Journal of Mathematical Sciences, Vol. 102, No. 2, 2000

LOG MODELS OF BIRATIONALLY RIGID VARIETIES

I. Cheltsov

UDC 512.774, 512.776, 512.76, 512.724

All varieties are assumed to be projective and to be defined over $\mathbb{C}$. The main definitions, notation, and notions are contained in [11].

The author thanks Professors V. A. Iskovskikh, A. V. Pukhlikov, and V. V. Shokurov for their useful conversations and helpful comments.

1. Introduction

The papers [5, 7–10, 15–17] are dedicated to algebraic varieties with a birational structure, now called birationally rigid varieties. Groups of birational automorphisms of such varieties were analyzed using the method of "untwisting maximal singularities" in these papers.

We note that the concept of birational rigidity can be considered in two ways: as in [7] or, more generally, as in [15]. From our standpoint, birational rigidity should mean an effective solution of the Fano–Iskovskikh problem (see below).

The results in the above-mentioned articles were generalized from the standpoint of the minimal model program (MMP) in [3]; they lead to the following problem.

Fano–Iskovskikh Problem. *Find all Mori fibrations birationally isomorphic to a given variety* X. (We assume that all fibrations have connected fibers and are nonbirational.)

In this paper, we describe methods that allow us to consider the Fano–Iskovskikh problem more naturally and to solve a more general problem. As a corollary, we find all fibrations with Kodaira dimension zero and all Fano varieties with arbitrary canonical singularities that are birationally isomorphic to such threefolds as the double cover of $\mathbb{P}^3$, the quartic threfold, the double cover of a quadric, and "very ramified" conic bundles.

We briefly describe the structure of this paper. In Sec. 2, we introduce the main objects of the paper (movable log pairs) and describe their general properties. In Secs. 3 and 4, we discuss the global and local properties of movable log pairs. In Secs. 5 and 6, we apply the results of Sec. 3 to Del Pezzo surfaces and two-dimensional conic bundles. In Secs. 7–10, we apply the results of Secs. 3 and 4 to the above-mentioned threefolds.

2. Movable Log Pairs

Definition 2.1. The movable log pair

$$(X, \mathcal{M}_X) = \left(X, \sum_{i=1}^{N} b_i \mathcal{M}_i\right) \tag{1}$$

is a variety X together with a formal finite linear combination of linear systems $\mathcal{M}_i$ without fixed components such that all b_i are nonnegative rational numbers.

We call movable log pairs simply log pairs. If necessary, we use the log canonical divisor $K_X + \mathcal{M}_X$ and the boundary $\mathcal{M}_X$ as divisors. (We do not distinguish between divisors and $\mathbb{Q}$-divisors.) We can define terminal, canonical, log terminal, and log canonical singularities for log pair (1) as for normal log pairs.

Definition 2.2. An irreducible subvariety $Y \subset X$ is a center of canonical singularities of log pair (1) if there is a birational morphism $f : W \to X$ and an f-exceptional divisor $E \subset W$ such that

$$a(X, \mathcal{M}_X, E) \leq 0 \qquad \text{and} \qquad f(E) = Y.$$

Definition 2.3. The set of all centers of canonical singularities of log pair (1) is denoted by $CS(X, \mathcal{M}_X)$.

Translated from Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory. Vol. 62, Algebraic Geometry–10, 1999.

Example 2.4. We consider the linear system $\mathcal{M}$ of lines in $\mathbb{P}^2$ passing through the point O. Then the log pair

$$(\mathbb{P}^2, b\mathcal{M})$$

is terminal for $b \in [0,1)$, canonical for $b = 1$, log terminal for $b \in [0,2)$, and log canonical for $b = 2$. Moreover,

$$CS(\mathbb{P}^2, b\mathcal{M}) = \begin{cases} \emptyset & \text{if } b < 1, \\ O & \text{if } \lambda \geq 1. \end{cases}$$

We note that

1. a strict transform of a boundary is naturally defined for any birational map,
2. the singularities of log pair (1) coincide with the singularities of the variety X outside $\bigcup\limits_{i=1}^{N} \mathrm{Bs}(\mathcal{M}_i)$, and
3. the log minimal model program (LMMP) preserves canonical and terminal singularities.

Definition 2.5. Log pair (1) has semiterminal singularities if it has canonical singularities and the set

$$\bigcup_{i=1}^{N} \mathrm{Bs}(\mathcal{M}_i)$$

does not contain elements of $CS(X, \mathcal{M}_X)$.

Example 2.6. We consider a quadric cone Q in $\mathbb{P}^3$ and a complete linear system $\mathcal{M}$ of hyperplane sections of Q. Then the log pair

$$(Q, b\mathcal{M})$$

is semiterminal for all $b \in \mathbb{Q}_{\geq 0}$.

Definition 2.7. The log pair $(V, \mathcal{M}_V)$ is a canonical, terminal, or weakly terminal model of log pair (1) if there is a birational map $\psi : X \dashrightarrow V$ such that

$$(V, \mathcal{M}_V) = (V, \psi(\mathcal{M}_X)),$$

the log pair $(V, \mathcal{M}_V)$ respectively has canonical, terminal and $\mathbb{Q}$-factorial, or canonical singularities, and the divisor $K_X + \mathcal{M}_X$ is respectively nef, nef, or ample.

We note that a log pair can have many terminal and weakly canonical models. Nevertheless, we have the following important theorem.

Theorem 2.8. *A canonical model is unique if it exists.*

We leave Theorem 2.8 without proof because its proof is very similar to the uniqueness proof for the log canonical model (see [18]).

Example 2.9. We consider a smooth Fano threefold V with $-K_V^3 = 16$, $\mathrm{Pic}(V) = \mathbb{Z}$, and a very ample anticanonical divisor (see [6]). Let $\mathcal{H}_C$ be a linear system of hyperplane sections of V passing doubly through a sufficiently general line $C \subset V$. Then the log pair

$$(V, b\mathcal{H}_C)$$

is not canonical for $b > 1/2$, and its canonical model is

$$(\mathbb{P}^3, b|\mathcal{O}_{\mathbb{P}^3}(1)|)$$

for $b > 4$.

We consider the birational morphism $f : W \to X$ such that the log pair

$$(W, \mathcal{M}_W) = (W, f^{-1}(\mathcal{M}_X))$$

has canonical singularities.

Definition 2.10. If the linear system $|n(K_W + \mathcal{M}_W)|$ is not empty for some $n \in \mathbb{N}$, then the map

$$I(X, \mathcal{M}_X) = \phi_{|n(K_W+\mathcal{M}_W)|} \circ f^{-1} \quad \text{for } n \gg 0$$

is called the Iitaka map of log pair (1), and the number

$$\kappa(X, \mathcal{M}_X) = \dim(I(X, \mathcal{M}_X)(X))$$

is called the Kodaira dimension of log pair (1). In the case where the linear system $|n(K_W + \mathcal{M}_W)|$ is empty for all $n \in \mathbb{N}$, we assume that the Iitaka map of log pair (1) is not everywhere defined, and we formally set $\kappa(X, \mathcal{M}_X) = -\infty$.

We note that $\kappa(X, \mathcal{M}_X)$ and $I(X, \mathcal{M}_X)$ do not depend on the choice of the morphism f.

We suppose that singularities of log pair (1) are terminal and $\mathbb{Q}$-factorial (canonical). Then we can apply the LMMP up to "flip conjectures." This gives us the birational map $\rho : X \dashrightarrow Y$ such that either the log pair

$$(Y, \mathcal{M}_Y) = (Y, \rho(\mathcal{M}_X))$$

is a terminal (weakly canonical) model or there is a fibration $\tau : Y \to Z$ with $\mathrm{Pic}(Y/Z) = \mathbb{Z}$ and the τ-ample divisor $-(K_Y + \mathcal{M}_Y)$. In the last case, $\kappa(X, \mathcal{M}_X) = -\infty$, and the fibration τ is called a log Fano fibration.

We suppose that the variety X is uniruled. Up to the MMP, we can assume that X admits a Fano fibration $\pi : X \to S$ with $\mathrm{Pic}(X/S) = \mathbb{Z}$ and terminal $\mathbb{Q}$-factorial singularities. In the last case, the morphism π is called a Mori fibration.

