Sextic Double Solids

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Summary. We study properties of double covers of $\mathbb{P}^3$ ramified along nodal sextic surfaces such as nonrationality, $\mathbb{Q}$-factoriality, potential density, and elliptic fibration structures. We also consider some relevant problems over fields of positive characteristic.

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All varieties are assumed to be projective, normal, and defined over the field $\mathbb{C}$ unless otherwise stated.

1 Introduction

For a given variety, one of the substantial questions is whether it is rational or not. Global holomorphic differential forms are natural birational invariants of smooth algebraic varieties which solve the rationality problem for algebraic curves and surfaces (see [147]). However, these birational invariants are not sensitive enough to tell whether a given higher dimensional algebraic variety is nonrational. There are only four known methods to prove the nonrationality of a higher dimensional algebraic variety (see [79]).

The nonrationality of a smooth quartic 3-fold was proved in [80] using the group of birational automorphisms as a birational invariant. The nonrationality of a smooth cubic 3-fold was proved in [39] through the study of its intermediate Jacobian. Birational invariance of the torsion subgroups of the 3rd integral cohomology groups was used in [4] to prove the nonrationality of some unirational varieties. The nonrationality of a wide class of rationally connected varieties was proved in [88] via reductions into fields of positive characteristic (see [34], [89], and [90]). Meanwhile, the method of intermediate Jacobians works only in 3-folds. In most of the interesting cases, the 3rd integral cohomology groups have no torsion. The method of paper [88] works in every dimension, but its direct application gives the nonrationality just for a very general element of a given family. Even though the method of [80] works in every dimension, the area of its application is not so broad.

For this paper we mainly use the method that has evolved out of [80]. The most significant concept in the method is the birational super-rigidity that was implicitly introduced in [80].

Definition 1.1. *A terminal $\mathbb{Q}$-factorial Fano variety V with* $\mathrm{Pic}(V) \cong \mathbb{Z}$ *is birationally super-rigid if the following three conditions hold:*

1. *the variety V cannot be birationally transformed into a fibration*[3] *whose general enough fiber is a smooth variety of Kodaira dimension $-\infty$;*
2. *the variety V cannot be birationally transformed into another terminal $\mathbb{Q}$-factorial Fano variety with Picard group $\mathbb{Z}$ that is not biregular to V;*
3. $\mathrm{Bir}(V) = \mathrm{Aut}(V)$.

Implicitly the paper [80] proved that all the smooth quartic 3-folds in $\mathbb{P}^4$ are birationally super-rigid. Moreover, some Fano 3-folds with nontrivial group of birational automorphisms were also handled by the technique of [80], which gave the following weakened version of the birational super-rigidity:

Definition 1.2. *A terminal $\mathbb{Q}$-factorial Fano variety V with* $\mathrm{Pic}(V) \cong \mathbb{Z}$ *is called birationally rigid if the first two conditions of Definition* 1.1 *are satisfied.*

It is clear that the birational rigidity implies the nonrationality. Initially the technique of [80] was applied only to smooth varieties such as quartic 3-folds, quintic 4-folds, certain complete intersections, double spaces, and so on, but later, to singular varieties in [44], [65], [66], [67], [103], [111], [113], and [119]. Moreover, similar results were proved for many higher-dimensional conic bundles (see [125] and [126]) and del Pezzo fibrations (see [115]). Recently, Shokurov's connectedness principle in [130] shed a new light on the birational rigidity, which simplified the proofs of old results and helped to obtain new results (see [25], [29], [42], [49], [114], and [118]).

A quartic 3-fold with a single simple double point is not birationally super-rigid, but it is proved in [111] to be birationally rigid (for a simple proof, see [42]). However, a quartic 3-fold with one nonsimple double point may not necessarily be birationally rigid as shown in [44]. On the other hand, $\mathbb{Q}$-factorial quartic 3-folds with only simple double points are birationally rigid (see [103]).

Double covers of $\mathbb{P}^3$ with at most simple double points, so-called double solids, were studied in [37] with a special regard to quartic double solids, i.e.,

[3] For every fibration $\tau : Y \to Z$, we assume that $\dim(Y) > \dim(Z) \neq 0$ and $\tau_*(\mathcal{O}_Y) = \mathcal{O}_Z$.

double covers of $\mathbb{P}^3$ ramified along quartic nodal surfaces. It is natural to ask whether a double solid is rational or not. We can immediately see that all double solids are nonrational when their ramification surfaces are of degree greater than six. However, if the ramification surfaces have lower degree, then the problem is not simple.

Smooth quartic double solids are known to be nonrational (see [38], [94], [133], [134], [135], [136], and [144]), but singular ones can be birationally transformed into conic bundles. Quartic double solids cannot have more than 16 simple double points (see [12], [53], [92], [106], and [123]) and in the case of one simple double point they are nonrational as well (see [13] and [138]). There are non-$\mathbb{Q}$-factorial quartic double solids with six simple double points that can be birationally transformed into smooth cubic 3-folds (see [91]) and therefore are not rational due to [39]. On the other hand, some quartic double solids with seven simple double points are rational (see [91]). In general, the rationality question of singular quartic double solids can be very subtle and must be handled through the technique of intermediate Jacobians (see [13], [128], and [129]).

In the present paper we will consider the remaining case—the nonrationality question of sextic double solids, *i.e.*, double covers of $\mathbb{P}^3$ ramified along sextic nodal surfaces. To generate various examples of sextic double solids, we note that a double cover $\pi : X \to \mathbb{P}^3$ ramified along a sextic surface $S \subset \mathbb{P}^3$ can be considered as a hypersurface

$$u^2 = f_6(x, y, z, w)$$

of degree 6 in the weighted projective space $\mathbb{P}(1,1,1,1,3)$, where x, y, z, and w are homogeneous coordinates of weight 1, u is a homogeneous coordinate of weight 3, and f_6 is a homogeneous polynomial of degree 6.

A smooth sextic double solid is proved to be birationally super-rigid in [77]. Moreover, a smooth double space of dimension $n \geq 3$ was considered in [110]. The birational super-rigidity of a double cover of $\mathbb{P}^3$ ramified along a sextic with one simple double point was proved in [113]. To complete the study in this direction, one needs to prove the following:

Theorem A. *Let $\pi : X \to \mathbb{P}^3$ be a $\mathbb{Q}$-factorial double cover ramified along a sextic nodal surface $S \subset \mathbb{P}^3$. Then X is birationally super-rigid.*

As an immediate consequence, we obtain:

Corollary A. *Every $\mathbb{Q}$-factorial double cover of $\mathbb{P}^3$ ramified along a sextic nodal surface is nonrational and not birationally isomorphic to a conic bundle.*

Remark 1.3. Our proof of Theorem A does not require the base field to be algebraically closed. Therefore, the statement of Theorem A is valid over an arbitrary field of characteristic zero.

One can try to prove the nonrationality of a sextic double solid using the technique of intermediate Jacobians (see [13], [128], and [129]), but it seems to be very hard and still undone even in the smooth case (see [22]) except for the nonrationality of a sufficiently general smooth sextic double solid via a degeneration technique (see [13], [36], and [138]).

It is worthwhile to put emphasis on the $\mathbb{Q}$-factoriality condition of Theorem A. Indeed, rational sextic double solids do exist if we drop the $\mathbb{Q}$-factoriality condition.

Example 1.4. Let X be the double cover of $\mathbb{P}^3$ ramified in the Barth sextic (see [6]) given by the equation

$$4(\tau^2x^2 - y^2)(\tau^2y^2 - z^2)(\tau^2z^2 - x^2) - w^2(1 + 2\tau)(x^2 + y^2 + z^2 - w^2)^2 = 0$$

in $\mathrm{Proj}(\mathbb{C}[x, y, z, w])$, where $\tau = \frac{1+\sqrt{5}}{2}$. Then X has only simple double points and the number of singular points is 65. Moreover, there is a determinantal quartic 3-fold $V \subset \mathbb{P}^4$ with 42 simple double points such that the diagram

$$\begin{tikzcd} V \arrow[r, hook, "f"] \arrow[d, dashed, "\rho"'] & \mathbb{P}^4 \arrow[d, dashed, "\gamma"] \\ X \arrow[r, "\pi"'] & \mathbb{P}^3 \end{tikzcd}$$

commutes (see [53] and [108]), where ρ is a birational map and γ is the projection from one simple double point of the quartic V. Therefore, the double cover X is rational because determinantal quartics are rational (see [103] and [108]). In particular, X is not $\mathbb{Q}$-factorial by Theorem A. Indeed, one can show that $\mathrm{Pic}(X) \cong \mathbb{Z}$ and $\mathrm{Cl}(X) \cong \mathbb{Z}^{14}$ (see Example 3.7 in [53]).

A point p on a double cover $\pi : X \to \mathbb{P}^3$ ramified along a sextic surface S is a simple double point on X if and only if the point $\pi(p)$ is a simple double point on S. Sextic surfaces cannot have more than 65 simple double points (see [7], [82], and [143]). Furthermore, for each positive integer m not exceeding 65 there is a sextic surface with m simple double points (see [6], [21], and [132]), but in many cases it is not clear whether the corresponding double cover is $\mathbb{Q}$-factorial or not (see [37], [46], and [53]).

Example 1.4 shows that the $\mathbb{Q}$-factoriality condition is crucial for Theorem A. Accordingly, it is worthwhile to study the $\mathbb{Q}$-factoriality of sextic double solids.

A variety X is called $\mathbb{Q}$-factorial if a multiple of each Weil divisor on the variety X is a Cartier divisor. The $\mathbb{Q}$-factoriality depends on both local types of singularities and their global position (see [35], [37], and [103]). Moreover, the $\mathbb{Q}$-factoriality of the variety X depends on the field of definition of the variety X as well. When X is a Fano 3-fold with mild singularities and defined over $\mathbb{C}$, the global topological condition

$$\mathrm{rank}(H^2(X,\mathbb{Z})) = \mathrm{rank}(H_4(X,\mathbb{Z}))$$

is equivalent to the $\mathbb{Q}$-factoriality. The following three examples are inspired by [5], [91], and [103].

Example 1.5. Let $\pi : X \to \mathbb{P}^3$ be the double cover ramified along a sextic S and given by

$$u^2 + g_3^2(x,y,z,w) = h_1(x,y,z,w)f_5(x,y,z,w) \subset \mathbb{P}(1,1,1,1,3),$$

where g_3, h_1, and f_5 are sufficiently general polynomials defined over $\mathbb{R}$ of degree 3, 1, and 5, respectively; x, y, z, w are homogeneous coordinates of weight 1; u is a homogeneous coordinate of weight 3. Then the double cover X is not $\mathbb{Q}$-factorial over $\mathbb{C}$ because the divisor $h_1 = 0$ splits into two non-$\mathbb{Q}$-Cartier divisors conjugated by $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and given by the equation

$$(u + \sqrt{-1}g_3(x,y,z,w))(u - \sqrt{-1}g_3(x,y,z,w)) = 0.$$

The sextic surface $S \subset \mathrm{Proj}(\mathbb{C}[x,y,z,w])$ has 15 simple double points at the intersection points of the three surfaces

$$\{h_1(x,y,z,w) = 0\} \cap \{g_3(x,y,z,w) = 0\} \cap \{f_5(x,y,z,w) = 0\},$$

which gives 15 simple double points of X. Introducing two new variables s and t of weight 2 defined by

$$\begin{cases} s = \dfrac{u + \sqrt{-1}g_3(x,y,z,w)}{h_1(x,y,z,w)} = \dfrac{f_5(x,y,z,w)}{u - \sqrt{-1}g_3(x,y,z,w)} \\ t = \dfrac{u - \sqrt{-1}g_3(x,y,z,w)}{h_1(x,y,z,w)} = \dfrac{f_5(x,y,z,w)}{u + \sqrt{-1}g_3(x,y,z,w)} \end{cases}$$

we can unproject $X \subset \mathbb{P}(1,1,1,1,3)$ in the sense of [121] into two complete intersections

$$\begin{cases} V_s = \left\{ \begin{array}{l} sh_1(x,y,z,w) = u + \sqrt{-1}g_3(x,y,z,w) \\ s(u - \sqrt{-1}g_3(x,y,z,w)) = f_5(x,y,z,w) \end{array} \right\} \subset \mathbb{P}(1,1,1,1,3,2) \\ V_t = \left\{ \begin{array}{l} th_1(x,y,z,w) = u - \sqrt{-1}g_3(x,y,z,w) \\ t(u + \sqrt{-1}g_3(x,y,z,w)) = f_5(x,y,z,w) \end{array} \right\} \subset \mathbb{P}(1,1,1,1,3,2), \end{cases}$$

respectively, which are not defined over $\mathbb{R}$. Eliminating variable u, we get

$$\begin{cases} V_s = \{s^2h_1 - 2\sqrt{-1}sg_3 - f_5 = 0\} \subset \mathbb{P}(1,1,1,1,2) \\ V_t = \{t^2h_1 + 2\sqrt{-1}tg_3 - f_5 = 0\} \subset \mathbb{P}(1,1,1,1,2) \end{cases}$$

and for the unprojections $\rho_s : X \dashrightarrow V_s$ and $\rho_t : X \dashrightarrow V_t$ we obtain a commutative diagram

$$\begin{tikzcd}[column sep=small] & Y_s \arrow[dl, "\psi_s"'] \arrow[dr, "\phi_s"] & & Y_t \arrow[dl, "\phi_t"'] \arrow[dr, "\psi_t"] & \\ V_s & & X \arrow[ll, dashed, "\rho_s"'] \arrow[rr, dashed, "\rho_t"] & & V_t \end{tikzcd}$$

with birational morphisms ϕ_s, ψ_s, ϕ_t, and ψ_t such that ψ_s and ψ_t are extremal contractions in the sense of [41], while ϕ_s and ϕ_t are flopping contractions. Both the weighted hypersurfaces V_s and V_t are quasi-smooth (see [75]) and $\mathbb{Q}$-factorial with Picard groups $\mathbb{Z}$ (Lemma 3.5 in [43], Lemma 3.2.2 in [50], Théoréme 3.13 of Exp. XI in [68], see also [20]). Moreover, V_s and V_t are projectively isomorphic in $\mathbb{P}(1,1,1,1,2)$ by the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}_2$. In particular,

$$\mathrm{Pic}(Y_s) \cong \mathrm{Pic}(Y_t) \cong \mathbb{Z} \oplus \mathbb{Z};$$

Y_s and Y_t are $\mathbb{Q}$-factorial; $\mathrm{Cl}(X) = \mathbb{Z} \oplus \mathbb{Z}$. However, the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariant part of the group $\mathrm{Cl}(X)$ is $\mathbb{Z}$. Thus the 3-fold X is $\mathbb{Q}$-factorial over $\mathbb{R}$. It is therefore birationally super-rigid and nonrational over $\mathbb{R}$ by Theorem A. It is also not rational over $\mathbb{C}$ because $V_s \cong V_t$ is birationally rigid (see [43]). Moreover, the involution of X interchanging fibers of π induces a nonbiregular involution $\tau \in \mathrm{Bir}(V_s)$ which is regularized by ρ_s, i.e., the self-map $\rho_s^{-1}\circ\tau\circ\rho_s : X \to X$ is biregular (see [32]).

Example 1.6. Let $V \subset \mathbb{P}^4$ be a quartic 3-fold with one simple double point o. Then the quartic V is $\mathbb{Q}$-factorial and $\mathrm{Pic}(V) \cong \mathbb{Z}$. In fact, V can be given by the equation

$$t^2 f_2(x,y,z,w)+tf_3(x,y,z,w)+f_4(x,y,z,w) = 0 \subset \mathbb{P}^4 = \mathrm{Proj}(\mathbb{C}[x,y,z,w,t]).$$

Here, the point o is located at $[0:0:0:0:1]$. It is well known that the quartic 3-fold V is birationally rigid and hence nonrational (see [42], [103], and [111]). However, the quartic V is not birationally super-rigid because $\mathrm{Bir}(V) \neq \mathrm{Aut}(V)$. Indeed, the projection $\phi : V \dashrightarrow \mathbb{P}^3$ from the point o has degree 2 at a generic point of V and induces a nonbiregular involution $\tau \in \mathrm{Bir}(V)$.

Let $f : Y \to V$ be the blowup at the point o. The linear system $|-nK_Y|$ is free for some natural number $n \gg 0$ and gives a birational morphism $g = \phi_{|-nK_Y|} : Y \to X$ contracting every curve $C_i \subset Y$ such that $f(C_i)$ is a line on V passing through the point o. We then obtain the double cover $\pi : X \to \mathbb{P}^3$ ramified along the sextic surface $S \subset \mathbb{P}^3$ given by the equation

$$f_3^2(x,y,z,w) - 4f_2(x,y,z,w)f_4(x,y,z,w) = 0.$$

The variety X, *a priori*, has canonical Gorenstein singularities.

We suppose that V is general enough. Each line $f(C_i)$ then corresponds to an intersection point of three surfaces

$$\{f_2(x,y,z,w) = 0\} \cap \{f_3(x,y,z,w) = 0\} \cap \{f_4(x,y,z,w) = 0\}$$

in $\mathbb{P}^3 = \operatorname{Proj}(\mathbb{C}[x,y,z,w])$ which gives 24 different smooth rational curves $C_1, C_2, \ldots, C_{24}$ on Y. For each curve C_i we have

$$\mathcal{N}_{Y/C_i} \cong \mathcal{O}_{C_i}(-1) \oplus \mathcal{O}_{C_i}(-1)$$

and hence the morphism g is a standard flopping contraction which maps every curve C_i to a simple double point of the 3-fold X. In particular, the sextic $S \subset \mathbb{P}^3$ has exactly 24 simple double points. However, the 3-fold X is not $\mathbb{Q}$-factorial and $\operatorname{Cl}(X) = \mathbb{Z} \oplus \mathbb{Z}$.

Put $\rho := g \circ f^{-1}$. Then the involution $\gamma = \rho \circ \tau \circ \rho^{-1}$ is biregular on X and interchanges the fibers of the double cover π. Thus the map ρ is a regularization of the birational nonbiregular involution τ in the sense of [32], while the commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "f"'] \arrow[dr, "g"] & & & Y \arrow[dl, "g"'] \arrow[dr, "f"] & \\
V \arrow[rr, dashed, "\rho"] & & X \arrow[r, "\gamma"'] & X & & V \arrow[ll, dashed, "\rho"']
\end{tikzcd}$$

is a decomposition of the birational involution $\tau \in \operatorname{Bir}(V)$ in a sequence of elementary links (or Sarkisov links) with a midpoint X (see [41], [43], and [78]).

Suppose that $f_2(x,y,z,w)$ and $f_4(x,y,z,w)$ are defined over $\mathbb{Q}$ and

$$f_3(x,y,z,w) = \sqrt{2} g_3(x,y,z,w),$$

where $g_3(x,y,z,t)$ is defined over $\mathbb{Q}$ as well. Then the quartic 3-fold V is defined over $\mathbb{Q}(\sqrt{2})$ and not invariant under the action of $\operatorname{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$. However, the sextic surface $S \subset \mathbb{P}^3$ is given by the equation

$$2g_3^2(x,y,z,w) - 4f_2(x,y,z,w)f_4(x,y,z,w) = 0 \subset \mathbb{P}^3 = \operatorname{Proj}(\mathbb{Q}[x,y,z,w]),$$

which implies that the 3-fold X is defined over $\mathbb{Q}$ as well. Moreover, the $\operatorname{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$-invariant part of the group $\operatorname{Cl}(X)$ is $\mathbb{Z}$. Therefore, the 3-fold X is $\mathbb{Q}$-factorial and birationally super-rigid over $\mathbb{Q}$ by Theorem A and Remark 1.3.

