

## Bounded Three-Dimensional Fano Varieties of Integer Index

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**ABSTRACT.** The cube of the anticanonical class of a three-dimensional Fano variety with canonical singularities and integer Fano index is effectively bounded.

**KEY WORDS:** Fano variety, anticanonical class, canonical singularity, integer Fano index, ample Cartier divisor, Gorenstein variety, terminal singularity, Gorenstein singularity, log pair.

All varieties under consideration are complex and projective. The basic definitions and notation can be found in [1] and [2].

### §1. Introduction

The main purpose of this paper is to prove the following theorem.

**Theorem 1.** *Suppose that  $X$  is a three-dimensional Fano variety with canonical singularities, and  $-K_X \sim_{\mathbb{Q}} H$ , where  $H$  is an ample Cartier divisor. Then  $H^3 \leq 184/I$ , where  $I$  is the Gorenstein index of  $X$ .*

**Remark 1.** In Theorem 1,  $I$  equals 1 or 2 (see [3]). Considering the global canonical covering of  $X$  (see [4]), we can assume that  $I = 1$ , i.e., that  $X$  is Gorenstein. It is easy to see that  $-K_X \sim H$  in this case.

**Remark 2.** The bound for  $H^3$  given by Theorem 1 is apparently far from being perfect. If  $X$  is smooth, then  $H^3 \leq 64$  (see [2]), and the equality is attained for  $\mathbb{P}^3$ . If  $X$  has terminal Gorenstein singularities,  $X$  can be deformed into a smooth variety (see [5]), and hence  $H^3 \leq 64$ . If  $X$  has terminal singularities, a consideration of the canonical covering of  $X$  (see Remark 1) implies that  $H^3 \leq 64/I$ .

There exist examples of three-dimensional Fano varieties with canonical Gorenstein singularities for which  $H^3 = 72$ , such as the cone over an anticanonically embedded del Pezzo surface of degree 9 (see [2]).

For  $X$  with non-Gorenstein terminal quotient singularities, the classification implies that  $H^3 \leq 24$ ; the equality is attained for the quotient of  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  by an involution with a finite number of fixed points (see [4]).

In [6, 7], Fano examined three-dimensional varieties whose hyperplane sections are  $K3$  and Enriques (with canonical singularities) surfaces, respectively. Assuming the normality of such varieties, we can show that they coincide with the varieties satisfying the assumptions of Theorem 1 and the additional condition that the divisor  $H$  is very ample. If  $H$  is a  $K3$  surface, we have  $I = 1$ , and if it is an Enriques surface, then  $I = 2$  (see [3]). Interestingly, Fano conjectured that  $H^3 \leq 72$  if  $X$  has Gorenstein singularities and  $H^3 \leq 24$  otherwise.

### §2. $X$ with $\text{Bs}|H| \neq \emptyset$

**Theorem 2.** *If  $I = 1$  and  $\text{Bs}|H| \neq \emptyset$  in Theorem 1, then  $H^3 \leq 46$ .*

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**Proof.** The results of [8] imply that, if  $H^3 > 2$ ,  $\text{Bs}|H|$  is a smooth rational curve  $Z \in X \setminus \text{Sing}(X)$ . Let  $\pi: \bar{X} \rightarrow X$  be a blow-up of  $Z$ . Then the following diagram is commutative:

$$\begin{array}{ccc} & \bar{X} & \\ \pi \swarrow & & \searrow f \\ X & \xrightarrow{\varphi|_H} & S \end{array},$$

where the morphism  $f$  is a fibration into elliptic curves and  $\pi^{-1}(Z) = \bar{Z}$  is a section of  $f$ . Moreover,  $f|_{\bar{Z}}: \bar{Z} \rightarrow S$  is either an isomorphism or a contraction of the exceptional section of the ruled surface  $\bar{Z} \cong \mathbb{F}_n$  ( $n \in \mathbb{N}_0$ ).

Let  $\bar{E}$  and  $\bar{F}$  be the surfaces swept out by the fibers of the morphism  $f$  and passing through the fiber and the exceptional section of the ruled surface  $\bar{Z}$ , respectively. Then

$$K_{\bar{X}} = -\bar{F} - \frac{m+n}{2}\bar{E},$$

where  $m = (H^3 + 2)/2$  and  $m = n$  if  $S$  is singular.

