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Sporadic simple groups and quotient singularities

I. A. Cheltsov and C. A. Shramov

Abstract. We show that if a faithful irreducible representation of a central extension of a sporadic simple group with centre contained in the commutator subgroup gives rise to an exceptional (resp. weakly exceptional but not exceptional) quotient singularity, then that simple group is the Hall–Janko group (resp. the Suzuki group).

Keywords: weakly exceptional singularities, log canonical threshold, sporadic simple groups.

To Igor Rostislavovich Shafarevich with deep respect

§ 1. Introduction

Finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ were classified more than a hundred years ago. The quotients of \mathbb{C}^2 by these groups are \mathbb{A} – \mathbb{D} – \mathbb{E} singularities (also known as Kleinian singularities, Du Val singularities, rational double points of surfaces, two-dimensional canonical singularities and so on). Shokurov suggested a higher-dimensional generalization of the singularities of type \mathbb{E} , and of both types \mathbb{D} and \mathbb{E} . He called them exceptional and weakly exceptional respectively. Precise definitions of exceptional and weakly exceptional singularities are rather technical (see [1], Definition 1.5 and [2], Definition 4.1 respectively). Surprisingly, they are connected with a wide range of algebraic and geometric questions.

It has turned out that exceptional and weakly exceptional singularities are related to the Calabi problem for orbifolds with positive first Chern class (see [3]).

Example 1.1. Let $(V \ni O)$ be a germ of three-dimensional isolated quasi-homogeneous hypersurface singularity given by the equation

$$\varphi(x, y, z, t) = 0 \subset \mathbb{C}^4 \cong \mathrm{Spec}(\mathbb{C}[x, y, z, t]),$$

where $\varphi(x, y, z, t)$ is a quasi-homogeneous polynomial of degree d in the weights $\mathrm{wt}(x) = a_0$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_2$, $\mathrm{wt}(t) = a_3$ which are such that $a_0 \leq a_1 \leq a_2 \leq a_3$ and $\mathrm{gcd}(a_0, a_1, a_2, a_3) = 1$. Let S be a weighted hypersurface of degree d in $\mathbb{P}(a_0, a_1, a_2, a_3)$ given by the same equation $\varphi(x, y, z, t) = 0$. We also suppose that $\sum_{i=0}^n a_i > d$ and the hypersurface S is well formed (see [4], Definition 6.9). Then S is a Del Pezzo surface with at most quotient singularities. Moreover, if the

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singularity $(V \ni O)$ is exceptional or weakly exceptional, then S admits an orbifold Kähler–Einstein metric (this follows, for example, from [5], [2], Theorem 4.9, [6], Theorem 2.1, [7], Theorem A.3).

Many old and still open group-theoretic questions have algebro-geometric counterparts related to the exceptionality of quotient singularities (see, for example, [8] and [3], Conjecture 1.25). It seems that the study of exceptional and weakly exceptional quotient singularities may shed new light on some group-theoretic problems.

Example 1.2. Let G and G' be finite subgroups of $\mathrm{GL}_{n+1}(\mathbb{C})$ containing no reflections,¹ and suppose that the projections of G' and G in $\mathrm{PGL}_{n+1}(\mathbb{C})$ coincide. Then it follows from [3], Theorems 3.15, 3.16, that the exceptionality (resp. weak exceptionality) of the singularity \mathbb{C}^{n+1}/G is equivalent to that of \mathbb{C}^{n+1}/G' . Moreover, it follows from [3], Theorems 1.30, 3.15, that if the singularity \mathbb{C}^{n+1}/G is weakly exceptional, then the subgroup $G \subset \mathrm{GL}_{n+1}(\mathbb{C})$ is transitive (that is, the corresponding $(n+1)$ -dimensional representation is irreducible). In a similar vein, it follows from [3], Theorem 1.29, that if \mathbb{C}^{n+1}/G is exceptional, then G must be primitive (see [9] or, for example, [3], Definition 1.21). Finally, it follows from [3], Theorem 3.16 (resp. [3], Theorem 3.15) that \mathbb{C}^{n+1}/G is not exceptional if G has a semi-invariant² of degree at most $n+1$ (resp. at most n).