Main Problem. *Describe log pairs on X.*

We note that the main problem seems too abstract. Nevertheless, it can be considered an analogue of the classification theory of algebraic varieties and can be split into the following steps:

1. describe log pairs (1) with $\kappa(X, \mathcal{M}_X) = -\infty$,
2. describe log pairs (1) with $\kappa(X, \mathcal{M}_X) \in [0, \dim(X))$ and their Iitaka maps, and
3. describe log pairs (1) with $\kappa(X, \mathcal{M}_X) = \dim(X)$ and their canonical models.

It is easy to see that log pairs (1) with $\kappa(X, \mathcal{M}_X) = \dim(X)$ form a huge class. This emphasizes an analogy with the classification theory of algebraic varieties. Mostly because of this, we say that such log pairs are of the general type.

A quite reasonable solution of the main problem can sometimes be obtained.

Example 2.11. Let X denote a double cover of $\mathbb{P}^3$ ramified in a smooth sextic. We consider $\lambda \in \mathbb{Q}_{>0} \cup \{+\infty\}$ such that

$$K_X + \lambda \mathcal{M}_X \sim_{\mathbb{Q}} 0$$

and $\lambda = +\infty$ for $\mathcal{M}_X = \emptyset$. Then

$$\kappa(X, \mathcal{M}_X) = \begin{cases} -\infty & \text{if } \lambda > 1, \\ 0 & \text{if } \lambda = 1, \\ 1 & \text{if } \lambda < 1 \text{ and all } \mathcal{M}_i \text{ are composed from one pencil } \mathcal{P} \subset |-K_X|, \\ 3 & \text{otherwise.} \end{cases}$$

Moreover, $I(X, \mathcal{M}_X) = \phi_{\mathcal{P}}$ when $\kappa(X, \mathcal{M}_X) = 1$.

All claims of Example 2.11 are proved in Sec. 7.

As is seen later, solving the main problem is implicitly based on solving the following problem.

Auxiliary Problem. *Describe $CS(X, \mathcal{M}_X)$ assuming*

$$K_X + \mathcal{M}_X \sim_{\mathbb{Q}} \pi^*(L),$$

where L is a $\mathbb{Q}$-Cartier divisor on S.

The following example gives a solution of the auxiliary problem.

Example 2.12. As in Example 2.11, let X be a double cover of $\mathbb{P}^3$ ramified in a smooth sextic. If

$$K_X + \mathcal{M}_X \sim_{\mathbb{Q}} 0,$$

then

$$CS(X, \mathcal{M}_X) = \begin{cases} \emptyset, \\ D_1 \cdot D_2, \text{ where } D_1 \text{ and } D_2 \text{ are two different surfaces in } |-K_X|. \end{cases}$$

The claim of Example 2.12 is also proved in Sec. 7.

3. Global Properties of Log Pairs

In this section, we consider general methods that are later applied to the solution of the main problem for several birationally rigid varieties.

We fix the Mori fibration $\pi : X \to S$ and the log pair

$$(X, \mathcal{M}_X) = \left(X, \sum_{i=1}^{N} b_i \mathcal{M}_i\right)$$

with $\kappa(X, \mathcal{M}_X) \in [0, \dim(X))$ and $\lambda \in \mathbb{Q} \cap (0, 1]$ such that

$$K_X + \lambda \mathcal{M}_X \sim_{\mathbb{Q}} \pi^*(L)$$

for the $\mathbb{Q}$-Cartier divisor L on the variety S.

We now obtain an analogue of the so-called Noether–Fano inequality.

Theorem 3.1. *Let $\dim(S) = 0$ and the log pair $(X, \lambda\mathcal{M}_X)$ be terminal. Then $\lambda = 1$, $\kappa(X, \mathcal{M}_X) = 0$, and log pair* (1) *has no weakly canonical models except itself.*

Proof. We suppose that $\lambda < 1$. We consider $\delta \in \mathbb{Q} \cap (\lambda, 1)$ such that the log pair $(X, \delta\mathcal{M}_X)$ is terminal. Then

$$\dim(X) = \kappa(X, \delta\mathcal{M}_X) \le \kappa(X, \mathcal{M}_X) < \dim(X).$$

Therefore, $\lambda = 1$, the log pair (1) is terminal, and $\kappa(X, \mathcal{M}_X) = 0$.

We assume the existence of the commutative diagram

$$\begin{array}{ccc} & W & \\ f \swarrow & & \searrow g \\ X & \overset{\rho}{\dashrightarrow} & Y \end{array}$$

such that the variety W is smooth, the morphisms f and g are birational, the log pair

$$(Y, \mathcal{M}_Y) = (Y, \rho(\mathcal{M}_X))$$

is canonical, and the divisor $K_Y + \mathcal{M}_Y$ is nef. Then

$$\sum_{j=1}^{k} a(X, \mathcal{M}_X, F_j) F_j \sim_{\mathbb{Q}} g^*(K_Y + \mathcal{M}_Y) + \sum_{i=1}^{l} a(Y, \mathcal{M}_Y, G_i) G_i,$$

where the divisors G_i and F_j are exceptional for the corresponding morphisms g and f. It follows from Lemma 2.19 in [13] that

$$a(X, \mathcal{M}_X, E) = a(Y, \mathcal{M}_Y, E)$$

for all divisors E on the variety W. In particular, $K_Y + \mathcal{M}_Y \sim_{\mathbb{Q}} 0$, the log pair $(Y, \mathcal{M}_Y)$ is terminal, and $k = l$.

Now, $\mathrm{Pic}(X) = \mathbb{Z}$ and the $\mathbb{Q}$-factoriality of the variety X imply

$$\mathrm{rk}(\mathrm{Pic}(W)) = 1 + k.$$

On the other hand,

$$\mathrm{rk}(\mathrm{Pic}(W)) \ge \mathrm{rk}(\mathrm{Pic}(Y)) + l.$$

Hence, the variety Y is $\mathbb{Q}$-factorial, and $\mathrm{Pic}(Y) = \mathbb{Z}$.

We should now consider $\zeta \in \mathbb{Q}_{>1}$ such that both $(X, \zeta\mathcal{M}_X)$ and $(Y, \zeta\mathcal{M}_Y)$ are canonical models. This implies (Theorem 2.8) that the map ρ is an isomorphism. $\square$

The conditions of Theorem 3.1 cannot be weakened.

Example 3.2. We consider a smooth quartic threefold V and a linear system of its hyperplane sections $\mathcal{H}_C$ passing through a line $C \subset X$. We set $\mathcal{M}_V = b\mathcal{H}_C$ for $b \in \mathbb{Q}_{>1}$. Then

$$K_V + \frac{1}{b}\mathcal{M}_V \sim_{\mathbb{Q}} 0,$$

$\kappa(V, \mathcal{M}_V) = 1$, and the log pair

$$\left(X, \frac{1}{b}\mathcal{M}_V\right)$$

is canonical.

Now, we need an analogue of Theorem 3.1 for the case $\dim(S) \neq 0$. It can be seen from the proof of Theorem 3.1 that some positivity restriction should be imposed on the divisor L.

Theorem 3.3. *If the log pair $(X, \lambda\mathcal{M}_X)$ is canonical and the divisor L is nef and big, then there is a dominant map*

$$\psi : I(X, \mathcal{M}_X)(X) \dashrightarrow S$$

such that $\pi = \psi \circ I(X, \mathcal{M}_X)$.

Proof. Let $\lambda = 1$. Then $I(X, \mathcal{M}_X) = \psi_{|n\pi^*(L)|}$ for $n \gg 0$. This implies the existence of a birational map ψ.

Now let $\lambda \in (0, 1)$. We consider a birational morphism $f : W \to X$ such that the log pair

$$(W, \mathcal{M}_W) = (W, f^{-1}(\mathcal{M}_X))$$

is terminal and $\mathbb{Q}$-factorial. Then

$$K_W + \mathcal{M}_W \sim_{\mathbb{Q}} f^*(\pi^*(L)) + \sum_{i=1}^{k} a(X, \lambda\mathcal{M}_X, F_i)F_i + (1-\lambda)\mathcal{M}_W,$$

where the divisors F_i are f-exceptional and all $a(X, \lambda\mathcal{M}_X, F_i) \geq 0$. By definition,

$$I(X, \mathcal{M}_X) = I(W, \mathcal{M}_W) \circ f^{-1},$$

and we immediately obtain the necessary map. $\square$

Corollary 3.4. *If the morphism π and the map $I(X, \mathcal{M}_X)$ in Theorem 3.3 are not birationally equivalent, then the log pair $(X, \lambda\mathcal{M}_X)$ is not terminal in the neighborhood of the general fiber of the morphism π.*

Proof. Theorem 3.1 is applied to the general fiber of the morphism π. $\square$

We leave the next remark without proof.