If the surface  $S$  is nonsingular, then  $\text{Bs}|\bar{E}| = \emptyset$ . If the surface  $S$  is singular and  $\text{Bs}|\bar{E}| \neq \emptyset$ , then  $\dim(\text{Bs}|\bar{E}|) = 0$ , because

$$\text{Bs}|\bar{E}| \cap \bar{Z} = \text{Bs}|\bar{E}| \cap \pi^{-1}(H) = \emptyset$$

and  $H$  is ample. Note that  $\text{Bs}|\bar{E}| \neq \emptyset$  only if  $\bar{E}$  is not a  $\mathbb{Q}$ -Cartier divisor. In the latter case, let  $g: \bar{X}' \rightarrow \bar{X}$  be a small morphism such that  $g^{-1}(\bar{E})$  is a  $g$ -ample  $\mathbb{Q}$ -Cartier divisor (see [9]). Then  $\text{Bs}|g^{-1}(\bar{E})| = \emptyset$ . Indeed, if  $\text{Bs}|g^{-1}(\bar{E})| \neq \emptyset$ , then  $\text{Bs}|g^{-1}(\bar{E})| = \bar{E}'_1 \cap \bar{E}'_2$  for sufficiently general divisors  $\bar{E}'_1$  and  $\bar{E}'_2$  from the linear system  $|g^{-1}(\bar{E})|$ , and  $\text{Bs}|g^{-1}(\bar{E})|$  lies in fibers of the morphism  $f \circ g|_{\bar{E}'_1}: \bar{E}'_1 \rightarrow \mathbb{P}^1$ . The intersection form is seminegative definite in the fibers of the morphism  $f \circ g|_{\bar{E}'_1}$ , and hence,  $g^{-1}(\bar{E})^3 \leq 0$ ; but  $\bar{E}'_1 \cap \bar{E}'_2$  lies in fibers of the morphism  $g$ , and  $g^{-1}(\bar{E})^3 > 0$ , because the divisor  $g^{-1}(\bar{E})$  is  $g$ -ample. Therefore,  $\text{Bs}|g^{-1}(\bar{E})| = \emptyset$ . Now, replacing  $\bar{X}$  with  $\bar{X}'$  where necessary, we can assume that  $\text{Bs}|\bar{E}| = \emptyset$ .

The assertion of the theorem follows from the inequality  $(m+n)/2 \leq 12$ . Suppose that this inequality does not hold. Consider the log pair

$$\left( K_{\bar{X}}, \gamma \left( \bar{Z} + \bar{F} + \frac{m+n}{4}\bar{E}_1 + \frac{m+n}{4}\bar{E}_2 \right) \right), \quad (*)$$

where  $\gamma \in \mathbb{Q}_{>0}$  and  $\bar{E}_1$  and  $\bar{E}_2$  are two sufficiently general divisors from the linear system  $|\bar{E}|$ . It is well known that, if  $\gamma < 1$ , the set of log-canonical singularities (see [10]) of the log pair  $(*)$  is connected. Indeed, consider the log resolution  $\tau: Y \rightarrow \bar{X}$ . The following relation holds:

$$K_Y + \tau^*(\pi^*((1-\gamma)H)) \equiv \tau^*(K_{\bar{X}} + \pi^*(H)) - \sum_{i=1}^k a_i E_i + \sum_{j=1}^l b_j D_j,$$

where  $E_i$  and  $D_j$  are irreducible divisors on  $Y$ ,  $a_i$  and  $b_j$  are positive rational numbers, and the divisors  $D_j$  are exceptional for the morphism  $\tau$ . We must show that the set  $V = -\left[-\sum_{i=1}^k a_i E_i\right]$  is connected. Note that

$$\pi^*(H) = \bar{Z} + \bar{F} + \frac{m+n}{2}\bar{E}.$$

In a neighborhood of  $\bar{Z}$ ,  $\tau$  can be assumed to be an isomorphism, and  $V$  is either  $\emptyset$ , if

$$\gamma < \frac{4}{m+n},$$

or  $\{\overline{E}_1, \overline{E}_2\}$ , otherwise. By the Kawamata–Viehweg vanishing theorem (see [1]),

$$H^1\left(\tau^*(K_{\overline{X}} + \pi^*(H)) + \left[-\sum_{i=1}^k a_i E_i\right] + \left[\sum_{j=1}^l b_j D_j\right]\right) = 0.$$

