

# Halphen pencils on quartic threefolds <sup>☆,☆☆</sup>

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## Abstract

For any smooth quartic threefold in  $\mathbb{P}^4$  we classify pencils on it whose general element is an irreducible surface birational to a surface of Kodaira dimension zero.

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## 1. Introduction

Let  $X$  be a smooth quartic threefold in  $\mathbb{P}^4$ . The following result is proved in [4].

**Theorem 1.1.** *The threefold  $X$  does not contain pencils whose general element is an irreducible surface that is birational to a smooth surface of Kodaira dimension  $-\infty$ .*

On the other hand, one can easily see that  $X$  contains infinitely many pencils whose general elements are irreducible surfaces of Kodaira dimension zero (cf. [1–3]).

**Definition 1.2.** A Halphen pencil is a one-dimensional linear system whose general element is an irreducible subvariety birational to a smooth variety of Kodaira dimension zero.

The following result is proved in [2].

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<sup>☆☆</sup> We assume that all varieties are projective, normal and defined over  $\mathbb{C}$ .

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**Theorem 1.3.** *Suppose that  $X$  is general. Then every Halphen pencil on  $X$  is cut out by*

$$\lambda l_1(x, y, z, t, w) + \mu l_2(x, y, z, t, w) = 0 \subset \text{Proj}(\mathbb{C}[x, y, z, t, w]) \cong \mathbb{P}^4,$$

where  $l_1$  and  $l_2$  are linearly independent linear forms, and  $(\lambda : \mu) \in \mathbb{P}^1$ .

The assertion of Theorem 1.3 is erroneously proved in [1] without the assumption that the threefold  $X$  is general. On the other hand, the following example is constructed in [3].

**Example 1.4.** Suppose that  $X$  is given by the equation

$$w^3x + w^2q_2(x, y, z, t) + wxp_2(x, y, z, t) + q_4(x, y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[x, y, z, t, w]) \cong \mathbb{P}^4,$$

where  $q_i$  and  $p_i$  are forms of degree  $i$ . Let  $\mathcal{P}$  be the pencil on  $X$  that is cut out by

$$\lambda x^2 + \mu(wx + q_2(x, y, z, t)) = 0,$$

where  $(\lambda : \mu) \in \mathbb{P}^1$ . Then  $\mathcal{P}$  is a Halphen pencil if  $q_2(0, y, z, t) \neq 0$  by [2, Theorem 2.3].

The purpose of this paper is to prove the following result.

**Theorem 1.5.** *Let  $\mathcal{M}$  be a Halphen pencil on the threefold  $X$ . Then*

- *either  $\mathcal{M}$  is cut out on  $X$  by the pencil*

$$\lambda l_1(x, y, z, t, w) + \mu l_2(x, y, z, t, w) = 0 \subset \text{Proj}(\mathbb{C}[x, y, z, t, w]) \cong \mathbb{P}^4,$$

where  $l_1$  and  $l_2$  are linearly independent linear forms, and  $(\lambda : \mu) \in \mathbb{P}^1$ ,

- *or the threefold  $X$  can be given by the equation*

$$w^3x + w^2q_2(x, y, z, t) + wxp_2(x, y, z, t) + q_4(x, y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[x, y, z, t, w]) \cong \mathbb{P}^4$$

such that  $q_2(0, y, z, t) \neq 0$ , and  $\mathcal{M}$  is cut out on the threefold  $X$  by the pencil

$$\lambda x^2 + \mu(wx + q_2(x, y, z, t)) = 0,$$

where  $q_i$  and  $p_i$  are forms of degree  $i$ , and  $(\lambda : \mu) \in \mathbb{P}^1$ .

Let  $P$  be an arbitrary point of the quartic hypersurface  $X \subset \mathbb{P}^4$ .

**Definition 1.6.** The mobility threshold of the threefold  $X$  at the point  $P$  is the number

$$\iota(P) = \sup\{\lambda \in \mathbb{Q} \text{ such that } |n(\pi^*(-K_X) - \lambda E)| \text{ has no fixed components for } n \gg 0\},$$

where  $\pi : Y \rightarrow X$  is the ordinary blow up of  $P$ , and  $E$  is the exceptional divisor of  $\pi$ .

Arguing as in the proof of Theorem 1.5, we obtain the following result.

**Theorem 1.7.** *The following conditions are equivalent:*

- the equality  $\iota(P) = 2$  holds,
- the threefold  $X$  can be given by the equation

$$w^3x + w^2q_2(x, y, z, t) + wxp_2(x, y, z, t) + q_4(x, y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[x, y, z, t, w]) \cong \mathbb{P}^4,$$

where  $q_i$  and  $p_i$  are forms of degree  $i$  such that

$$q_2(0, y, z, t) \neq 0,$$

and  $P$  is given by the equations  $x = y = z = t = 0$ .

One can easily check that  $2 \geq \iota(P) \geq 1$ . Similarly, one can show that

- $\iota(P) = 1 \iff$  the hyperplane section of  $X$  that is singular at  $P$  is a cone,
- $\iota(P) = 3/2 \iff$  the threefold  $X$  contains no lines passing through  $P$ .

The proof of Theorem 1.5 is completed on board of IL-96-300 *Valery Chkalov* while flying from Seoul to Moscow. We thank Aeroflot Russian Airlines for good working conditions.

## 2. Important lemma

Let  $S$  be a normal surface, let  $O$  be a smooth point of  $S$ , let  $R$  be an effective divisor on the surface  $S$ , and let  $\mathcal{D}$  be a linear system on the surface  $S$  that has no fixed components.

**Lemma 2.1.** *Let  $D_1$  and  $D_2$  be general curves in  $\mathcal{D}$ . Then*

$$\text{mult}_O(D_1 \cdot R) = \text{mult}_O(D_2 \cdot R) \leq \text{mult}_O(R) \text{mult}_O(D_1 \cdot D_2).$$

**Proof.** Put  $S_0 = S$  and  $O_0 = O$ . Let us consider the sequence of blow ups

$$S_n \xrightarrow{\pi_n} S_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} S_0$$

such that  $\pi_1$  is a blow up of the point  $O_0$ , and  $\pi_i$  is a blow up of the point  $O_{i-1}$  that is contained in the curve  $E_{i-1}$ , where  $E_{i-1}$  is the exceptional curve of  $\pi_{i-1}$ , and  $i = 2, \dots, n$ .

Let  $D_j^i$  be the proper transform of  $D_j$  on  $S_i$  for  $i = 0, \dots, n$  and  $j = 1, 2$ . Then

$$D_1^i \equiv D_2^i \equiv \pi_i^*(D_1^{i-1}) - \text{mult}_{O_{i-1}}(D_1^{i-1})E_i \equiv \pi_i^*(D_2^{i-1}) - \text{mult}_{O_{i-1}}(D_2^{i-1})E_i$$

for  $i = 1, \dots, n$ . Put  $d_i = \text{mult}_{O_{i-1}}(D_1^{i-1}) = \text{mult}_{O_{i-1}}(D_2^{i-1})$  for  $i = 1, \dots, n$ .

Let  $R^i$  be the proper transform of  $R$  on the surface  $S_i$  for  $i = 0, \dots, n$ . Then

$$R^i \equiv \pi_i^*(R^{i-1}) - \text{mult}_{O_{i-1}}(R^{i-1})E_i$$

for  $i = 1, \dots, n$ . Put  $r_i = \text{mult}_{O_{i-1}}(R^{i-1})$  for  $i = 1, \dots, n$ . Then  $r_1 = \text{mult}_O(R)$ .

We may choose the blow ups  $\pi_1, \dots, \pi_n$  in a way such that  $D_1^n \cap D_2^n$  is empty in the neighborhood of the exceptional locus of  $\pi_1 \circ \pi_2 \circ \dots \circ \pi_n$ . Then

$$\text{mult}_O(D_1 \cdot D_2) = \sum_{i=1}^n d_i^2.$$

We may choose the blow ups  $\pi_1, \dots, \pi_n$  in a way such that  $D_1^n \cap R^n$  and  $D_2^n \cap R^n$  are empty in the neighborhood of the exceptional locus of  $\pi_1 \circ \pi_2 \circ \dots \circ \pi_n$ . Then

$$\text{mult}_O(D_1 \cdot R) = \text{mult}_O(D_2 \cdot R) = \sum_{i=1}^n d_i r_i,$$

where some numbers among  $r_1, \dots, r_n$  may be zero. Then

$$\text{mult}_O(D_1 \cdot R) = \text{mult}_O(D_2 \cdot R) = \sum_{i=1}^n d_i r_i \leq \sum_{i=1}^n d_i r_1 \leq \sum_{i=1}^n d_i^2 r_1 = \text{mult}_O(R) \text{mult}_O(D_1 \cdot D_2),$$

because  $d_i \leq d_i^2$  and  $r_i \leq r_1 = \text{mult}_O(R)$  for every  $i = 1, \dots, n$ .  $\square$

The assertion of Lemma 2.1 is a cornerstone of the proof of Theorem 1.5.

