $d>a_2a_3$, a contradiction. The proposition is proved.

Corollary 9.2. *We have* $\operatorname{lct}(X)=1$ *in the case when* $\gimel\geqslant 17$ *and*

$$\gimel\notin\{18,\dots,25,28,29,30,32,33,34,35,37,38,42,43,46,47,50,52,\\ 55,56,57,62,63,67,82\}.$$

A proof of Proposition 9.1 was communicated to the author by J. Kollár.

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