Corollary 3.5. *The following statements are equivalent under the conditions of Theorem 3.3:*

1. *The map ψ is birational.*
2. $\lambda = 1$.
3. *The general fiber of the map $I(X, \mathcal{M}_X)$ is uniruled.*
4. $\kappa(X, \mathcal{M}_X) = \dim(S)$.

We give one generalization of Theorem 3.3 without proof.

Theorem 3.6. *Let the log pair $(X, \lambda\mathcal{M}_X)$ be canonical. Then there is a dominant map*

$$\psi : I(X, \mathcal{M}_X)(X) \dashrightarrow \phi_{|nL|}(S)$$

such that $\psi \circ I(X, \mathcal{M}_X) = \phi_{|nL|} \circ \pi$ for $n \gg 0$.

The conditions of Theorem 3.6 cannot be weakened.

Example 3.7. We consider a linear system of cubics $\mathcal{C}$ on a surface $S \cong \mathbb{P}^2$ and identify its elements with points on the surface $Z \cong \mathbb{P}^2$. The relation

$$(x, y) \in V \iff \text{cubic } y \text{ contains point } x$$

defines a smooth threefold V in $S \times Z$.

Let $\pi : V \to S$ and $\tau : V \to Z$ be natural projections. Then τ is an elliptic fibration, and π is a $\mathbb{P}^1$-bundle with a section. In particular, V is rational.

The log pair

$$(V, \mathcal{M}_V) = (V, b|\tau^*(\mathcal{O}_{\mathbb{P}^2}(1)|)$$

is terminal for all $b > 0$, and

$$K_V + \frac{2}{b}\mathcal{M}_V \sim_{\mathbb{Q}} 0.$$

4. Local Properties of Log Pairs

Many examples show that the global methods in Sec. 3 are not sufficient for the solution of the main problem for threefolds. The principally new (global) method of the "test class" was presented in the classic paper [8]. This allowed solving several problems.

We note that the first counterexample to the Lüroth problem was found because of the "test class" method.

Recently, a local analogue of the "test class," which can be called the Iskovskikh–Pukhlikov inequality, was presented in [15]. This section is dedicated to the generalization of this inequality. We use arguments from [15].

We fix the threefold X, the log pair

$$(X, \mathcal{M}_X) = \left(X, \sum_{i=1}^{N} b_i \mathcal{M}_i\right), \tag{2}$$

and the smooth point $O \in X$.

The main result in this section is the following theorem.

Theorem 4.1. *Let $O \in CS(X, \mathcal{M}_X)$. Then*

$$\mathrm{mult}_O(\mathcal{M}_X^2) \geq 4,$$

and the equality holds if

$$a(X, \mathcal{M}_X, E) = 0,$$

where E is an exceptional divisor of the blowup of the point O.

To prove Theorem 4.1, we use one version of Theorem 3.1 in [13]. As usual, LCS denotes the set of log canonical singularities (see [14]).

Lemma 4.2. *Let O be a smooth point of the surface H, and for some nonnegative rational numbers a_1 and a_2, let*

$$O \in LCS(H, (1 - a_1)\Delta_1 + (1 - a_2)\Delta_2 + M_H),$$

where the boundary M_H is movable and the irreducible reduced curves Δ_1 and Δ_2 intersect normally in the point O. Then

$$\mathrm{mult}_O(M_H^2) \geq \begin{cases} 4a_1a_2 & \text{if } a_1 \leq 1 \text{ or } a_2 \leq 1, \\ 4(a_1 + a_2 - 1) & \text{if } a_1 > 1 \text{ and } a_2 > 1. \end{cases}$$

Lemma 4.2 can be proved inductively in the same way as Theorem 3.1 in [3]. We now consider one corollary of Lemma 4.2.

Lemma 4.3. *If*

$$O \in LCS(H, M_H)$$

for the smooth point O on the surface H and the movable boundary M_H, then

$$\mathrm{mult}_O(M_H^2) \geq 4,$$

and the equality holds if

$$a(H, \mathcal{M}_H, E) = -1,$$

where E is an exceptional divisor of the blowup of the point O.

We leave Lemma 4.3 without proof.

Proof of Theorem 4.1. Let H be a sufficiently general very ample divisor on X passing through the point O. Then

$$\mathrm{mult}_O(M_X^2) = \mathrm{mult}_O((M_X|_H)^2),$$

and

$$O \in LCS(X, H + M_X).$$

V. V. Shokurov's theorem on the connectedness of the locus of log canonical singularities (see [14]) implies

$$O \in LCS(H, M_X|_H).$$

Applying Lemma 4.3 to the log pair

$$(H, M_H) = (H, M_X|_H),$$

we obtain

$$\mathrm{mult}_O(M_X^2) \geq 4,$$

and the equality holds if

$$a(H, M_H, F) = -1,$$

where F is an exceptional divisor of the blowup of the point O on the surface H. It is easy to see that

$$a(H, M_H, F) = a(X, M_X, E) - 1,$$

where E is an exceptional divisor of the blowup of the point O on the threefold X. □

5. Del Pezzo Surfaces

In this section, we apply the results of Sec. 3 to smooth del Pezzo surfaces. We consider surfaces over an arbitrary field $\mathbb{F}$ with the algebraic closure $\bar{\mathbb{F}}$. All the results of Sec. 3 remain valid despite the arbitrariness of the field $\mathbb{F}$. We call zero-dimensional scheme points simply points. An $\mathbb{F}$-point means a geometrically irreducible point.

We fix a smooth del Pezzo surface X with $\mathrm{Pic}(X) = \mathbb{Z}$ and $K_X^2 \leq 3$.

Remark 5.1. It was shown in [14] that

$$X \cong \begin{cases} \text{hypersurface of degree 6 in } \mathbb{P}(1,1,2,3) & \text{if } K_X^2 = 1, \\ \text{hypersurface of degree 4 in } \mathbb{P}(1,1,1,2) & \text{if } K_X^2 = 2, \\ \text{cubic in } \mathbb{P}^3 & \text{if } K_X^2 = 3. \end{cases}$$

We consider the log pair

$$(X, \mathcal{M}_X) = (X, \sum_{i=1}^{N} b_i \mathcal{M}_i) \tag{3}$$

and $\lambda \in \mathbb{Q} \cup \{+\infty\}$ such that

$$K_X + \lambda \mathcal{M}_X \sim_{\mathbb{Q}} 0.$$

In the last equivalence, $\lambda = +\infty$ if $\mathcal{M}_X = \emptyset$.

We want to study how the properties of log pair (3) depend on λ.

Remark 5.2. It is well known that the surface X is not birationally isomorphic to a conic bundle and a smooth del Pezzo surface with the Picard group $\mathbb{Z}$ different from X (see [9, 10]).

Into which del Pezzo surfaces with elliptic fibrations can we birationally transform the surface X?

We consider one construction of a birational map between X and del Pezzo sufaces with canonical singularities. We suppose that $f : W \to X$ is a birational morphism such that the surface W is smooth and $K_W^2 > 0$.

Lemma 5.3. *The divisor $-K_W$ is nef and big.*

Proof. The claim easily follows from Remark 5.1 and $\mathrm{Pic}(X) \cong \mathbb{Z}$. □

The linear system $|-nK_W|$ is free for $n \gg 0$ (see [11]) and gives a birational morphism to the del Pezzo surface V with canonical singularities.

Definition 5.4. We say that the birational transformation $\phi_{|-nK_W|} \circ f^{-1}$ of the del Pezzo surface X is standard.

In general, the surface X has many nonstandard birational transformations to del Pezzo surfaces with canonical singularities.

We now consider one construction of an elliptic fibration that is birationally isomorphic to the surface X.