This implies surjectivity:

$$H^0\left(\tau^*(\overline{Z}) + \left[\sum_{j=1}^l b_j D_j\right]\right) \rightarrow H^0\left(\left(\tau^*(\overline{Z}) + \left[\sum_{j=1}^l b_j D_j\right]\right)\Big|_V\right) \rightarrow 0.$$

Since the divisors  $D_j$  are  $\tau$ -exceptional and the divisor  $\overline{Z}$  is fixed, we have

$$H^0\left(\tau^*(\overline{Z}) + \left[\sum_{j=1}^l b_j D_j\right]\right) = \mathbb{C}.$$

Now let us show that, if  $(m+n)/2 > 12$  and  $\gamma = 1/(6+\varepsilon)$  ( $0 < \varepsilon \ll 1$ ), then the set of log-canonical singularities of the log pair  $(*)$  is disconnected. First, this set contains the divisors  $\overline{E}_1$  and  $\overline{E}_2$ . Secondly, the assertion is trivially true in a neighborhood of  $\overline{Z}$ . Therefore, it is sufficient to show that the set of log-canonical singularities of the log pair

$$\left(K_{\overline{X}}, \frac{1}{6+\varepsilon}\overline{F}\right) \quad (**)$$

contains no elements of codimension two that do not lie in fibers of the morphism  $f$ . It is easy to see that, in codimension two, the singularities of the pair  $(**)$  that do not lie in fibers of  $f$  coincide with the singularities of the log pair

$$\left(K_{\overline{E}_1}, \frac{1}{6+\varepsilon}\overline{F} \cap \overline{E}_1\right). \quad (***)$$

Note that  $\overline{F} \cap \overline{E}_1$  is a nonmultiple irreducible fiber of the elliptic fibration  $f|_{\overline{E}_1}: \overline{E}_1 \rightarrow \mathbb{P}^1$ , and the surface  $\overline{E}_1$  has canonical singularities, because the linear system  $|\overline{E}|$  can be assumed free. It is easy to calculate directly that the log pair  $(***)$  is log-terminal if  $\gamma < 1, 5/6, 3/4, 2/3, 1/2, 1/3, 1/4$ , or  $1/6$  and the preimage of  $\overline{F} \cap \overline{E}_1$  on the minimal resolution of the surface  $\overline{E}_1$  is a degenerate fiber of an elliptic surface of the type of stable fiber, *II*, *III*, *IV*,  $I_{b \geq 0}^*$ ,  $IV^*$ ,  $III^*$ , or  $II^*$ , respectively.  $\square$

**Remark 3.** If  $I = 1$  and  $X$  has terminal singularities in Theorem 1, then  $H^3 \leq 6$ . Indeed, this can be shown by setting  $\gamma = 5/(6+\varepsilon)$  ( $0 < \varepsilon \ll 1$ ) in the proof of Theorem 2.

**Remark 4.** If  $I = 1$  and  $H^3 \geq 48$  in Theorem 1, then there are the following possibilities:

- (1)  $H$  is very ample;
- (2) the linear system  $|H|$  determines a double covering of a variety of minimal degree ( $X$  is hyperelliptic).

This follows from Theorem 2 and the properties of movable linear systems on  $K3$  surfaces (see [2, 8]).