Example 1.7. Let V be a smooth divisor of bidegree $(2,3)$ in $\mathbb{P}^1 \times \mathbb{P}^3$. The 3-fold V is then defined by the bihomogeneous equation

$$f_3(x,y,z,w)s^2 + g_3(x,y,z,w)st + h_3(x,y,z,w)t^2 = 0,$$

where f_3, g_3, and h_3 are homogeneous polynomials of degree 3. In addition, we denote the natural projection of V to $\mathbb{P}^3$ by $\pi : V \longrightarrow \mathbb{P}^3$. Suppose that the polynomials f_3, g_3, and h_3 are general enough. The 3-fold V then has exactly 27 lines $C_1, C_2, \cdots, C_{27}$ such that $-K_V \cdot C_i = 0$ because the intersection

$$\{f_3(x,y,z,w) = 0\} \cap \{g_3(x,y,z,w) = 0\} \cap \{h_3(x,y,z,w) = 0\}$$

in $\mathbb{P}^3$ consists of exactly 27 points. The projection π has degree 2 in the outside of the 27 points $\pi(C_i)$. The anticanonical model

$$\mathrm{Proj}\left(\bigoplus_{n\geq 0} H^0\left(V, \mathcal{O}_V(-nK_V)\right)\right)$$

of V is the double cover X of $\mathbb{P}^3$ ramified along the nodal sextic S defined by

$$g_3^2(x,y,z,w) - 4f_3(x,y,z,w)h_3(x,y,z,w) = 0.$$

It has exactly 27 simple double points each of which comes from each line C_i. The morphism $\phi_{|-K_V|} : V \longrightarrow X$ given by the anticanonical system of V contracts these 27 lines to the simple double points. Therefore, it is a small contraction and hence the double cover X cannot be $\mathbb{Q}$-factorial. A generic divisor of bidegree $(2,3)$ in $\mathbb{P}^1 \times \mathbb{P}^3$ over $\mathbb{C}$ is known to be nonrational (see [5], [33], and [131]), and hence the double cover X is also nonrational.

As shown in Examples 1.5, 1.6, and 1.7, there are non-$\mathbb{Q}$-factorial sextic double solids with 15, 24, and 27 simple double points. However, we will prove the following:

Theorem B. *Let $\pi : X \to \mathbb{P}^3$ be a double cover ramified along a nodal sextic surface $S \subset \mathbb{P}^3$. Then the 3-fold X is $\mathbb{Q}$-factorial when $\#|\mathrm{Sing}(S)| \leq 14$ and it is not $\mathbb{Q}$-factorial when $\#|\mathrm{Sing}(S)| \geq 57$.*

Using Theorem A with the theorem above, we immediately obtain:

Corollary B. *Let $\pi : X \to \mathbb{P}^3$ be a double cover ramified along a sextic $S \subset \mathbb{P}^3$ with at most 14 simple double points. Then X is birationally super-rigid. In particular, X is not rational and not birationally isomorphic to a conic bundle.*

In [21], there are explicit constructions of sextic surfaces in $\mathbb{P}^3$ with each number of simple double points not exceeding 64, which give us many examples of nonrational singular sextic double solids with at most 14 simple double points.

Besides the birational super-rigidity, a $\mathbb{Q}$-factorial double cover of $\mathbb{P}^3$ ramified in a sextic nodal surface has other interesting properties. Implicitly the method of [80] to prove the birational (super-)rigidity also gives us information on birational transformations to elliptic fibrations and Fano varieties with canonical singularities.

Construction A. *Consider a double cover $\pi : X \to \mathbb{P}^3$ ramified along a sextic $S \subset \mathbb{P}^3$ with a simple double point o. Let $f : W \to X$ be the blowup at the point o. Then the anticanonical linear system $|-K_W|$ is free and the morphism $\phi_{|-K_W|} : W \to \mathbb{P}^2$ is an elliptic fibration such that the diagram*

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "\phi_{|-K_W|}"'] & X \arrow[d, "\pi"] \\ \mathbb{P}^2 & \mathbb{P}^3 \arrow[l, dashed, "\gamma"'] \end{tikzcd}$$

is commutative, where $\gamma : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ *is the projection from the point* $\pi(o)$.

It is a surprise that some double covers of $\mathbb{P}^3$ ramified in nodal sextics can be birationally transformed into elliptic fibrations in a way very different from the one described in Construction A.

Construction B. *Let* $\pi : X \to \mathbb{P}^3$ *be a double cover ramified along a sextic* $S \subset \mathbb{P}^3$ *such that the surface* S *contains a line* $L \subset \mathbb{P}^3$ *and the line* L *passes through exactly four simple double points of* S. *For a general enough point* $p \in X$, *there is a unique hyperplane* $H_p \subset \mathbb{P}^3$ *containing* $\pi(p)$ *and* L. *The set* $L \cap (C \setminus \mathrm{Sing}(S))$ *consists of a single point* q_p, *where* $C \subset H_p$ *is the quintic curve given by* $S \cap H_p = L \cup C$. *The two points* $\pi(p)$ *and* q_p *determine a line* L_p *in* $\mathbb{P}^3$. *Define a rational map* $\Xi_L : X \dashrightarrow Grass(2,4)$ *by* $\Xi_L(p) = L_p$. *The image of the map* Ξ_L *is isomorphic to* $\mathbb{P}^2$, *hence we may assume that the map* Ξ_L *is a rational map of* X *onto* $\mathbb{P}^2$. *Obviously the map* Ξ_L *is not defined over* L, *the normalization of its general fiber is an elliptic curve, and a resolution of indeterminacy of the map* Ξ_L *birationally transforms the* 3*-fold* X *into an elliptic fibration.*

In this paper we will prove that these two constructions are essentially the only ways to transform X birationally into an elliptic fibration when X is $\mathbb{Q}$-factorial.

Theorem C. *Let* $\pi : X \to \mathbb{P}^3$ *be a* $\mathbb{Q}$*-factorial double cover ramified along a nodal sextic* S. *Suppose that we have a birational map* $\rho : X \dashrightarrow Y$, *where* $\tau : Y \to Z$ *is an elliptic fibration. Then one of the following holds:*

1. *There are a simple double point* o *on* X *and a birational map* $\beta : \mathbb{P}^2 \dashrightarrow Z$ *such that the projection* γ *from the point* $\pi(o)$ *makes the diagram*

$$\begin{tikzcd} X \arrow[rr, dashed, "\rho"] \arrow[d, "\pi"'] & & Y \arrow[d, "\tau"] \\ \mathbb{P}^3 \arrow[r, dashed, "\gamma"] & \mathbb{P}^2 \arrow[r, dashed, "\beta"] & Z \end{tikzcd}$$

   commute.
2. *The sextic* S *contains a line* $L \subset \mathbb{P}^3$ *with* $\#|\mathrm{Sing}(S) \cap L| = 4$ *and there is a birational map* $\beta : \mathbb{P}^2 \dashrightarrow Z$ *such that the diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\Xi_L"'] & Y \arrow[d, "\tau"] \\ \mathbb{P}^2 \arrow[r, dashed, "\beta"] & Z \end{tikzcd}$$

is commutative, where Ξ_L is the rational map defined in Construction B.

In the case of one simple double point, Theorem C was proved in [27].

Corollary C1. *All birational transformations of a $\mathbb{Q}$-factorial double cover of $\mathbb{P}^3$ ramified along a sextic nodal surface into elliptic fibrations*[4] *are described by Constructions A and B.*

The following result was also proved in [24].

Corollary C2. *A smooth double cover X of $\mathbb{P}^3$ ramified along a sextic surface $S \subset \mathbb{P}^3$ cannot be birationally transformed into any elliptic fibration.*

Remark 1.8. Let X be a double cover of $\mathbb{P}^3$ ramified in a sextic surface $S \subset \mathbb{P}^3$ such that the surface S has a double line (see [67]). Then the set of birational transformations of X into elliptic fibrations is infinite and cannot be effectively described (see [31]).

The statement of Theorem C is valid over an arbitrary field $\mathbb{F}$ of characteristic zero, but in Construction A the singular point must be defined over $\mathbb{F}$ as we see in the example below. Similarly the same has to be satisfied for Theorem D, but the total number of singular points on a line must be counted in geometric sense (over the algebraic closure of $\mathbb{F}$).

Example 1.9. Let X be the double cover of $\mathbb{P}^3$ ramified in a sextic $S \subset \mathbb{P}^3$ and defined by the equation

$$u^2 = x^6 + xy^5 + y^6 + (x+y)(z^5 - 2zw^4) + y(z^4 - 2w^4)(z - 3w)$$

in $\mathbb{P}(1,1,1,1,3)$. Then X is smooth in the outside of four simple double points given by $x = y = z^4 - 2w^4 = 0$. Hence, X is $\mathbb{Q}$-factorial, birationally superrigid, and nonrational over $\mathbb{C}$ by Theorems A and B. Moreover, $x = y = 0$ cuts a curve $C \subset X$ such that $-K_X \cdot C = 1$ and $\pi(C) \subset S$ is a line. Therefore, X can be birationally transformed over $\mathbb{C}$ into exactly five elliptic fibrations given by Constructions A and B. However, the 3-fold X defined over $\mathbb{Q}$ is birationally isomorphic to only one elliptic fibration given by Construction B.

Birational transformations of other higher-dimensional algebraic varieties into elliptic fibrations were studied in [24], [25], [26], [28], [29], [30], [31], and [124]. It turns out that classification of birational transformations into elliptic fibrations implicitly gives classification of birational transformations into canonical Fano 3-folds.

In the present paper we will prove the following result.

[4]Fibrations $\tau_1 : U_1 \to Z_1$ and $\tau_2 : U_2 \to Z_2$ can be identified if there are birational maps $\alpha : U_1 \dashrightarrow U_2$ and $\beta : Z_1 \dashrightarrow Z_2$ such that $\tau_2 \circ \alpha = \beta \circ \tau_1$ and the map α induces an isomorphism between generic fibers of τ_1 and τ_2.

Theorem D. *Let $\pi : X \to \mathbb{P}^3$ be a $\mathbb{Q}$-factorial double cover ramified in a nodal sextic $S \subset \mathbb{P}^3$. Then X is birationally isomorphic to a Fano 3-fold with canonical singularities that is not biregular to X if and only if the sextic S contains a line L passing through five simple double points of the surface $S \subset \mathbb{P}^3$.*

During the proof of Theorem D, we will explicitly describe the constructions of all possible birational transformations of sextic double solids into Fano 3-folds with canonical singularities.

Example 1.10. Let X be the double cover of $\mathbb{P}^3$ ramified in a sextic $S \subset \mathbb{P}^3$ and defined by the equation

$$u^2 = x^6 + xy^5 + y^6 + (x+y)(z^5 - zw^4)$$

in $\mathbb{P}(1,1,1,1,3)$. Then X is smooth in the outside of five simple double points given by $x = y = z(z^4 - w^5) = 0$. For the same reason as in Example 1.9, the double cover X is $\mathbb{Q}$-factorial, birationally super-rigid, and nonrational. As for elliptic fibrations, it can be birationally transformed into five elliptic fibrations given by Construction A. Also, the 3-fold X is birationally isomorphic to a unique Fano 3-fold with canonical singularities that is not biregular to X.

The statements of Theorems A, C, and D are valid over all fields of characteristic zero, but over fields of positive characteristic some difficulties may occur. Indeed, the vanishing theorem of Y. Kawamata and E. Viehweg (see [84], [142]) is no longer true in positive characteristic. Even though there are some vanishing theorems over fields of positive characteristic (see [55], [127]), they are not applicable to our case. A smooth resolution of indeterminacy of a birational map may fail as well because it implicitly uses resolution of singularities (see [74]) which is completely proved only in characteristic zero. However, resolution of singularities for 3-folds is proved in [1] for the case of characteristic > 5 (see also [45]).

Consider the following very special example.

Example 1.11. Suppose that the base field is $\mathbb{F}_5 = \mathbb{Z}/5\mathbb{Z}$. Let X be the double cover of $\mathbb{P}^3 = \mathrm{Proj}(\mathbb{F}_5[x,y,z,w])$ ramified along the sextic S given by the equation

$$x^5y + x^4y^2 + x^2y^3z - y^5z - 2x^4z^2 + xz^5 + yz^5 + x^3y^2w + 2x^2y^3w$$
$$-xyz^3w - xyz^2w^2 - x^2yw^3 + xy^2w^3 + x^2zw^3 + xyw^4 + xw^5 + 2yw^5 = 0.$$

Then X is smooth (see [52] and [63]) and $\mathrm{Pic}(X) \cong \mathbb{Z}$ by Lemma 3.2.2 in [50] or Lemma 3.5 in [43] (see [20] and [68]). Moreover, X contains a curve C given by the equations $x = y = 0$ whose image in $\mathbb{P}^3$ is a line L contained in the sextic $S \subset \mathbb{P}^3$. For a general enough point $p \in X$, there is a unique hyperplane $H_p \subset \mathbb{P}^3$ containing $\pi(p)$ and L. The residual quintic curve $Q \subset H_p$ given by

$S \cap H_p = L \cup Q$ intersects L at a single point q_p with $\mathrm{mult}_{q_p}(Q|_L) = 5$. The two points $\pi(p)$ and q_p determine a line L_p in $\mathbb{P}^3$. As in Construction B we can define a rational map $\Psi : X \dashrightarrow \mathbb{P}^2$ by the lines L_p. As we see, the situation is almost the same as that of Construction B. We, at once, see that a resolution of indeterminacy of the map Ψ birationally transforms the 3-fold X into an elliptic fibration.

Therefore, Theorem C and even Corollary C2 are not valid over some fields of positive characteristic. We will, however, prove the following result:

Theorem E. *Let $\pi : X \to \mathbb{P}^3$ be a double cover defined over a perfect field $\mathbb{F}$ and ramified along a sextic nodal surface $S \subset \mathbb{P}^3$. Suppose that X is $\mathbb{Q}$-factorial and* $\mathrm{Pic}(X) \cong \mathbb{Z}$. *Then X is birationally super-rigid and birational maps of X into elliptic fibrations are described by Constructions A and B if* $\mathrm{char}(\mathbb{F}) > 5$.

Nonrationality and related questions like nonruledness or birational rigidity over fields of positive characteristic may be interesting in the following cases:

1. arithmetics of algebraic varieties over finite fields (see [54], [93], and [107]);
2. classification of varieties over fields of positive characteristic (see [102] and [127]);
3. algebro-geometric coding theory (see [18], [61], [62], [73], [137], and [140]);
4. proofs of the nonrationality of certain higher-dimensional varieties by means of reduction into fields of positive characteristic (see [34], [88], [89], and [90]), where even nonperfect fields may appear in some very subtle questions as in [90].

In arithmetic geometry, it is an important and difficult problem to measure the size of the set of rational points on a given variety defined over a number field $\mathbb{F}$. One of the most profound works in this area is, for example, Faltings' theorem that a smooth curve of genus at least two defined over a number field $\mathbb{F}$ has finitely many $\mathbb{F}$-rational points (see [56]). One of the higher-dimensional generalizations of the theorem is the Weak Lang Conjecture that the set of rational points of a smooth variety of general type defined over a number field is not Zariski dense, which is still far away from proofs.

A counterpart of the Weak Lang Conjecture is the conjecture that for a smooth variety X with ample $-K_X$ defined over a number field $\mathbb{F}$ there is a finite field extension of the field $\mathbb{F}$ over which the set of rational points of X is Zariski dense. We can easily check that this conjecture is true for curves and surfaces, where the condition implies that X is rational over some finite field extension. Therefore, smooth Fano 3-folds are the first nontrivial cases testing the conjecture.

Definition 1.12. *The set of rational points of a variety X defined over a number field $\mathbb{F}$ is said to be potentially dense if for some finite field extension $\mathbb{K}$ of the field $\mathbb{F}$ the set of $\mathbb{K}$-rational points of X is Zariski dense in X.*

Using elliptic fibrations, [15] and [70] have proved:

Theorem 1.13. *The set of rational points is potentially dense on all smooth Fano 3-folds defined over a number field $\mathbb{F}$ possibly except double covers of $\mathbb{P}^3$ ramified along smooth sextics.*

Arithmetic properties of algebraic varieties are closely related to their biregular and birational geometry (see [8], [9], [10], [11], [58], [95], [96], [97], [98], [99], [100], and [101]). For example, the possible exception appears in Theorem 1.13 because smooth double covers of $\mathbb{P}^3$ ramified in sextics are the only smooth Fano 3-folds that are not birationally isomorphic to elliptic fibrations (see [81]). Besides Fano varieties, on several other classes of algebraic varieties the potential density of rational points has been proved (see [15], [16], and [17]).

In Section 8 we prove the following result:

Theorem F. *Let $\pi : X \to \mathbb{P}^3$ be a double cover defined over a number field $\mathbb{F}$ and ramified along a sextic nodal surface $S \subset \mathbb{P}^3$. If $\mathrm{Sing}(X) \neq \varnothing$, then the set of rational points on X is potentially dense.*

As shown in Theorem C, the sextic double solid can be birationally transformed into an elliptic fibration if it has a simple double point. Therefore, we can adopt the methods of [15] and [70] in this case.

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2 Movable log pairs and Nöther–Fano inequalities

To study sextic double solids we frequently use movable log pairs introduced in [2]. In this section we overview their properties and Nöther–Fano inequalities that are the most important tools for birational (super-)rigidity.

Definition 2.1. *On a variety X a movable boundary $\mathcal{M}_X = \sum_{i=1}^{n} a_i \mathcal{M}_i$ is a formal finite $\mathbb{Q}$-linear combination of linear systems $\mathcal{M}_i$ on X such that the base locus of each $\mathcal{M}_i$ has codimension at least two and each coefficient a_i is nonnegative. A movable log pair $(X, \mathcal{M}_X)$ is a variety X with a movable boundary $\mathcal{M}_X$ on X.*

Every movable log pair can be considered as a usual log pair by replacing each linear system by its general element. In particular, for a given movable log pair $(X, \mathcal{M}_X)$ we may handle the movable boundary $\mathcal{M}_X$ as an effective divisor. We can also consider the self-intersection $\mathcal{M}_X^2$ of $\mathcal{M}_X$ as a well-defined effective codimension-two cycle when X is $\mathbb{Q}$-factorial. We call $K_X + \mathcal{M}_X$ the log canonical divisor of the movable log pair $(X, \mathcal{M}_X)$. Throughout the rest of this section, we will assume that log canonical divisors are $\mathbb{Q}$-Cartier divisors.

Definition 2.2. *Movable log pairs $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ are birationally equivalent if there is a birational map $\rho : X \dashrightarrow Y$ such that $\mathcal{M}_Y = \rho(\mathcal{M}_X)$.*

The notions such as discrepancies, (log) terminality, and (log) canonicity can be defined for movable log pairs as for usual log pairs (see [86]).

Definition 2.3. *A movable log pair $(X, \mathcal{M}_X)$ has canonical (terminal, resp.) singularities if for every birational morphism $f : W \to X$ each discrepancy $a(X, \mathcal{M}_X, E)$ in*

$$K_W + f^{-1}(\mathcal{M}_X) \sim_{\mathbb{Q}} f^*(K_X + \mathcal{M}_X) + \sum_{E:\ f\text{-exceptional divisor}} a(X, \mathcal{M}_X, E)E$$

is nonnegative (positive, resp.).

Example 2.4. Let $\mathcal{M}$ be a linear system on a 3-fold X with no fixed components. Then the log pair $(X, \mathcal{M})$ has terminal singularities if and only if the linear system $\mathcal{M}$ has only isolated simple base points which are smooth points on the 3-fold X.

The Log Minimal Model Program holds good for three-dimensional movable log pairs with canonical (terminal) singularities (see [2] and [86]). In particular, it preserves their canonicity (terminality).

Every movable log pair is birationally equivalent to a movable log pair with canonical or terminal singularities. Away from the base loci of the components of its boundary, the singularities of a movable log pair coincide with those of its variety.

Definition 2.5. *A proper irreducible subvariety $Y \subset X$ is called a center of the canonical singularities of a movable log pair $(X, \mathcal{M}_X)$ if there are a birational morphism $f : W \to X$ and an f-exceptional divisor $E \subset W$ such that the discrepancy $a(X, \mathcal{M}_X, E) \leq 0$ and $f(E) = Y$. The set of all the centers of the canonical singularities of the movable log pair $(X, \mathcal{M}_X)$ will be denoted by $\mathbb{CS}(X, \mathcal{M}_X)$.*

Note that a log pair $(X, \mathcal{M}_X)$ is terminal if and only if $\mathbb{CS}(X, \mathcal{M}_X) = \varnothing$.