Definition 5.5. A pencil $\mathcal{P}$ in a linear system $|-nK_X|$ is called an Alphan pencil (see [4]) if a map $\phi_{\mathcal{P}}$ can be given by the commutative diagram

$$\begin{array}{ccc} & W & \\ f\swarrow & & \searrow \phi_{|-nK_W|} \\ X & \overset{\phi_{\mathcal{P}}}{\dashrightarrow} & Z, \end{array}$$

where f is a blowup of K_X^2 points and the linear system $|-nK_W|$ is free.

It is easy to see that the morphism $\phi_{|-nK_W|}$ in Definition 5.5 is an elliptic fibration.

Definition 5.6. An elliptic fibration birationally equivalent to $\phi_{|-nK_W|}$ is called the elliptic fibration given by the pencil $\mathcal{P}$.

Remark 5.7. We note that every pencil in the linear system $|-K_X|$ is an Alphan pencil and gives an elliptic fibration without a multiple fiber.

It is easy to see that the curve Z is rational over the field $\bar{\mathbb{F}}$, but we can say more in this case.

Lemma 5.8. *In the given definition of Alphan pencil, the curve Z is rational.*

Proof. We use the notation in Definition 5.5. Because the case $n = 1$ is trivial, we assume that $n > 1$.

The Riemann–Roch theorem implies

$$H^0(-K_W) = \mathbb{F}.$$

The unique curve F in the linear system $|-K_W|$ is a multiple fiber of the fibration $\phi_{|-nK_W|}$. The curve $F \otimes \bar{\mathbb{F}}$ is invariant under the action of $\mathrm{Gal}(\bar{\mathbb{F}}/\mathbb{F})$. Therefore, the curve Z contains the $\mathbb{F}$-point $\phi_{|-nK_W|}(F)$. □

We consider the natural action of $\mathrm{Bir}(X)$ on log pairs of the surface X. For every map $g \in \mathrm{Bir}(X)$, there is $\lambda(g) \in \mathbb{Q}_{>0} \cup \{+\infty\}$ such that

$$K_X + \lambda(g)g(\mathcal{M}_X) \sim_{\mathbb{Q}} 0.$$

The set $\{\lambda(g)\}$ satisfies the ascending chain condition because

$$\frac{1}{\lambda(g)} = \sum_{i=1}^{N} b_i d_i(g),$$

where $g(\mathcal{M}_i) \sim_{\mathbb{Q}} -d_i(g)K_X$ and $d_i(g) \in \mathbb{N}$. In particular, it has the maximal element $\lambda(g_{\max})$.

Definition 5.9. We call the log pair

$$(X, g_{\max}(\mathcal{M}_X))$$

maximal.

The following theorem is the main one in this section.

Theorem 5.10. *If log pair* (3) *is maximal and* $\lambda = 1$, *then* $\kappa(X, \mathcal{M}_X) = 0$ *and log pair* (3) *is canonical. Moreover, if log pair* (3) *is not terminal, then either all linear systems* $\mathcal{M}_i$ *are composed of one Alphan pencil* $\mathcal{P}$ *or there is a standard birational transformation* ρ *of the surface* X *to a del Pezzo surface* V *such that the log pair*

$$(V, \mathcal{M}_V) = (V, \rho(\mathcal{M}_X))$$

is semiterminal.

We deduce a couple of theorems from Theorem 5.10 and then split its proof into several lemmas.

Theorem 5.11. *Let log pair* (3) *be maximal and* $\lambda > 1$. *Then* $\kappa(X, \mathcal{M}_X) = -\infty$, *and log pair* (3) *is terminal.*

Proof. We can assume that $\lambda \neq +\infty$. Multiplication by a positive number does not change the maximality of the log pair. Therefore, Theorem 5.10 implies that the log pair $(X, \lambda\mathcal{M}_X)$ is canonical. Therefore, log pair (3) is terminal, and $\kappa(X, \mathcal{M}_X) = -\infty$ by definition. □

Theorem 5.12. *If log pair* (3) *is maximal and* $\lambda < 1$, *then either* $\kappa(X, \mathcal{M}_X) = 2$ *or log pair* (3) *is not canonical, all linear systems* $\mathcal{M}_i$ *are composed of one Alphan pencil* $\mathcal{P}$, $I(X, \mathcal{M}_X) = \phi_{\mathcal{P}}$, *and* $\kappa(X, \mathcal{M}_X) = 1$.

Proof. Let $\kappa(X, \mathcal{M}_X) \neq 2$. As in the proof of Theorem 5.11, it follows from Theorem 5.10 that the log pair $(X, \lambda\mathcal{M}_X)$ is canonical and $\kappa(X, \lambda\mathcal{M}_X) = 0$. We note that

$$\kappa(X, \mathcal{M}_X) \geq \kappa(X, \lambda\mathcal{M}_X).$$

Therefore, the log pair $(X, \lambda\mathcal{M}_X)$ is not terminal by Theorem 3.1, and log pair (3) is not canonical.

We suppose that not all linear systems $\mathcal{M}_i$ are composed of one Alphan pencil. Theorem 5.10 implies the inequality $K_X^2 > 1$ and the existence of a standard birational transformation ρ of the surface X to a del Pezzo surface V with canonical singularities such that the log pair

$$(V, \lambda\mathcal{M}_V) = (V, \lambda\rho(\mathcal{M}_X))$$

is semiterminal. We choose $\zeta \in \mathbb{Q} \cap (1, \frac{1}{\lambda})$ such that the log pair $(V, \zeta\lambda\mathcal{M}_V)$ is canonical. Then

$$2 > \kappa(X, \mathcal{M}_X) \geq \kappa(X, \zeta\lambda\mathcal{M}_X) = 2.$$

□

For a given point O of the surface X under certain conditions, we can construct the birational involution $\psi(O)$ of the surface X. Such an involution is called a Bertini involution or a Geizer involution depending on the geometrical irreducibility of the point O and K_X^2 (for details, see [9, 10]).

We now prove the first part of Theorem 5.10.

Lemma 5.13. *In Theorem* 5.10, *log pair* (3) *is canonical.*

Proof. We assume that log pair (3) is not canonical in the point O. This means that

$$\mathrm{mult}_O(\mathcal{M}_X) > 1.$$

Let

$$O \otimes \bar{\mathbb{F}} = \{\bar{O}_1, \dots, \bar{O}_k\}.$$

Then

$$K_X^2 = \mathcal{M}_X^2 \geq \mathrm{mult}_O(\mathcal{M}_X)^2 k.$$

This inequality leads to a contradiction in the case $K_X^2 = 1$, to the equality $k = 1$ in the case $K_X^2 = 2$, and to the inequality $k < 2$ in the case $K_X^2 = 3$.

We choose a curve C on the surface $X \otimes \bar{\mathbb{F}}$ containing the point $\bar{O}_i$, $i \in [1, k]$. We consider the log pair

$$(X \otimes \bar{\mathbb{F}}, \mathcal{M}_{X\otimes\bar{\mathbb{F}}})$$

induced by log pair (3). Then

$$-C \cdot K_{X\otimes\bar{\mathbb{F}}} = C \cdot \mathcal{M}_{X\otimes\bar{\mathbb{F}}} \geq \mathrm{mult}_O(\mathcal{M}_X)\,\mathrm{mult}_{\bar{O}_i}(C) > \mathrm{mult}_{\bar{O}_i}(C).$$

The obtained inequalities imply the existence of a Bertini or Geizer involution $\psi(O)$. It is easy to calculate that

$$\lambda(\psi(O)) > 1,$$

which contradicts the maximality of log pair (3).

Proof of Theorem 5.10. By Lemma 5.13, log pair (3) is canonical. We suppose that log pair (3) is not canonical and consider its terminal modification $f : W \to X$. The log pair

$$(W, \mathcal{M}_W) = (W, f^{-1}(\mathcal{M}_X))$$

has terminal singularities, and

$$K_W + \mathcal{M}_W \sim_{\mathbb{Q}} f^*(K_X + \mathcal{M}_X) \sim_{\mathbb{Q}} 0.$$

In particular, $-K_W$ is nef.