### §3. $X$ is hyperelliptic or trigonal

**Lemma 1** (M. Reid). *Consider*

$$V \cong \text{Proj}\left(\bigoplus_{i=1}^m \mathcal{O}_{\mathbb{P}^1}(d_i)\right) \quad \text{and} \quad Y_j \cong \text{Proj}\left(\bigoplus_{i=j}^m \mathcal{O}_{\mathbb{P}^1}(d_i)\right),$$

where  $d_1 \geq \dots \geq d_m$ ,  $d_1 > d_m$ , and  $m \geq j > 1$ . Let us identify  $Y_j$  with the subvariety in  $V$  determined by the natural projection  $\bigoplus_{i=1}^m \mathcal{O}_{\mathbb{P}^1}(d_i) \rightarrow \bigoplus_{i=j}^m \mathcal{O}_{\mathbb{P}^1}(d_i)$ . Suppose that  $s \in H^0(\mathcal{O}_{V/\mathbb{P}^1}(a) \otimes f^*(\mathcal{O}_{\mathbb{P}^1}(b)))$ , where  $f: V \rightarrow \mathbb{P}^1$  is the natural projection and  $a$  and  $b$  are integers. Then  $s$  has a zero on  $Y_j$  of order no smaller than  $q$  if and only if

$$ad_j + b + (d_1 - d_j)(q - 1) < 0.$$

**Proof.** See [2].  $\square$

**Lemma 2.** *If, in Theorem 1,  $I = 1$  and  $X$  is hyperelliptic (see Remark 4), then  $H^3 \leq 16$ .*

**Proof** (In the smooth case, see [2]). It is easy to see that  $\varphi_{|H|}(X)$  is a variety of “minimal degree;” taking into account the required inequality, we can assume that

$$\varphi_{|H|}(X) \cong \varphi_{|\mathcal{O}_{V/\mathbb{P}^1(1)}|}(V), \quad V \cong \text{Proj} \left( \bigoplus_{i=1}^3 \mathcal{O}_{\mathbb{P}^1}(d_i) \right), \quad d_1 \geq d_2 \geq d_3 \geq 0.$$

If  $d_1 = d_2 = d_3$ , then  $H^3 \leq 16$ . Therefore, we can assume that  $d_1 > d_3$ . If  $\varphi_{|H|}(X)$  is singular in codimension 2, then [11]  $H^3 \leq 8$ . Hence we can assume that  $d_2 \neq 0$ . Note that  $d_3 = 0$  if  $\varphi_{|H|}(X)$  is singular, and, in the notations of Lemma 1,  $\varphi_{|\mathcal{O}_{V/\mathbb{P}^1(1)}|}$  contracts the curve  $Y_3$ .

Let  $D$  be the proper preimage on  $V$  of the ramification divisor of the double covering  $\varphi_{|H|}$ . In the notations of Lemma 1, we have

$$D \sim \mathcal{O}_{V/\mathbb{P}^1}(4) - f^*(\mathcal{O}_{\mathbb{P}^1}(2(d_1 + d_2 + d_3 - 2))).$$

Considering the normalization of the fiber product  $X \times_{\varphi_{|H|}(X)} V$ , we readily see that  $D$  contains  $Y_3$  with multiplicity no larger than 2. Since  $D$  is reduced, it contains  $Y_2$  with multiplicity no larger than 1. Lemma 1 implies the inequalities

$$d_2 - d_1 - 2d_3 + 4 \geq 0, \quad 4 - 2d_2 \geq 0,$$

whence  $H^3 = 2(d_1 + d_2 + d_3) \leq 16$ .  $\square$

**Lemma 3.** *Suppose that, in Theorem 1,  $I = 1$  and  $X$  is trigonal, i.e.,  $H$  is very ample, and a general element of the family  $\{H^2\}$  of curves is an irreducible smooth trigonal curve. Then  $H^3 \leq 54$ .*

**Proof.** Let us identify  $X$  with its anticanonical image. As in the case of a smooth  $X$  (see [2]), we can show that the intersection of the quadrics in  $\mathbb{P}^{H^3/2+2}$  that contain  $X$  is  $W \cong \varphi_{|\mathcal{O}_{V/\mathbb{P}^1(1)}|}(V)$ , where

$$V \cong \text{Proj} \left( \bigoplus_{i=1}^4 \mathcal{O}_{\mathbb{P}^1}(d_i) \right), \quad d_1 \geq d_2 \geq d_3 \geq d_4 \geq 0.$$