Let $(X, \mathcal{M}_X)$ be a movable log pair and $Z \subset X$ be a proper irreducible subvariety such that X is smooth along the subvariety Z. Then elementary properties of blowups along smooth subvarieties of smooth varieties imply that

$$Z \in \mathbb{CS}(X, \mathcal{M}_X) \Rightarrow \mathrm{mult}_Z(\mathcal{M}_X) \geq 1$$

and in the case when $\mathrm{codim}(Z \subset X) = 2$ we have

$$Z \in \mathbb{CS}(X, \mathcal{M}_X) \iff \mathrm{mult}_Z(\mathcal{M}_X) \geq 1.$$

For a movable log pair $(X, \mathcal{M}_X)$ we consider a birational morphism $f : W \to X$ such that the log pair $(W, \mathcal{M}_W := f^{-1}(\mathcal{M}_X))$ has canonical singularities.

Definition 2.6. *The number $\kappa(X, \mathcal{M}_X) = \dim(\phi_{|nm(K_W+\mathcal{M}_W)|}(W))$ for $n \gg 0$ is called the Kodaira dimension of the movable log pair $(X, \mathcal{M}_X)$, where m is a natural number such that $m(K_W + \mathcal{M}_W)$ is a Cartier divisor. When $|nm(K_W + \mathcal{M}_W)| = \varnothing$ for all $n \in \mathbb{N}$, the Kodaira dimension $\kappa(X, \mathcal{M}_X)$ is defined to be $-\infty$.*

Proposition 2.7. *The Kodaira dimension of a movable log pair is well-defined. In particular, it does not depend on the choice of the birationally equivalent movable log pair with canonical singularities.*

Proof. Let $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ be movable log pairs with canonical singularities such that there is a birational map $\rho : Y \dashrightarrow X$ with $\mathcal{M}_X = \rho(\mathcal{M}_Y)$. Choose positive integers m_X and m_Y such that both $m_X(K_X + \mathcal{M}_X)$ and $m_Y(K_Y + \mathcal{M}_Y)$ are Cartier divisors. We must show that either

$$|nm_X(K_X + \mathcal{M}_X)| = |nm_Y(K_Y + \mathcal{M}_Y)| = \varnothing \text{ for all } n \in \mathbb{N}$$

or

$$\dim(\phi_{|nm_X(K_X+\mathcal{M}_X)|}(X)) = \dim(\phi_{|nm_Y(K_Y+\mathcal{M}_Y)|}(Y)) \text{ for } n \gg 0.$$

We consider a Hironaka hut of $\rho : Y \dashrightarrow X$, i.e., a smooth variety W with birational morphisms $g : W \to X$ and $f : W \to Y$ such that the diagram

$$\begin{tikzcd} & W \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y \arrow[rr, dashed, "\rho"] & & X \end{tikzcd}$$

commutes. We then obtain

$$K_W + \mathcal{M}_W \sim_{\mathbb{Q}} g^*(K_X + \mathcal{M}_X) + \Sigma_X \sim_{\mathbb{Q}} f^*(K_Y + \mathcal{M}_Y) + \Sigma_Y,$$

where $\mathcal{M}_W = g^{-1}(\mathcal{M}_X)$, Σ_X and Σ_Y are the exceptional divisors of g and f, respectively. Because the movable log pairs $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ have canonical singularities, the exceptional divisors Σ_X and Σ_Y are effective and hence the linear systems $|n(K_W+\mathcal{M}_W)|$, $|g^*(n(K_X+\mathcal{M}_X))|$, and $|f^*(n(K_Y+\mathcal{M}_Y))|$ have the same dimension for a big and divisible enough natural number n. Moreover, if these linear systems are not empty, then we have

$$\phi_{|n(K_W+\mathcal{M}_W)|} = \phi_{|g^*(n(K_X+\mathcal{M}_X))|} = \phi_{|f^*(n(K_Y+\mathcal{M}_Y))|},$$

which implies the claim. $\square$

By definition, the Kodaira dimension of a movable log pair is a birational invariant and a nondecreasing function of the coefficients of the movable boundary.

Definition 2.8. *A movable log pair $(V, \mathcal{M}_V)$ is called a canonical model of a movable log pair $(X, \mathcal{M}_X)$ if there is a birational map $\psi : X \dashrightarrow V$ such that $\mathcal{M}_V = \psi(\mathcal{M}_X)$, the movable log pair $(V, \mathcal{M}_V)$ has canonical singularities, and the divisor $K_V + \mathcal{M}_V$ is ample.*

Proposition 2.9. *A canonical model of a movable log pair is unique if it exists.*

Proof. Let $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ be canonical models such that there is a birational map $\rho : Y \dashrightarrow X$ with $\mathcal{M}_X = \rho(\mathcal{M}_Y)$. Take a smooth variety W with birational morphisms $g : W \to X$ and $f : W \to Y$ such that the diagram

$$\begin{tikzcd} & W \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y \arrow[rr, dashed, "\rho"] & & X \end{tikzcd}$$

commutes. We have

$$K_W + \mathcal{M}_W \sim_{\mathbb{Q}} g^*(K_X + \mathcal{M}_X) + \Sigma_X \sim_{\mathbb{Q}} f^*(K_Y + \mathcal{M}_Y) + \Sigma_Y,$$

where $\mathcal{M}_W = g^{-1}(\mathcal{M}_X) = f^{-1}(\mathcal{M}_Y)$, Σ_X and Σ_Y are the exceptional divisors of birational morphisms g and f, respectively. Let $n \in \mathbb{N}$ be a big and divisible enough number such that $n(K_W + \mathcal{M}_W)$, $n(K_X + \mathcal{M}_X)$, and $n(K_Y + \mathcal{M}_Y)$ are Cartier divisors. For the same reason as in the proof of Proposition 2.7 we obtain

$$\phi_{|n(K_W+\mathcal{M}_W)|} = \phi_{|g^*(n(K_X+\mathcal{M}_X))|} = \phi_{|f^*(n(K_Y+\mathcal{M}_Y))|}.$$

Therefore, the birational map ρ is an isomorphism because $K_X + \mathcal{M}_X$ and $K_Y + \mathcal{M}_Y$ are ample. □

The existence of the canonical model of a movable log pair implies that its Kodaira dimension is equal to the dimension of the variety.

Nöther–Fano inequalities can be immediately reinterpreted in terms of canonical singularities of movable log pairs. For reader's understanding, we give the theorems and their proofs on the relation between singularities of movable log pairs and birational (super-)rigidity. In addition, with del Pezzo surfaces of Picard number 1 defined over nonclosed fields, we demonstrate how to apply the theorems, which is so simple that one can easily understand.

The following result is known as a classical Nöther-Fano inequality.

Theorem 2.10. *Let X be a terminal $\mathbb{Q}$-factorial Fano variety with* $\mathrm{Pic}(X) \cong \mathbb{Z}$. *If every movable log pair $(X, \mathcal{M}_X)$ with $K_X + \mathcal{M}_X \sim_{\mathbb{Q}} 0$ has canonical singularities, then X is birationally super-rigid.*

Proof. Suppose that there is a birational map $\rho : X \dashrightarrow V$ such that V is a Fano variety with $\mathbb{Q}$-factorial terminal singularities and $\mathrm{Pic}(V) \cong \mathbb{Z}$. We are to show that ρ is an isomorphism. Let $\mathcal{M}_V = r| - nK_V|$ and $\mathcal{M}_X = \rho^{-1}(\mathcal{M}_V)$ for a natural number $n \gg 0$ and a rational number $r > 0$ such that $K_X + \mathcal{M}_X \sim_{\mathbb{Q}} 0$. Because $| - nK_V|$ is free for $n \gg 0$ and V has at worst terminal singularities, the log pair $(V, \mathcal{M}_V)$ has terminal singularities. In addition, the equality

$$\kappa(X, \mathcal{M}_X) = \kappa(V, \mathcal{M}_V) = 0$$

implies that the divisor $K_V + \mathcal{M}_V$ is nef; otherwise the Kodaira dimension $\kappa(V, \mathcal{M}_V)$ would be $-\infty$.

Let $f : W \to X$ be a birational morphism of a smooth variety W such that $g = \rho \circ f$ is a morphism. Then

$$\begin{aligned} K_W + \mathcal{M}_W &= f^*(K_X + \mathcal{M}_X) + \sum_{i=1}^{l_1} a(X, \mathcal{M}_X, F_i)F_i + \sum_{k=1}^{m} a(X, \mathcal{M}_X, E_k)E_k \\ &= g^*(K_V + \mathcal{M}_V) + \sum_{j=1}^{l_2} a(V, \mathcal{M}_V, G_j)G_j + \sum_{k=1}^{m} a(V, \mathcal{M}_V, E_k)E_k, \end{aligned}$$

where $\mathcal{M}_W = f^{-1}(\mathcal{M}_X)$, each divisor F_i is f-exceptional but not g-exceptional, each divisor G_j is g-exceptional but not f-exceptional, and each E_k is both f-exceptional and g-exceptional. Applying Lemma 2.19 in [87], we obtain

$$a(X, \mathcal{M}_X, E_k) = a(V, \mathcal{M}_V, E_k)$$

for each k and we see that there is no g-exceptional but not f-exceptional divisor, i.e., $l_2 = 0$ because the log pair $(V, \mathcal{M}_V)$ has terminal singularities. Furthermore, there exits no f-exceptional but not g-exceptional divisor, i.e., $l_1 = 0$ because the Picard numbers of V and X are the same. Therefore, the log pair $(X, \mathcal{M}_X)$ has at worst terminal singularities. For some $d \in \mathbb{Q}_{>1}$, both the movable log pairs $(X, d\mathcal{M}_X)$ and $(V, d\mathcal{M}_V)$ are canonical models. Hence, ρ is an isomorphism by Proposition 2.9.

We now suppose that we have a birational map $\chi : X \dashrightarrow Y$ of X into a fibration $\tau : Y \to Z$, where Y is smooth and a general fiber of τ is a smooth variety of Kodaira dimension $-\infty$. Let $\mathcal{M}_Y = c|\tau^*(H)|$ and $\mathcal{M}_X = \chi^{-1}(\mathcal{M}_Y)$, where H is a very ample divisor on Z and c is a positive rational number such that $K_X + \mathcal{M}_X \sim_{\mathbb{Q}} 0$. Then the Kodaira dimension $\kappa(X, \mathcal{M}_X)$ is zero because the log pair $(X, \mathcal{M}_X)$ has at worst canonical singularities and $K_X + \mathcal{M}_X \sim_{\mathbb{Q}} 0$. However, the Kodaira dimension $\kappa(Y, \mathcal{M}_Y) = -\infty$. This contradiction completes the proof. $\square$

The proof of Theorem 2.10 shows a condition for the Fano variety X to be birationally rigid as follows:

Corollary 2.11. *Let X be a terminal $\mathbb{Q}$-factorial Fano variety with* $\mathrm{Pic}(X) \cong \mathbb{Z}$. *Suppose that for every movable log pair $(X, \mathcal{M}_X)$ with $K_X + \mathcal{M}_X \sim_{\mathbb{Q}} 0$ either the singularities of the log pair $(X, \mathcal{M}_X)$ are canonical or the divisor $-(K_X + \rho(\mathcal{M}_X))$ is ample for some birational automorphism $\rho \in \mathrm{Bir}(X)$. Then X is birationally rigid.*

The Log Minimal Model Program tells us that the condition in Theorem 2.10 is a necessary and sufficient one for X to be birationally super-rigid.

Proposition 2.12. *Let X be a terminal $\mathbb{Q}$-factorial Fano 3-fold with* $\mathrm{Pic}(X) \cong \mathbb{Z}$. *Then X is birationally super-rigid if and only if every movable log pair $(X, \mathcal{M}_X)$ with $K_X + \mathcal{M}_X \sim_{\mathbb{Q}} 0$ has at worst canonical singularities.*

Proof. Suppose that X is birationally super-rigid. In addition, we suppose that there is a movable log pair $(X, \mathcal{M}_X)$ with noncanonical singularities such that $K_X + \mathcal{M}_X \sim_{\mathbb{Q}} 0$. Let $f : W \to X$ be a birational morphism such that the log pair $(W, \mathcal{M}_W := f^{-1}(\mathcal{M}_X))$ has canonical singularities. Then

$$K_W + \mathcal{M}_W = f^*(K_X + \mathcal{M}_X) + \sum_{i=1}^{k} a(X, \mathcal{M}_X, E_i)E_i \sim_{\mathbb{Q}} \sum_{i=1}^{k} a(X, \mathcal{M}_X, E_i)E_i,$$

where E_i is an f-exceptional divisor and $a(X, \mathcal{M}_X, E_j) < 0$ for some j.

Applying the relative Log Minimal Model Program to the log pair $(W, \mathcal{M}_W)$ over X we may assume $K_W + \mathcal{M}_W$ is f-nef. Then, Lemma 2.19 in [87] immediately implies that $a(X, \mathcal{M}_X, E_i) \leq 0$ for all i. The Log Minimal Model Program for $(W, \mathcal{M}_W)$ gives a birational map ρ of W into a Mori fibration space Y, i.e., a fibration $\pi : Y \to Z$ such that $-K_Y$ is π-ample, the variety Y has $\mathbb{Q}$-factorial terminal singularities, and $\mathrm{Pic}(Y/Z) \cong \mathbb{Z}$. However, the birational map $\rho \circ f^{-1}$ is not an isomorphism. □

Despite its formal appearance, Theorem 2.10 can be effectively applied in many different cases. For example, the following result in [95] and [96] is an application of Theorem 2.10.

Theorem 2.13. *Let X be a smooth del Pezzo surface defined over a perfect field $\mathbb{F}$ with* $\mathrm{Pic}(X) \cong \mathbb{Z}$ *and $K_X^2 \leq 3$. Then X is birationally rigid and nonrational over $\mathbb{F}$.*

Proof. Suppose that X is not birationally rigid. Then there is a movable log pair $(X, \mathcal{M}_X)$ defined over $\mathbb{F}$ such that $K_X + \mathcal{M}_X \sim_{\mathbb{Q}} 0$ and that is not canonical at some smooth point $o \in X$. Therefore, $\mathrm{mult}_o(\mathcal{M}_X) > 1$ and

$$3 \geq K_X^2 = M_X^2 \geq \mathrm{mult}_o^2(\mathcal{M}_X) \deg(o \otimes \bar{\mathbb{F}}) > \deg(o \otimes \bar{\mathbb{F}}),$$

where $\bar{\mathbb{F}}$ is the algebraic closure of the field $\mathbb{F}$. In the case $K_X^2 = 1$, the strict inequality is a contradiction. Moreover, if $K_X^2 = 2$, then the point o is defined over $\mathbb{F}$, and if $K_X^3 = 3$, then the point o splits into no more than two points over the field $\bar{\mathbb{F}}$.

Suppose that K_X^2 is either 2 or 3. Let $f : V \to X$ be the blowup at the point o. Then

$$K_V^2 = K_X^2 - \deg(o \otimes \bar{\mathbb{F}})$$

and V is a smooth del Pezzo surface because $\mathrm{Pic}(X) = \mathbb{Z}$, the inequality $\mathrm{mult}_o(\mathcal{M}_X) > 1$ holds, and the boundary $\mathcal{M}_X$ is movable. The double cover $\phi_{|-K_V|}$ induces an involution $\tau \in \mathrm{Bir}(X)$ that is classically known as Bertini or Geizer involution. Simple calculations show the ampleness of divisor $-(K_X + \tau(\mathcal{M}_X))$, which contradicts Corollary 2.11. □

The proofs of Theorems 2.10 and 2.13 and Lemma 5.3.1 in [90] imply that a result similar to Theorem 2.13 holds over a nonperfect field as well. Indeed, one can prove that a nonsingular del Pezzo surface X defined over nonperfect field $\mathbb{F}$ is nonrational over $\mathbb{F}$ and is not birationally isomorphic over $\mathbb{F}$ to any nonsingular del Pezzo surface Y with $\mathrm{Pic}(Y) = \mathbb{Z}$, which is smooth in codimension one, if $\mathrm{Pic}(X) \cong \mathbb{Z}$ and $K_X^2 \leq 3$.

Most applications of Theorem 2.10 have the pattern of the proof of Theorem 2.13 implicitly.

The following result can be considered as a weak Nöther–Fano inequality.

Theorem 2.14. *Let X be a terminal $\mathbb{Q}$-factorial Fano variety with* $\mathrm{Pic}(X) \cong \mathbb{Z}$, *$\rho : X \dashrightarrow Y$ a birational map, and $\pi : Y \to Z$ a fibration. Suppose that a general enough fiber of π is a smooth variety of Kodaira dimension zero. Then the singularities of the movable log pair $(X, \mathcal{M}_X)$ are not terminal, where $\mathcal{M}_X = r\rho^{-1}(|\pi^*(H)|)$ for a very ample divisor H on Z and $r \in \mathbb{Q}_{>0}$ such that $K_X + \mathcal{M}_X \sim_{\mathbb{Q}} 0$.*

Proof. Suppose $\mathbb{CS}(X, \mathcal{M}_X) = \varnothing$. Let $\mathcal{M}_Y = r|\pi^*(H)|$. Then we see

$$\kappa(X, c\mathcal{M}_X) = \kappa(Y, c\mathcal{M}_Y) \leq \dim(Z) < \dim(X).$$

However, $\mathbb{CS}(X, c\mathcal{M}_X) = \varnothing$ for small $c > 1$ and hence $\kappa(X, c\mathcal{M}_X) = \dim(X)$, which is a contradiction. □

The easy result below shows how to apply Theorem 2.14.

Proposition 2.15. *Let X be a smooth del Pezzo surface of degree one with* $\mathrm{Pic}(X) \cong \mathbb{Z}$ *defined over a perfect field $\mathbb{F}$ and o the unique base point of the anticanonical linear system of the surface X. Let $\rho : X \dashrightarrow Y$ be a birational map, where Y is a smooth surface. Suppose that $\pi : Y \to Z$ is a relatively minimal elliptic fibration with connected fibers such that a general enough fiber of π is smooth. Then the birational map ρ is the blowup at some $\mathbb{F}$-rational point p on the del Pezzo surface X and the morphism π is induced by $|-nK_Y|$ for some $n \in \mathbb{N}$. Furthermore, $p \in \hat{C}$ and the equality $p^n = \mathrm{id}_{\hat{C}}$ holds, where $\hat{C}$ is the smooth part of the unique curve C of arithmetic genus one in $|-K_X|$ passing through the point p and considered as a group scheme with the identity* $\mathrm{id}_{\hat{C}} = o$.

Proof. Let $\mathcal{M}_X = c\rho^{-1}(|\pi^*(H)|)$, where H is a very ample on curve Z and $c \in \mathbb{Q}_{>0}$, such that the equivalence $K_X + \mathcal{M}_X \sim_{\mathbb{Q}} 0$ holds. Then the set $\mathbb{CS}(X, \mathcal{M}_X)$ contains a point p on the surface X by Theorem 2.14. In particular, $\mathrm{mult}_p(\mathcal{M}_X) \geq 1$, but

$$1 = K_X^2 = M_X^2 \geq \mathrm{mult}_p^2(\mathcal{M}_X)\deg(p \otimes \bar{\mathbb{F}}) \geq \deg(p \otimes \bar{\mathbb{F}}) \geq 1,$$

where $\bar{\mathbb{F}}$ is the algebraic closure of the field $\mathbb{F}$. Hence, $\mathrm{mult}_p(\mathcal{M}_X) = 1$ and the point p is defined over the field $\mathbb{F}$. Let $f : V \to X$ be the blow up at the point p. Then $K_V^2 = 0$ and

$$-K_V \sim_{\mathbb{Q}} \mathcal{M}_V = f^{-1}(\mathcal{M}_X),$$

which implies that the linear system $|-rK_V|$ is free for a natural number $r \gg 0$. The morphism $\phi_{|-rK_V|}$ is a relatively minimal elliptic fibration and $\mathcal{M}_V \cdot E = 0$ for a general enough fiber E of the elliptic fibration $\phi_{|-rK_V|}$. Therefore the linear system $(\rho \circ f)^{-1}(|\pi^*(H)|)$ is contained in the fibers of the fibration $\phi_{|-rK_V|}$. Relative minimality of the fibrations π and $\phi_{|-rK_V|}$ implies $\rho \circ f$ is an isomorphism.