Let $K_W^2 > 0$. Then the linear system $|-nK_W|$ is base-point free for $n \gg 0$, and the surface $\phi_{|-nK_W|}(W)$ is normal (see [11]). If

$$\rho = \phi_{|-nK_W|} \circ f^{-1} \qquad \text{and} \qquad V = \phi_{|-nK_W|}(W),$$

then V is a del Pezzo surface with canonical singularities, and the birational transformation ρ is standard.

We must prove the semiterminality of the log pair

$$(V, \mathcal{M}_V) = (V, \rho(\mathcal{M}_X)),$$

which has canonical singularities by construction. For this, we choose $\zeta \in \mathbb{Q}_{>1}$ such that the log pair $(W, \zeta\mathcal{M}_W)$ is terminal. The morphism $\phi_{|-nK_W|}$ is crepant for the log pair $(W, \zeta\mathcal{M}_W)$. Hence, the log pair $(V, \zeta\mathcal{M}_V)$ is canonical. This implies the semiterminality of the log pair $(V, \mathcal{M}_V)$.

Now let $K_X^2 = 0$. Then

$$0 = \mathcal{M}_W^2 = \sum_{i=1,j=1}^{N} b_i b_j f^{-1}(\mathcal{M}_i) \cdot f^{-1}(\mathcal{M}_j).$$

It easily follows from the last equality that all linear systems $\mathcal{M}_i$ are composed of one Alphan pencil. ☐

We now deduce one well-known statement from Theorem 5.12.

Corollary 5.14. *The surface X is not birationally isomorphic to a conic bundle.*

Proof. We suppose that the surface X is birationally isomorphic to the conic bundle $\tau : Y \to Z$ via the map θ. We consider the log pair

$$(X, \mathcal{M}_X) = (X, \gamma\theta^{-1}(|\tau^*(D)|)) \quad \text{for } \deg(D) \gg 0.$$

We can consider it to be maximal. Then for all γ, $\kappa(X, \mathcal{M}_X) = -\infty$, which contradicts Theorem 5.12. ☐

We now show how Theorem 5.10 describes del Pezzo surfaces with canonical singularities, which are birationally isomorphic to the surface X.

Corollary 5.15. *Up to the action of the group* Bir(X), *all birational transformations of the surface X into a del Pezzo surface with canonical singularities are standard.*

Proof. Let θ birationally map the surface X to the del Pezzo surface Y with canonical singularities. We consider the log pair

$$(X, \mathcal{M}_X) = \left(X, \frac{1}{n}\theta^{-1}(|-nK_Y|)\right) \quad \text{for } n \gg 0$$

with $\kappa(X, \mathcal{M}_X) = 0$. We can assume that the log pair $(X, \mathcal{M}_X)$ is maximal.

Theorem 5.10 implies the existence of a birational map $\rho : X \dashrightarrow V$ such that

$$(V, \mathcal{M}_V) = \left(V, \frac{1}{n}\rho \circ \theta^{-1}(|-nK_Y|)\right)$$

is semiterminal, V is a del Pezzo surface with canonical singularities, and the map ρ is standard.

We consider $\zeta \in \mathbb{Q}_{>1}$ such that the log pair $(V, \zeta\mathcal{M}_V)$ is canonical. We can let the linear system $|-nK_Y|$ be free. Therefore, the log pair

$$(Y, \frac{\zeta}{n}(|-nK_Y|)$$

is canonical, too. The uniqueness of the canonical model (Theorem 2.8) implies that $\rho\circ\theta^{-1}$ is an isomorphism. □

We now describe elliptic fibrations that are birationally isomorphic to the surface X.

Corollary 5.16. *Up to the action of* $\mathrm{Bir}(X)$, *all elliptic fibrations that are birationally isomorphic to the surface X can be given by Alphan pencils.*

Proof. We suppose that X is birationally isomorphic to the elliptic fibration $\tau : Y \to Z$ via the map θ. We consider the log pair

$$(X, \mathcal{M}_X) = (X, \theta^{-1}(|\tau^*(D)|)) \quad \text{for } \deg(D) \gg 0.$$

By construction, $\kappa(X, \mathcal{M}_X) = 1$ and $I(X, \mathcal{M}_X) = \tau \circ \theta$. Up to the action of the group $\mathrm{Bir}(X)$, we can assume that the log pair $(X, \mathcal{M}_X)$ is maximal. Then by Theorem 5.12, $I(X, \mathcal{M}_X) = \phi_{\mathcal{P}}$ for some Alphan pencil $\mathcal{P}$ on the surface X. □

We have not yet proved the existence of a surface X. We show one example over the field $\mathbb{Q}$.

Example 5.17. Let a surface X be given over $\mathbb{Q}$ by the equation

$$2x_0^4 + 3x_1^4 + 5x_2^4 = x_3^2$$

of degree 4 in $\mathbb{P}(1, 1, 1, 2)$. It is easy to see that X is a smooth del Pezzo surface with $K_X^2 = 2$.

To show that $\mathrm{Pic}(X) \cong \mathbb{Z}$, we consider the action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on exceptional curves of the surface $X \otimes \bar{\mathbb{Q}}$.

The surface $X \otimes \bar{\mathbb{Q}}$ contains exactly 56 exceptional curves, and each equation

$$2^{\frac{1}{4}}x_0 + 3^{\frac{1}{4}}x_1 + 5^{\frac{1}{4}}x_2 = 0, \quad 2^{\frac{1}{4}}x_0 + (-3)^{\frac{1}{4}}x_1 = 0, \quad 2^{\frac{1}{4}}x_0 + (-5)^{\frac{1}{4}}x_2 = 0, \quad \text{and} \quad 3^{\frac{1}{4}}x_1 + (-5)^{\frac{1}{4}}x_2 = 0,$$

gives a couple of exceptional curves on the surface X. From the last statement, it is easy to deduce that the group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on these curves such that every orbit contains at least three mutually intersecting curves. This implies $\mathrm{Pic}(X) = \mathbb{Z}$.

6. Two-Dimensional Conic Bundles

In this section, we continue applying the results obtained in Sec. 3 to surfaces over an arbitrary field $\mathbb{F}$ with the algebraic closure $\bar{\mathbb{F}}$. Points and $\mathbb{F}$-points denote the same objects as in Sec. 5.

We fix a conic bundle $\pi : X \to S$ on a smooth surface with $\mathrm{Pic}(X/S) = \mathbb{Z}$ and $K_X^2 < 0$. The geometry of such surfaces was studied in [5].

We consider the log pair

$$(X, \mathcal{M}_X) = \left(X, \sum_{i=1}^{N} b_i \mathcal{M}_i\right) \tag{4}$$

and $\lambda \in \mathbb{Q}_{>0} \cup \{+\infty\}$ such that

$$K_X + \lambda\mathcal{M}_X \sim_{\mathbb{Q}} \pi^*(L),$$

where L is a divisor on the curve S. If the boundary $\mathcal{M}_X$ lies in the fibers of π, then we formally set $\lambda = +\infty$ and $L = f(\mathcal{M}_X)$.

How do the properties of log pair (4) depend on λ?

Example 6.1. We suppose that all components of $\mathcal{M}_X$ are free and "sufficiently big" linear systems. Then log pair (4) is terminal and

$$\kappa(X, \mathcal{M}_X) = \begin{cases} -\infty & \text{if } \lambda > 1, \\ 1 & \text{if } \lambda = 1, \\ 2 & \text{if } \lambda < 1. \end{cases}$$

We now consider one class of birational maps.

Definition 6.2. The class $\mathcal{U}$ consists of the birational maps

$$\psi : X/S \dashrightarrow W/S$$

such that the surface W is smooth, $\mathrm{Pic}(W/S) = \mathbb{Z}$, and $K_X^2 = K_W^2$.

We note that the class $\mathcal{U}$ is not empty.

The inequality $K_X^2 < 0$ means that the morphism π is "strongly ramified." We see later that in this case, modulo maps of the class $\mathcal{U}$ in Example 6.1 reflect general properties of general log pairs on the surface X.

Theorem 6.3. *Let $\lambda = 1$. Then $\kappa(X, \mathcal{M}_X) = 1$, $I(X, \mathcal{M}_X) = \pi$, and there is a birational map ρ of the class $\mathcal{U}$ such that $(\rho(X), \rho(\mathcal{M}_X))$ has canonical singularities.*

To prove Theorem 6.3, we introduce one function on the class $\mathcal{U}$ with values in $\mathbb{Z}_{\geq 0}$.