### 3. Curves

Let  $X$  be a smooth quartic threefold in  $\mathbb{P}^4$ , let  $\mathcal{M}$  be a Halphen pencil on  $X$ . Then

$$\mathcal{M} \sim -nK_X,$$

since  $\text{Pic}(X) = \mathbb{Z}K_X$ . Put  $\mu = 1/n$ . Then

- the log pair  $(X, \mu\mathcal{M})$  is canonical by [3, Theorem A],
- the log pair  $(X, \mu\mathcal{M})$  is not terminal by [2, Theorem 2.1].

Let  $\mathbb{CS}(X, \mu\mathcal{M})$  be the set of non-terminal centers of  $(X, \mu\mathcal{M})$  (see [2]). Then

$$\mathbb{CS}(X, \mu\mathcal{M}) \neq \emptyset,$$

because  $(X, \mu\mathcal{M})$  is not terminal. Let  $M_1$  and  $M_2$  be two general surfaces in  $\mathcal{M}$ .

**Lemma 3.1.** *Suppose that  $\mathbb{CS}(X, \mu\mathcal{M})$  contains a point  $P \in X$ . Then*

$$\text{mult}_P(M) = n \text{mult}_P(T) = 2n,$$

where  $M$  is any surface in  $\mathcal{M}$ , and  $T$  is the surface in  $|-K_X|$  that is singular at  $P$ .

**Proof.** It follows from [5, Proposition 1] that the inequality

$$\mathrm{mult}_P(M_1 \cdot M_2) \geq 4n^2$$

holds. Let  $H$  be a general surface in  $|-K_X|$  such that  $P \in H$ . Then

$$4n^2 = H \cdot M_1 \cdot M_2 \geq \mathrm{mult}_P(M_1 \cdot M_2) \geq 4n^2,$$

which gives  $(M_1 \cdot M_2)_P = 4n^2$ . Arguing as in the proof of [5, Proposition 1], we see that

$$\mathrm{mult}_P(M_1) = \mathrm{mult}_P(M_2) = 2n,$$

because  $(M_1 \cdot M_2)_P = 4n^2$ . Similarly, we see that

$$4n = H \cdot T \cdot M_1 \geq \mathrm{mult}_P(T) \mathrm{mult}_P(M_1) = 2n \mathrm{mult}_P(T) \geq 4n,$$

which implies that  $\mathrm{mult}_P(T) = 2$ . Finally, we also have

$$4n^2 = H \cdot M \cdot M_1 \geq \mathrm{mult}_P(M) \mathrm{mult}_P(M_1) = 2n \mathrm{mult}_P(M) \geq 4n^2,$$

where  $M$  is any surface in  $\mathcal{M}$ , which completes the proof.  $\square$

**Lemma 3.2.** Suppose that  $\mathbb{CS}(X, \mu\mathcal{M})$  contains a point  $P \in X$ . Then

$$M_1 \cap M_2 = \bigcup_{i=1}^r L_i,$$

where  $L_1, \dots, L_r$  are lines on the threefold  $X$  that pass through the point  $P$ .

**Proof.** Let  $H$  be a general surface in  $|-K_X|$  such that  $P \in H$ . Then

$$4n^2 = H \cdot M_1 \cdot M_2 = \mathrm{mult}_P(M_1 \cdot M_2) = 4n^2$$

by Lemma 3.1. Then  $\mathrm{Supp}(M_1 \cdot M_2)$  consists of lines on  $X$  that pass through  $P$ .  $\square$

**Lemma 3.3.** Suppose that  $\mathbb{CS}(X, \mu\mathcal{M})$  contains a point  $P \in X$ . Then

$$n/3 \leq \mathrm{mult}_L(\mathcal{M}) \leq n/2$$

for every line  $L \subset X$  that passes through the point  $P$ .

**Proof.** Let  $D$  be a general hyperplane section of  $X$  through  $L$ . Then we have

$$M|_D = \mathrm{mult}_L(\mathcal{M})L + \Delta,$$

where  $M$  is a general surface in  $\mathcal{M}$  and  $\Delta$  is an effective divisor such that

$$\mathrm{mult}_P(\Delta) \geq 2n - \mathrm{mult}_L(\mathcal{M}).$$

On the surface  $D$  we have  $L \cdot L = -2$ . Then

$$n = (\text{mult}_L(\mathcal{M})L + \Delta) \cdot L = -2 \text{mult}_L(\mathcal{M}) + \Delta \cdot L$$

on the surface  $D$ . But  $\Delta \cdot L \geq \text{mult}_P(\Delta) \geq 2n - \text{mult}_L(\mathcal{M})$ . Thus, we get

$$n \geq -2 \text{mult}_L(\mathcal{M}) + \text{mult}_P(\Delta) \geq 2n - 3 \text{mult}_L(\mathcal{M}),$$

which implies that  $n/3 \leq \text{mult}_L(\mathcal{M})$ .

Let  $T$  be the surface in  $|-K_X|$  that is singular at  $P$ . Then  $T \cdot D$  is reduced and

$$T \cdot D = L + Z,$$

where  $Z$  is an irreducible plane cubic curve such that  $P \in Z$ . Then

$$3n = (\text{mult}_L(\mathcal{M})L + \Delta) \cdot Z = 3 \text{mult}_L(\mathcal{M}) + \Delta \cdot Z$$

on the surface  $D$ . The set  $\Delta \cap Z$  is finite by Lemma 3.2. In particular, we have

$$\Delta \cdot Z \geq \text{mult}_P(\Delta) \geq 2n - \text{mult}_L(\mathcal{M}),$$

because  $\text{Supp}(\Delta)$  does not contain the curve  $Z$ . Thus, we get

$$3n \geq 3 \text{mult}_L(\mathcal{M}) + \text{mult}_P(\Delta) \geq 2n + 2 \text{mult}_L(\mathcal{M}),$$

which implies that  $\text{mult}_L(\mathcal{M}) \leq n/2$ .  $\square$

In the rest of this section we prove the following result.

**Proposition 3.4.** *Suppose that  $\mathbb{CS}(X, \mu\mathcal{M})$  contains a curve. Then  $n = 1$ .*

Suppose that the set  $\mathbb{CS}(X, \mu\mathcal{M})$  contains a curve  $Z$ . Then  $\mathbb{CS}(X, \mu\mathcal{M})$  does not contain points of the threefold  $X$  by Lemmas 3.2 and 3.3. Then

$$\text{mult}_Z(\mathcal{M}) = n, \tag{3.5}$$

because the log pair  $(X, \mu\mathcal{M})$  is canonical. Then  $\deg(Z) \leq 4$  by [2, Lemma 2.1].

**Lemma 3.6.** *Suppose that  $\deg(Z) = 1$ . Then  $n = 1$ .*

**Proof.** Let  $\pi : V \rightarrow X$  be the blow up of  $X$  along the line  $Z$ . Let  $\mathcal{B}$  be the proper transform of the pencil  $\mathcal{M}$  on the threefold  $V$ , and let  $B$  be a general surface in  $\mathcal{B}$ . Then

$$B \sim -nK_V \tag{3.7}$$

by (3.5). There is a commutative diagram

$$\begin{array}{ccc} & V & \\ \pi \swarrow & & \searrow \eta \\ X & \dashrightarrow & \mathbb{P}^2, \\ & \psi & \end{array}$$

where  $\psi$  is the projection from the line  $Z$  and  $\eta$  is the morphism induced by the complete linear system  $|-K_V|$ . It follows from (3.7) that  $\mathcal{B}$  is the pull-back of a pencil  $\mathcal{P}$  on  $\mathbb{P}^2$  by  $\eta$ .

We see that the base locus of  $\mathcal{B}$  is contained in the union of fibers of  $\eta$ .

The set  $\mathbb{CS}(V, \mu\mathcal{B})$  is not empty by [2, Theorem 2.1]. Thus, it easily follows from (3.5) that the set  $\mathbb{CS}(V, \mu\mathcal{B})$  does not contain points because  $\mathbb{CS}(X, \mu\mathcal{M})$  contains no points.

We see that there is an irreducible curve  $L \subset V$  such that

$$\text{mult}_L(\mathcal{B}) = n$$

and  $\eta(L)$  is a point  $Q \in \mathbb{P}^2$ . Let  $C$  be a general curve in  $\mathcal{P}$ . Then  $\text{mult}_Q(C) = n$ . But

$$C \sim \mathcal{O}_{\mathbb{P}^2}(n)$$

by (3.7). Thus, we see that  $n = 1$ , because general surface in  $\mathcal{M}$  is irreducible.  $\square$

Thus, we may assume that the set  $\mathbb{CS}(X, \mu\mathcal{M})$  does not contain lines.

**Lemma 3.8.** *The curve  $Z \subset \mathbb{P}^4$  is contained in a plane.*

**Proof.** Suppose that  $Z$  is not contained in any plane in  $\mathbb{P}^4$ . Let us show that this assumption leads to a contradiction. It follows from [2, Lemma 2.1] that  $\deg(Z) \leq 4$ . Then

$$\deg(Z) \in \{3, 4\},$$

and  $Z$  is smooth if  $\deg(Z) = 3$ . If  $\deg(Z) = 4$ , then  $Z$  may have at most one double point.