Suppose $p \neq o$. Let $C \in |-K_X|$ be a curve passing through p. Because

$$1 = K_X^2 = C \cdot \mathcal{M}_X \geq \mathrm{mult}_p(\mathcal{M}_X)\,\mathrm{mult}_p(C) = \mathrm{mult}_p(C) \geq 1,$$

the curve C is smooth at the point p. Let $\tilde{C} = f^{-1}(C) \sim -K_V$. Then $h^0(V, \mathcal{O}_V(\tilde{C})) = 1$ and the curve $\tilde{C}$ is $\mathrm{Gal}(\bar{\mathbb{F}}/\mathbb{F})$-invariant. In particular, the curve Z has an $\mathbb{F}$-point $\phi_{|-rK_V|}(\tilde{C})$ and we have $Z \cong \mathbb{P}^1$. Take the smallest natural n such that $h^0(V, \mathcal{O}_V(n\tilde{C})) > 1$. The exact sequence

$$0 \to \mathcal{O}_V((n-1)\tilde{C}) \to \mathcal{O}_V(n\tilde{C}) \to \mathcal{O}_{\tilde{C}}(n\tilde{C}|_{\tilde{C}}) \to 0$$

implies $h^0(C, \mathcal{O}_C(n(p-o))) = h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(n\tilde{C}|_{\tilde{C}})) \neq 0$, which implies the claim. □

Corollary 2.16. *Let X be a terminal $\mathbb{Q}$-factorial Fano variety with* $\mathrm{Pic}(X) \cong \mathbb{Z}$ *such that every movable log pair $(X, \mathcal{M}_X)$ with $K_X + \mathcal{M}_X \sim_{\mathbb{Q}} 0$ has terminal singularities. Then X is not birationally isomorphic to a fibration of varieties of Kodaira dimension zero.*

Unfortunately, Corollary 2.16 is almost impossible to use. As far as we know, there are no known examples of Fano varieties that are not birationally isomorphic to fibrations of varieties of Kodaira dimension zero. The only known example of a rationally connected variety that cannot be birationally transformed into a fibration of varieties of Kodaira dimension zero is a conic bundle with a big enough discriminant locus in [30].

Theorem 2.17. *Let X be a terminal $\mathbb{Q}$-factorial Fano variety with* $\mathrm{Pic}(X) \cong \mathbb{Z}$ *and $\rho : X \dashrightarrow Y$ be a nonbiregular birational map onto a Fano variety Y with canonical singularities. Then $K_X + \mathcal{M}_X \sim_{\mathbb{Q}} 0$ and*

$$\mathbb{CS}(X, \mathcal{M}_X) \neq \varnothing,$$

where $\mathcal{M}_X = \frac{1}{n}\rho^{-1}(|-nK_Y|)$ *for a natural number* $n \gg 0$.

Proof. Let $\mathcal{M}_Y = \frac{1}{n}|-nK_Y|$. We then see

$$\kappa(X, \mathcal{M}_X) = \kappa(Y, \mathcal{M}_Y) = 0,$$

which implies $K_X + \mathcal{M}_X \sim_{\mathbb{Q}} 0$. Suppose $\mathbb{CS}(X, \mathcal{M}_X) = \varnothing$. Both the log pair $(X, r\mathcal{M}_X)$ and $(Y, r\mathcal{M}_Y)$ are canonical models for a rational number $r > 1$ sufficiently close to 1. It is a contradiction that ρ is an isomorphism by Proposition 2.9. □

The following easy result shows how to apply Theorem 2.17.

Proposition 2.18. *Let X be a smooth del Pezzo surface of degree one with* $\mathrm{Pic}(X) \cong \mathbb{Z}$ *defined over an arbitrary perfect field* $\mathbb{F}$. *Then the surface X is not birationally isomorphic to a del Pezzo surface with du Val singularities which is not isomorphic to the surface X.*

Proof. Let $\rho : X \dashrightarrow Y$ be a birational map over the field $\mathbb{F}$ and $\mathcal{M}_X = \frac{1}{n}\rho^{-1}(|-nK_Y|)$ for a natural number $n \gg 0$, where Y is a del Pezzo surface with du Val singularities and ρ is not an isomorphism. Then $K_X + \mathcal{M}_X \sim_{\mathbb{Q}} 0$ and $\mathbb{CS}(X, \mathcal{M}_X)$ contains some smooth point o on the del Pezzo surface X by Theorem 2.17. In particular, $\mathrm{mult}_o(\mathcal{M}_X) \geq 1$, but

$$1 = K_X^2 = M_X^2 \geq \mathrm{mult}_o^2(\mathcal{M}_X)\deg(o \otimes \bar{\mathbb{F}}) \geq \deg(o \otimes \bar{\mathbb{F}}) \geq 1,$$

where $\bar{\mathbb{F}}$ is the algebraic closure of the field $\mathbb{F}$. Hence, $\mathrm{mult}_o(\mathcal{M}_X) = 1$ and the point o is defined over the field $\mathbb{F}$. Let $f : V \to X$ be the blow up at the point o. Then $K_V^2 = 0$ and

$$-K_V \sim_{\mathbb{Q}} \mathcal{M}_V = f^{-1}(\mathcal{M}_X),$$

which implies freeness of the linear system $|-rK_V|$ for a natural number $r \gg 0$. The morphism $\phi_{|-rK_V|}$ is an elliptic fibration and $\mathcal{M}_V \cdot E = 0$ for a general enough fiber E of $\phi_{|-rK_V|}$. Therefore, the linear system $(\rho \circ f)^{-1}(|-nK_Y|)$ is compounded from a pencil, which is impossible. □

The paper [80] by V. Iskovskikh and Yu. Manin was based on the idea of G. Fano that can be summarized by Nöther–Fano inequalities. Since 1971 the method of Iskovskikh and Manin has evolved to show birational rigidity of various Fano varieties. Recently, Shokurov's connectedness principle improved the method so that one can extremely simplify the proof of the result of Iskovskikh and Manin (see [42]). Furthermore, it also made it possible to prove the birational super-rigidity of smooth hypersurfaces of degree n in $\mathbb{P}^n$, $n \geq 4$ (see [118]). In what follows we will explain Shokurov's connectedness principle and how it can be applied to birational rigidity.

Movable boundaries always can be considered as effective divisors and movable log pairs as usual log pairs. Therefore, we may use compound log pairs that contain both movable and fixed components. From now on, we will not assume any restrictions on the coefficients of boundaries. In particular, boundaries may not be effective unless otherwise stated.

Definition 2.19. *A log pair (V, B^V) is called the log pullback of a log pair (X, B_X) with respect to a birational morphism $f : V \to X$ if*

$$B^V = f^{-1}(B_X) - \sum_{i=1}^{n} a(X, B_X, E_i)E_i,$$

where $a(X, B_X, E_i)$ is the discrepancy of an f-exceptional divisor E_i over (X, B_X). In particular, it satisfies $K_V + B^V \sim_{\mathbb{Q}} f^*(K_X + B_X)$.

Definition 2.20. *A proper irreducible subvariety $Y \subset X$ is called a center of the log canonical singularities of (X, B_X) if there are a birational morphism $f : W \to X$ and a divisor $E \subset W$ such that E is contained in the support of the effective part of the divisor $\lfloor B^W \rfloor$ and $f(E) = Y$. The set of all the centers of the log canonical singularities of a log pair (X, B_X) will be denoted by $\mathbb{LCS}(X, B_X)$. In addition, the union of all the centers of log canonical singularities of $(X, \mathcal{M}_X)$ will be denoted by $LCS(X, B_X)$.*

Consider a log pair (X, B_X), where $B_X = \sum_{i=1}^{k} a_i B_i$ is effective and B_i's are prime divisors on X. Choose a birational morphism $f : Y \to X$ such that Y is smooth and the union of all the proper transforms of the divisors B_i and all f-exceptional divisors forms a divisor with simple normal crossing. The morphism f is called a log resolution of the log pair (X, B_X). By definition, the equality

$$K_Y + B^Y \sim_{\mathbb{Q}} f^*(K_X + B_X)$$

holds, where (Y, B^Y) is the log pullback of the log pair (X, B_X) with respect to the birational morphism f.

Definition 2.21. *The subscheme $\mathcal{L}(X, B_X)$ associated with the ideal sheaf $\mathcal{I}(X, B_X) = f_*(\mathcal{O}_Y(\lceil -B^Y \rceil))$ is called the log canonical singularity subscheme of the log pair (X, B_X).*

The support of the subscheme $\mathcal{L}(X, B_X)$ is exactly the locus of $LCS(X, B_X)$. The following result is called Shokurov vanishing (see [130]).

Theorem 2.22. *Let (X, B_X) be a log pair with an effective divisor B_X. Suppose that there is a nef and big $\mathbb{Q}$-divisor H on X such that $D = K_X + B_X + H$ is Cartier. Then $H^i(X, \mathcal{I}(X, B_X) \otimes \mathcal{O}_X(D)) = 0$ for $i > 0$.*

Proof. Let $f : W \longrightarrow X$ be a log resolution of (X, B_X). Because f^*H is nef and big on W and $f^*D = K_W + B^W + f^*H$, we obtain

$$R^i f_*(f^*\mathcal{O}_X(D)\otimes\mathcal{O}_W(\lceil -B^W\rceil))=0$$

for $i>0$ from relative Kawamata–Viehweg vanishing (see [84] and [142]). The degeneration of local-to-global spectral sequence and

$$R^0 f_*(f^*\mathcal{O}_X(D)\otimes\mathcal{O}_W(\lceil -B^W\rceil))=\mathcal{I}(X,B_X)\otimes\mathcal{O}_X(D)$$

imply that for all i

$$H^i(X,\mathcal{I}(X,B_X)\otimes\mathcal{O}_X(D))=H^i(W,f^*\mathcal{O}_X(D)\otimes\mathcal{O}_W(\lceil -B^W\rceil)),$$

while $H^i(W,f^*\mathcal{O}_X(D)\otimes\mathcal{O}_W(\lceil -B^W\rceil))=0$ for $i>0$ by Kawamata–Viehweg vanishing. □

Consider the following application of Theorem 2.22.

Lemma 2.23. *Let V be a variety isomorphic to $\mathbb{P}^1\times\mathbb{P}^1$. Let B_V be an effective $\mathbb{Q}$-divisor on V of type (a,b), where a and $b\in\mathbb{Q}\cap[0,1)$. Then $\mathbb{LCS}(V,B_V)=\varnothing$.*

Proof. Intersecting the boundary B_V with the rulings of V, we see that the set $\mathbb{LCS}(V,B_V)$ does not contain a curve on V. Suppose that the set $\mathbb{LCS}(V,B_V)$ contains a point o. There is a $\mathbb{Q}$-divisor H on V of type $(1-a,1-b)$ such that the divisor

$$D=K_V+B_V+H$$

is Cartier. Since the divisor H is ample, Theorem 2.22 implies the sequence

$$H^0(V,\mathcal{O}_V(D))\to H^0(\mathcal{L}(V,B_V),\mathcal{O}_{\mathcal{L}(V,B_V)}(D))\to 0$$

is exact. However, $H^0(V,\mathcal{O}_V(D))=0$, which is a contradiction. □

For every Cartier divisor D on X, the sequence

$$0\to\mathcal{I}(X,B_X)\otimes D\to\mathcal{O}_X(D)\to\mathcal{O}_{\mathcal{L}(X,B_X)}(D)\to 0$$

is exact and Theorem 2.22 implies the following two connectedness theorems of Shokurov.

Theorem 2.24. *Let (X,B_X) be a log pair with an effective boundary B_X. If the divisor $-(K_X+B_X)$ is nef and big, then the locus $LCS(X,B_X)$ is connected.*

Theorem 2.25. *Let (X,B_X) be a log pair with an effective boundary. Let $g: X\longrightarrow Z$ be a contraction. If the divisor $-(K_X+B_X)$ is g-nef and g-big, then $LCS(X,B_X)$ is connected in a neighborhood of each fiber of the contraction g.*

The following result is Theorem 17.4 of [87].

Theorem 2.26. *Let $g : X \to Z$ be a contraction, where the varieties X and Z are normal. Let $D_X = \sum_{i=1}^{m} d_i D_i$ be a $\mathbb{Q}$-divisor on X such that the divisor $-(K_X + D_X)$ is g-nef and g-big. Suppose that* $\operatorname{codim}(g(D_i) \subset Z) \geq 2$ *whenever $d_i < 0$. Then, for a log resolution $h : V \to X$ of the log pair (X, D_X), the locus $\cup_{a_E \leq -1} E$ is connected in a neighborhood of every fiber of the morphism $g \circ h$, where E is a divisor on V and the rational number a_E is the discrepancy of E with respect to (X, D_X).*

Proof. Let $f = g \circ h$, $A = \sum_{a_E > -1} a_E E$, and $B = \sum_{a_E \leq -1} -a_E E$. Then

$$\lceil A \rceil - \lfloor B \rfloor = K_V - h^*(K_X + D_X) + \{-A\} + \{B\}$$

and $R^1 f_*(\mathcal{O}_V(\lceil A \rceil - \lfloor B \rfloor)) = 0$ by Kawamata–Viehweg vanishing. Hence, the map

$$f_*(\mathcal{O}_V(\lceil A \rceil)) \to f_*(\mathcal{O}_{\lfloor B \rfloor}(\lceil A \rceil))$$

is surjective. Every irreducible component of $\lceil A \rceil$ is either h-exceptional or the proper transform of some D_j with $d_j < 0$. Thus $h_*(\lceil A \rceil)$ is g-exceptional and $f_*(\mathcal{O}_V(\lceil A \rceil)) = \mathcal{O}_Z$. Consequently, the map

$$\mathcal{O}_Z \to f_*(\mathcal{O}_{\lfloor B \rfloor}(\lceil A \rceil))$$

is surjective, which implies the connectedness of $\lfloor B \rfloor$ in a neighborhood of every fiber of the morphism f because the divisor $\lceil A \rceil$ is effective and has no common component with $\lfloor B \rfloor$. $\square$

We defined the notions of centers of canonical singularities and the set of centers of canonical singularities for movable log pairs. However, the movability of boundaries has nothing to do with all these notions. So we are free to use them for usual log pairs as well.

The following theorem, frequently referred to as adjunction, leads us to the bridge between Shokurov's connectedness principle and Nöther–Fano inequalities.

Theorem 2.27. *Let (X, B_X) be a log pair with an effective divisor B_X, Z an element in $\mathbb{CS}(X, B_X)$, and H an effective irreducible Cartier divisor on X. Suppose that both the varieties X and H are smooth at a generic point of Z and $Z \subset H \not\subset$* $\operatorname{Supp}(B_X)$*. Then, the set $\mathbb{LCS}(H, B_X|_H)$ is not empty.*

Proof. Let $f : W \to X$ be a log resolution of $(X, B_X + H)$. Put $\hat{H} = f^{-1}(H)$. Then

$$K_W + \hat{H} = f^*(K_X + B_X + H) + \sum_{E \neq \hat{H}} a(X, B_X + H, E)E$$

and by our assumption the subvarieties Z and H are centers of the log canonical singularities of the pair $(X, B_X + H)$. Therefore, applying Theorem 2.26

to the log pullback of $(X, B_X + H)$ on W, we obtain a divisor $E \neq \hat{H}$ on W such that $f(E) = Z$, $a(X, B_X, E) \leq -1$, and $\hat{H} \cap E \neq \varnothing$. Now the equalities

$$K_{\hat{H}} = (K_W + \hat{H})|_{\hat{H}} = f|_{\hat{H}}^{*}(K_H + B_X|_H) + \sum_{E \neq \hat{H}} a(X, B_X + H, E)E|_{\hat{H}}$$

imply the claim. □

By taking Nöther–Fano inequalities into consideration, it is significant for us to study the singularities of certain movable log pairs on Fano varieties. It requires us to investigate the multiplicities of certain movable boundaries or their self-intersections.

The following result is Theorem 3.1 of [42].

Theorem 2.28. *Let S be a smooth surface and $\mathcal{M}_S$ an effective movable boundary on the surface S. Suppose that there is a point o in $\mathbb{LCS}(S, (1-a_1)B_1 + (1-a_2)B_2 + \mathcal{M}_S)$, where a_i's are nonnegative rational numbers and B_i's are irreducible and reduced curves on S intersecting normally at the point o. Then, we have*

$$\operatorname{mult}_o(\mathcal{M}_S^2) \geq \begin{cases} 4a_1a_2 \ \text{if}\ a_1 \leq 1 \ \text{or}\ a_2 \leq 1 \\ 4(a_1 + a_2 - 1) \ \text{if}\ a_1 > 1 \ \text{and}\ a_2 > 1. \end{cases}$$

Furthermore, the inequality is strict if the singularities of the log pair $(S, (1-a_1)B_1 + (1-a_2)B_2 + \mathcal{M}_S)$ are not log canonical in a neighborhood of the point o.

Proof. Let $D = (1-a_1)B_1 + (1-a_2)B_2 + \mathcal{M}_S$ and $f : S' \to S$ be a birational morphism such that the surface S' is smooth. We consider

$$K_{S'} + f^{-1}(D) = E_i)E_i,$$

where E_i is an f-exceptional curve. We suppose that $a(S, D, E_1) \leq -1$ and the curve E_1 is contracted to the point o. Then the birational morphism f is a composition of k blowups at smooth points.

Claim 1. The statement is true when $a_1 > 1$ and $a_2 > 1$ if the statement holds when $a_1 \leq 1$ or $a_2 \leq 1$.

Define the numbers $a(S, E_i)$, $m(S, \mathcal{M}_S, E_i)$, and $m(S, B_j, E_i)$ as follows:

$$\sum_{i=1}^{k} a(S, E_i)E_i = K_{S'} - f^*(K_S),$$

$$\sum_{i=1}^{k} m(S, \mathcal{M}_S, E_i)E_i = f^{-1}(\mathcal{M}_S) - f^*(\mathcal{M}_S),$$

$$\sum_{i=1}^{k} m(S, B_j, E_i)E_i = f^{-1}(B_j) - f^*(B_j).$$

We then observe that the equality

$$a(S,D,E_i) = a(S,E_i) - m(S,\mathcal{M}_S,E_i) + m(S,B_1,E_i)(a_1-1) + m(S,B_2,E_i)(a_2-1)$$

holds. We may assume that $m(S,B_1,E_1) \geq m(S,B_2,E_1)$. Then,

$$-1 \geq a(S,D,E_1) \geq a(S,E_1) - m(S,\mathcal{M}_S,E_1) + m(S,B_2,E_1)(a_1+a_2-2)$$

and hence $o \in \mathbb{LCS}(S,(2-a_1-a_2)B_2+\mathcal{M}_S)$. Because the log pair $(S,(2-a_1-a_2)B_2+\mathcal{M}_S)$ satisfies our assumption, we obtain $\mathrm{mult}_o(\mathcal{M}_S^2) \geq 4(a_1+a_2-1)$.

Claim 2. The statement holds when $a_1 \leq 1$ or $a_2 \leq 1$.

We may assume that $a_1 \leq 1$. Let $h : T \to S$ be the blowup at the point o and E be an exceptional curve of h. Then f factors through h such that $f = g \circ h$ for some birational morphism $g : S' \to T$ which is a composition of $k-1$ blowups at smooth points. Then

$$K_T + (1-a_1)\bar{B}_1 + (1-a_2)\bar{B}_2 + (1-a_1-a_2+m)E + \mathcal{M}_T = h^*(K_S+D),$$

where $\bar{B}_j = h^{-1}(B_j)$, $m = \mathrm{mult}_o(\mathcal{M}_S)$, and $\mathcal{M}_T = h^{-1}(\mathcal{M}_S)$.

We are to use the induction on k. In the case $k=1$, we have $S' = T$, $E_1 = E$, and $a(S,D,E_1) = a_1+a_2-m-1 \leq -1$. Thus

$$\mathrm{mult}_o(\mathcal{M}_S^2) \geq m^2 \geq (a_1+a_2)^2 \geq 4a_1a_2$$

and we are done.