Definition 6.4. For $\psi \in \mathcal{U}$, let

$$q(\psi) = \text{number of points in } CS(\psi(X), \psi(\mathcal{M}_X)).$$

The proof of Theorem 6.3 follows from the lemma below.

Lemma 6.5. *If $\rho \in \mathcal{U}$ minimizes the function q, then the log pair*

$$(Y, \mathcal{M}_Y) = (\rho(X), \rho(\mathcal{M}_X))$$

has canonical singularities.

Proof. We suppose that the log pair $(Y, \mathcal{M}_Y)$ is not canonical in some point $y \in Y$.

We introduce the notation

$$\bar{Y} = Y \otimes \bar{\mathbb{F}}, \qquad \bar{S} = S \otimes \bar{\mathbb{F}}, \qquad \mathcal{M}_{\bar{Y}} = \mathcal{M}_Y \otimes \bar{\mathbb{F}}.$$

We consider the induced morphism

$$\bar{\tau} : \bar{Y} \to \bar{S}.$$

If

$$y \otimes \bar{\mathbb{F}} = \{\bar{y}_1, \dots, \bar{y}_k\},$$

then the multiplicity of the boundary $\mathcal{M}_{\bar{Y}}$ in every point $\bar{y}_i$ is strictly greater than one.

Let $\bar{F}_i$ be a fiber of the morphism $\bar{\tau}$ containing the point $\bar{y}_i$. We want to show that $\bar{F}_i$ is irreducible and does not contain the point $\bar{y}_j$ if $j \neq i$. First,

$$2 = \bar{B}_Y \cdot \bar{F}_i = \sum_{\bar{y}_j \in \bar{F}_i} (\bar{B}_Y \cdot \bar{F}_i)_{\bar{y}_j} \geq \sum_{\bar{y}_j \in \bar{F}_i} \mathrm{mult}_{\bar{y}_j}(\bar{\mathcal{M}}_Y)\, \mathrm{mult}_{\bar{y}_j}(\bar{F}_i) > \#\{\bar{y}_j \in \bar{F}_i\}.$$

Hence, the fiber $\bar{F}_i$ contains exactly one point among $\{\bar{y}_1, \dots, \bar{y}_k\}$ that is smooth on $\bar{F}_i$. Second, $\mathrm{Pic}(Y/S) = \mathbb{Z}$ implies the nonexistence of irreducible components of the reducible fibers of the morphism $\bar{\tau}$ that are invariant via the action of $\mathrm{Gal}(\bar{\mathbb{F}}/\mathbb{F})$. Therefore, if the fiber $\bar{F}_i$ is reducible, then it must have at least two points among $\{\bar{y}_1, \dots, \bar{y}_k\}$, but, as already proved, this is impossible.

We consider the fiber F of the morphism τ containing the point y. In our notation,

$$F \otimes \bar{\mathbb{F}} = \{\bar{F}_1, \dots, \bar{F}_k\}.$$

We define the birational map $\mu : Y/S \dashrightarrow V/S$ as a composition of the blowup of the point y and the blow-down of the direct image of the fiber F. The previous arguments show that the map μ is well defined. Then

$$K_V^2 = K_X^2 \qquad \text{and} \qquad q(\mu \circ \rho) < q(\rho).$$

This contradicts our assumption on the birational map ρ. □

Proof of Theorem 6.3. Using Lemma 6.5, we obtain a birational map $\rho \in \mathcal{U}$ such that the log pair

$$(Y, \mathcal{M}_Y) = (Y, \rho(\mathcal{M}_X))$$

has canonical singularities and on the surface Y, the relation

$$K_Y + \mathcal{M}_Y \sim_{\mathbb{Q}} \tau^*(D)$$

holds for some divisor D on the curve S.

We must show that $\deg(D) > 0$. But $\deg(D) \leq 0$ implies

$$0 > K_Y^2 = \mathcal{M}_Y^2 - 2\mathcal{M}_Y\tau^*(D) \geq 0.$$

□

The case $\lambda \neq 1$ can be reduced to Theorem 6.3.

Theorem 6.6. *Let $\lambda > 1$. Then $\kappa(X, \mathcal{M}_X) = -\infty$, and a birational map $\rho \in \mathcal{U}$ can be found such that the log pair $(\rho(X), \rho(\mathcal{M}_X))$ is terminal.*

Proof. If $\lambda = +\infty$, then we can set $\rho = id_X$. Otherwise, we can apply Theorem 6.3 to the log pair $(X, \lambda\mathcal{M}_X)$. □

Theorem 6.7. *If $\lambda < 1$, then $\kappa(X, \mathcal{M}_X) = 2$.*

Proof. By Theorem 6.3, $\kappa(X, \lambda\mathcal{M}_X) = 1$, and we can assume that the log pair $(X, \lambda\mathcal{M}_X)$ is canonical and the divisor L is ample. The inequality

$$\kappa(X, \mathcal{M}_X) \geq \kappa(X, \lambda\mathcal{M}_X)$$

and Corollary 3.5 imply $\kappa(X, \mathcal{M}_X) = 2$. □

The proofs of Lemma 5.13 and Corollaries 5.14 and 5.15 together with the results of Theorems 6.6, 6.7, and 6.3 lead to the following result.

Corollary 6.8. *The surface X is not birationally isomorphic to a del Pezzo surface with canonical singularities and elliptic fibration and is nonequivalent to the conic bundle π.*

The product of rational and elliptic curves shows that the conditions of Corollary 6.8 cannot be weakened.

We note that Corollary 6.8 generalizes the classical result for the birational rigidity of extremal "strongly ramified" conic bundles.

We now present an example of a surface X that satisfies our conditions.

Example 6.9. We consider a surface W over a field $\mathbb{Q}$ given by zeros of the polynomial

$$t_0(t_0^2 - 4t_1^2)(t_0^2 - 25t_1^2)x_0^2 + (t_0 - 4t_1)(t_0^2 - 9t_1^2)(t_0^2 - t_1^2)x_1^2 + (t_0 + 4t_1)(t_0^2 - 36t_1^2)(t_0^2 - 49t_1^2)x_2^2$$

of bi-degree $(5, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^2$, where t_i and x_j denote homogeneous coordinates in $\mathbb{P}^1$ and $\mathbb{P}^2$ respectively. It can be shown that W is smooth and $K_X^2 = -7$. The projection $\theta : W \to \mathbb{P}^1$ gives a conic-bundle structure on W with five reducible fibers. After contraction of exceptional curves in the fibers of θ, we obtain a smooth conic bundle $\pi : X \to \mathbb{P}^1$ with $\mathrm{Pic}(X/\mathbb{P}^1) = \mathbb{Z}$ and $K_X^2 = -2$.

7. Double Cover of $\mathbb{P}^3$

In this section, we apply the global results of Sec. 3 and the local results of Sec. 4 to the solution of the main problem for a double cover of $\mathbb{P}^3$.

Let X be a smooth Fano threefold with $-K_X^3 = 2$. Then $\mathrm{Pic}(X) = \mathbb{Z}$ and the linear system $|-K_X|$ is free and gives the morphism

$$\phi_{|-K_X|} : X \to \mathbb{P}^3,$$

which is a double cover ramified in the smooth sextic S. In the following, the morphism $\phi_{|-K_X|}$ is denoted by the letter θ.

We consider the log pair

$$(X, \mathcal{M}_X) = \left(X, \sum_{i=1}^{N} b_i \mathcal{M}_i\right). \tag{5}$$

If $\mathcal{M}_X \neq \emptyset$, then there exists $\lambda \in \mathbb{Q}_{>0}$ such that

$$K_X + \lambda \mathcal{M}_X \sim_{\mathbb{Q}} 0.$$

For $\mathcal{M}_X = \emptyset$, we formally set $\lambda = +\infty$.

In which Fano fibrations and fibrations on varieties with Kodaira dimension zero can X be birationally transformed? First, it is well known that X is not birationally isomorphic to a Mori fibration except itself (see [7, 13]). Second, pencils in the linear system $|-K_X|$ naturally give birational transformations of X into fibrations on K3 surfaces.

We now state the main result of this section.