Suppose that  $Z$  is smooth. Let  $\alpha: U \rightarrow X$  be the blow up at  $Z$ , and let  $F$  be the exceptional divisor of the morphism  $\alpha$ . Then the base locus of the linear system

$$|\alpha^*(-\deg(Z)K_X) - F|$$

does not contain any curve. Let  $D_1$  and  $D_2$  be the proper transforms on  $U$  of two sufficiently general surfaces in the linear system  $\mathcal{M}$ . Then it follows from (3.5) that

$$(\alpha^*(-\deg(Z)K_X) - F) \cdot D_1 \cdot D_2 = n^2(\alpha^*(-\deg(Z)K_X) - F) \cdot (\alpha^*(-K_X) - F)^2 \geq 0,$$

because the cycle  $D_1 \cdot D_2$  is effective. On the other hand, we have

$$(\alpha^*(-\deg(Z)K_X) - F) \cdot (\alpha^*(-K_X) - F)^2 = (3 \deg(Z) - (\deg(Z))^2 - 2) < 0,$$

which is a contradiction. Thus, the curve  $Z$  is not smooth.

Thus, we see that  $Z$  is a quartic curve with a double point  $O$ .

Let  $\beta : W \rightarrow X$  be the composition of the blow up of the point  $O$  with the blow up of the proper transform of the curve  $Z$ . Let  $G$  and  $E$  be the exceptional surfaces of the birational morphism  $\beta$  such that  $\beta(E) = Z$  and  $\beta(G) = O$ . Then the base locus of the linear system

$$|\beta^*(-4K_X) - E - 2G|$$

does not contain any curve. Let  $R_1$  and  $R_2$  be the proper transforms on  $W$  of two sufficiently general surfaces in  $\mathcal{M}$ . Put  $m = \text{mult}_O(\mathcal{M})$ . Then it follows from (3.5) that

$$(\beta^*(-4K_X) - E - 2G) \cdot R_1 \cdot R_2 = (\beta^*(-4K_X) - E - 2G) \cdot (\beta^*(-nK_X) - nE - mG)^2 \geq 0,$$

and  $m < 2n$ , because the set  $\mathbb{CS}(X, \mu\mathcal{M})$  does not contain points. Then

$$(\beta^*(-4K_X) - E - 2G) \cdot (\beta^*(-nK_X) - nE - mG)^2 = -8n^2 + 6mn - m^2 < 0,$$

which is a contradiction.  $\square$

If  $\deg(Z) = 4$ , then  $n = 1$  by Lemma 3.8 and [2, Theorem 2.2].

**Lemma 3.9.** *Suppose that  $\deg(Z) = 3$ . Then  $n = 1$ .*

**Proof.** Let  $\mathcal{P}$  be the pencil in  $|-K_X|$  that contains all hyperplane sections of  $X$  that pass through the curve  $Z$ . Then the base locus of  $\mathcal{P}$  consists of the curve  $Z$  and a line  $L \subset X$ .

Let  $D$  be a sufficiently general surface in the pencil  $\mathcal{P}$ , and let  $M$  be a sufficiently general surface in the pencil  $\mathcal{M}$ . Then  $D$  is a smooth surface, and

$$M|_D = nZ + \text{mult}_L(\mathcal{M})L + B \equiv nZ + nL, \quad (3.10)$$

where  $B$  is a curve whose support does not contain neither  $Z$  nor  $L$ .

On the surface  $D$ , we have  $Z \cdot L = 3$  and  $L \cdot L = -2$ . Intersecting (3.10) with  $L$ , we get

$$n = (nZ + nL) \cdot L = 3n - 2 \text{mult}_L(\mathcal{M}) + B \cdot L \geq 3n - 2 \text{mult}_L(\mathcal{M}),$$

which easily implies that  $\text{mult}_L(\mathcal{M}) \geq n$ . But the inequality  $\text{mult}_L(\mathcal{M}) \geq n$  is impossible, because we assumed that  $\mathbb{CS}(X, \mu\mathcal{M})$  contains no lines.  $\square$

**Lemma 3.11.** *Suppose that  $\deg(Z) = 2$ . Then  $n = 1$ .*

**Proof.** Let  $\alpha : U \rightarrow X$  be the blow up of the curve  $Z$ . Then  $|-K_U|$  is a pencil, whose base locus consists of a smooth irreducible curve  $L \subset U$ .

Let  $D$  be a general surface in  $|-K_U|$ . Then  $D$  is a smooth surface.

Let  $\mathcal{B}$  be the proper transform of the pencil  $\mathcal{M}$  on the threefold  $U$ . Then

$$-nK_U|_D \equiv B|_D \equiv nL,$$



where  $B$  is a general surface in  $\mathcal{B}$ . But  $L^2 = -2$  on the surface  $D$ . Then

$$L \in \mathbb{CS}(U, \mu\mathcal{B})$$

which implies that  $\mathcal{B} = |-K_U|$  by [2, Theorem 2.2]. Then  $n = 1$ .  $\square$

The assertion of Proposition 3.4 is proved.

#### 4. Points

Let  $X$  be a smooth quartic threefold in  $\mathbb{P}^4$ , let  $\mathcal{M}$  be a Halphen pencil on  $X$ . Then

$$\mathcal{M} \sim -nK_X,$$

since  $\text{Pic}(X) = \mathbb{Z}K_X$ . Put  $\mu = 1/n$ . Then

- the log pair  $(X, \mu\mathcal{M})$  is canonical by [3, Theorem A],
- the log pair  $(X, \mu\mathcal{M})$  is not terminal by [2, Theorem 2.1].

**Remark 4.1.** To prove Theorem 1.5, it is enough to show that  $X$  can be given by

$$w^3x + w^2q_2(x, y, z, t) + wxp_2(x, y, z, t) + q_4(x, y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[x, y, z, t, w]) \cong \mathbb{P}^4,$$

where  $q_i$  and  $p_i$  are homogeneous polynomials of degree  $i \geq 2$  such that  $q_2(0, y, z, t) \neq 0$ .

Let  $\mathbb{CS}(X, \mu\mathcal{M})$  be the set of non-terminal centers of  $(X, \mu\mathcal{M})$  (see [2]). Then

$$\mathbb{CS}(X, \mu\mathcal{M}) \neq \emptyset,$$

because  $(X, \mu\mathcal{M})$  is not terminal. Suppose that  $n \neq 1$ . There is a point  $P \in X$  such that

$$P \in \mathbb{CS}(X, \mu\mathcal{M})$$

by Proposition 3.4. It follows from Lemmas 3.1, 3.2 and 3.3 that

- the equality  $\text{mult}_P(T) = 2$  holds, where  $T \in |-K_X|$  such that  $\text{mult}_P(T) \geq 2$ ,
- there are finitely many distinct lines  $L_1, \dots, L_r \subset X$  containing  $P \in X$ ,
- the equality  $\text{mult}_P(M) = 2n$  holds, and

$$n/3 \leq \text{mult}_{L_i}(M) \leq n/2,$$

where  $M$  is a general surface in the pencil  $\mathcal{M}$ ,

- the base locus of the pencil  $\mathcal{M}$  consists of the lines  $L_1, \dots, L_r$ , and

$$\text{mult}_P(M_1 \cdot M_2) = 4n^2,$$

where  $M_1$  and  $M_2$  are sufficiently general surfaces in  $\mathcal{M}$ .

**Lemma 4.2.** *The equality  $\mathbb{CS}(X, \mu\mathcal{M}) = \{P\}$  holds.*

**Proof.** The set  $\mathbb{CS}(X, \mu\mathcal{M})$  does not contain curves by Proposition 3.4.

Suppose that  $\mathbb{CS}(X, \mu\mathcal{M})$  contains a point  $Q \in X$  such that  $Q \neq P$ . Then  $r = 1$ .

Let  $D$  be a general hyperplane section of  $X$  that passes through  $L_1$ . Then

$$M|_D = \text{mult}_{L_1}(\mathcal{M})L_1 + \Delta,$$

where  $M$  is a general surface in  $\mathcal{M}$  and  $\Delta$  is an effective divisor such that

$$\text{mult}_P(\Delta) \geq 2n - \text{mult}_{L_1}(\mathcal{M}) \leq \text{mult}_Q(\Delta).$$

On the surface  $D$ , we have  $L_1^2 = -2$ . Then

$$\begin{aligned} n &= (\text{mult}_{L_1}(\mathcal{M})L_1 + \Delta) \cdot L_1 = -2\text{mult}_{L_1}(\mathcal{M}) + \Delta \cdot L_1 \\ &\geq -2\text{mult}_{L_1}(\mathcal{M}) + 2(2n - \text{mult}_{L_1}(\mathcal{M})), \end{aligned}$$

which gives  $\text{mult}_{L_1}(\mathcal{M}) \geq 3n/4$ . But  $\text{mult}_{L_1}(\mathcal{M}) \leq n/2$  by Lemma 3.3.  $\square$

The quartic threefold  $X$  can be given by the equation

$$w^3x + w^2q_2(x, y, z, t) + wq_3(x, y, z, t) + q_4(x, y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[x, y, z, t, w]) \cong \mathbb{P}^4,$$

where  $q_i$  is a homogeneous polynomial of degree  $i \geq 2$ .