We therefore suppose that $k > 1$ and $g(E_1)$ is a point $p \in E$. We see

$$p \in \mathbb{LCS}(T,(1-a_1)\bar{B}_1 + (1-a_2)\bar{B}_2 + (1-a_1-a_2+m)E + \mathcal{M}_T).$$

There are three possible cases: $p \in E \cap \bar{B}_1$, $p \in E \cap \bar{B}_2$, and $p \notin \bar{B}_1 \cup \bar{B}_2$. By the induction hypothesis, the statement holds for the log pair

$$(T,(1-a_1)\bar{B}_1 + (1-a_1-a_2+m)E + \mathcal{M}_T)$$

in the case $p \in E \cap \bar{B}_1$, for the log pair

$$(T,(1-a_2)\bar{B}_2 + (1-a_1-a_2+m)E + \mathcal{M}_T)$$

in the case $p \in E \cap \bar{B}_2$, and for the log pair

$$(T,(1-a_1-a_2+m)E + \mathcal{M}_T)$$

in the case $p \notin \bar{B}_1 \cup \bar{B}_2$ because all conditions of the theorem are satisfied in these cases and the morphism g consists of $k-1$ blowups at smooth points. Also we have

$$\mathrm{mult}_o(\mathcal{M}_S^2) \geq m^2 + \mathrm{mult}_p(\mathcal{M}_T^2).$$

In the case $p \in E \cap \bar{B}_1$, we obtain

$$\mathrm{mult}_o(\mathcal{M}_S^2) \geq m^2 + 4a_1(a_1 + a_2 - m) = (2a_1 - m)^2 + 4a_1a_2 \geq 4a_1a_2.$$

Consider the case $p \in E \cap \bar{B}_2$. If either $a_2 \leq 1$ or $a_1 + a_2 - m \leq 1$, then we can proceed as in the previous case. If not, then we have

$$\mathrm{mult}_o(\mathcal{M}_S^2) \geq m^2 + 4(a_1 + 2a_2 - m - 1) > 4a_2 \geq 4a_1a_2.$$

If $p \notin \bar{B}_1 \cup \bar{B}_2$, then we obtain

$$\mathrm{mult}_o(\mathcal{M}_S^2) \geq m^2 + 4(a_1 + a_2 - m) > m^2 + 4a_1(a_1 + a_2 - m) \geq 4a_1a_2,$$

which completes the proof. □

Instead of Theorem 2.28, the following simplified version, which is a special case of Theorem 2.1 in [49], is more often applied.

Theorem 2.29. *Let S be a smooth surface, o a point on S, and $\mathcal{M}_S$ an effective movable boundary on S such that $o \in \mathbb{LCS}(S, \mathcal{M}_S)$. Then $\mathrm{mult}_o(\mathcal{M}_S^2) \geq 4$. Moreover, if the equality holds, then $\mathrm{mult}_o(\mathcal{M}_S) = 2$.*

Even though Theorems 2.28 and 2.29 are results on surfaces, they can be applied to 3-folds via Theorem 2.27. The following result is Corollary 7.3 of [116], which holds even over fields of positive characteristic and implicitly goes back to the classical paper [80].

Theorem 2.30. *Let X be a smooth 3-fold and $\mathcal{M}_X$ an effective movable boundary on X. Suppose that a point o belongs to $\mathbb{CS}(X, \mathcal{M}_X)$. Then the inequality $\mathrm{mult}_o(\mathcal{M}_X^2) \geq 4$ holds, with equality only when $\mathrm{mult}_o(\mathcal{M}_X) = 2$.*

Proof. Let H be a general very ample divisor on X containing o. Then the point o is a center of log canonical singularities of the log pair $(H, \mathcal{M}_X|_H)$ by Theorem 2.27. On the other hand,

$$\mathrm{mult}_o(\mathcal{M}_X^2) = \mathrm{mult}_o((\mathcal{M}_X|_H)^2)$$

and $\mathrm{mult}_o(\mathcal{M}_X) = \mathrm{mult}_o(\mathcal{M}_X|_H)$. Hence, the claim follows from Theorem 2.29. □

As a matter of fact, Theorem 2.30 can be proved in a more geometric way.

Lemma 2.31. *Let X be a smooth 3-fold and $\mathcal{M}_X$ an effective movable boundary on X. Suppose that the log pair $(X, \mathcal{M}_X)$ has canonical singularities and $\mathbb{CS}(X, \mathcal{M}_X)$ contains a point o. Then there is a birational morphism $f : V \to X$ such that V has $\mathbb{Q}$-factorial terminal singularities, f contracts exactly one exceptional divisor E to the point o, and*

$$K_V + \mathcal{M}_V = f^*(K_X + \mathcal{M}_X),$$

where $\mathcal{M}_V = f^{-1}(\mathcal{M}_X)$.

Proof. Because the log pair $(X, \mathcal{M}_X)$ has at worst canonical singularities, there are finitely many divisorial discrete valuations ν of the field of rational functions of X whose centers on X are the point o and whose discrepancies $a(X, \mathcal{M}_X, \nu)$ are nonpositive. Therefore, we may consider a birational morphism $g : W \to X$ such that the 3-fold W is smooth, g contracts k divisors,

$$K_W + \mathcal{M}_W = g^*(K_X + \mathcal{M}_X) + \sum_{i=1}^{k} a_i E_i,$$

the movable log pair $(W, \mathcal{M}_W)$ has canonical singularities, and the set $\mathbb{CS}(W, \mathcal{M}_W)$ does not contain subvarieties of $\cup_{i=1}^{k} E_i$, where $\mathcal{M}_W = g^{-1}(\mathcal{M}_X)$, $g(E_i) = o$, and $a_i \in \mathbb{Q}$. Applying the relative version of the Log Minimal Model Program (see [86]) to the movable log pair $(W, \mathcal{M}_W)$ over X, we may assume that W has $\mathbb{Q}$-factorial terminal singularities and

$$K_W + \mathcal{M}_W = g^*(K_X + \mathcal{M}_X)$$

because of the canonicity of $(X, \mathcal{M}_X)$. Applying the relative Minimal Model Program to W over the variety X, we get the necessary 3-fold and the birational morphism. □

The following result was conjectured in [41] and proved in [83].

Theorem 2.32. *Let X be a smooth 3-fold and $f : V \to X$ be a birational morphism of a 3-fold V with $\mathbb{Q}$-factorial terminal singularities. Suppose that the morphism f contracts exactly one exceptional divisor E and contracts it to a point o. Then the morphism f is the weighted blowup at the point o with weights $(1, n_1, n_2)$ in suitable local coordinates on X, where the natural numbers n_1 and n_2 are coprime.*

With Theorem 2.32, Theorem 2.30 was proved in [41] in the following way, which explains the geometrical nature of the inequality in Theorem 2.30.

Proposition 2.33. *Let X be a smooth 3-fold with an effective movable boundary $\mathcal{M}_X$ on X. Suppose that $\mathbb{CS}(X, \mathcal{M}_X)$ contains a point o. Let $f : V \to X$ be the weighted blowup at the point o with weights $(1, n_1, n_2)$ in suitable local coordinates on X such that*

$$K_V + \mathcal{M}_V = f^*(K_X + \mathcal{M}_X),$$

where natural numbers n_1 and n_2 are coprime and $\mathcal{M}_V = f^{-1}(\mathcal{M}_X)$. Then

$$\mathrm{mult}_o(\mathcal{M}_X^2) \geq \frac{(n_1 + n_2)^2}{n_1 n_2} = 4 + \frac{(n_1 - n_2)^2}{n_1 n_2} \geq 4.$$

Moreover, if $n_1 = n_2$, then f is the regular blowup at the point o and $\mathrm{mult}_o(\mathcal{M}_X) = 2$.

Proof. Let $E \subset V$ be the f-exceptional divisor. Then

$$K_V = f^*(K_X) + (n_1 + n_2)E$$

and $\mathcal{M}_V = f^*(\mathcal{M}_X) - mE$ for some $m \in \mathbb{Q}_{>0}$. Thus, $m = n_1 + n_2$ and

$$\mathrm{mult}_o(\mathcal{M}_X^2) \geq m^2E^3 = \frac{(n_1+n_2)^2}{n_1n_2}.$$

□

The following application of Theorem 2.27 is Theorem 3.10 in [42].

Theorem 2.34. *Let X be a 3-fold with a simple double point o and B_X an effective boundary on X such that $o \in \mathbb{CS}(X, B_X)$. Then the inequality* $\mathrm{mult}_o(B_X) \geq 1$ *holds.*

Proof. Let $f : W \to X$ be the blowup at the point o and E be the f-exceptional divisor. Then

$$K_W + B_W = f^*(K_X + B_X) + (1 - \mathrm{mult}_o(B_X))E,$$

where $B_W = f^{-1}(B_X)$. Suppose that $\mathrm{mult}_o(B_X) < 1$. Then, there is a center $Z \in \mathbb{CS}(W, B_W)$ that is contained in E, and hence

$$\mathbb{LCS}(E, B_W|_E) \neq \varnothing$$

by Theorem 2.27. But it is impossible because of Lemma 2.23. □

3 Birational super-rigidity

The goal of this section is to prove Theorem A.

Let $\pi : X \to \mathbb{P}^3$ be a $\mathbb{Q}$-factorial double cover ramified along a nodal sextic $S \subset \mathbb{P}^3$. We then see that $\mathrm{Pic}(X) \cong \mathbb{Z}K_X$, $-K_X \sim \pi^*(\mathcal{O}_{\mathbb{P}^3}(1))$, and $-K_X^3 = 2$. Consider an arbitrary movable boundary $\mathcal{M}_X$ on the 3-fold X such that the divisor $-(K_X + \mathcal{M}_X)$ is ample. To prove Theorem A we must show that $\mathbb{CS}(X, \mathcal{M}_X) = \varnothing$ and then apply Theorem 2.10.

We suppose that $\mathbb{CS}(X, \mathcal{M}_X) \neq \varnothing$. In what follows, we will derive a contradiction.

Lemma 3.1. *Smooth points of the* 3*-fold X are not contained in* $\mathbb{CS}(X, \mathcal{M}_X)$.

Proof. Suppose that $\mathbb{CS}(X, \mathcal{M}_X)$ has a smooth point o on X. Let H be a general enough divisor in the linear system $|-K_X|$ passing through the point o. We then obtain

$$2 = H \cdot K_X^2 > H \cdot \mathcal{M}_X^2 \geq \mathrm{mult}_o(\mathcal{M}_X^2) \geq 4$$

from Theorem 2.30, which is absurd. □

Lemma 3.2. *Singular points of the* 3*-fold* X *are not contained in* $\mathbb{CS}(X, \mathcal{M}_X)$.

Proof. If $\mathbb{CS}(X, \mathcal{M}_X)$ contains a singular point o on X, then Theorem 2.34 gives us

$$2 = H \cdot K_X^2 > H \cdot \mathcal{M}_X^2 \geq 2\,\mathrm{mult}_o^2(\mathcal{M}_X) \geq 2,$$

where H is a general enough divisor in $|-K_X|$ passing through the point o. It is absurd. □

Lemmas 3.1 and 3.2 together show that any element of the set $\mathbb{CS}(X, \mathcal{M}_X)$ cannot be a point of X. Therefore, it must contain a curve $C \subset X$. To complete the proof of Theorem A it is enough to show that the set $\mathbb{CS}(X, \mathcal{M}_X)$ cannot contain a curve.

Lemma 3.3. *The intersection number* $-K_X \cdot C$ *is* 1.

Proof. Let H be a general enough divisor in the anticanonical linear system $|-K_X|$. Then

$$2 = H \cdot K_X^2 > H \cdot \mathcal{M}_X^2 \geq \mathrm{mult}_C(\mathcal{M}_X^2) H \cdot C \geq -K_X \cdot C,$$

which implies $-K_X \cdot C = 1$. □

Corollary 3.4. *The curve* $\pi(C) \subset \mathbb{P}^3$ *is a line and* $C \cong \mathbb{P}^1$.

Lemma 3.5. *The curve* C *is not contained in the smooth locus of the* 3*-fold* X.

Proof. Suppose that the curve C lies on the smooth locus of the 3-fold X. Let $f : W \to X$ be the blowup along the curve C and E be the f-exceptional divisor. We then get $\mathrm{mult}_C(\mathcal{M}_X) \geq 1$ and

$$\mathcal{M}_W = f^{-1}(\mathcal{M}_X) = f^*(\mathcal{M}_X) - \mathrm{mult}_C(\mathcal{M}_X)E.$$

The linear system $|-K_W| = |f^*(-K_X) - E|$ has just one base curve $\tilde{C}$ such that

$$\pi \circ f(\tilde{C}) = \pi(C) \subset \mathbb{P}^3.$$

We see that $\tilde{C} \subset E$ if and only if $\pi(C) \subset S$.

Let $H = f^*(-K_X)$. Then the divisor $3H - E$ has nonnegative intersection with all the curves on W possibly except $\tilde{C}$. We are to show that the divisor $3H - E$ is nef. We obtain $(3H - E) \cdot \tilde{C} = 0$ unless $\tilde{C}$ is contained in E. Therefore, we suppose that the curve $\tilde{C}$ is contained in E.

The normal bundle $\mathcal{N}_{X/C}$ of the curve $C \cong \mathbb{P}^1$ on the 3-fold X splits into

$$\mathcal{N}_{X/C} \cong \mathcal{O}_C(a) \oplus \mathcal{O}_C(b)$$

for some integers $a \geq b$. The exact sequence

$$0 \to \mathcal{T}_C \to \mathcal{T}_X|_C \to \mathcal{N}_{X/C} \to 0$$

shows $\deg(\mathcal{N}_{X/C}) = a + b = -K_X \cdot C + 2g(C) - 2 = -1$.

On the other hand, the curve C is contained in the smooth locus of the proper transform $\hat{S} \cong S$ of the sextic $S \subset \mathbb{P}^3$. The exact sequence

$$0 \to \mathcal{N}_{\hat{S}/C} \to \mathcal{N}_{X/C} \to \mathcal{N}_{X/\hat{S}}|_C \to 0$$

and $\mathcal{N}_{\hat{S}/C} \cong \mathcal{O}_C(-4)$ imply $b \geq -4$. In particular, $a - b \leq 7$.

Let s_∞ be the exceptional section of the ruled surface $f|_E : E \to C$. Because $E^3 = -\deg(\mathcal{N}_{X/C}) = 1$ and $-K_X \cdot C = 1$, we obtain

$$(3H - E) \cdot s_\infty = \frac{7 + b - a}{2} \geq 0,$$

which implies that the divisor $3H - E$ is nef.

Because $3H - E$ is nef, we get $(3H - E) \cdot \mathcal{M}_W^2 \geq 0$, but

$$(3H - E) \cdot \mathcal{M}_W^2 = 6r^2 - 4\,\mathrm{mult}_C^2(\mathcal{M}_X) - 2r\,\mathrm{mult}_C(\mathcal{M}_X) < 0,$$

where $r \in \mathbb{Q} \cap (0,1)$ such that $\mathcal{M}_X \sim_{\mathbb{Q}} -rK_X$. □

Corollary 3.6. *The curve C contains a simple double point of the 3-fold X.*

Lemma 3.7. *The line $\pi(C)$ is contained in the sextic surface S.*

Proof. Suppose $\pi(C) \not\subset S$. Let $\mathcal{H}$ be the linear subsystem in $|-K_X|$ of surfaces containing the curve C. The base locus of $\mathcal{H}$ consists of the curve C and the curve $\widetilde{C}$ such that $\pi(C) = \pi(\widetilde{C})$. Choose a general enough surface D in the pencil $\mathcal{H}$. The restriction $\mathcal{M}_X|_D$ is not movable, but

$$\mathcal{M}_X|_D = \mathrm{mult}_C(\mathcal{M}_X)C + \mathrm{mult}_{\tilde{C}}(\mathcal{M}_X)\tilde{C} + \mathcal{R}_D,$$

where $\mathcal{R}_D$ is a movable boundary. The surface D is smooth outside of the singular points p_i of the 3-fold X which are contained in the curve C. Moreover, each point p_i is a simple double point on the surface D. Thus, on the surface D, we have

$$C^2 = \tilde{C}^2 = -2 + \frac{k}{2},$$

where k is the number of the points p_i on C. Hence, we obtain $C^2 = \widetilde{C}^2 < 0$ on the surface D because $k \leq 3$. Immediately, the inequality

$$(1 - \mathrm{mult}_{\tilde{C}}(\mathcal{M}_X))\widetilde{C}^2 \geq (\mathrm{mult}_C(\mathcal{M}_X) - 1)C \cdot \widetilde{C} + \mathcal{R}_D \cdot \widetilde{C} \geq 0$$

follows, which implies $\mathrm{mult}_{\tilde{C}}(\mathcal{M}_X) \geq 1$. Therefore, for a general member $H \in |-K_X|$ we have a contradiction

$$2 = H \cdot K_X^2 > H \cdot \mathcal{M}_X^2 \geq \mathrm{mult}_C^2(\mathcal{M}_X) H \cdot C + \mathrm{mult}_{\tilde{C}}^2(\mathcal{M}_X) H \cdot \tilde{C} \geq 2.$$

□

Lemma 3.8. *The line $\pi(C)$ is not contained in the sextic surface S.*

Proof. Suppose $\pi(C) \subset S$. Let p be a general enough point on the curve C and $L \subset \mathbb{P}^3$ be a general line tangent to S at the point $\pi(p)$. Then the proper transform $\tilde{L} \subset X$ of L is an irreducible curve which is singular at the point p. By construction, the curve L is not contained in the base locus of the components of the movable boundary $\mathcal{M}_X$. Thus, we obtain contradictory inequalities

$$2 > \tilde{L} \cdot \mathcal{M}_X \geq \operatorname{mult}_p \tilde{L} \operatorname{mult}_p(\mathcal{M}_X) \geq 2 \operatorname{mult}_C(\mathcal{M}_X) \geq 2.$$

□

We have shown that the set $\mathbb{CS}(X, \mathcal{M}_X)$ is empty. Now, we can immediately obtain Theorem A from Theorem 2.10.

4 $\mathbb{Q}$-factoriality

In this section we study the $\mathbb{Q}$-factoriality on double covers of $\mathbb{P}^3$ ramified along nodal sextics and prove Theorem B.

The $\mathbb{Q}$-factoriality depends both on local types of singularities and on their global position. In the case of Fano 3-folds, the $\mathbb{Q}$-factoriality is equivalent to the global topological condition

$$\operatorname{rank}(H^2(X, \mathbb{Z})) = \operatorname{rank}(H_4(X, \mathbb{Z})).$$

In the case of the double solids, the condition means the 4th integral homology group of X generated by the class of the pullback of a hyperplane in $\mathbb{P}^3$ via the covering morphism.

Using the method in [37], we study the $\mathbb{Q}$-factoriality on a double cover X of $\mathbb{P}^3$ ramified along a sextic S. As before, we assume that X has only simple double points. Note that $\operatorname{Pic}(X) \cong H^2(X, \mathbb{Z})$ when X has at worst rational singularities.

For us in order to see whether a double solid X is $\mathbb{Q}$-factorial, the main job is to compute the rank of the group $H_4(X, \mathbb{Z})$. Indeed, the double solid X is $\mathbb{Q}$-factorial if and only if $\operatorname{rank}(H_4(X, \mathbb{Z})) = 1$ because $\operatorname{rank} H^2(X, \mathbb{Z}) = 1$. The paper [37] gives us a method to compute it by studying the number of singularities of S, their position in $\mathbb{P}^3$, and the sheaf $\mathcal{I} \otimes \mathcal{O}_{\mathbb{P}^3}(5)$, where $\mathcal{I}$ is the ideal sheaf of the set Σ of singular points of S in $\mathbb{P}^3$. The following result was proved in [37] (see also [48] and [47]).

Theorem 4.1. *Under the same notation, we have*

$$\operatorname{rank}(H_4(X, \mathbb{Z})) = \#(\Sigma) - I + 1,$$

where I is the number of independent conditions which vanishing on Σ imposes on homogeneous forms of degree 5 on $\mathbb{P}^3$.

We define the *defect* of X to be the nonnegative integer $\#(\Sigma) - I$. Then we can restate the $\mathbb{Q}$-factoriality as follows:

Corollary 4.2. *The double cover X is $\mathbb{Q}$-factorial if and only if the defect of X is* 0.