If  $d_1 = d_2 = d_3 = d_4$ , then  $H^3 \leq 18$ . Therefore, we can assume that  $d_1 > d_4$ . Suppose that  $Z = \text{Sing}(W) \cong \mathbb{P}^k$  ( $k = 0, 1, 2$ ); then  $W$  is a cone with vertex  $Z$ . As in the smooth case,  $Z \cap X \subset \text{Sing}(X)$ . Note that, if  $\dim(Z) = 2$ , then  $\dim(Z \cap X) = 1$ ,  $d_2 = d_3 = d_4 = 0$ , and, in the notations of Lemma 1,  $\varphi_{|\mathcal{O}_{V/\mathbb{P}^1(1)}|}$  contracts  $Y_2$  onto  $Z$  and is a blow-up at a generic point of  $Z$ .

Suppose that  $\overline{X} = \varphi_{|\mathcal{O}_{V/\mathbb{P}^1(1)}|}^{-1}(X)$ . The subadjunction formula (see [1]) and the smoothness of  $V$  imply that  $\overline{X}$  has canonical singularities; in the notations of Lemma 1, we have

$$\overline{X} \sim \mathcal{O}_{V/\mathbb{P}^1}(3) + f^*(\mathcal{O}_{\mathbb{P}^1}(2 - d_1 - d_2 - d_3 - d_4)).$$

Since  $\overline{X}$  is irreducible,  $\overline{X}$  does not contain  $Y_2$ ; it contains  $Y_3$  with multiplicity no larger than 1 and  $Y_4$  with multiplicity no larger than 2. Lemma 1 implies the inequalities

$$2d_2 - d_1 - d_3 - d_4 + 2 \geq 0, \quad d_3 - d_2 + 2 \geq 0, \quad 2 - d_2 - d_3 + d_1 \geq 0,$$

whence  $H^3 = 2 + 2(d_1 + d_2 + d_3 + d_4) \leq 54$ .  $\square$

#### §4. $X$ is swept out by “straight lines”

**Lemma 4.** *Suppose that  $x$  is a closed smooth point of a  $k$ -dimensional variety  $X$  and  $H$  is an ample Cartier divisor on  $X$ . If a generic point  $v \in X$  is connected with  $x$  by an irreducible curve  $C_v$  such that  $HC_v \leq d$ , then  $H^k \leq d^k$ .*

**Proof.** We have

$$h^0(\mathcal{O}_X(mH)) = m^k H^k / k! + O(m^{k-1})$$

for  $m \in \mathbb{N}_{\gg 0}$ . If  $H^k > d^k$ , then, for  $m \gg 0$ , there exists a divisor  $D \in H^0(\mathcal{O}_X(mH))$  of multiplicity no smaller than  $(md + 1)$  at the point  $x$ . Therefore,  $D$  contains all curves  $C_v$ , which cannot be.  $\square$

**Lemma 5.** *If, in Theorem 1,  $I = 1$  and there exists an irreducible curve  $C_v$  passing through a generic point  $v \in X$  and such that  $HC_v = 1$ , then  $H^3 \leq 46$ .*

**Proof.** If  $\text{Bs}|H| \neq \emptyset$ , then Theorem 2 implies that  $H^3 \leq 46$ . Suppose that  $\text{Bs}|H| = \emptyset$  and consider the family  $\{C\}$  of irreducible reduced curves such that  $HC = 1$  and  $\{C\}$  contains a curve passing through a generic point of  $X$ . Remark 4 implies that  $C \cong \mathbb{P}^1$ .

Consider the RC-fibration (see [12, 13])  $\varphi: X \dashrightarrow W$  associated with  $\{C\}$ .

If  $\dim(W) = 0$ , then two generic points of  $X$  can be connected by a chain of no more than three curves from  $\{C\}$ ; we can glue together these curves and obtain a new family  $\{C'\}$  such that two generic points of  $X$  can be connected by one curve from the family  $\{C'\}$  (see [13]). The application of Lemma 4 to  $\{C'\}$  yields the inequality  $H^3 \leq 26$  ( $H^3$  is even).