In what follows we consider only quotient singularities.

The study of exceptional and weakly exceptional quotient singularities in low dimensions is closely related to the classification of finite collineation groups (see [9]–[14]). Classical results of Blichfeldt, Brauer and Lindsey enabled Markushevich, Prokhorov and the authors to classify exceptional quotient singularities in dimensions 3–6 (see [3], [15], [16]). Moreover, we used a classification obtained by Wales (see [12] and [13]) to prove that there are no seven-dimensional exceptional quotient singularities (see [16]). Sakovics classified weakly exceptional quotient singularities in dimensions 3 and 4 (see [17]). Higher-dimensional weakly exceptional quotient singularities were studied in [18]. Unfortunately, we still have no clear picture of those finite subgroups of $\mathrm{GL}_{n+1}(\mathbb{C})$ that give rise to exceptional or weakly exceptional singularities for $n \gg 0$.

A surprising fact observed in [16] is that among the (very few) groups corresponding to exceptional six-dimensional quotient singularities there appears a central extension $2.J_2$ of the Hall–Janko sporadic simple group (see [19]). This property is actually very rare for the projective representations of sporadic simple groups: we have only one other example of such behaviour among them. It is related to the Suzuki sporadic simple group (see [20]).

In this paper we prove the following result.

Theorem 1.3 (compare [21], Theorem 14). *Let G be a sporadic simple group or a central extension of one with centre contained in the commutator subgroup.³ Let $G \hookrightarrow \mathrm{GL}(U)$ be a faithful finite-dimensional complex representation of G . Then the*

¹We recall that an element $g \in \mathrm{GL}_{n+1}(\mathbb{C})$ is called a *reflection* (or sometimes a *quasi-reflection*) if it has exactly one eigenvalue different from 1.

²We recall that a *semi-invariant* of degree d of a group $G \subset \mathrm{GL}_{n+1}(\mathbb{C})$ is a one-dimensional subrepresentation in $\mathrm{Sym}^d(\mathbb{C}^{n+1})$.

³We recall that for a perfect group G , every such extension is a quotient of a universal central extension of G by the Schur multiplier (see, for example, [22], § 9.4).

singularity U/G is exceptional if and only if $G \cong 2.J_2$ and U is a 6-dimensional irreducible representation of G . The singularity U/G is weakly exceptional but not exceptional if and only if $G \cong 6.Suz$ is the universal perfect central extension of the Suzuki simple group and U is a 12-dimensional irreducible representation of G .

Theorem 1.3 shows that among the sporadic simple groups, the groups J_2 and Suz are somehow distinguished from a geometric point of view. This motivates the following question.

Question 1.4. Is there a group-theoretic property that distinguishes the groups J_2 and Suz among the sporadic simple groups?

One can see from the appendix (§ 3) that the groups $2.J_2$ and $6.Suz$ are characterized by the property of having irreducible representations without semi-invariants of low degrees. But this property is not equivalent to the weak exceptionality of the corresponding quotient singularity: a geometric characterization via weak exceptionality requires a further series of coincidences. On the other hand, it would be interesting to know whether there is an intrinsic characterization of the groups J_2 and Suz that goes beyond an observation about semi-invariants and, possibly, involves no representation theory at all. One of the goals of this paper is to attract the attention of experts in group theory to Question 1.4 and, more generally, to a broader range of questions on the possible interplay between properties of groups and the geometric properties of the corresponding quotient singularities.⁴