Theorem 7.1. *Let $\lambda = 1$. Then $\kappa(X, \mathcal{M}_X) = 0$, and log pair (5) is canonical. Moreover, if log pair (5) is not terminal, then all linear systems $\mathcal{M}_i$ are composed from one pencil $\mathcal{P}$ in $|-K_X|$.*

We now state two theorems, which follow from Theorem 7.1 and describe log pairs with $\lambda \neq 1$.

Theorem 7.2. *Let $\lambda > 1$. Then $\kappa(X, \mathcal{M}_X) = -\infty$, and log pair (5) is terminal.*

Proof. We can assume that $\lambda \neq +\infty$. Theorem 7.1 implies the canonicity of the log pair $(X, \lambda\mathcal{M}_X)$ and the claim of the theorem. □

Theorem 7.3. *Let $\lambda < 1$. Then either $\kappa(X, \mathcal{M}_X) = 3$ or $\kappa(X, \mathcal{M}_X) = 1$, log pair (5) is not canonical, all linear systems $\mathcal{M}_i$ are composed from one pencil $\mathcal{P}$ in the linear system $|-K_X|$, and $I(X, \mathcal{M}_X) = \psi_{\mathcal{P}}$.*

Proof. Theorem 7.1 implies

$$\kappa(X, \mathcal{M}_X) \geq \kappa(X, \lambda\mathcal{M}_X) = 0.$$

We can assume that the log pair is not of the general type. Then Theorem 3.1 implies that the log pair $(X, \lambda\mathcal{M}_X)$ is not terminal. In particular, log pair (5) is not canonical. The claim now follows from Theorem 7.1. □

The proof of Theorem 7.1 follows from several auxiliary lemmas.

Lemma 7.4. *In Theorem 7.1, $CS(X, \mathcal{M}_X)$ does not contain points.*

Proof. We suppose that $CS(X, \mathcal{M}_X)$ contains the point $O \in X$. We consider the sufficiently general divisor H_O in $|-K_X|$ passing through the point O. Then

$$2 = H_O \cdot \mathcal{M}_X^2 \geq \mathrm{mult}_O(\mathcal{M}_X^2),$$

which contradicts Theorem 4.1. □

Lemma 7.5. *If $CS(X, \mathcal{M}_X)$ in Theorem 7.1 contains the reduced irreducible curve C, then $\theta(C)$ is a line.*

Proof. We take a sufficiently general divisor H in the linear system $|-K_X|$. Then

$$2 = H \cdot \mathcal{M}_X^2 \geq \operatorname{mult}_C(\mathcal{M}_X^2) H \cdot C \geq H \cdot C = -K_X \cdot C$$

because

$$\operatorname{mult}_C(\mathcal{M}_X) \geq 1.$$

We suppose that $\theta(C)$ is not a line. Then $\theta(C)$ is a conic, and $\theta|_C$ is anisomorphism.

We consider the blowup $f : W \to X$ of the curve C and set $E = f^{-1}(C)$. We now show that the divisor $f^*(-3K_X) - E$ is nef.

We suppose that $\theta(C) \not\subset S$. Then

$$\theta^{-1}(\theta(C)) = C \cup \tilde{C}.$$

It is easy to see that

$$\operatorname{Bs}(|f^*(-2K_X) - E|) = f^{-1}(\tilde{C}).$$

This implies that the divisor $f^*(-2K_X) - E$ has a nonnegative intersection with every curve on W except $f^{-1}(\tilde{C})$. It is easy to verify the equality

$$(f^*(-2K_X) - E) \cdot f^{-1}(\tilde{C}) = -2,$$

which implies the nefness of the divisor $f^*(-3K_X) - E$.

Therefore, we can assume that the conic $\theta(C) \subset S$. Then

$$\operatorname{Bs}(|f^*(-2K_X) - E|) \subset E.$$

If s_∞ is an exceptional section of the ruled surface $f|_E : E \to C$, then the nefness of the divisor $f^*(-3K_X) - E$ follows from the inequality

$$(f^*(-3K_X) - E)|_E \cdot s_\infty \geq 0.$$

Elementary blowup properties imply $E^3 = 0$ and

$$(f^*(-3K_X) - E)|_E \cdot s_\infty = 6 + \frac{s_\infty^2}{2}.$$

Therefore, we must show that $s_\infty^2 \geq -12$. Let

$$N_{X/C} \cong \mathcal{O}_C(m) \oplus \mathcal{O}_C(n) \quad \text{for } m \geq n.$$

Then

$$m + n = \deg(N_{X/C}) = c_1(X) \cdot C - c_1(C) = 0,$$

and the exact sequence

$$0 \to N_{\theta^{-1}(S)/C} \to N_{X/C} \to N_{X/\theta^{-1}(S)}|_C \to 0$$

implies $n \geq \deg(N_{\theta^{-1}(S)/C}) = -6$. Therefore,

$$s_\infty^2 = n - m = 2n \geq -12.$$

Therefore, the divisor $f^*(-3K_X) - E$ is nef. We consider the log pair

$$(W, \mathcal{M}_W) = (W, f^{-1}(\mathcal{M}_X)).$$

Then

$$(f^*(3H) - E) \cdot \mathcal{M}_W^2 \geq 0.$$

On the other hand,

$$(f^*(-3K_X) - E) \cdot \mathcal{M}_W^2 = (f^*(-3K_X) - E) \cdot (f^*(-K_X) - \operatorname{mult}_C(\mathcal{M}_X) E)^2 = 6 - \operatorname{mult}_C(\mathcal{M}_X)(6 \operatorname{mult}_C(\mathcal{M}_X) + 4),$$

which contradicts the inequality

$$6 - \operatorname{mult}_C(\mathcal{M}_X)(6 \operatorname{mult}_C(\mathcal{M}_X) + 4) < 0.$$

□

Proof of Theorem 7.1. We suppose that log pair (5) is not terminal. Then it follows from Lemmas 7.4 and 7.5 that $CS(X, \mathcal{M}_X)$ contains a reduced irreducible curve C such that $\theta(C)$ is a line.

For the curve C, we have three possibilities:

a. $-K_X \cdot C = 2$, $\mathcal{M}_X^2 = C$, and $\theta|_C$ is a double cover;
b. $-K_X \cdot C = 1$, $\theta|_C$ is an isomorphism, and the sextic S does not contain the line $\theta(C)$, and
c. $-K_X \cdot C = 1$, $\theta|_C$ is an isomorphism, and $\theta(C) \in S$.

We consider the pencil $\mathcal{H}_C$ consisting of surfaces in the linear system $|-K_X|$ that contain the curve C. We note that the pencil $\mathcal{H}_C$ is the inverse image via morphism θ of the pencil of planes in $\mathbb{P}^3$ that contain the line $\theta(C)$.

We consider case a. We resolve the indeterminacy of the rational map $\phi_{\mathcal{H}_C}$ using the commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, dashed, "\phi_{\mathcal{H}_C}"] & & \mathbb{P}^1 \end{tikzcd}$$

such that the variety W is smooth. We can assume that the variety W contains exactly one divisor E lying over a general point of the curve C and f is an isomorphism outside the curve C.

We consider a general fiber D of the morphism g, which is a smooth K3 surface,

$$D \sim f^*(-K_X) - E - \sum_{i=1}^{k} a_i F_i,$$

and for every divisor F_i, $f(F_i)$ is a point on the curve C. We consider the log pair

$$(D, \mathcal{M}_D) = (D, f^{-1}(\mathcal{M}_X)|_D).$$

We have

$$\mathcal{M}_D \sim_{\mathbb{Q}} \left((1 - \mathrm{mult}_C(\mathcal{M}_X))E + \sum_{i=1}^{k} c_i F_i \right) \Bigg|_D$$

for some rational numbers c_i. Therefore, $\mathrm{mult}_C(\mathcal{M}_X) = 1$ and $\mathcal{M}_D = \emptyset$. This implies that all linear systems $\mathcal{M}_i$ are composed from the pencil $\mathcal{H}_C$.