Consider the commutative diagram

$$\begin{array}{ccc} & \tilde{X} & \\ \pi \swarrow & & \searrow \tilde{\varphi} \\ X & \xrightarrow{\varphi} & W \end{array}$$

where  $\tilde{X}$  is a smooth variety,  $\pi$  is birational, and  $\tilde{\varphi}$  is a morphism.

Suppose that  $\dim(W) = 1$ . Consider a generic fiber  $S$  of the morphism  $\tilde{\varphi}$  and a generic curve  $C$  from  $\{C\}$ . The family  $\{\pi^{-1}(C)\}$  of curves determines a family  $\{C_S\} \in S$  of curves on  $S$ . The generic curve  $C_S$  from  $\{C_S\}$  is irreducible, smooth, and rational; we have  $-K_S C_S \leq 1$  and  $C_S^2 > 0$ , which contradicts the adjunction formula.

If  $\dim(W) = 2$ , we consider a generic curve  $C$  from  $\{C\}$ ;  $\pi^{-1}(C)$  is a generic fiber of the morphism  $\tilde{\varphi}$  and  $-K_{\tilde{X}} \pi^{-1}(C) \leq 1$ , which contradicts the relation  $-K_{\tilde{X}} \pi^{-1}(C) = 2$ , because the generic fiber of  $\tilde{\varphi}$  is  $\mathbb{P}^1$ .  $\square$

#### §5. Double projection from a generic point of $X$

**Lemma 6.** *Suppose that, in Theorem 1,  $I = 1$ ,  $H^3 \geq 56$ , and  $\pi: \hat{X} \rightarrow X$  is a blow-up of a sufficiently generic point  $v \in X$  with exceptional divisor  $E$ . Then the divisor  $\pi^*(H) - 2E$  is numerically effective and volume.*

**Proof.** It is sufficient to prove that  $\pi^*(H) - 2E$  is numerically effective. Remark 4 and Lemma 2 imply that  $H$  is very ample. Let us identify  $X$  with its anticanonical image. If  $(\pi^*(H) - 2E)C < 0$  for a curve  $C \in \hat{X}$ , then  $v \in \pi(C) \in T_v(X) \cap X$ , where  $T_v(X)$  is the tangent space to  $X$  at the point  $v$ . Lemma 3, the Noether–Enriques–Peetre Theorem (see [14]), and the results of [2] imply that  $X$  is cut out by quadrics. Therefore,  $C$  is a straight line on  $X$  passing through the point  $v$ ; by Lemma 5, this cannot be.  $\square$

**Proof of Theorem 1.** According to Remark 1, it is sufficient to prove Theorem 1 under the condition  $I = 1$ . Let  $\pi: \tilde{X} \rightarrow X$  be a blow-up of a generic point  $v \in X$ . If  $H^3 \geq 56$ , then, by Lemma 6, the divisor  $K_{\tilde{X}} \sim \pi^*(H) - 2E$  is numerically effective and volume. By the base point free theorem (see [1]), there

exists  $N \in \mathbb{N}$  such that, for  $n \in \mathbb{N}_{\geq N}$ , the linear system  $|n(\pi^*(H) - 2E)|$  is base-point-free. Therefore, for  $n \gg 0$ , there exists a morphism  $\varphi: \tilde{X} \rightarrow X'$  such that  $X'$  has canonical singularities,

$$\pi^*(H) - 2E = \varphi^*(H'), \quad -K_{X'} \sim H', \quad \text{and} \quad H'^3 = H^3 - 8,$$

where  $H'$  is an ample Cartier divisor.

Under the assumptions of Theorem 1, this surgery can be repeated 17 times. Thus we can assume that  $\pi$  is a blow-up of 17 points of  $X$  in general position.

The general divisor  $\tilde{D} \in |-K_{\tilde{X}}|$  is reduced and irreducible, it has only canonical singularities, and it is a  $K3$  surface (see [15]). Since the divisor  $-K_{\tilde{X}}$  is  $\pi$ -ample, the morphism  $\pi$  determines on  $\tilde{D}$  a contraction of 17 pairwise disjoint curves; according to [16], this cannot be.  $\square$

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