On the other hand, from the exact sequence

$$0 \to \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}^3}(5) \to \mathcal{O}_{\mathbb{P}^3}(5) \to \bigoplus_{p\in\Sigma} \mathbb{C} \to 0$$

we obtain a long exact sequence

$$H^0(\mathbb{P}^3, \mathcal{I}\otimes\mathcal{O}_{\mathbb{P}^3}(5)) \hookrightarrow H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5)) \to H^0(\mathbb{P}^3, \bigoplus_{p\in\Sigma} \mathbb{C}) \to H^1(\mathbb{P}^3, \mathcal{I}\otimes\mathcal{O}_{\mathbb{P}^3}(5)) \to 0,$$

which tells us

$$\text{defect of } X = \dim(H^1(\mathbb{P}^3, \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}^3}(5))).$$

An immediate application of this method is the second part of Theorem B. Since $\dim(H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5))) = 56$, the defect of X is positive if $\#(\Sigma) \geq 57$.

We can easily observe that if $\#(\Sigma) \leq 6$, then the sequence

$$0 \to H^0(\mathbb{P}^3, \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}^3}(5)) \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5)) \to H^0(\mathbb{P}^3, \bigoplus_{p\in\Sigma} \mathbb{C}) \to 0$$

is exact regardless of their position. Therefore, when $\#(\Sigma) \leq 6$ the defect of X is trivially 0, i.e., the sextic double solid X is $\mathbb{Q}$-factorial. As a matter of fact, we can go farther. As Theorem B states, if the surface S has at most 14 nodes, then the 3-fold X is $\mathbb{Q}$-factorial regardless of their position. In what follows, we prove the first part of Theorem B.

Definition 4.3. *We say that a set of points Γ on $\mathbb{P}^3$ is on sextic-node position if no $5k+1$ points of Γ can lie on a curve of degree k in $\mathbb{P}^3$ for every positive integer k.*

Lemma 4.4. *Let Σ be the set of singular points of the sextic S. Then the set Σ is on sextic-node position.*

Proof. Suppose that the surface S is defined by a homogeneous polynomial equation $F(x_0, x_1, x_2, x_3) = 0$ of degree six. We consider the linear system

$$\mathcal{L} := \left| \sum_{i=0}^{3} \lambda_i \frac{\partial F}{\partial x_i} = 0 \right| .$$

The base locus of the linear system $\mathcal{L}$ is exactly the singular locus of the surface S. A curve of degree k in $\mathbb{P}^3$ intersects a generic member of the linear system $\mathcal{L}$ exactly $5k$ times since $\mathcal{L} \subset |\mathcal{O}_{\mathbb{P}^3}(5)|$. Therefore, the set Σ is on sextic-node position. $\square$

For convenience, we state an elementary lemma.

Lemma 4.5. *Let $\Gamma = \{q_1, \ldots, q_s\}$ be a set of $s \geq 4$ points in $\mathbb{P}^3$. For a given point $q \notin \Gamma$, there is a hyperplane H which contains at least three points of Γ but not the point q unless all the points $q, q_1, \ldots, q_s$ lie on a single hyperplane.*

Proof. Because not all the points $q, q_1, \cdots, q_s$ lie on a single hyperplane, we may assume there are two distinct hyperplanes H_1 and H_2 such that $H_1 \cup H_2$ contains the point q and four points, say q_1, q_2, q_3, and q_4, of Γ, $q_1 \in H_1 \setminus H_2$, and $q_2 \in H_2 \setminus H_1$. Then one of the hyperplanes generated by $\{q_1, q_2, q_3\}$ and $\{q_1, q_2, q_4\}$ must not pass through the point q; otherwise all of the five points $q, q_1, \ldots, q_4$ would be on a single hyperplane. □

Also, the following result of [14] is useful.

Theorem 4.6. *Let $\pi : Y \to \mathbb{P}^2$ be the blowup at points $p_1, \ldots, p_s \in \mathbb{P}^2$. Then the linear system $|\pi^*(\mathcal{O}_{\mathbb{P}^2}(d)) - \sum_{i=1}^s E_i|$ is base-point-free for all*

$$s \leq \frac{1}{3}(h^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d+3)) - 5),$$

where $d \geq 3$ and $E_i = \pi^{-1}(p_i)$, if at most $k(d+3-k) - 2$ of the points p_i lie on a curve of degree $1 \leq k \leq \frac{1}{2}(d+3)$.

Theorem 4.1 tells us that the first part of Theorem B immediately follows from the lemma below.

Lemma 4.7. *Let $\gamma : V \to \mathbb{P}^3$ be the blowup at k different points $\Gamma = \{p_1, \ldots, p_k\}$ and p be a point in $V \setminus \cup_{i=1}^k E_i$ such that the set $\Gamma \cup \{\gamma(p)\}$ is on sextic-node position, where $E_i = \gamma^{-1}(p_i)$. If $k \leq 13$, then the linear system $|\gamma^*(\mathcal{O}_{\mathbb{P}^3}(5)) - \sum_{i=1}^k E_i|$ is base-point-free at the point p.*

Proof. It is enough to find a quintic hypersurface in $\mathbb{P}^3$ that passes through all the points of Γ but not the point $q := \gamma(p)$. We may assume that $k = 13$. Let r be the maximal number of points of Γ that belong to a single hyperplane of $\mathbb{P}^3$ together with the point q. Note that $2 \leq r \leq 13$. Without loss of generality, we may also assume that the first r points of Γ, i.e., $p_1, \ldots, p_r$, are contained in a hyperplane H together with the point q.

We prove the statement case by case.

Case 1. ($r = 2$)

We divide the set Γ into five subsets of Γ such that each subset contains exactly three points of Γ and the union of all the five subsets is Γ. Because $r = 2$, the hyperplane generated by each subset cannot contain the point q. The product of these five hyperplanes is what we want.

Before we proceed, we note that the points q and $p_1, \ldots, p_r$ do not lie on a single line. If they do, then the hyperplane H must contain more than r points of Γ.

Case 2. $(r = 3)$

By Lemma 4.5, we can find three points of Γ outside of H such that generate a hyperplane not passing through the point q. Since $r = 3$, we can repeat this procedure two more times with the remaining points of Γ in the outside of H. Only one point, say p_{13}, then remains in the outside of H. Because the four points, q, p_1, p_2, p_3, cannot lie on a line, there is a quadric hypersurface passing through the points p_1, p_2, p_3, p_{13}, but not the point q.

Case 3. $(r = 4)$

As in the previous case, we can find two hyperplanes which together contains six points of Γ in the outside of H but not q. We then have three remaining points of Γ in the outside of H. There is a line passing though two points, say p_1, p_2, of $p_1, \ldots, p_4$, but not the point q. Then these two points together with one of the remaining points in the outside of H generate a hyperplane not containing the point q. Now, we have four points, two of them are on H and the others not on H. Obviously, these four points belong to a quadric hypersurface not passing through the point q. Therefore, the product of the quadric hypersurface and the hyperplanes gives us a quintic hypersurface that we are looking for.

Case 4. $(r = 5)$

First of all, by Lemma 4.5, we find a hyperplane which contains three points, say p_6, p_7, p_8, of Γ in the outside of H but not the point q. We split the case into two subcases.

Subcase 4.1. When four points of Γ on H together with the point q lie on a line.

Assume that the points q and $p_1, \ldots, p_4$ lie on a single line. The hyperplane generated by the points p_4, p_5, and p_9 cannot contain q. The hyperplane generated by $\{p_3, p_{10}, p_{11}\}$ cannot pass through the point q; otherwise the number r would be bigger than five. By the same reason, we can find a hyperplane which contains $\{p_2, p_{12}, p_{13}\}$ but not the point q. Choose a hyperplane which passes through the point p_1 but not the point q. Then we are done.

Subcase 4.2. When no four points of Γ on H lie on a line together with the point q.

In this case, two pairs of points of Γ on H give two lines which do not contain the point q. Therefore, we can find a quadric hypersurface containing six points of Γ, four from H and two from $\Gamma \setminus (H \cup \{p_6, p_7, p_8\})$, but not the point q. Furthermore, because the number r is five we may choose the two points from $\Gamma \setminus (H \cup \{p_6, p_7, p_8\})$ so that the other three points in the outside of H cannot belong to a single line together with the point q. We then have four points which we have not covered yet, three points, say p_{11}, p_{12}, p_{13}, from the outside of H, and one point, say p_1, on H. Because the points p_{11}, p_{12}, p_{13} and q do not lie on a line, we can easily find a quadric hypersurface passing through all the four points but not the point q.

Case 5. $(r = 6)$

Again, by Lemma 4.5, we find a hyperplane which contains three points, say p_7, p_8, p_9, of Γ in the outside of H but not the point q. By the sextic-node position condition, we can find two lines on H which together contain four points of Γ on H but not the point q. They give us a quadric hypersurface in $\mathbb{P}^3$ which pass though six points of $\Gamma \setminus \{p_7, p_8, p_9\}$. Among these six points, two points are from the outside of H and the others from H. Therefore, we have four points that have not been yet covered. Because two of them are in the outside of H, we can easily find a quadric hypersurface which contains these four points but not the point q.

Case 6. $(r = 7)$

In this case, we can find three pairs of points of Γ on H such that each pair gives us a line not passing through the point q. It implies that we can construct a cubic hypersurface which passes through six points of Γ on H and three points of Γ in the outside of H but not the point q. Moreover, we may assume that the remaining three points in the outside of H do not lie on a single line together with the point q due to the sextic-node position condition. It is easy to find a quadric hypersurface containing the remaining points of Γ but not q. So we are done.

Case 7. $(r = 8 \text{ or } 9)$

We can find four pairs of points of Γ on H such that each pair gives us a line not passing through the point q. From this fact, we easily obtain a quartic hypersurface passing eight points of Γ on H and four points of Γ outside of H but not the point q. We then have only one point of Γ that is not covered. Just take a hyperplane passing through this point but not the point q, and we are done.

Case 8. $(r = 10)$

Because of the sextic-node position condition, we can find three pairs, say $\{p_1, p_2\}$, $\{p_3, p_4\}$, $\{p_5, p_6\}$, of points of Γ on H such that each pair gives us a line not passing through the point q and, in addition, no three of the points p_7, p_8, p_9, p_{10} cannot lie on a line passing through point q. This shows there is a quintic hypersurface which passes through Γ but not the point q.

Case 9. $(r = 11)$

We have eleven points of Γ on H and two points, p_{12}, p_{13}, of Γ in the outside of H. We can find a quintic curve C on H which passes through the eleven points on H but not the point q by Theorem 4.6. Note that the support of the curve C is not a line because of the sextic-node position condition. A generic hyperplane passing through p_{12}, p_{13} meets C at more than two points. Choose two points p' and p'' among these intersection points. Let v be the point at which two lines $\overline{p_{12}, p'}$ and $\overline{p_{13}, p''}$ intersect. Then the cone over the curve C with vertex v has all the points of Γ but not the point q.

Case 10. ($r = 12$)

All the points except one point, p_{13}, lie on the hyperplane H. It immediately follows from Theorem 4.6 that we can find a plane quintic curve which passing $\{p_1, \ldots, p_{12}\}$ but not the point q. Taking the cone over the plane quintic curve with vertex p_{13}, we obtain a quintic hypersurface that we want.

Case 11. ($r = 13$)

In this case, all the points lie on the hyperplane H. It immediately follows from Theorem 4.6 that we can find a plane quintic passing all the points except the point q, which gives us a quintic hypersurface in $\mathbb{P}^3$ that we want.

Consequently, we complete the proof. $\square$

Therefore, the first part of Theorem B has been proved.

The three-dimensional conjecture of Fujita (see [51], [85], and [122]) implies Lemma 4.7 in the case when the points in Γ are in *very general* position. Moreover, in the case when points in Σ are in *very general* position, the $\mathbb{Q}$-factoriality of X follows from Lefschetz theory (see Theorem 1.34 in [37]). However, neither the three-dimensional nor the two-dimensional conjecture of Fujita can be applied, in general, to an appropriate adjoint linear system in our case. The crucial point here is that the proof of Theorem 4.6 is based on the vanishing theorem of Ramanujam (see [19] and [120]) for 2-connected effective divisors on an algebraic surface (see Proposition 2 in [141]) which is slightly stronger in some cases than the vanishing theorem of Kawamata and Viehweg (see [84] and [142]).

The method of [37] also explains the non-$\mathbb{Q}$-factoriality of Examples 1.5, 1.6, and 1.7 over $\mathbb{C}$. Let $X \longrightarrow \mathbb{P}^3$ be a double cover ramified along a sextic S. Suppose that the sextic $S \subset \mathbb{P}^3$ is given by the equation

$$g_3^2(x,y,z,w) + h_r(x,y,z,w) f_{6-r}(x,y,z,w) = 0,$$

where g_3, h_r, and f_{6-r}, $1 \leq r \leq 3$, are generic homogeneous polynomials over $\mathbb{C}$ of degree 3, r, and $6-r$, respectively. Then the number of singular points is $18r - 3r^2$. All of them are simple double points. The defect of V is

$$\begin{aligned} h^1(\mathbb{P}^3, \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}^3}(5)) &= h^0(\mathbb{P}^3, \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}^3}(5)) - h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(5)) + h^0(\mathbb{P}^3, \bigoplus_{p \in \Sigma} \mathbb{C}) \\ &= h^0(\mathbb{P}^3, \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}^3}(5)) - 56 + 18r - 3r^2. \end{aligned}$$

Let H be the hypersurface of degree r defined by $h_r = 0$. Then it is easy to check $h^0(\mathbb{P}^3, \mathcal{I} \otimes \mathcal{O}_{\mathbb{P}^3}(5))$ is bigger than or equal to

$$\begin{aligned} h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(4)) + h^0(H, \mathcal{O}_H(2)) + h^0(H, \mathcal{O}_H) &= 42 \text{ when } r = 1, \\ h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) + h^0(H, \mathcal{O}_H(2)) + h^0(H, \mathcal{O}_H(1)) &= 33 \text{ when } r = 2, \\ h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) + h^0(H, \mathcal{O}_H(2)) + h^0(H, \mathcal{O}_H(2)) &= 30 \text{ when } r = 3. \end{aligned}$$

In all the cases, the defect of V is positive. Therefore, the double cover X is not $\mathbb{Q}$-factorial.

5 Elliptic fibrations

This section is devoted to Theorem C.

Let $\pi : X \to \mathbb{P}^3$ be a $\mathbb{Q}$-factorial double cover ramified along a nodal sextic $S \subset \mathbb{P}^3$. Consider a fibration $\tau : Y \to Z$ whose general enough fiber is a smooth elliptic curve. Suppose that we have a birational map ρ of X onto Y. We then put $\mathcal{M}_X = \frac{1}{n}\mathcal{M}$ with $\mathcal{M} = \rho^{-1}(|\tau^*(H_Z)|)$, where H_Z is a very ample divisor on Z and n is the natural number such that $\mathcal{M} \subset |-nK_X|$.

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, "\pi"'] & Y \arrow[d, "\tau"] \\ \mathbb{P}^3 & Z \end{tikzcd}$$

It immediately follows from Theorem 2.14 that the set $\mathbb{CS}(X, \mathcal{M}_X)$ is nonempty.

Remark 5.1. The linear system $\mathcal{M}$ is not composed from a pencil and cannot be contained in the fibers of any dominant rational map $\chi : X \dashrightarrow \mathbb{P}^1$.

Using the proof of Lemma 3.1, we can easily show that the set $\mathbb{CS}(X, \mathcal{M}_X)$ does not contain any smooth point of X.

Lemma 5.2. *Let o be a simple double point on X that belongs to $\mathbb{CS}(X, \mathcal{M}_X)$. Then there is a birational map $\beta : \mathbb{P}^2 \dashrightarrow Z$ such that the diagram*

$$\begin{tikzcd} X \arrow[rr, dashed, "\rho"] \arrow[d, "\pi"'] & & Y \arrow[d, "\tau"] \\ \mathbb{P}^3 \arrow[r, dashed, "\gamma"] & \mathbb{P}^2 \arrow[r, dashed, "\beta"] & Z \end{tikzcd}$$

commutes, where γ is the projection from the point $\pi(o)$.

Proof. Let $f : W \to X$ be the blowup at the point o and C be a general enough fiber of the elliptic fibration $\phi_{|-K_W|} : W \to \mathbb{P}^2$. Then for a general surface D in $f^{-1}(\mathcal{M})$,

$$2(n - \mathrm{mult}_o(\mathcal{M})) = C \cdot D \geq 0,$$

while $\mathrm{mult}_o(\mathcal{M}) \geq n$ by Theorem 2.34. We can therefore conclude that $\mathrm{mult}_o(\mathcal{M}) = n$ and $f^{-1}(\mathcal{M})$ lies in the fibers of the elliptic fibration $\phi_{|-K_W|} : W \to \mathbb{P}^2$, which implies the claim. $\square$

Corollary 5.3. *The set $\mathbb{CS}(X, \mathcal{M}_X)$ cannot contain two singular points of the 3-fold X.*

We assume that $\mathbb{CS}(X, \mathcal{M}_X)$ does not contain any point and that it contains a curve $C \subset X$.

Lemma 5.4. *The intersection number $-K_X \cdot C$ is 1.*

Proof. Let H be a general enough divisor in the linear system $|-K_X|$. Then we have

$$2 = H \cdot K_X^2 = H \cdot \mathcal{M}_X^2 \geq \mathrm{mult}_C(\mathcal{M}_X^2) H \cdot C \geq -K_X \cdot C,$$

which implies $-K_X \cdot C \leq 2$.

Suppose $-K_X \cdot C = 2$. Then $\mathrm{Supp}(\mathcal{M}_X^2) = C$ and

$$\mathrm{mult}_C(\mathcal{M}_X^2) = \mathrm{mult}_C^2(\mathcal{M}_X) = 1,$$

which means that for two different divisors M_1 and M_2 in the linear system $\mathcal{M}$ we have

$$\mathrm{mult}_C(M_1 \cdot M_2) = n^2, \quad \mathrm{mult}_C(M_1) = \mathrm{mult}_C(M_2) = n,$$

and set-theoretically $M_1 \cap M_2 = C$. However, the linear system $\mathcal{M}$ is not composed from a pencil. Therefore, for a general enough point $p \notin C$ the linear subsystem $\mathcal{D}$ of $\mathcal{M}$ passing through the point p has no base components. Let D_1 and D_2 be general enough divisors in $\mathcal{D}$. Then in set-theoretic sense

$$p \in D_1 \cap D_1 = M_1 \cap M_2 = C,$$

which is a contradiction. □

Corollary 5.5. *The curve $\pi(C) \subset \mathbb{P}^3$ is a line and $C \cong \mathbb{P}^1$.*

Remark 5.6. In the second part of the proof of Lemma 5.4, we have never used the irreducibility of the curve C. Hence, we may assume $\mathbb{CS}(X, \mathcal{M}_X) = \{C\}$. Moreover, the same arguments imply $\mathrm{mult}_C(\mathcal{M}^2) < 2n^2$.

Lemma 5.7. *The line $\pi(C)$ is contained in the sextic surface S.*

Proof. It follows from the proof of Lemma 3.7 and Remark 5.6. □

Before we proceed, we observe

$$\#|\mathrm{Sing}(X) \cap C| \leq \begin{cases} 3, & \pi(C) \not\subset S \\ 5, & \pi(C) \subset S, \end{cases}$$

by intersecting S with either the line $\pi(C)$ or a hyperplane in $\mathbb{P}^3$ passing through $\pi(C)$. Furthermore, when $\pi(C) \subset S$, the equality $\#|\mathrm{Sing}(X) \cap C| = 5$ holds if and only if all the hyperplanes tangent to the sextic surface S at points of $\pi(C \setminus \mathrm{Sing}(X))$ coincide.

Lemma 5.8. *The curve C passes through at least four singular points of X.*

Proof. Let H be a general hyperplane in $\mathbb{P}^3$ containing the line $\pi(C)$. Then the curve

$$D = H \cap S = \pi(C) \cup Q$$

is reduced, where Q is a quintic curve. The curve D is singular at each singular point p_i of S such that $p_i \in \pi(C)$ for $i \in \{1, \dots, k\}$. The set $\pi(C) \cap Q$ consists of at most five points and $\mathrm{Sing}(D) \cap \pi(C) \subset \pi(C) \cap Q$. Thus $k = \#|\mathrm{Sing}(X) \cap C| \leq 5$.