**Remark 4.3.** The lines  $L_1, \dots, L_r \subset \mathbb{P}^4$  are given by the equations

$$x = q_2(x, y, z, t) = q_3(x, y, z, t) = q_4(x, y, z, t) = 0,$$

the surface  $T$  is cut out on  $X$  by  $x = 0$ , and  $\text{mult}_P(T) = 2 \iff q_2(0, y, z, t) \neq 0$ .

Let  $\pi : V \rightarrow X$  be the blow up of the point  $P$ , let  $E$  be the  $\pi$ -exceptional divisor. Then

$$\mathcal{B} \equiv \pi^*(-nK_X) - 2nE \equiv -nK_V,$$

where  $\mathcal{B}$  is the proper transform of the pencil  $\mathcal{M}$  on the threefold  $V$ .

**Remark 4.4.** The pencil  $\mathcal{B}$  has no base curves in  $E$ , because

$$\text{mult}_P(M_1 \cdot M_2) = \text{mult}_P(M_1)\text{mult}_P(M_2).$$

Let  $\bar{L}_i$  be the proper transform of the line  $L_i$  on the threefold  $V$  for  $i = 1, \dots, r$ . Then

$$B_1 \cdot B_2 = \sum_{i=1}^r \text{mult}_{\bar{L}_i}(B_1 \cdot B_2)\bar{L}_i,$$

where  $B_1$  and  $B_2$  are proper transforms of  $M_1$  and  $M_2$  on the threefold  $V$ , respectively.

**Lemma 4.5.** *Let  $Z$  be an irreducible curve on  $X$  such that  $Z \notin \{L_1, \dots, Z_r\}$ . Then*

$$\deg(Z) \geq 2 \operatorname{mult}_P(Z),$$

*and the equality  $\deg(Z) = 2 \operatorname{mult}_P(Z)$  implies that*

$$\bar{Z} \cap \left( \bigcup_{i=1}^r \bar{L}_i \right) = \emptyset,$$

*where  $\bar{Z}$  is a proper transform of the curve  $Z$  on the threefold  $V$ .*

**Proof.** The curve  $\bar{Z}$  is not contained in the base locus of the pencil  $\mathcal{B}$ . Then

$$0 \leq B_i \cdot \bar{Z} \leq n(\deg(Z) - 2 \operatorname{mult}_P(Z)),$$

which implies the required assertions.  $\square$

To conclude the proof of Theorem 1.5, it is enough to show that

$$q_3(x, y, z, t) = xp_2(x, y, z, t) + q_2(x, y, z, t)p_1(x, y, z, t),$$

where  $p_1$  and  $p_2$  are some homogeneous polynomials of degree 1 and 2, respectively.

## 5. Good points

Let us use the assumptions and notation of Section 4. Suppose that the conic

$$q_2(0, y, z, t) = 0 \subset \operatorname{Proj}(\mathbb{C}[y, z, t]) \cong \mathbb{P}^2$$

is reduced and irreducible. In this section we prove the following result.

**Proposition 5.1.** *The polynomial  $q_3(0, y, z, t)$  is divisible by  $q_2(0, y, z, t)$ .*

Let us prove Proposition 5.1. Suppose that  $q_3(0, y, z, t)$  is not divisible by  $q_2(0, y, z, t)$ .

Let  $\mathcal{R}$  be the linear system on the threefold  $X$  that is cut out by quadrics

$$xh_1(x, y, z, t) + \lambda(wx + q_2(x, y, z, t)) = 0,$$

where  $h_1$  is an arbitrary linear form and  $\lambda \in \mathbb{C}$ . Then  $\mathcal{R}$  does not have fixed components.

**Lemma 5.2.** *Let  $R_1$  and  $R_2$  be general surfaces in the linear system  $\mathcal{R}$ . Then*

$$\sum_{i=1}^r \operatorname{mult}_{L_i}(R_1 \cdot R_2) \leq 6.$$

**Proof.** We may assume that  $R_1$  is cut out by the equation

$$wx + q_2(x, y, z, t) = 0,$$

and  $R_2$  is cut out by  $xh_1(x, y, z, t) = 0$ , where  $h_1$  is sufficiently general. Then

$$\text{mult}_{L_i}(R_1 \cdot R_2) = \text{mult}_{L_i}(R_1 \cdot T).$$

Put  $m_i = \text{mult}_{L_i}(R_1 \cdot T)$ . Then

$$R_1 \cdot T = \sum_{i=1}^r m_i L_i + \Delta,$$

where  $m_i \in \mathbb{N}$ , and  $\Delta$  is a cycle, whose support contains no lines passing through  $P$ .

Let  $\bar{R}_1$  and  $\bar{T}$  be the proper transforms of  $R_1$  and  $T$  on  $V$ , respectively. Then

$$\bar{R}_1 \cdot \bar{T} = \sum_{i=1}^r m_i \bar{L}_i + \Omega,$$

where  $\Omega$  is an effective cycle, whose support contains no lines passing through  $P$ .

The support of the cycle  $\Omega$  does not contain curves that are contained in the exceptional divisor  $E$ , because  $q_3(0, y, z, t)$  is not divisible by  $q_2(0, y, z, t)$  by our assumption. Then

$$6 = E \cdot \bar{R}_1 \cdot \bar{T} = \sum_{i=1}^r m_i (E \cdot \bar{L}_i) + E \cdot \Omega \geq \sum_{i=1}^r m_i (E \cdot \bar{L}_i) = \sum_{i=1}^r m_i,$$

which is exactly what we want.  $\square$

Let  $M$  and  $R$  be general surfaces in  $\mathcal{M}$  and  $\mathcal{R}$ , respectively. Put

$$M \cdot R = \sum_{i=1}^r m_i L_i + \Delta,$$

where  $m_i \in \mathbb{N}$ , and  $\Delta$  is a cycle, whose support contains no lines passing through  $P$ .

**Lemma 5.3.** *The cycle  $\Delta$  is not trivial.*

**Proof.** Suppose that  $\Delta = 0$ . Then  $\mathcal{M} = \mathcal{R}$  by [2, Theorem 2.2]. But  $\mathcal{R}$  is not a pencil.  $\square$

We have  $\deg(\Delta) = 8n - \sum_{i=1}^r m_i$ . On the other hand, the inequality

$$\text{mult}_P(\Delta) \geq 6n - \sum_{i=1}^r m_i$$

holds, because  $\text{mult}_P(M) = 2n$  and  $\text{mult}_P(R) \geq 3$ . It follows from Lemma 4.5 that

$$\deg(\Delta) = 8n - \sum_{i=1}^r m_i \geq 2 \text{mult}_P(\Delta) \geq 2 \left( 6n - \sum_{i=1}^r m_i \right),$$

which implies that  $\sum_{i=1}^r m_i \geq 4n$ . But it follows from Lemmas 2.1 and 3.3 that

$$m_i \leq \text{mult}_{L_i}(R_1 \cdot R_2) \text{mult}_{L_i}(M) \leq \text{mult}_{L_i}(R_1 \cdot R_2)n/2$$

for every  $i = 1, \dots, r$ , where  $R_1$  and  $R_2$  are general surfaces in  $\mathcal{R}$ . Then

$$\sum_{i=1}^r m_i \leq \sum_{i=1}^r \text{mult}_{L_i}(R_1 \cdot R_2)n/2 \leq 3n$$

by Lemma 5.2, which is a contradiction.

The assertion of Proposition 5.1 is proved.

## 6. Bad points

Let us use the assumptions and notation of Section 4. Suppose that the conic

$$q_2(0, y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[y, z, t]) \cong \mathbb{P}^2$$

is reduced and reducible. Therefore, we have

$$q_2(x, y, z, t) = (\alpha_1 y + \beta_1 z + \gamma_1 t)(\alpha_2 y + \beta_2 z + \gamma_2 t) + x p_1(x, y, z, t)$$

where  $p_1(x, y, z, t)$  is a linear form, and  $(\alpha_1 : \beta_1 : \gamma_1) \in \mathbb{P}^2 \ni (\alpha_2 : \beta_2 : \gamma_2)$ .

**Proposition 6.1.** *The polynomial  $q_3(0, y, z, t)$  is divisible by  $q_2(0, y, z, t)$ .*

Suppose that  $q_3(0, y, z, t)$  is not divisible by  $q_2(0, y, z, t)$ . Then without loss of generality, we may assume that  $q_3(0, y, z, t)$  is not divisible by  $\alpha_1 y + \beta_1 z + \gamma_1 t$ .

Let  $Z$  be the curve in  $X$  that is cut out by the equations

$$x = \alpha_1 y + \beta_1 z + \gamma_1 t = 0.$$

**Remark 6.2.** The equality  $\text{mult}_P(Z) = 3$  holds, but  $Z$  is not necessary reduced.

Hence, it follows from Lemma 4.5 that  $\text{Supp}(Z)$  contains a line among  $L_1, \dots, L_r$ .