When studying the exceptionality or weak exceptionality of a singularity \mathbb{C}^{n+1}/G for a finite subgroup $G \subset GL_{n+1}(\mathbb{C})$, one can always assume that G does not contain reflections (compare Example 1.2 and [16], Remark 1.16). Example 1.2 also shows that it suffices to prove Theorem 1.3 in the case of irreducible representations. Moreover, Example 1.2 enables one to exclude all groups that have semi-invariants of low degree for the corresponding representations. This is done by a straightforward case-by-case study. The results of the corresponding computations are listed in § 3. They were obtained by A. Zavaritsyn using the GAP software (see [25]) and the classification of finite simple groups (see [26]). Hence it remains to consider only two cases: the group $2.J_2$ acting on $U \cong \mathbb{C}^6$ and the group $6.Suz$ acting on $U \cong \mathbb{C}^{12}$. The exceptionality of the quotient singularity corresponding to the first case was proved in [16]. Therefore the only new geometric result in this paper is the following theorem that is proved in § 2.

Theorem 1.5. *Suppose that $G \cong 6.Suz$ and U is an irreducible 12-dimensional representation of G . Then the singularity U/G is weakly exceptional but not exceptional.*

§ 2. The Suzuki group

Here we prove Theorem 1.5 using methods that first appeared in [21] and the following assertion.

⁴There is an interesting characterization of the groups $2.J_2$ and $6.Suz$, which simultaneously distinguishes the representations arising in Theorem 1.3, in terms of the irreducibility of symmetric powers. It was obtained in [23], Theorem 1.1, and applied to stable vector bundles in [23], Corollary 1.3 and [24].

Theorem 2.1 ([18], Theorem 1.12). *Let G be a finite subgroup of $\mathrm{GL}_{n+1}(\mathbb{C})$ containing no reflections, and let \overline{G} be the image of G in $\mathrm{PGL}_{n+1}(\mathbb{C})$. If the singularity \mathbb{C}^{n+1}/G is not weakly exceptional, then there is a \overline{G} -invariant projectively normal subvariety $V \subset \mathbb{P}^n$ of Fano type⁵ such that*

$$\deg(V) \leq \binom{n}{\dim(V)},$$

we have $h^i(\mathcal{O}_{\mathbb{P}^n}(m) \otimes \mathcal{I}_V) = h^i(\mathcal{O}_V(m)) = 0$ for all $i \geq 1$ and $m \geq 0$, and

$$h^0(\mathcal{O}_{\mathbb{P}^n}(\dim(V) + 1) \otimes \mathcal{I}_V) \geq \binom{n}{\dim(V) + 1},$$

where \mathcal{I}_V is the ideal sheaf of the subvariety $V \subset \mathbb{P}^n$. Let Π be a general linear subspace of codimension $k \leq \dim(V)$ in \mathbb{P}^n . We put $X = V \cap \Pi$. Then

$$h^i(\mathcal{O}_{\Pi}(m) \otimes \mathcal{I}_X) = 0$$

for all $i \geq 1$ and $m \geq k$, where \mathcal{I}_X is the ideal sheaf of the subvariety $X \subset \Pi$. Moreover, if $k = 1$ and $\dim(V) \geq 2$, then X is irreducible and projectively normal and we have $h^i(\mathcal{O}_X(m)) = 0$ for all $i \geq 1$ and $m \geq 1$.

Let $G \cong 6.\mathrm{Suz}$ be the central extension of the Suzuki sporadic simple group. Then there is an embedding $G \hookrightarrow \mathrm{SL}_{12}(\mathbb{C})$ corresponding to an irreducible 12-dimensional representation U of G .

Let Δ_k be the tuple of dimensions of the irreducible subrepresentations of $\mathrm{Sym}^k(U^\vee)$. We shall use the following notation. We write $\Delta_k = [\dots, r \times m, \dots]$ to mean that the decomposition of $\mathrm{Sym}^k(U^\vee)$ into a sum of irreducible subrepresentations contains exactly r summands of dimension m (not necessarily isomorphic to each other). Furthermore, we write Σ_k for the set of partial sums of Δ_k , that is, the set of numbers of the form $s = \sum_i r'_i m_i$, where $\Delta_k = [r_1 \times m_1, r_2 \times m_2, \dots, r_i \times m_i, \dots]$ and $0 \leq r'_i \leq r_i$ for all i . We shall also abbreviate $1 \times m_i$ to m_i .