Suppose $k \leq 3$. Then the intersection $\pi(C) \cap Q$ contains two points o_1 and o_2 different from p_i due to the generality in the choice of H. The hyperplane H is therefore tangent to the sextic S at o_1 and o_2. Hyperplanes passing through the line $\pi(C)$ form a pencil whose proper transforms on the 3-fold X are K3 surfaces in $|-K_X|$ passing through C. Hence, the lines tangent to the sextic surface S at a general point of the line $\pi(C)$ span whole $\mathbb{P}^3$. Note that this is no longer true in the case $k = 5$ as we mentioned right before the lemma.

Let L_1 and L_2 be general enough lines in H passing through the points o_1 and o_2, respectively. Then L_j is tangent to the sextic surface S at the point o_j. Therefore, the proper transform $\tilde{L}_j \subset X$ of the curve L_j is an irreducible curve such that $-K_X \cdot \tilde{L}_j = 2$. Also, it is singular at the point $\tilde{o}_j = \pi^{-1}(o_j)$. Consider the proper transform $\tilde{H}$ of the surface H on X and a general surface M in the linear system $\mathcal{M}$. Then

$$M|_{\tilde{H}} = \mathrm{mult}_C(\mathcal{M})C + R,$$

where R is an effective divisor on $\tilde{H}$ such that $C \not\subset \mathrm{Supp}(R)$. Moreover,

$$2n = M \cdot \tilde{L}_j \geq \mathrm{mult}_{\tilde{o}_j}(\tilde{L}_j)\,\mathrm{mult}_C(M) + \sum_{p \in (M \setminus C) \cap \tilde{L}_j} \mathrm{mult}_p(M) \cdot \mathrm{mult}_p(\tilde{L}_j) \geq 2n,$$

which implies $M \cap \tilde{L}_j \subset C$ set-theoretically. However, on $\tilde{H}$ the curves $\tilde{L}_1$ and $\tilde{L}_2$ span two pencils with the base loci consisting of the points $\tilde{o}_1$ and $\tilde{o}_2$, respectively. Therefore, we see $R = \varnothing$ due to the generality in the choice of two curves L_1 and L_2. Note that if $k = 4$, then this is not true.

Hence, set-theoretically $M \cap \tilde{H} = C$ for a general divisor $\tilde{H} \in |-K_X|$ passing through the curve C and a divisor $M \in \mathcal{M}$ with $\tilde{H} \not\subset \mathrm{Supp}(M)$. Let $\tilde{p}$ be a general point on the surface $\tilde{H}$ and $\mathcal{M}_{\tilde{p}}$ be the linear system of surfaces in $\mathcal{M}$ containing $\tilde{p}$. Then $\mathcal{M}_{\tilde{p}}$ has no base components due to Remark 5.1. Therefore, for a general divisor $\tilde{M}$ in $\mathcal{M}_{\tilde{p}}$

$$\tilde{p} \in \tilde{M} \cap \tilde{H} = C$$

because $\tilde{H} \not\subset \mathrm{Supp}(\tilde{M})$, which contradicts the generality of the point $\tilde{p} \in \tilde{H}$. ∎

Lemma 5.9. *Suppose that the curve C contains exactly four singular points of the 3-fold X. Then there is a birational map $\beta : \mathbb{P}^2 \dashrightarrow Z$ such that the diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, dashed, "\Xi_{\pi(C)}"'] & Y \arrow[d, "\tau"] \\ \mathbb{P}^2 \arrow[r, dashed, "\beta"] & Z \end{tikzcd}$$

is commutative, where $\Xi_{\pi(C)}$ is a rational map defined as in Construction B.

Proof. By our assumption, the curve C passes through four singular points p_1, p_2, p_3, p_4 of X. We consider the blowup $g_1 : \tilde{X} \to X$ at the points $p_1, \ldots, p_4$ and the blowup $g_2 : W \to \tilde{X}$ along the proper transform of the curve C on $\tilde{X}$. Put $g := g_2 \circ g_1 : W \to X$. We then get

$$-K_W = g^*(-K_X) - \sum_{i=1}^{4} E_i - F,$$

where E_i and F are the g-exceptional divisors such that $g(E_i) = p_i$ and $g(F) = C$. Let L be a curve on W such that $\pi \circ g(L)$ is a line tangent to S at some general point of $\pi(C)$. Then

$$\mathcal{M}_W \cdot L \leq 2 - 2\,\mathrm{mult}_C(\mathcal{M}_X) \leq 0,$$

where $\mathcal{M}_W = g^{-1}(\mathcal{M}_X)$. Because such curves as L span a Zariski dense subset in W, we obtain $\mathrm{mult}_C(\mathcal{M}_X) = 1$. Each elliptic curve L is a fiber of the elliptic fibration $\Xi_{\pi(C)} \circ g : W \to \mathbb{P}^2$. Thus $\mathcal{M}_W$ lies in the fibers of $\Xi_{\pi(C)} \circ g$, which implies the claim. □

Lemma 5.10. *The curve C passes through at most four singular points of X.*

Proof. Suppose that the curve C passes through five singular points $p_1, \ldots, p_5$ of X. Again, we consider the blowup $g_1 : \tilde{X} \to X$ at the points $p_1, \ldots, p_5$ and the blowup $g_2 : W \to \tilde{X}$ along the proper transform of the curve C on $\tilde{X}$. Put $g := g_2 \circ g_1 : W \to X$. Then we obtain

$$-K_W = g^*(-K_X) - \sum_{i=1}^{5} E_i - F,$$

where E_i and F are the g-exceptional divisors such that $g(E_i) = p_i$ and $g(F) = C$. Let $f : U \to W$ be a birational morphism such that $h = \rho \circ g \circ f$ is a morphism. Then we obtain

$$K_U + \mathcal{M}_U = (g \circ f)^*(K_X + \mathcal{M}_X) + \sum_{i=0}^{r} a_i G_i,$$

where $\mathcal{M}_U = (g \circ f)^{-1}(\mathcal{M}_X)$, G_i are the $(g \circ f)$-exceptional divisors, and $a_i \in \mathbb{Q}$. Whenever $a_i \leq 0$, we have $g \circ f(G_i) = C$. But $\mathrm{mult}_C(\mathcal{M}_X) < 2$ by Remark 5.6 and hence there is exactly one i, say $i = 0$, such that $a_0 \leq 0$. It implies $f(G_0) = F$ and $a_0 = 0$.

Consider a general enough fiber $\hat{L}$ of the morphism $\tau \circ h : U \to Z$. Then $K_U \cdot \hat{L} = 0$ because the curve $\hat{L}$ is elliptic. However, $\mathcal{M}_U \cdot \hat{L} = 0$ by construction. So we see $G_i \cdot \hat{L} = 0$ for $i \neq 0$, which means that f is an isomorphism near $\hat{L}$. Thus $\mathcal{M}_W \cdot \tilde{L} = 0$, where $\mathcal{M}_W = f^{-1}(\mathcal{M}_X)$ and $\tilde{L} = f(\hat{L})$.

There is a surface $D \subset W$ such that $\pi \circ g(D) \subset \mathbb{P}^3$ is the plane tangent to the sextic surface S along the whole line $\pi(C)$. The surface D is the closure of the set spanned by curves whose images via $\pi \circ g$ are lines tangent to the surface S at some point of $\pi(C)$.

By the same argument as in the proof of Lemma 5.9, we obtain that $\mathrm{mult}_C(\mathcal{M}_X) = 1$, and hence

$$D \sim \mathcal{M}_W - F + \sum_{i=1}^{5} b_i E_i$$

for some $b_i \in \mathbb{Z}$. On the other hand, because $\hat{L} \cdot G_i = 0$ for $i \neq 0$, we get

$$E_j \cdot \tilde{L} = f^*(E_j) \cdot \hat{L} = \sum_{i=1}^{r} c_{ij} G_i \cdot \hat{L} = 0$$

where $c_{ij} \in \mathbb{N}$. Therefore, $\tilde{L} \cdot D < 0$, which means $\tilde{L} \subset D$. This is impossible because the curves $\tilde{L}$ span a Zariski dense subset in W. $\square$

Therefore, Theorem C is proven.

6 Canonical Fano 3-folds

To prove Theorem D, we let $\pi : X \to \mathbb{P}^3$ be a $\mathbb{Q}$-factorial double cover ramified in a nodal sextic $S \subset \mathbb{P}^3$. We then suppose that there is a nonbiregular birational map $\rho : X \dashrightarrow Y$ of X onto a Fano 3-fold Y with canonical singularities. We are to show that there is a curve $C \subset X$ such that $\pi(C)$ is a line on the surface S passing through five nodes of the sextic S.

We put $\mathcal{M} = \rho^{-1}(|-nK_Y|)$ and $\mathcal{M}_X = \frac{1}{n}\mathcal{M}$ for a natural number $n \gg 0$. We then see that $K_X + \mathcal{M}_X \sim_{\mathbb{Q}} 0$ and the singularities of the movable log pair $(X, \mathcal{M}_X)$ are not terminal by Theorem 2.17. By our construction, the linear system $\mathcal{M}$ cannot be contained in the fibers of any dominant rational map $\chi : X \dashrightarrow Z$ with $0 < \dim(Z) \leq 2$.

Proposition 6.1. *The set $\mathbb{CS}(X, \mathcal{M}_X)$ consists of a single curve $C \subset X$ which satisfies*

1. $-K_X \cdot C = 1$,

2. $\pi(C) \subset S$,
3. $\#|\operatorname{Sing}(X) \cap C| = 5$.

Proof. For the proof, we literally repeat the proofs in Section 5 except those of Lemmas 5.2 and 5.9. $\square$

Let $p_1, p_2, p_3, p_4, p_5 \in C$ be singular points of X. We consider the blowup $f_1 : \tilde{X} \to X$ at all the points p_i and the blowup $f_2 : W \to \tilde{X}$ along the proper transform of the curve C on $\tilde{X}$. Put $f = f_2 \circ f_1 : W \to X$. We then note that W is smooth and

$$-K_W \sim f^*(-K_X) - \sum_{i=1}^{5} E_i - G,$$

where E_i and G are the f-exceptional divisors with $f(E_i) = p_i$ and $f(G) = C$. Each surface E_i is isomorphic to the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at one point. We have the proper transforms F_1^i and F_2^i of two rulings of the quadric $\mathbb{P}^1 \times \mathbb{P}^1$ with self-intersection -1 on each surface E_i.

The normal bundle $\mathcal{N}_{W/F_j^i}$ of the curve $F_j^i \cong \mathbb{P}^1$ in the 3-fold W splits into

$$\mathcal{N}_{W/F_j^i} \cong \mathcal{O}_{F_j^i}(a) \oplus \mathcal{O}_{F_j^i}(b)$$

for some integers $a \geq b$. The exact sequence

$$0 \to \mathcal{T}_{F_j^i} \to \mathcal{T}_W|_{F_j^i} \to \mathcal{N}_{W/F_j^i} \to 0$$

implies $\deg(\mathcal{N}_{W/F_j^i}) = a + b = -K_W \cdot F_j^i + 2g(F_j^i) - 2 = -2$. On the other hand, the exact sequence

$$0 \to \mathcal{N}_{E_i/F_j^i} \to \mathcal{N}_{W/F_j^i} \to \mathcal{N}_{W/E_i}|_{F_j^i} \to 0$$

together with $\mathcal{N}_{E_i/F_j^i} \cong \mathcal{O}_{F_j^i}(-1)$ implies $b \geq -1$. Therefore, $a = b = -1$ and we can make a standard flop for each curve F_j^i. Indeed, we let $h : \widetilde{W} \to W$ be the blowup along all the curves F_j^i and R_j^i be the h-exceptional divisor dominating the curve F_j^i. Then $R_j^i \cong \mathbb{P}^1 \times \mathbb{P}^1$ and there is a birational morphism $\hat{h} : \widetilde{W} \to \hat{W}$ which contracts each surface R_j^i to a curve $\hat{F}_j^i \subset \hat{W}$ and for which $\hat{h} \circ h^{-1}$ is not an isomorphism in a neighborhood of each curve F_j^i.

Let $\hat{E}_i = \hat{h} \circ h^{-1}(E_i) \subset \hat{W}$. Then $\hat{E}_i \cong \mathbb{P}^2$ and

$$\hat{E}_i|_{\hat{E}_i} \cong \mathcal{O}_{\mathbb{P}^2}(-2),$$

which implies that each divisor $\hat{E}_i$ can be contracted to a terminal cyclic quotient singularity of type $\frac{1}{2}(1,1,1)$. Let $\hat{f} : \hat{W} \to V$ be the contraction of all the $\hat{E}_i$. Then V has exactly five singular points o_i of type $\frac{1}{2}(1,1,1)$, it is $\mathbb{Q}$-factorial, and $\operatorname{Pic}(V) \cong \mathbb{Z} \otimes \mathbb{Z}$.

Let $F = \hat{f} \circ \hat{h} \circ h^{-1}(G)$. Then there is a birational morphism $g : V \to X$ contracting the surface F to the curve C.

$$\begin{tikzcd}
 & \widetilde{W} \arrow[dl, "h"'] \arrow[dr, "\hat{h}"] & \\
W \arrow[rr, dashed, "\hat{h}\circ h^{-1}"] \arrow[d, "f"'] & & \hat{W} \arrow[d, "\hat{f}"] \\
X \arrow[d, dashed, "\rho"'] & & V \arrow[ll, "g"'] \arrow[d, "\phi_{|-rK_V|}"] \\
Y \arrow[rr, dashed] & & U
\end{tikzcd}$$

At a generic point of C the morphism g is a blowup. In fact, the morphism g is the blowup of the ideal sheaf of the curve $C \subset X$ by Proposition 1.2 in [139]. Moreover, the proof of Lemma 3.8 implies $\mathrm{mult}_C(\mathcal{M}_X) = 1$. Hence,

$$-K_V \sim_{\mathbb{Q}} \mathcal{M}_V \sim_{\mathbb{Q}} g^*(-K_X) - F,$$

where $\mathcal{M}_V = g^{-1}(\mathcal{M}_X)$. The morphism $g|_F : F \to C$ has five reducible fibers consisting of two copies of $\mathbb{P}^1$ intersecting transversally at the corresponding singular point o_i that is a simple double point on the surface F.

Let $\tilde{C} \subset F$ be the unique base curve of the pencil $|-K_V|$. Then the numerical equivalence $\tilde{C} \equiv K_V^2$ holds. Therefore, we have

$$-K_V \text{ is nef } \iff -K_V \cdot \tilde{C} \geq 0 \iff -K_V^3 \geq 0.$$

Because elementary calculations imply $-K_V^3 = \frac{1}{2}$, the anticanonical divisor $-K_V$ is nef and big. Hence, $|-rK_V|$ is base-point-free for a natural number $r \gg 0$ by the Base Point Freeness theorem (see [86]). The morphism $\phi_{|-rK_V|} : V \to U$ is birational and U is a canonical Fano 3-fold with $-K_U^3 = \frac{1}{2}$.

The image of every element in the set $\mathbb{CS}(V, \mathcal{M}_V)$ on the 3-fold X is an element in $\mathbb{CS}(X, \mathcal{M}_X)$ because $K_V + \mathcal{M}_V = g^*(K_X + \mathcal{M}_X)$. Hence, every element in $\mathbb{CS}(V, \mathcal{M}_V)$ must be a curve dominating the curve C due to the assumption made in Remark 5.6, which implies $\mathrm{mult}_C(\mathcal{M}) \geq 2n^2$. However, it is impossible because of Remark 5.6. Therefore, the set $\mathbb{CS}(V, \mathcal{M}_V) = \varnothing$.

For a rational number c slightly bigger than 1, the singularities of the log pair $(V, c\mathcal{M}_V)$ are still terminal and the equivalence

$$K_V + c\mathcal{M}_V = \phi^*_{|-rK_V|}(K_U + c\mathcal{M}_U)$$

holds, where $\mathcal{M}_U = \phi_{|-rK_V|}(\mathcal{M}_V)$. Hence, the movable log pair $(U, c\mathcal{M}_U)$ is a canonical model. On the other hand, the movable log pair $(Y, \frac{c}{n}|-nK_Y|)$ is a canonical model as well. Consequently, the map

$$\phi_{|-rK_V|} \circ (\rho \circ g)^{-1} : Y \dashrightarrow U$$

is an isomorphism by Proposition 2.9.

All the statements above do not depend on the existence of a birational map ρ of X onto Y. They depend only on the condition that X has a curve C such that $\pi(C) \subset S$ is a line passing through five nodes of the sextic surface S. Once such a curve $C \subset X$ exists, we can construct a birational transformation of X into a canonical Fano 3-fold by means of blowing up the ideal sheaf of the curve $C \subset X$ and the birational morphism given by a plurianticanonical linear system.

We have proved Theorem D. In addition, we have obtained explicit classification of all birational transformations of a double cover X into Fano 3-folds with canonical singularities.

As we mentioned before, five singular points of the surface S lying on the line $\pi(C) \subset S$ force every hyperplane in $\mathbb{P}^3$ tangent to S at some point of $\pi(C)$ smooth on S to be tangent to the surface S along the whole line $\pi(C)$. Such a tangent hyperplane is unique and its proper transform on V is the only divisor in the linear system $|-K_V - F|$ which is contracted by the birational morphism $\phi_{|-rK_V|}$ to a nonterminal point of the canonical Fano 3-fold U.

7 Sextic double solids over finite fields

We consider a double cover $\pi : X \to \mathbb{P}^3$ defined over a perfect field $\mathbb{F}$ of characteristic $\mathrm{char}(\mathbb{F}) > 5$. Suppose that the 3-fold X is $\mathbb{Q}$-factorial and that it is ramified along a nodal sextic surface $S \subset \mathbb{P}^3$. Actually, we may assume that the field $\mathbb{F}$ is algebraically closed because $\mathbb{F}$ is perfect. We are to adjust the proofs of both Theorems A and C to the case $\mathrm{char}(\mathbb{F}) > 5$.

We first list valid statements in Sections 3 and 5 in the case $\mathrm{char}(\mathbb{F}) > 5$. The following remain valid:

1. Propositions 2.7, 2.9, and Theorem 2.30;
2. negativity of exceptional loci (see [3] and Lemma 2.19 in [87]);
3. resolution of singularities of 3-folds (see [1] and [45]);
4. numerical intersection theory on smooth 3-folds (see [59]);
5. elementary properties of blowups (see [71]).

Lemma 7.1. *Theorems* 2.10 *and* 2.14 *are valid in the case* $\mathrm{char}(\mathbb{F}) > 5$.

Proof. The proofs for the case $\mathrm{char}(\mathbb{F}) = 0$ depend only on the facts listed above. □

The following may not remain valid in the case $\mathrm{char}(\mathbb{F}) \neq 0$:

1. Theorem 2.34;
2. special cases of Bertini's theorem (see [64]).

For the birational super-rigidity, we need Theorem 2.34 and Bertini's theorem.

The characteristic-free method for the proof of Theorem 2.30 in [116] can be used to prove Theorem 2.34. However, we used Theorem 2.34 just to prove Lemma 3.2. So instead of proving Theorem 2.34 in the case $\mathrm{char}(\mathbb{F}) > 5$, we prove Lemma 3.2 only with Theorem 2.30, which is enough for the birational super-rigidity.

Lemma 7.2. *Let $(X, \mathcal{M}_X)$ be a movable log pair such that $-(K_X + \mathcal{M}_X)$ is ample and let $o \in X$ be a simple double point. Then the point o does not belong to $\mathbb{CS}(X, \mathcal{M}_X)$.*

Proof. Suppose that the point o belongs to the set $\mathbb{CS}(X, \mathcal{M}_X)$. Let $f : W \to X$ be the blowup at the point o and C be a general enough fiber of the elliptic fibration $\phi_{|-K_W|} : W \to \mathbb{P}^2$. Then

$$2(1 - \mathrm{mult}_o(\mathcal{M}_X)) > C \cdot \mathcal{M}_W \geq 0,$$

where $\mathcal{M}_W = f^{-1}(\mathcal{M}_X)$. This implies $\mathrm{mult}_o(\mathcal{M}_X) < 1$.