**Lemma 6.3.** *The support of the curve  $Z$  does not contain an irreducible conic.*

**Proof.** Suppose that  $\text{Supp}(Z)$  contains an irreducible conic  $C$ . Then

$$Z = C + L_i + L_j$$

for some  $i \in \{1, \dots, r\} \ni j$ . Then  $i = j$ , because otherwise the set

$$(C \cap L_i) \cup (C \cap L_j)$$

contains a point that is different from  $P$ , which is impossible by Lemma 4.5. We see that

$$Z = C + 2L_i,$$

and it follows from Lemma 4.5 that  $C \cap L_i = P$ . Then  $C$  is tangent to  $L_i$  at the point  $P$ .

Let  $\bar{C}$  be a proper transform of the curve  $C$  on the threefold  $V$ . Then

$$\bar{C} \cap \bar{L}_i \neq \emptyset,$$

which is impossible by Lemma 4.5. The assertion is proved.  $\square$

**Lemma 6.4.** *The support of the curve  $Z$  consists of lines.*

**Proof.** Suppose that  $\text{Supp}(Z)$  does not consist of lines. It follows from Lemma 6.3 that

$$Z = L_i + C,$$

where  $C$  is an irreducible cubic curve. But  $\text{mult}_P(Z) = 3$ . Then

$$\text{mult}_P(C) = 2,$$

which is impossible by Lemma 4.5.  $\square$

We may assume that there is a line  $L \subset X$  such that  $P \notin L$  and

$$Z = a_1 L_1 + \dots + a_k L_k + L,$$

where  $a_1, a_2, a_3 \in \mathbb{N}$  are such that  $a_1 \geq a_2 \geq a_3$  and  $\sum_{i=1}^k a_i = 3$ .

**Remark 6.5.** We have  $L_i \neq L_j$  whenever  $i \neq j$ .

Let  $H$  be a sufficiently general surface of  $X$  that is cut out by the equation

$$\lambda x + \mu(\alpha_1 y + \beta_1 z + \gamma_1 t) = 0,$$

where  $(\lambda : \mu) \in \mathbb{P}^1$ . Then  $H$  has at most isolated singularities.

**Remark 6.6.** The surface  $H$  is smooth at the points  $P$  and  $L \cap L_i$ , where  $i = 1, \dots, k$ .

Let  $\bar{H}$  and  $\bar{L}$  be the proper transforms of  $H$  and  $L$  on the threefold  $V$ , respectively.

**Lemma 6.7.** *The inequality  $k \neq 3$  holds.*

**Proof.** Suppose that the equality  $k = 3$  holds. Then  $H$  is smooth. Put

$$B|_{\bar{H}} = m_1 \bar{L}_1 + m_2 \bar{L}_2 + m_3 \bar{L}_3 + \Omega,$$

where  $B$  is a general surface in  $\mathcal{B}$ , and  $\Omega$  is an effective divisor on  $\bar{H}$  whose support does not contain any of the curves  $\bar{L}_1$ ,  $\bar{L}_2$  and  $\bar{L}_3$ . Then

$$\bar{L} \not\subseteq \text{Supp}(\Omega) \not\supseteq \bar{H} \cap E,$$

because the base locus of the pencil  $\mathcal{B}$  consists of the curves  $\bar{L}_1, \dots, \bar{L}_r$ . Then

$$n = \bar{L} \cdot (m_1 \bar{L}_1 + m_2 \bar{L}_2 + m_3 \bar{L}_3 + \Omega) = \sum_{i=1}^3 m_i + \bar{L} \cdot \Omega \geq \sum_{i=1}^3 m_i,$$

which implies that  $\sum_{i=1}^3 m_i \leq n$ . On the other hand, we have

$$-n = \bar{L}_i \cdot (m_1 \bar{L}_1 + m_2 \bar{L}_2 + m_3 \bar{L}_3 + \Omega) = -3m_i + L_i \cdot \Omega \geq -3m_i,$$

which implies that  $m_i \geq n/3$ . Thus, we have  $m_1 = m_2 = m_3 = n/3$  and

$$\Omega \cdot \bar{L} = \Omega \cdot \bar{L}_1 = \Omega \cdot \bar{L}_2 = \Omega \cdot \bar{L}_3 = 0,$$

which implies that  $\text{Supp}(\Omega) \cap \bar{L}_1 = \text{Supp}(\Omega) \cap \bar{L}_2 = \text{Supp}(\Omega) \cap \bar{L}_3 = \emptyset$ .

Let  $B'$  be another general surface in  $\mathcal{B}$ . Arguing as above, we see that

$$B'|_{\bar{H}} = \frac{n}{3}(\bar{L}_1 + \bar{L}_2 + \bar{L}_3) + \Omega',$$

where  $\Omega'$  is an effective divisor on the surface  $\bar{H}$  such that

$$\text{Supp}(\Omega') \cap \bar{L}_1 = \text{Supp}(\Omega') \cap \bar{L}_2 = \text{Supp}(\Omega') \cap \bar{L}_3 = \emptyset.$$

One can easily check that  $\Omega \cdot \Omega' = n^2 \neq 0$ . Then

$$\text{Supp}(\Omega) \cap \text{Supp}(\Omega') \neq \emptyset,$$

because  $|\text{Supp}(\Omega) \cap \text{Supp}(\Omega')| < +\infty$  due to generality of the surfaces  $B$  and  $B'$ .

The base locus of the pencil  $\mathcal{B}$  consists of the curves  $\bar{L}_1, \dots, \bar{L}_r$ . Hence, we have

$$\text{Supp}(B) \cap \text{Supp}(B') = \bigcup_{i=1}^r \bar{L}_i,$$

but  $\bar{L}_i \cap \bar{H} = \emptyset$  whenever  $i \notin \{1, 2, 3\}$ . Hence, we have

$$\bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3 \cup (\text{Supp}(\Omega) \cap \text{Supp}(\Omega')) = \text{Supp}(B) \cap \text{Supp}(B') \cap \bar{H} = \bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3,$$

which implies that  $\text{Supp}(\Omega) \cap \text{Supp}(\Omega') \subset \bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3$ . In particular, we see that

$$\text{Supp}(\Omega) \cap (\bar{L}_1 \cup \bar{L}_2 \cup \bar{L}_3) \neq \emptyset,$$

because  $\text{Supp}(\Omega) \cap \text{Supp}(\Omega') \neq \emptyset$ . But  $\text{Supp}(\Omega) \cap \bar{L}_i = \emptyset$  for  $i = 1, 2, 3$ .  $\square$

**Lemma 6.8.** *The inequality  $k \neq 2$  holds.*

**Proof.** Suppose that the equality  $k = 2$  holds. Then  $Z = 2L_1 + L_2 + L$ . Put

$$B|_{\bar{H}} = m_1 \bar{L}_1 + m_2 \bar{L}_2 + \Omega,$$

where  $B$  is a general surface in  $\mathcal{B}$ , and  $\Omega$  is an effective divisor on  $\bar{H}$  whose support does not contain the curves  $\bar{L}_1$  and  $\bar{L}_2$ . Then  $\bar{L} \not\subseteq \text{Supp}(\Omega) \not\supseteq \bar{H} \cap E$  and

$$n = \bar{L} \cdot (m_1 \bar{L}_1 + m_2 \bar{L}_2 + \Omega) = m_1 + m_2 + \bar{L} \cdot \Omega \geq m_1 + m_2,$$

which implies that  $m_1 + m_2 \leq n$ . On the other hand, we have

$$\bar{T}|_{\bar{H}} = 2\bar{L}_1 + \bar{L}_2 + \bar{L} + E|_{\bar{H}} \equiv (\pi^*(-K_X) - 2E)|_{\bar{H}},$$

where  $\bar{T}$  is the proper transform of the surface  $T$  on the threefold  $V$ . Then

$$-1 = \bar{L}_1 \cdot (2\bar{L}_1 + \bar{L}_2 + \bar{L} + E|_{\bar{H}}) = 2(\bar{L}_1 \cdot \bar{L}_1) + 2,$$

which implies that  $\bar{L}_1 \cdot \bar{L}_1 = -3/2$  on the surface  $\bar{H}$ . Then

$$-n = \bar{L}_1 \cdot (m_1 \bar{L}_1 + m_2 \bar{L}_2 + \Omega) = -3m_1/2 + L_1 \cdot \Omega \geq -3m_1/2,$$

which gives  $m_1 \geq 2n/3$ . Similarly, we see that  $\bar{L}_2 \cdot \bar{L}_2 = -3$  on the surface  $\bar{H}$ . Then

$$-n = \bar{L}_2 \cdot (m_1 \bar{L}_1 + m_2 \bar{L}_2 + \Omega) = -3m_2/2 + L_2 \cdot \Omega \geq -3m_2/2,$$

which implies that  $m_2 \leq n/3$ . Thus, we have  $m_1 = 2m_2 = 2n/3$  and

$$\Omega \cdot \bar{L} = \Omega \cdot \bar{L}_1 = \Omega \cdot \bar{L}_2 = 0,$$

which implies that  $\text{Supp}(\Omega) \cap \bar{L}_1 = \text{Supp}(\Omega) \cap \bar{L}_2 = \emptyset$ .