The following properties of the representation U can be verified by direct computation. We used the GAP software (see [25]) to carry them out.

Lemma 2.2. *The representations $\mathrm{Sym}^k(U^\vee)$ are irreducible for $2 \leq k \leq 5$ (and have dimensions 78, 364, 1365 and 4368 respectively). Furthermore,*

$$\Delta_6 = [364, 12012], \quad \Delta_7 = [4368, 27456],$$

$$\Delta_8 = [1365, 4290, 27027, 42900], \quad \Delta_9 = [2 \times 364, 2 \times 16016, 35100, 100100],$$

$$\Delta_{10} = [78, 1365, 3003, 4290, 2 \times 27027, 2 \times 75075, 139776],$$

$$\Delta_{11} = [12, 924, 2 \times 4368, 2 \times 12012, 2 \times 27456, 112320, 144144, 2 \times 180180],$$

$$\Delta_{12} = [1, 143, 2 \times 364, 1001, 2 \times 5940, 2 \times 12012, 2 \times 14300, 2 \times 15015, 15795, 25025, \\ 2 \times 40040, 54054, 75075, 88452, 2 \times 93555, 2 \times 100100, 163800, 168960, 197120].$$

⁵A variety V is said to be of *Fano type* if it is irreducible, normal, and there is an effective \mathbb{Q} -divisor Δ_V on V such that $-(K_V + \Delta_V)$ is an ample \mathbb{Q} -Cartier divisor and the log pair (V, Δ_V) has at most Kawamata log terminal singularities.

Corollary 2.3. *The group G has no semi-invariants of degree $d \leq 11$ but does have a semi-invariant of degree $d = 12$.*

Proof of Theorem 1.5. Suppose that the singularity U/G is not weakly exceptional. Let \overline{G} be the image of G in $\mathrm{PGL}_{12}(\mathbb{C})$. By Theorem 2.1 there is a \overline{G} -invariant projectively normal subvariety $V \subset \mathbb{P}^{11}$ of Fano type such that

$$\deg(V) \leq \binom{11}{\dim(V)}$$

and we have $h^i(\mathcal{O}_{\mathbb{P}^{11}}(m) \otimes \mathcal{I}_V) = h^i(\mathcal{O}_V(m)) = 0$ for all $i \geq 1$ and $m \geq 0$, where \mathcal{I}_V is the ideal sheaf of the subvariety $V \subset \mathbb{P}^{11}$. Put $n = \dim(V)$.

Lemma 2.4. *We have $1 \leq n \leq 9$.*

Proof. We have $n \neq 0$ since U is an irreducible representation of G . On the other hand, if $n = 10$, then V is a \overline{G} -invariant hypersurface with $\deg(V) \leq 11$, which contradicts Corollary 2.3. \square

We put $h_m = h^0(\mathcal{O}_V(m))$ and $q_m = h^0(\mathcal{O}_{\mathbb{P}^{11}}(m) \otimes \mathcal{I}_V)$ for all $m \in \mathbb{Z}$. Then

$$q_m = h^0(\mathcal{O}_{\mathbb{P}^{11}}(m)) - h_m = \binom{11+m}{m} - h_m$$

for all $m \geq 1$ since $h^1(\mathcal{O}_{\mathbb{P}^{11}}(m) \otimes \mathcal{I}_V) = 0$ for all $m \geq 0$.

Let H be a general hyperplane section of V . We put $d = H^n = \deg(V)$ and $H_V(m) = \chi(\mathcal{O}_V(m))$. Then $H_V(m) = h_m$ for all $m \geq 1$ since $h^i(\mathcal{O}_V(mH)) = 0$ for all $i \geq 1$ and $m \geq 0$. We recall that $H_V(m)$ is the Hilbert polynomial of V . This is a polynomial of degree n in m with leading coefficient $d/n!$.