We consider

$$K_W + \mathcal{M}_W = f^*(K_X + \mathcal{M}_X) + (1 - \mathrm{mult}_o(\mathcal{M}_X))G,$$

where G is the f-exceptional divisor. We then see that there is a center $B \in \mathbb{CS}(W, \mathcal{M}_W)$ with $B \subset G$.

The intersection number of $\mathcal{M}_W$ with each ruling of $G \cong \mathbb{P}^1 \times \mathbb{P}^1$ is $\mathrm{mult}_o \mathcal{M}_X < 1$. On the other hand, we have $\mathrm{mult}_B(\mathcal{M}_W) \geq 1$. Therefore, the center B must be a point and

$$\mathrm{mult}_B(\mathcal{M}_W^2) \geq 4$$

by Theorem 2.30.

Let H_1 and H_2 be two general surfaces in $|-K_W|$ passing through the point B. Then $H_1 \cap H_2$ consists of the fiber E of the elliptic fibration $\phi_{|-K_W|}$ with $B \in E$. Consider general enough divisors $D \in |-2K_W|$ and $F_1, F_2 \in |f^*(-K_X)|$. Then the divisors D, F_1, and F_2 do not pass through the point B at all. The divisors $H_1 + F_1$, $H_2 + F_2$, and $D + G$ are elements of the linear subsystem $\mathcal{H} \subset |f^*(-2K_X) - G|$ of surfaces passing B. The intersection

$$\mathrm{Supp}(H_1 + F_1) \cap \mathrm{Supp}(H_2 + F_2) \cap \mathrm{Supp}(D + G)$$

contains B and consists of a finite number of points. In particular, the linear system $\mathcal{H}$ has no base curves but B is a base point of $\mathcal{H}$. Let H be a general surface in $\mathcal{H}$. Then we obtain

$$4 > H \cdot \mathcal{M}_W^2 \geq \mathrm{mult}_B(H)\,\mathrm{mult}_B(\mathcal{M}_W^2) \geq 4,$$

which is absurd. $\square$

During excluding a one-dimensional member of $\mathbb{CS}(X, \mathcal{M}_X)$, we implicitly used Bertini's theorem only one time just for the following special case.

Lemma 7.3. *Let $C \subset X$ be a curve with $-K_X \cdot C = 1$ and $\pi(C) \not\subset S$. Then a general enough surface $H \in |-K_X|$ passing through C is smooth along $C \setminus \mathrm{Sing}(X)$.*

Proof. The simple double points of the 3-fold X correspond to the simple double points of the sextic surface S because $\mathrm{char}(\mathbb{F}) \neq 2$. Meanwhile, the curve $L := \pi(C)$ on $\mathbb{P}^3$ is a line. The line L cannot pass through more than three singular points of S; otherwise it would be contained in S. The surface $D = \pi(H) \subset \mathbb{P}^3$ is a plane containing L. The singularities of surface H correspond to the singularities of the curve $D \cap S$ which is the ramification divisor of the double cover $\pi : H \to D$. For a general enough surface $H \in |-K_X|$, the plane D is not tangent to the sextic S at the points of $L \setminus \mathrm{Sing}(S)$, which implies the claim. □

Therefore, the birational super-rigidity remains true over the field $\mathbb{F}$.

Now, we consider the statements in Section 5 over the field $\mathbb{F}$. They also require both Theorem 2.34 and Bertini's theorem.

The reason why Theorem 2.34 is required again is the lemma below. However, it can be proved only with Theorem 2.30.

Lemma 7.4. *Let $\rho : X \dashrightarrow Y$ be a birational map and $\tau : Y \to Z$ be a fibration whose general fiber is a smooth elliptic curve. Let $(X, \mathcal{M}_X)$ be the movable log pair such that $\mathcal{M} := \rho^{-1}(|\tau^*(H)|)$ and $\mathcal{M}_X = \frac{1}{n}\mathcal{M}$, where H is a very ample divisor on surface Z and n is the natural number such that $\mathcal{M} \subset |-nK_X|$. Suppose that the set $\mathbb{CS}(X, \mathcal{M}_X)$ contains a singular point $o \in X$. Then there is a birational map $\beta : \mathbb{P}^2 \dashrightarrow Z$ such that the diagram*

$$\begin{tikzcd} X \arrow[rr, dashed, "\rho"] \arrow[d, "\pi"'] & & Y \arrow[d, "\tau"] \\ \mathbb{P}^3 \arrow[r, dashed, "\gamma"] & \mathbb{P}^2 \arrow[r, dashed, "\beta"] & Z \end{tikzcd}$$

commutes, where γ is the projection from the point $\pi(o)$.

Proof. Consider the blowup $f : W \to X$ at the point o. Let C be a general fiber of $\phi_{|-K_W|}$. Then

$$2n - 2\,\mathrm{mult}_o(\mathcal{M}) = C \cdot f^{-1}(\mathcal{M}) \geq 0,$$

which implies $\mathrm{mult}_o(\mathcal{M}_X) \leq 1$. Furthermore, the multiplicity $\mathrm{mult}_o(\mathcal{M}_X)$ cannot be less than 1. Indeed, if $\mathrm{mult}_o(\mathcal{M}_X) < 1$, then the proof of Lemma 7.2 shows contradictory inequalities

$$4 \geq H \cdot \mathcal{M}_W^2 \geq \mathrm{mult}_B(H)\,\mathrm{mult}_B(\mathcal{M}_W^2) > 4,$$

where $\mathcal{M}_W = f^{-1}(\mathcal{M}_X)$, B is a center of $\mathbb{CS}(W, \mathcal{M}_W)$, and H is a general surface in $|f^*(-2K_X) - E|$ passing through B.

In the case $\mathrm{mult}_o(\mathcal{M}_X) = 1$, the linear system $f^{-1}(\mathcal{M})$ lies in the fibers of the elliptic fibration $\phi_{|-K_W|} : W \to \mathbb{P}^2$, which implies the claim. □

Bertini's Theorem is required again only for the following statement that can be proved without using Bertini's theorem.

Lemma 7.5. *Let C be a curve on X such that $-K_X \cdot C = 1$ and $\pi(C) \subset S$. Suppose that $\#|\operatorname{Sing}(X) \cap C| \leq 3$. Then a general surface $H \in |-K_X|$ passing through curve C has at least two different simple double points on the curve $C \subset X$ at which the 3-fold X is smooth.*

Proof. The surface $\pi(H) \subset \mathbb{P}^3$ is a plane passing through the line $L := \pi(C) \subset S$. Therefore,

$$\pi(H) \cap S = L \cup Q,$$

where Q is a plane quintic. Whenever H moves in the pencil of surfaces in $|-K_X|$ passing through C, the quintic Q moves in a pencil on S whose base locus is $\operatorname{Sing}(S) \cap L$. It gives a finite morphism $\gamma : L \to \mathbb{P}^1$ of degree $5 - \#|\operatorname{Sing}(S) \cap L|$ such that in the outside of the set $\operatorname{Sing}(S) \cap L$ the morphism γ is ramified at the points where $L \cup Q$ is not a normal crossing divisor on the plane $\pi(H)$. These points correspond to nonsimple double points of the surface H contained in the curve C and different from $\operatorname{Sing}(X) \cap C$. However, this morphism cannot be ramified everywhere because we assumed $\operatorname{char}(\mathbb{F}) > 5$. □

Corollary 7.6. *Lemma 5.8 remains true in the case* $\operatorname{char}(\mathbb{F}) > 5$.

Proof. Apply Lemma 7.5 to the proof of Lemma 5.8. □

Because the proofs of Lemmas 5.9 and 5.10 are characteristic-free, Theorem E is true.

8 Potential density

Now, we prove Theorem F.

Consider a double cover $\pi : X \to \mathbb{P}^3$ defined over a number field $\mathbb{F}$ and ramified along a nodal sextic surface $S \subset \mathbb{P}^3$. We suppose that $\operatorname{Sing}(X) \neq \varnothing$. We will show that the set of rational points of the 3-fold X is potentially dense, which means that there exists a finite extension $\mathbb{K}$ of the field $\mathbb{F}$ such that the set of all $\mathbb{K}$-rational points of the 3-fold X is Zariski dense.

The rationality and the unirationality of the 3-fold X over the field $\overline{\mathbb{Q}}$ would automatically imply potential density of rational points on X. However, the 3-fold X is nonrational in general due to Theorem A and the unirationality of the 3-fold X is unknown. Moreover, X is expected to be nonunirational in general. Actually, the degree of a rational dominant map from $\mathbb{P}^3$ to a double cover of $\mathbb{P}^3$ ramified in a very generic smooth sextic surface must be divisible by 2 and 3 due to [89] and [90].

The following result was proved in [16]:

Theorem 8.1. *Let $\tau : D \to \mathbb{P}^2$ be a double cover defined over a number field $\mathbb{F}$ and ramified along a reduced sextic curve $R \subset \mathbb{P}^2$. Suppose* $\mathrm{Sing}(D) \neq \varnothing$. *Then the set of rational points on the surface D is potentially dense if and only if the curve $R \subset \mathbb{P}^2$ is not a union of six lines intersecting at a single point.*

Actually, Theorem 8.1 is a special case of the following result in [17].

Theorem 8.2. *Let D be a K3 surface defined over a number field $\mathbb{F}$ such that D has either a structure of an elliptic fibration or an infinite group of automorphisms. Then the set of rational points on D is potentially dense.*

Hence, taking Theorem C into consideration, we see that Theorem F is a three-dimensional analogue of Theorem 8.1.

When singularities of the sextic surface S are worse than simple double points but are not too bad, the double cover X tends to be more rational (see [23]). Thus Theorem F must be true for sextic surfaces with any singularities possibly except cones over sextic curves. If the sextic surface $S \subset \mathbb{P}^3$ is a reduced union of six hyperplanes passing through one line, the set of rational points on X is not potentially dense due to Faltings' theorem ([56] and [57]) because the 3-fold X is birationally isomorphic to a product $\mathbb{P}^2 \times C$, where C is a smooth curve of genus 2.

As a matter of fact, the sets of rational points are potentially dense on double covers of $\mathbb{P}^n$ ramified along general enough sextic hypersurfaces for $n \gg 0$ due to the following result ([40]):

Theorem 8.3. *Let V be a double cover of $\mathbb{P}^n$ ramified in a sufficiently general hypersurface of degree $2d > 4$. Then V is unirational if $n \geq c(d)$, where $c(d) \in \mathbb{N}$ depends only on d.*

We will prove the potential density of the set of rational points on X using the technique of [15], [16], and [70] which relies on the following result proved in [104].

Theorem 8.4. *Let $\mathbb{F}$ be a number field. Then there is an integer $n_{\mathbb{F}}$ such that no elliptic curve defined over $\mathbb{F}$ has an $\mathbb{F}$-rational torsion point of order $n > n_{\mathbb{F}}$.*

Let o be a simple double point on X. The point $\pi(o)$ is a node of the sextic surface S. Replacing the field $\mathbb{F}$ by a finite extension of $\mathbb{F}$, we may assume that the point o and some other finitely many points that we will need in the sequel are defined over $\mathbb{F}$. Let $f : V \to X$ be the blowup at the point o with f-exceptional divisor E. Then the linear system $|-K_V|$ is free and the morphism

$$\phi_{|-K_V|} : V \to \mathbb{P}^2$$

is an elliptic fibration. The surface E is a multisection of $\phi_{|-K_V|}$ of degree 2. Let H be a general surface in $|-f^*(K_X)|$. Then H is a multisection of $\phi_{|-K_V|}$ of degree 2 as well.

The following lemma is a corollary of Proposition 2.4 in [15].

Lemma 8.5. *Suppose that there is a multisection M of $\phi_{|-K_V|}$ of degree $d \geq 2$ such that the morphism $\phi_{|-K_V|}|_M$ is branched at a point $p \in M$ which is contained in a smooth fiber of the elliptic fibration $\phi_{|-K_V|}$. Then the divisor $p_1 - p_2 \in \mathrm{Pic}(C_b)$ is not a torsion for some distinct points p_1 and p_2 of the intersection $M \cap C_b$, where $C_b = \phi^{-1}_{|-K_V|}(b)$ and b is a $\mathbb{C}$-rational point in the complement to a countable union of proper Zariski closed subsets in $\mathbb{P}^2$.*

Proof. See [15]. □

Lemma 8.6. *Let $M \in |H|$ be an irreducible multisection of $\phi_{|-K_V|}$ of degree 2 defined over $\mathbb{F}$ such that the set of rational points on M is potentially dense in M and $\phi_{|-K_V|}|_M$ is branched at a point contained in a smooth fiber of $\phi_{|-K_V|}$. Then the set of rational points on X is potentially dense.*

Proof. For each $n \in \mathbb{N}$, we let Φ_n be the set of points p of M satisfying the following two conditions:

1. the point p is contained in a smooth fiber C_p of the elliptic fibration $\phi_{|-K_V|}$;
2. $2np = nH|_{C_p}$ in $\mathrm{Pic}(C_p)$.

Let $\overline{\Phi}_n$ be the Zariski closure of the set Φ_n in M.

Suppose $\overline{\Phi}_n = M$ for some n. Take a very general fiber C of $\phi_{|-K_V|}$ and let

$$C \cap M = \{p_1, p_2\},$$

where $p_1 \neq p_2$. Then either $2np_1 \sim nH|_C$ or $2np_2 \sim nH|_C$ because $\overline{\Phi}_n = M$. However, $p_1 + p_2 \sim H|_C$. Thus

$$2np_1 \sim 2np_2 \sim nH|_C$$

and the element $p_1 - p_2$ is a torsion in $\mathrm{Pic}(C)$. Therefore, the $\mathbb{C}$-rational point $\phi_{|-K_V|}(C)$ is contained in the countable union of proper Zariski closed subsets in $\mathbb{P}^2$ of Lemma 8.5, which contradicts the very general choice of the fiber C. Accordingly, the set Φ_n is not Zariski dense in M for any $n \in \mathbb{N}$. Moreover, it follows from Theorem 8.4 that each set Φ_n for $n > n_{\mathbb{F}}$, where $n_{\mathbb{F}}$ is the number defined in Theorem 8.4, is disjoint from the set of $\mathbb{F}$-rational points on M.

Because of the assumption on the multisection M, we may assume that the set of $\mathbb{F}$-rational points on the surface M is Zariski dense. Take an $\mathbb{F}$-rational point

$$q \in M' := M \backslash (Z \cup_{i=1}^{n_{\mathbb{F}}} \overline{\Phi}_i),$$

where the set $Z \subset M$ consists of points contained in singular fibers of $\phi_{|-K_V|}$. Let C_q be the fiber of $\phi_{|-K_V|}$ passing through q. Then both the curve C_q and the point $\phi_{|-K_V|}(q)$ are defined over the field $\mathbb{F}$. The divisor $2q - H|_{C_q} \in \mathrm{Pic}(C_q)$ is defined over $\mathbb{F}$ as well. Moreover, $2q - H|_{C_q}$ is not a torsion divisor. By Riemann–Roch theorem, for each $n \in \mathbb{N}$ there is a unique $\mathbb{F}$-rational point $q_n \in C_q$ such that

$$q_n + (2n-1)q = nH|_{C_q}$$

in $\mathrm{Pic}(C_q)$. Because $2q - H|_{C_q}$ is not a torsion divisor, we see that $q_i \neq q_j$ if and only if $i \neq j$. We obtain an infinite collection of $\mathbb{F}$-rational points on C_q. Consequently, for each $\mathbb{F}$-rational point q in M', the curve C_q is contained in the Zariski closure of the set of $\mathbb{F}$-rational points of V. Because the set M' is a Zariski dense subset of M, the set of rational points on the 3-fold X is potentially dense. □

In order to prove Theorem F, it is enough to find an element in $|H|$ satisfying the conditions of Lemma 8.6. To find such an element, we let T be the set of points $(p,q) \in S \times S$ satisfying the following conditions:

1. $p \neq q$;
2. the points p and q are smooth points on the sextic surface S;
3. the point q is contained in the hyperplane $D \subset \mathbb{P}^3$ tangent to S at p;
4. the point q is a smooth point of the intersection $S \cap D$;
5. the intersection $S \cap D$ is reduced.

We also let $\psi : T \to S$ be the projection on the second factor.

Lemma 8.7. *The image $\psi(T)$ contains a Zariski open subset of the sextic $S \subset \mathbb{P}^3$.*

Proof. Let p be a general point on the sextic $S \subset \mathbb{P}^3$ and D be the hyperplane tangent to the surface S at the point p in $\mathbb{P}^3$. To prove the claim we just need to show that $D \cap S$ is reduced, which is nothing but the finiteness of the Gauss map at a generic point of S.

When the surface S is smooth, the intersection $D \cap S$ is known to be reduced (see [60], [76], or [112]). Even though S can have double points in our case, the intersection $D \cap S$ is reduced because S is not ruled (see [105]). Here, we prove it only with simple calculation.

Suppose that $D \cap S$ is not reduced and

$$D \cap S = mC + F \subset D \cong \mathbb{P}^2,$$

where $m \geq 2$. Then C is a line, a conic, or a plane cubic curve. Let $\gamma : \tilde{S} \to S$ be the blowup at the double points of S and $\tilde{C} = \gamma^{-1}(C)$. Then S is a surface of general type,

$$K_{\tilde{S}} = \gamma^*(\mathcal{O}_{\mathbb{P}^3}(2)|_S),$$

and $\tilde{C}$ is either a rational curve or an elliptic curve. Moreover, the self-intersection number $\tilde{C}^2$ of $\tilde{C}$ is negative by adjunction formula, but $\tilde{C}$ moves in a family on the surface $\tilde{S}$ when we move the point p in S, which is a contradiction. □

Therefore, by Lemma 8.7 we can find a hyperplane $D \subset \mathbb{P}^3$ such that $D \cap S$ is reduced and singular at some smooth point of S. Moreover, we may assume that D does not contain the point $\pi(o)$ and there is a line $L \subset \mathbb{P}^3$ passing through the point $\pi(o)$ such that

$$L \cap D \cap S \neq \varnothing$$

and L intersects the sextic S transversally at four different smooth points of S. Let $\tilde{D}$ be the surface in the linear system $|H|$ such that $\pi \circ f(\tilde{D}) = D$. Then $\tilde{D}$ is an irreducible multisection of the elliptic fibration $\phi_{|-K_V|}$ of degree 2 such that $\phi_{|-K_V|}|_{\tilde{D}}$ is branched at a point $q \in \tilde{D}$ contained in the fiber C of $\phi_{|-K_V|}$ such that $\pi \circ f(C) = L$. By construction, the fiber C is a smooth elliptic curve, $\pi \circ f(q) = L \cap D \cap S$, and q is a smooth point on $\tilde{D}$. Moreover, extending the field $\mathbb{F}$ we can assume that $\tilde{D}$ is defined over $\mathbb{F}$. Hence, the set of rational points is potentially dense on $\tilde{D}$ by Theorem 8.1. Theorem F is proven.

It would be natural to prove Theorem F in the case when the sextic S is singular and reduced (see Theorem 8.1). Most of the arguments in this section work for any reduced singular sextic surface. Actually, in the case when the sextic S has nonisolated singularities (for example, when it is reducible) we do not need to use Lemma 8.7 at all, but in the case when the sextic S is irreducible and has isolated singularities we can prove Lemma 8.7 using the finiteness of the Gauss map for curves (see [69]) in the assumption S is not a scroll (see [105], [145], and [146]), which is satisfied automatically if S is not a cone. Moreover, in general the proof of Theorem F must be simpler for bad singularities. For instance, in the case when the sextic S has a singular point of multiplicity 4, the double cover X is unirational and nonrational in general due to [138], but it is rational when S has a singular point of multiplicity 5. However, when S is a cone over a smooth sextic curve $R \subset \mathbb{P}^2$, the double cover X is birationally equivalent to $\mathbb{P}^1 \times D$, where D is a double cover of $\mathbb{P}^2$ ramified along R. The potential density of rational points on X is therefore equivalent to the potential density of rational points on D, which is still unknown in general (see [17]).

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