Let  $B'$  be another general surface in  $\mathcal{B}$ . Arguing as above, we see that

$$B'|_{\bar{H}} = \frac{2n}{3} \bar{L}_1 + \frac{n}{3} \bar{L}_2 + \Omega',$$

where  $\Omega'$  is an effective divisor on  $\bar{H}$  whose support does not contain  $\bar{L}_1$  and  $\bar{L}_2$  such that

$$\text{Supp}(\Omega') \cap \bar{L}_1 = \text{Supp}(\Omega') \cap \bar{L}_2 = \emptyset,$$



which implies that  $\Omega \cdot \Omega' = n^2$ . In particular, we see that

$$\text{Supp}(\Omega) \cap \text{Supp}(\Omega') \neq \emptyset,$$

and arguing as in the proof of Lemma 6.7 we obtain a contradiction.  $\square$

It follows from Lemmas 6.7 and 6.8 that  $Z = 3L_1 + L$ . Put

$$B|_{\bar{H}} = m_1 \bar{L}_1 + \Omega,$$

where  $B$  is a general surface in  $\mathcal{B}$ , and  $\Omega$  is a curve such that  $\bar{L}_1 \not\subseteq \text{Supp}(\Omega)$ . Then

$$\bar{L} \not\subseteq \text{Supp}(\Omega) \not\supseteq \bar{H} \cap E,$$

because the base locus of  $\mathcal{B}$  consists of the curves  $\bar{L}_1, \dots, \bar{L}_r$ . Then

$$n = \bar{L} \cdot (m_1 \bar{L}_1 + \Omega) = m_1 + \bar{L} \cdot \Omega \geq m_1,$$

which implies that  $m_1 \leq n$ . On the other hand, we have

$$\bar{T}|_{\bar{H}} = 3\bar{L}_1 + \bar{L} + E|_{\bar{H}} \equiv (\pi^*(-K_X) - 2E)|_{\bar{H}},$$

where  $\bar{T}$  is the proper transform of the surface  $T$  on the threefold  $V$ . Then

$$-1 = \bar{L}_1 \cdot (3\bar{L}_1 + \bar{L} + E|_{\bar{H}}) = 3\bar{L}_1 \cdot \bar{L}_1 + 2,$$

which implies that  $\bar{L}_1 \cdot \bar{L}_1 = -1$  on the surface  $\bar{H}$ . Then

$$-n = \bar{L}_1 \cdot (m_1 \bar{L}_1 + \Omega) = -m_1 + L_1 \cdot \Omega \geq -m_1,$$

which gives  $m_1 \geq n$ . Thus, we have  $m_1 = n$  and  $\Omega \cdot \bar{L} = \Omega \cdot \bar{L}_1 = 0$ . Then  $\text{Supp}(\Omega) \cap \bar{L}_1 = \emptyset$ .

Let  $B'$  be another general surface in  $\mathcal{B}$ . Arguing as above, we see that

$$B'|_{\bar{H}} = n\bar{L}_1 + \Omega',$$

where  $\Omega'$  is an effective divisor on  $\bar{H}$  whose support does not contain  $\bar{L}_1$  such that

$$\text{Supp}(\Omega') \cap \bar{L}_1 = \emptyset,$$

which implies that  $\Omega \cdot \Omega' = n^2$ . In particular, we see that  $\text{Supp}(\Omega) \cap \text{Supp}(\Omega') \neq \emptyset$ .

The base locus of the pencil  $\mathcal{B}$  consists of the curves  $\bar{L}_1, \dots, \bar{L}_r$ . Hence, we have

$$\text{Supp}(B) \cap \text{Supp}(B') = \bigcup_{i=1}^r \bar{L}_i,$$

but  $\bar{L}_i \cap \bar{H} = \emptyset$  whenever  $\bar{L}_i \neq \bar{L}_1$ . Then  $\text{Supp}(\Omega) \cap \bar{L}_1 \neq \emptyset$ , because

$$\bar{L}_1 \cup (\text{Supp}(\Omega) \cap \text{Supp}(\Omega')) = \text{Supp}(B) \cap \text{Supp}(B') \cap \bar{H} = \bar{L}_1,$$

which is a contradiction. The assertion of Proposition 6.1 is proved.

## 7. Very bad points

Let us use the assumptions and notation of Section 4. Suppose that  $q_2 = y^2$ .

The proof of Proposition 6.1 implies that  $q_3(0, y, z, t)$  is divisible by  $y$ . Then

$$q_3 = yf_2(z, t) + xh_2(z, t) + x^2a_1(x, y, z, t) + xyb_1(x, y, z, t) + y^2c_1(y, z, t)$$

where  $a_1, b_1, c_1$  are linear forms,  $f_2$  and  $h_2$  are homogeneous polynomials of degree two.

**Proposition 7.1.** *The equality  $f_2(z, t) = 0$  holds.*

Let us prove Proposition 7.1 by reductio ad absurdum. Suppose that  $f_2(z, t) \neq 0$ .

**Remark 7.2.** By choosing suitable coordinates, we may assume that  $f_2 = zt$  or  $f_2 = z^2$ .

We must use smoothness of the threefold  $X$  by analyzing the shape of  $q_4$ . We have

$$q_4 = f_4(z, t) + xu_3(z, t) + yv_3(z, t) + x^2a_2(x, y, z, t) + xyb_2(x, y, z, t) + y^2c_2(y, z, t),$$

where  $a_2, b_2, c_2$  are homogeneous polynomials of degree two,  $u_3$  and  $v_3$  are homogeneous polynomials of degree three, and  $f_4$  is a homogeneous polynomial of degree four.

**Lemma 7.3.** *Suppose that  $f_2(z, t) = zt$  and*

$$f_4(z, t) = t^2g_2(z, t)$$

*for some  $g_2(z, t) \in \mathbb{C}[z, t]$ . Then  $v_3(z, 0) \neq 0$ .*

**Proof.** Suppose that  $v_3(z, 0) = 0$ . The surface  $T$  is given by the equation

$$\begin{aligned} w^2y^2 + yzt + y^2c_1(x, y, z, t) + t^2g_2(z, t) + yv_3(z, t) + y^2c_2(x, y, z, t) &= 0 \\ \subset \text{Proj}(\mathbb{C}[y, z, t, w]) &\cong \mathbb{P}^3 \end{aligned}$$

because  $T$  is cut out on  $X$  by the equation  $x = 0$ . Then  $T$  has non-isolated singularity along the line  $x = y = t = 0$ , which is impossible because  $X$  is smooth.  $\square$

Arguing as in the proof of Lemma 7.3, we obtain the following corollary.

**Corollary 7.4.** *Suppose that  $f_2(z, t) = zt$  and*

$$f_4(z, t) = z^2g_2(z, t)$$

*for some  $g_2(z, t) \in \mathbb{C}[z, t]$ . Then  $v_3(0, t) \neq 0$ .*

**Lemma 7.5.** *Suppose that  $f_2(z, t) = zt$ . Then  $f_4(0, t) = f_4(z, 0) = 0$ .*

**Proof.** We may assume that  $f_4(z, 0) \neq 0$ . Let  $\mathcal{H}$  be the linear system on  $X$  that is cut out by

$$\lambda x + \mu y + \nu t = 0,$$

where  $(\lambda : \mu : \nu) \in \mathbb{P}^2$ . Then the base locus of  $\mathcal{H}$  consists of the point  $P$ .

Let  $\mathcal{R}$  be the proper transform of  $\mathcal{H}$  on the threefold  $V$ . Then the base locus of  $\mathcal{R}$  consists of a single point that is not contained in any of the curves  $\bar{L}_1, \dots, \bar{L}_r$ .

The linear system  $\mathcal{R}|_B$  has no base points, where  $B$  is a general surface in  $\mathcal{B}$ . But

$$R \cdot R \cdot B = 2n > 0,$$

where  $R$  is a general surface in  $\mathcal{R}$ . Then  $\mathcal{R}|_B$  is not composed from a pencil, which implies that the curve  $R \cdot B$  is irreducible and reduced by the Bertini theorem.

Let  $H$  and  $M$  be general surfaces in  $\mathcal{H}$  and  $\mathcal{M}$ , respectively. Then  $M \cdot H$  is irreducible and reduced. Thus, the linear system  $\mathcal{M}|_H$  is a pencil.

The surface  $H$  contains no lines passing through  $P$ , and  $H$  can be given by

$$\begin{aligned} w^3 x + w^2 y^2 + w(y^2 l_1(x, y, z) + x l_2(x, y, z)) + l_4(x, y, z) = 0 \\ \subset \text{Proj}(\mathbb{C}[x, y, z, w]) \cong \mathbb{P}^3, \end{aligned}$$

where  $l_i(x, y, z)$  is a homogeneous polynomial of degree  $i$ .

Arguing as in Example 1.4, we see that there is a pencil  $\mathcal{Q}$  on the surface  $H$  such that

$$\mathcal{Q} \sim \mathcal{O}_{\mathbb{P}^3}(2)|_H,$$

general curve in  $\mathcal{Q}$  is irreducible, and  $\text{mult}_P(\mathcal{Q}) = 4$ . Arguing as in the proof of Lemma 3.1, we see that  $\mathcal{M}|_H = \mathcal{Q}$  by [2, Theorem 2.2]. Let  $M$  be a general surface in  $\mathcal{M}$ . Then

$$M \equiv -2K_X,$$

and  $\text{mult}_P(M) = 4$ . The surface  $M$  is cut out on  $X$  by the equation

$$\lambda x^2 + x(A_0 + A_1(y, z, t)) + B_2(y, z, t) + B_1(y, z, t) + B_0 = 0,$$

where  $A_i$  and  $B_i$  are homogeneous polynomials of degree  $i$ , and  $\lambda \in \mathbb{C}$ .