Theorem 2.1 yields another useful property of V . Let $\Lambda_1, \Lambda_2, \dots, \Lambda_n$ be general hyperplanes in \mathbb{P}^{11} . We put

$$\Pi_j = \Lambda_1 \cap \dots \cap \Lambda_j, \quad V_j = V \cap \Pi_j, \quad H_j = V_j \cap H$$

for all $j \in \{1, \dots, n\}$ and also define $V_0 = V$, $H_0 = H$, $\Pi_0 = \mathbb{P}^{11}$. For every $j \in \{0, 1, \dots, n\}$ let \mathcal{I}_{V_j} be the ideal sheaf of the subvariety $V_j \subset \Pi_j$. Then it follows from Theorem 2.1 that $h^i(\mathcal{O}_{\Pi_j}(m) \otimes \mathcal{I}_{V_j}) = 0$ for all $i \geq 1$ and $m \geq j$.

We recall that $\Pi_j \cong \mathbb{P}^{11-j}$ and put $q_i(V_j) = h^0(\mathcal{O}_{\Pi_j}(i) \otimes \mathcal{I}_{V_j})$ for all $j \in \{0, 1, \dots, n\}$.

Lemma 2.5. *Suppose that $i \geq j + 1$ and $j \in \{1, \dots, n\}$. Then*

$$q_i(V_j) = q_i - \binom{j}{1} q_{i-1} + \binom{j}{2} q_{i-2} - \dots + (-1)^j q_{i-j}.$$

Proof. See the proof of Lemma 27 in [21]. \square

Since the representation U is irreducible, we have $q_1 = 0$. Moreover, $q_i = 0$ for $2 \leq i \leq 5$ by Lemma 2.2. Thus we get the following corollary.

Corollary 2.6. *If $n = 9$, then $q_9 - 8q_8 + 28q_7 - 56q_6 = q_9(V_8) \geq 0$.*

It is also easy to obtain the following assertion.

Lemma 2.7 (compare [21], Lemma 35). *We have*

$$\binom{12}{n} - \frac{(n+1)d}{2} > q_n(V_{n-1}) \geq \binom{12}{n} - nd - 1.$$

Proof. The subvariety $V_{n-1} \subset \Pi_{n-1} \cong \mathbb{P}^{12-n}$ is a smooth curve of degree d because V is normal. Moreover, V_{n-1} is irreducible since V is irreducible. We denote the genus of the curve V_{n-1} by g . It follows from the adjunction formula that

$$2g - 2 = (K_V + (n-1)) \cdot H^{n-1} = K_V \cdot H^{n-1} + (n-1)d < (n-1)d$$

since $K_V \cdot H^{n-1} < 0$ (this inequality holds because $-K_V$ is big). On the other hand, for all $m \geq n$ we have

$$q_m(V_{n-1}) = \binom{11-n+1+m}{m} - h^0(\mathcal{O}_{V_{n-1}}(mH_{n-1}))$$

because $h^1(\mathcal{O}_{\Pi_{n-1}}(m) \otimes \mathcal{I}_{V_{n-1}}) = 0$ for all $m \geq n-1$. Since $2g-2 < nd$, the divisor nH_{n-1} is non-special. Using the Riemann–Roch theorem, we get

$$q_n(V_{n-1}) = \binom{12}{n} - nd + g - 1.$$

This yields the desired inequalities since $2g-2 < (n-1)d$ and $g \geq 0$. \square

Combining Lemma 2.7 and Corollary 2.6, we get the following corollary.

Corollary 2.8. *If $n = 9$, then*

$$\max(0, 219 - 9d) \leq q_9 - 8q_8 + 28q_7 - 56q_6 < 220 - 5d.$$

Here is a by-product of Corollary 2.8.

Corollary 2.9. *If $n = 9$, then $1 \leq d \leq 43$.*

The resulting restrictions reduce the proof of the theorem to the combinatorial question of finding all polynomials H_V of degree n with leading coefficient $d/n!$ such that $h_m = H_V(m)$ lies in Σ_m for sufficiently many values of $m \geq 1$ and such that the numbers $h_m, q_m = h^0(\mathcal{O}_{\mathbb{P}^{11}}(m)) - h_m$ and d satisfy the conditions imposed by Corollaries 2.8, 2.9. This can be done in a straightforward way, although the number of cases to be considered is so large that it requires some checks to be done by a computer. This yields the following results, which will be stated without proof.