It follows from  $\text{mult}_P(M) = 4$  that  $B_1(y, z, t) = B_0 = 0$ .

The coordinates  $(y, z, t)$  are also local coordinates on  $X$  near the point  $P$ . Then

$$x = -y^2 - y(z t + y p_1(y, z, t)) + \text{higher order terms},$$

which is a Taylor power series for  $x = x(y, z, t)$ , where  $p_1(y, z, t)$  is a linear form.

The surface  $M$  is locally given by the analytic equation

$$\lambda y^4 + (-y^2 - y z t - y^2 p_1(y, z, t))(A_0 + A_1(y, z, t)) + B_2(y, z, t) + \text{higher order terms} = 0,$$

and  $\text{mult}_P(M) = 4$ . Hence, we see that  $B_2(y, z, t) = A_0 y^2$  and

$$A_1(y, z, t) y^2 + A_0 y(z t + y p_1(y, z, t)) = 0,$$

which implies that  $A_0 = A_1(y, z, t) = B_2(y, z, t) = 0$ . Hence, we see that a general surface in the pencil  $\mathcal{M}$  is cut out on  $X$  by the equation  $x^2 = 0$ , which is absurd.  $\square$

Arguing as in the proof of Lemma 7.5, we obtain the following corollary.

**Corollary 7.6.** *Suppose that  $f_2(z, t) = z^2$ . Then  $f_4(0, t) = 0$ .*

Let  $\mathcal{R}$  be the linear system on the threefold  $X$  that is cut out by cubics

$$xh_2(x, y, z, t) + \lambda(w^2x + wy^2 + q_3(x, y, z, t)) = 0,$$

where  $h_2$  is a form of degree 2, and  $\lambda \in \mathbb{C}$ . Then  $\mathcal{R}$  has no fixed components.

Let  $M$  and  $R$  be general surfaces in  $\mathcal{M}$  and  $\mathcal{R}$ , respectively. Put

$$M \cdot R = \sum_{i=1}^r m_i L_i + \Delta,$$

where  $m_i \in \mathbb{N}$ , and  $\Delta$  is a cycle, whose support contains no lines among  $L_1, \dots, L_r$ .

**Lemma 7.7.** *The cycle  $\Delta$  is not trivial.*

**Proof.** Suppose that  $\Delta = 0$ . Then  $\mathcal{M} = \mathcal{R}$  by [2, Theorem 2.2]. But  $\mathcal{R}$  is not a pencil.  $\square$

We have  $\text{mult}_P(\Delta) \geq 8n - \sum_{i=1}^r m_i$ , because  $\text{mult}_P(\mathcal{M}) = 2n$  and  $\text{mult}_P(\mathcal{R}) \geq 4$ . Then

$$\deg(\Delta) = 12n - \sum_{i=1}^r m_i \geq 2 \text{mult}_P(\Delta) \geq 2 \left( 8n - \sum_{i=1}^r m_i \right)$$

by Lemma 4.5, because  $\text{Supp}(\Delta)$  does not contain any of the lines  $L_1, \dots, L_r$ .

**Corollary 7.8.** *The inequality  $\sum_{i=1}^r m_i \geq 4n$  holds.*

Let  $R_1$  and  $R_2$  be general surfaces in the linear system  $\mathcal{R}$ . Then

$$m_i \leq \text{mult}_{L_i}(R_1 \cdot R_2) \text{mult}_{L_i}(M) \leq \text{mult}_{L_i}(R_1 \cdot R_2)n/2$$

for every  $1 \leq i \leq 4$  by Lemmas 2.1 and 3.3. Then

$$4n \leq \sum_{i=1}^r m_i \leq \sum_{i=1}^r \text{mult}_{L_i}(R_1 \cdot R_2)n/2.$$

**Corollary 7.9.** *The inequality  $\sum_{i=1}^r \text{mult}_{L_i}(R_1 \cdot R_2) \geq 8$  holds.*

Now we suppose that  $R_1$  is cut out on the quartic  $X$  by the equation

$$w^2x + wy^2 + q_3(x, y, z, t) = 0,$$

and  $R_2$  is cut out by  $xh_2(x, y, z, t) = 0$ , where  $h_2$  is sufficiently general. Then

$$\sum_{i=1}^r \text{mult}_{L_i}(R_1 \cdot T) = \sum_{i=1}^r \text{mult}_{L_i}(R_1 \cdot R_2) \geq 8,$$

where  $T$  is the hyperplane section of the hypersurface  $X$  that is cut out by  $x = 0$ . But

$$R_1 \cdot T = Z_1 + Z_2,$$

where  $Z_1$  and  $Z_2$  are cycles on  $X$  such that  $Z_1$  is cut out by  $x = y = 0$ , and  $Z_2$  is cut out by

$$x = wy + f_2(z, t) + yc_1(x, y, z, t) = 0.$$

**Lemma 7.10.** *The equality  $\sum_{i=1}^r \text{mult}_{L_i}(Z_1) = 4$  holds.*

**Proof.** The lines  $L_1, \dots, L_r \subset \mathbb{P}^4$  are given by the equations

$$x = y = q_4(x, y, z, t) = 0,$$

which implies that  $\sum_{i=1}^r \text{mult}_{L_i}(Z_1) = 4$ .  $\square$

Hence, we see that  $\sum_{i=1}^r \text{mult}_{L_i}(Z_2) \geq 4$ . But  $Z_2$  can be considered as a cycle

$$\begin{aligned} wy + f_2(z, t) + yc_1(y, z, t) &= f_4(z, t) + yv_3(z, t) + y^2c_2(y, z, t) \\ &= 0 \subset \text{Proj}(\mathbb{C}[y, z, t, w]) \cong \mathbb{P}^3, \end{aligned}$$

and, putting  $u = w + c_1(y, z, t)$ , we see that  $Z_2$  can be considered as a cycle

$$uy + f_2(z, t) = f_4(z, t) + yv_3(z, t) + y^2c_2(y, z, t) = 0 \subset \text{Proj}(\mathbb{C}[y, z, t, u]) \cong \mathbb{P}^3,$$

and we can consider the set of lines  $L_1, \dots, L_r$  as the set of curves in  $\mathbb{P}^3$  given by  $y = f_4(z, t) = 0$ .

**Lemma 7.11.** *The inequality  $f_2(z, t) \neq zt$  holds.*

**Proof.** Suppose that  $f_2(z, t) = zt$ . Then it follows from Lemma 7.5 that

$$f_4(z, t) = zt(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t)$$

for some  $(\alpha_1 : \beta_1) \in \mathbb{P}^1 \ni (\alpha_2 : \beta_2)$ . Then  $Z_2$  can be given by

$$\begin{aligned} uy + zt &= yv_3(z, t) + y^2c_2(y, z, t) - uy(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t) \\ &= 0 \subset \text{Proj}(\mathbb{C}[y, z, t, u]) \cong \mathbb{P}^3, \end{aligned}$$

which implies  $Z_2 = Z_2^1 + Z_2^2$ , where  $Z_2^1$  and  $Z_2^2$  are cycles in  $\mathbb{P}^3$  such that  $Z_2^1$  is given by

$$y = uy + zt = 0,$$

and  $Z_2^2$  is given by  $uy + zt = v_3(z, t) + yc_2(y, z, t) - u(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t) = 0$ .

We may assume that  $L_1$  is given by  $y = z = 0$ , and  $L_2$  is given by  $y = t = 0$ . Then

$$Z_2^1 = L_1 + L_2,$$

which implies that  $\sum_{i=1}^r \text{mult}_{L_i}(Z_2^2) \geq 2$ .

Suppose that  $r = 4$ . Then  $\alpha_1 \neq 0$ ,  $\beta_1 \neq 0$ ,  $\alpha_2 \neq 0$ ,  $\beta_2 \neq 0$ . Hence, we see that

$$L_1 \not\subseteq \text{Supp}(Z_2^2) \not\subseteq L_2,$$

because  $v_3(z, t) + yc_2(y, z, t) - u(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t)$  does not vanish on  $L_1$  and  $L_2$ . But

$$L_3 \not\subseteq \text{Supp}(Z_2^2) \not\subseteq L_4,$$

because  $zt$  does not vanish on  $L_3$  and  $L_4$ . Then  $\sum_{i=1}^r \text{mult}_{L_i}(Z_2^2) = 0$ , which is impossible.

Suppose that  $r = 3$ . We may assume that  $(\alpha_1, \beta_1) = (1, 0)$ , but  $\alpha_2 \neq 0 \neq \beta_2$ . Then

$$L_2 \not\subseteq \text{Supp}(Z_2^2),$$

because  $v_3(z, t) + yc_2(y, z, t) - uz(\alpha_2 z + \beta_2 t)$  does not vanish on  $L_2$ . We have

$$f_4(z, t) = z^2 t(\alpha_2 z + \beta_2 t),$$

which implies that  $v_3(0, t) \neq 0$  by Corollary 7.4. Hence, we see that

$$L_1 \not\subseteq \text{Supp}(Z_2^2) \not\subseteq L_3,$$

because  $v_3(z, t) + yc_2(y, z, t) - uz(\alpha_2 z + \beta_2 t)$  and  $zt$  do not vanish on  $L_1$  and  $L_3$ , respectively, which implies that  $\sum_{i=1}^r \text{mult}_{L_i}(Z_2^2) = 0$ . The latter is a contradiction.

We see that  $r = 2$ . We may assume that  $(\alpha_1, \beta_1) = (1, 0)$ , and either  $\alpha_2 = 0$  or  $\beta_2 = 0$ .

Suppose that  $\alpha_2 = 0$ . Then  $f_4(z, t) = \beta_2 z^2 t^2$ . By Lemma 7.3 and Corollary 7.4, we get

$$v_3(0, t) \neq 0 \neq v_3(z, 0),$$

which implies that  $v_3(z, t) + yc_2(y, z, t) - \beta_2 zt$  does not vanish on neither  $L_1$  nor  $L_2$ . Then

$$L_1 \not\subseteq \text{Supp}(Z_2^2) \not\subseteq L_2,$$

which implies that  $\sum_{i=1}^r \text{mult}_{L_i}(Z_2^2) = 0$ , which is a contradiction.

We see that  $\alpha_2 \neq 0$  and  $\beta_2 = 0$ . We have  $f_4(z, t) = \alpha_2 z^3 t$ . Then

$$v_3(0, t) \neq 0$$

by Corollary 7.4. Then  $L_1 \not\subseteq \text{Supp}(Z_2^2)$  because the polynomial

$$v_3(z, t) + yc_2(y, z, t) - \alpha_2 z^2$$

does not vanish on  $L_1$ .

The line  $L_2$  is given by the equations  $y = t = 0$ . But  $Z_2$  is given by the equations

$$uy + zt = v_3(z, t) + yc_2(y, z, t) - \alpha_2 uz^2 = 0,$$

which implies that  $L_2 \not\subseteq \text{Supp}(Z_2^2)$ . Then  $\sum_{i=1}^r \text{mult}_{L_i}(Z_2^2) = 0$ , which is a contradiction.  $\square$

Therefore, we see that  $f_2(z, t) = z^2$ . It follows from Corollary 7.6 that

$$f_4(z, t) = zg_3(z, t)$$

for some  $g_3(z, t) \in \mathbb{C}[z, t]$ . We may assume that  $L_1$  is given by  $y = z = 0$ .

**Lemma 7.12.** *The equality  $g_3(0, t) = 0$  holds.*

**Proof.** Suppose that  $g_3(0, t) \neq 0$ . Then  $\text{Supp}(Z_2) = L_1$ , because  $Z_2$  is given by

$$uy + z^2 = zg_3(z, t) + yv_3(z, t) + y^2c_2(y, z, t) = 0,$$

and the lines  $L_2, \dots, L_r$  are given by the equations  $y = g_3(z, t) = 0$ .

The cycle  $Z_2 + L_1$  is given by the equations

$$uy + z^2 = z^2g_3(z, t) + zyv_3(z, t) + zy^2c_2(y, z, t) = 0,$$

which implies that the cycle  $Z_2 + L_1$  can be given by the equations

$$uy + z^2 = zyv_3(z, t) + zy^2c_2(y, z, t) - uyg_3(z, t) = 0.$$

We have  $Z_2 + L_1 = C_1 + C_2$ , where  $C_1$  and  $C_2$  are cycles in  $\mathbb{P}^3$  such that  $C_1$  is given by

$$y = uy + z^2 = 0,$$

and the cycle  $C_2$  is given by the equations

$$uy + z^2 = zv_3(z, t) + zyc_2(y, z, t) - ug_3(z, t) = 0.$$

We have  $C_1 = 2L_2$ . But  $L_1 \not\subseteq \text{Supp}(C_2)$  because the polynomial

$$zv_3(z, t) + zyc_2(y, z, t) - ug_3(z, t)$$

does not vanish on  $L_1$ , because  $g_3(0, t) \neq 0$ . Then

$$Z_2 + L_1 = 2L_2,$$

which implies that  $Z_2 = L_1$ . Then  $\sum_{i=1}^r \text{mult}_{L_i}(Z_2) = 1$ , which is a contradiction.  $\square$

Thus, we see that  $r \leq 3$  and

$$f_4(z, t) = z^2(\alpha_1 z + \beta_1 t)(\alpha_2 z + \beta_2 t)$$

for some  $(\alpha_1 : \beta_1) \in \mathbb{P}^1 \ni (\alpha_2 : \beta_2)$ . Then

$$v_3(0, t) \neq 0$$

by Corollary 7.4. But  $Z_2$  can be given by the equations

$$\begin{aligned} uy + z^2 &= yv_3(z, t) + y^2c_2(y, z, t) - uy(\alpha_1z + \beta_1t)(\alpha_2z + \beta_2t) \\ &= 0 \subset \text{Proj}(\mathbb{C}[y, z, t, u]) \cong \mathbb{P}^3, \end{aligned}$$

which implies  $Z_2 = Z_2^1 + Z_2^2$ , where  $Z_2^1$  and  $Z_2^2$  are cycles on  $\mathbb{P}^3$  such that  $Z_2^1$  is given by

$$y = uy + z^2 = 0,$$

and the cycle  $Z_2^2$  is given by the equations

$$uy + z^2 = v_3(z, t) + yc_2(y, z, t) - u(\alpha_1z + \beta_1t)(\alpha_2z + \beta_2t) = 0,$$

which implies that  $Z_2^1 = 2L_1$ . Thus, we see that  $\sum_{i=1}^r \text{mult}_{L_i}(Z_2^2) \geq 2$ .

**Lemma 7.13.** *The inequality  $r \neq 3$  holds.*

**Proof.** Suppose that  $r = 3$ . Then  $\beta_1 \neq 0 \neq \beta_2$ , which implies that

$$L_1 \not\subseteq \text{Supp}(Z_2^2),$$

because  $v_3(z, t) + yc_2(y, z, t) - u(\alpha_1z + \beta_1t)(\alpha_2z + \beta_2t)$  does not vanish on  $L_1$ . But

$$L_2 \not\subseteq \text{Supp}(Z_2^2) \not\supseteq L_3,$$

because  $\beta_1 \neq 0 \neq \beta_2$ . Then  $\sum_{i=1}^r \text{mult}_{L_i}(Z_2^2) = 0$ , which is a contradiction.  $\square$

**Lemma 7.14.** *The inequality  $r \neq 2$  holds.*

**Proof.** Suppose that  $r = 2$ . We may assume that

- either  $\beta_1 \neq 0 = \beta_2$ ,
- or  $\alpha_1 = \alpha_2$  and  $\beta_1 = \beta_2 \neq 0$ .

Suppose that  $\beta_2 = 0$ . Then  $f_4(z, t) = \alpha_2z^3(\alpha_1z + \beta_1t)$  and

$$L_1 \not\subseteq \text{Supp}(Z_2^2),$$

because  $v_3(z, t) + yc_2(y, z, t) - \alpha_2uz(\alpha_1z + \beta_2t)$  does not vanish on  $L_1$ . But  $L_2$  is given by

$$y = \alpha_1z + \beta_1t = 0,$$



which implies that  $z^2$  does not vanish on  $L_2$ , because  $\beta_1 \neq 0$ . Then

$$L_2 \not\subseteq \text{Supp}(Z_2^2),$$

which implies that  $\sum_{i=1}^r \text{mult}_{L_i}(Z_2^2) = 0$ , which is a contradiction.

Hence, we see that  $\alpha_1 = \alpha_2$  and  $\beta_1 = \beta_2 \neq 0$ . Then  $L_1 \not\subseteq \text{Supp}(Z_2^2)$ , because

$$v_3(z, t) + yc_2(y, z, t) - u(\alpha_1 z + \beta_1 t)^2$$

does not vanish on  $L_1$ . But  $L_2 \not\subseteq \text{Supp}(Z_2^2)$ , because  $z^2$  does not vanish on  $L_2$ . Then

$$\sum_{i=1}^r \text{mult}_{L_i}(Z_2^2) = 0,$$

which is a contradiction.  $\square$

We see that  $f_2(z, t) = z^2$  and  $f_4(z, t) = \mu z^4$  for some  $0 \neq \mu \in \mathbb{C}$ . Then  $Z_2^2$  is given by

$$uy + z^2 = v_3(z, t) + yc_2(y, z, t) - \mu z^2 = 0,$$

where  $v_3(0, t) \neq 0$  by Corollary 7.4. Thus, we see that  $L_1 \not\subseteq \text{Supp}(Z_2^2)$ , because

$$v_3(z, t) + yc_2(y, z, t) - \mu z^2$$

does not vanish on  $L_1$ . Then  $\sum_{i=1}^r \text{mult}_{L_i}(Z_2^2) = 0$ , which is a contradiction.

The assertion of Proposition 7.1 is proved.

The assertion of Theorem 1.5 follows from Propositions 3.4, 5.1, 6.1, 7.1.

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