Lemma 2.10. *There are no polynomials $H(m)$ of degree $n \leq 8$ such that the values $h_m = H(m)$ lie in Σ_m for $1 \leq m \leq 12$.*

Lemma 2.11. *There are no polynomials $H(m)$ of degree $n = 9$ with leading coefficient $d/n!$, where $d \in \mathbb{Z}$ and $1 \leq d \leq 43$, such that the values $h_m = H(m)$ lie in Σ_m for $1 \leq m \leq 12$ and the numbers h_m and $q_m = \binom{11+m}{m} - h_m$ satisfy the restrictions imposed in Corollary 2.8.*

This completes the proof of Theorem 1.5. \square

§ 3. Appendix. Semi-invariants of low degrees

In this section we list the results (communicated to us by A. Zavaritsyn) of the computation of low-degree semi-invariants for irreducible representations of the groups considered above. The computation was done using the GAP software (see [25]).

The following tables contain information on those representations (of central extensions of sporadic groups with centre contained in the commutator subgroup) that provide the least possible value of the ratio $\mu(U) = d(U)/\dim(U)$ among all irreducible representations U of the corresponding group G , where $d(U)$ is the minimal degree of a semi-invariant⁶ of G for the representation U . In each case we list the values of $d(U)$ and $\dim(U)$ at which the ratio $\mu(U)$ attains its minimum (except for the groups $2.J_2$ and $6.Suz$, where we give other information; see below). It turns out that for each of our groups there is a unique value of $\dim(U)$ (and hence of $d(U)$) at which $\mu(U)$ attains its minimum.

Here is a table with the information about the Mathieu groups.

G	M_{11}	M_{12}	$2.M_{12}$	M_{22}	$2.M_{22}$	$3.M_{22}$	$4.M_{22}$	$6.M_{22}$	$12.M_{22}$	M_{23}	M_{24}
$d(U)$	4	3	6	2	4	3	4	6	12	2	4
$\dim(U)$	10	16	10	21	10	21	56	66	120	22	45

The following table contains the information about the Conway groups.

G	Co_1	$2.Co_1$	Co_2	Co_3
$d(U)$	2	2	2	2
$\dim(U)$	276	24	23	23

The following table contains the information on groups related to the Leech lattice, excluding the Conway groups (we omit the calculation of d and n for the 6-dimensional representations of $2.J_2$ and the 12-dimensional representations of $6.Suz$ since these representations yield weakly exceptional singularities and their values of d and n satisfy $d/n \geq 1$).

G	HS	2.HS	J_2	$2.J_2$	McL	3.McL	Suz	2.Suz	3.Suz	6.Suz
$d(U)$	2	2	2	4	2	3	2	4	6	6
$\dim(U)$	22	56	14	14	22	126	143	220	78	780

Here is a table with the information about the Fischer groups.

G	Fi_{22}	$2.Fi_{22}$	$3.Fi_{22}$	$6.Fi_{22}$	Fi_{23}	Fi'_{24}	$3.Fi'_{24}$
$d(U)$	2	2	3	6	2	2	3
$\dim(U)$	78	352	351	1728	782	8671	783

⁶Actually, the semi-invariants for irreducible representations of such groups are easily seen to be invariants.

The following table contains the information on the other sections of the Monster.

G	He	HN	Th	B	2.B	M
$d(U)$	3	2	2	2	2	2
$\dim(U)$	51	133	248	4371	96256	196883

The last table contains the information about the Tits groups and the pariahs.

G	T	J ₁	O'N	3.O'N	J ₃	3.J ₃	Ru	2.Ru	J ₄	Ly
$d(U)$	6	2	4	6	3	6	4	4	4	6
$\dim(U)$	26	56	13376	342	85	18	378	28	1333	2480

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