We now consider case b. Let

$$\theta^{-1}(\theta(C)) = C \cup \tilde{C}.$$

We consider a sufficiently general divisor D in the linear system $\mathcal{H}_C$. The divisor D is a smooth K3 surface containing the curve $\tilde{C}$, and

$$\mathcal{M}_X|_D = \mathrm{mult}_C(\mathcal{M}_X)C + \mathrm{mult}_{\tilde{C}}(\mathcal{M}_X) + R,$$

where R is an effective divisor on the surface D, whose support does not contain the curves C and $\tilde{C}$. On the surface D,

$$C^2 = \tilde{C}^2 = -2 \qquad \text{and} \qquad C \cdot \tilde{C} = 3.$$

Hence,

$$1 = \mathcal{M}_X|_D \cdot \tilde{C} = 3\,\mathrm{mult}_C(\mathcal{M}_X) - 2\,\mathrm{mult}_{\tilde{C}}(\mathcal{M}_X) + R \cdot \tilde{C}.$$

Therefore, $\mathrm{mult}_{\tilde{C}}(\mathcal{M}_X) \geq 1$, and $CS(X, \mathcal{M}_X)$ contains the curve $\tilde{C}$.

We note that in case a, we did not use the irreducibility of the curve C. We used the fact that $\mathcal{H}_C^2 = C$. Therefore, the statement that $CS(X, \mathcal{M}_X)$ contains the curve $\tilde{C}$ implies that all linear systems $\mathcal{M}_i$ are composed from the pencil H_C.

We now consider case c. As in the previous cases a and b, $\mathrm{mult}_C(\mathcal{H}_C) = 1$, but $\mathcal{H}_C^2 = 2C$. Let $f : W \to X$ be a blowup of the curve C and $E = f^{-1}(C)$. Then the base locus of the linear system $f^{-1}(\mathcal{H}_C)$ consists of the smooth rational curve $\tilde{C}$, which is a section of the ruled surface $f|_E : E \to C$. We consider the log pair

$$(W, \mathcal{M}_W) = (W, f^{-1}(\mathcal{M}_X))$$

and a sufficiently general divisor D in the linear system $f^{-1}(\mathcal{H}_C)$. The divisor D is a smooth K3 surface, and

$$\mathcal{M}_W|_D = \mathrm{mult}_{\widetilde{C}}(\mathcal{M}_W)\widetilde{C} + R,$$

where R is an effective divisor whose support does not contain the curve $\widetilde{C}$. On the other hand, $\widetilde{C}^2 = -2$ on the surface D, and

$$\mathcal{M}_W|_D \sim_{\mathbb{Q}} \widetilde{C} + (1 - \mathrm{mult}_C(\mathcal{M}_X))E|_D.$$

This implies

$$\mathrm{mult}_{\widetilde{C}}(\mathcal{M}_W) = \mathrm{mult}_C(\mathcal{M}_X) = 1.$$

We can now repeat all arguments used in case a and find that all linear systems $\mathcal{M}_i$ are composed from the pencil $\mathcal{H}_C$.

In all cases, the multiplicity $\mathrm{mult}_C(\mathcal{M}_X)$ is equal to one. The generality of the curve C and Lemma 7.4 implies the canonicity of log pair (5) and $\kappa(X, \mathcal{M}_X) = 0$. □

We now state three corollaries of Theorems 7.1–7.3. Applying the constructions from the proofs of Lemma 5.13 and Corollaries 5.14 and 5.15 to the variety X, we obtain three corollaries of Theorems 7.1–7.3.

Corollary 7.6. *The variety X cannot be birationally transformed into a fibration on rational surfaces or on rational or elliptic curves.*

Corollary 7.7. *If X is birationally isomorphic to a fibration on surfaces with Kodaira dimension zero $\tau : Y \to Z$ via a birational map ρ, then we can find a pencil in the linear system $|-K_X|$ such that $\tau \circ \rho = \phi_{\mathcal{P}}$.*

Corollary 7.8. $\mathrm{Bir}(X) = \mathrm{Aut}(X)$, *and a threefold X is not birationally isomorphic to a Fano threefold with canonical singularities except itself.*

8. Quartic Threefold

We now state and prove claims for a quartic threefold similar to the claims in Sec. 7. We fix a smooth quartic threefold X in $\mathbb{P}^4$. Then $\mathrm{Pic}(X) = \mathbb{Z}$ and

$$-K_X \sim \mathcal{O}_{\mathbb{P}^4}(1)|_X.$$

We consider the log pair

$$(X, \mathcal{M}_X) = \left(X, \sum_{i=1}^{N} b_i \mathcal{M}_i\right) \tag{6}$$

and $\lambda \in \mathbb{Q}_{>0} \cup \{+\infty\}$ such that we have the relation

$$K_X + \lambda \mathcal{M}_X \sim_{\mathbb{Q}} 0,$$

where $\lambda = +\infty$ for $\mathcal{M}_X = \emptyset$.

As for the threefold in Sec. 7, the quartic X cannot be birationally isomorphic to a Mori fibration except itself (see [7, 13]), $\mathrm{Bir}(X) = \mathrm{Aut}(X)$ (see [7]), and pencils in the linear system $|-K_X|$ give fibrations on K3 surfaces that are birationally isomorphic to X.

We note that the projection from every line on the quartic gives an elliptic fibration on the blowup of X in this line.

The main results in this section are deduced from the following theorem.

Theorem 8.1. *Let $\lambda = 1$. Then $\kappa(X, \mathcal{M}_X) = 0$, and log pair* (6) *is canonical. If log pair* (6) *is not terminal, then one of the following holds:*

1. *all linear systems $\mathcal{M}_i$ are composed from one pencil $\mathcal{P}$ in $|-K_X|$ or*
2. *for the blowup $f : W \to X$ of some line on X, all linear systems $f^{-1}(\mathcal{M}_i)$ are contained in fibers of the elliptic fibration*

$$\psi_{|-K_W|} : W \to \mathbb{P}^2.$$

We prove Theorem 8.1 step by step.

Lemma 8.2. *In the theorem, $CS(X, \mathcal{M}_X)$ does not contain points.*

Proof. We suppose that $CS(X, \mathcal{M}_X)$ contains the point O. Let H_O be a sufficiently general hyperplane section of X passing through O. Then

$$4 = H_O \cdot \mathcal{M}_X^2 \geq \mathrm{mult}_O(\mathcal{M}_X^2).$$

We blow up the point O via $f : W \to X$ and set $E = f^{-1}(O)$. This inequality and Theorem 4.1 imply

$$a(X, \mathcal{M}_X, E) = 0.$$

Therefore,

$$f^{-1}(\mathcal{M}_X) \sim_{\mathbb{Q}} f^*(-K_X) - 2E \sim -K_W.$$

In particular, the linear system $|-nK_W|$ has no fixed components for $n \gg 0$.

Let S be the inverse image of the surface H_O on the threefold W. The linear system $|-K_W|_S|$ contains exactly one effective divisor D. On the other hand, $D^2 = 0$ implies that for $n \gg 0$, the linear system $|nD|$ is free, and

$$\dim(\phi_{|nD|}(S)) = 1.$$

Moreover,

$$\phi_{|nD|}(S) = \mathbb{P}^1$$

because the curve $E|_S$ does not lie in the fibers of the fibration $\phi_{|nD|}$. This implies that for some $k \in (1, n]$, kD is a multiple fiber of $\phi_{|-nD|}$. Therefore, $\phi_{|-nD|}$ should be an elliptic fibration, but the arithmetic genus of the curve D is two. $\square$

Lemma 8.3. *We suppose that $CS(X, \mathcal{M}_X)$ in Theorem* 8.1 *contains the irreducible reduced curve C. Then* $\deg(C) \leq 4$.

Proof. For the curve C,

$$\mathrm{mult}_C(\mathcal{M}_X) \geq 1.$$

We consider a general hyperplane section H of the variety X. Then

$$4 = H \cdot \mathcal{M}_X^2 \geq \mathrm{mult}_C(\mathcal{M}_X^2) H \cdot C \geq \deg(C).$$

$\square$

Lemma 8.4. *In Lemma* 8.3, *the curve C is a plane.*

Proof. We suppose that the curve C is not a plane. Then Lemma 8.3 implies that the curve C is one of the following curves:

a. a smooth rational curve of degree 3,
b. a smooth rational curve of degree 4,
c. a smooth elliptic curve of degree 4, or
d. a rational curve of degree 4 with one double point P.

We consider cases a, b, and c. Let $f : W \to X$ be a blowup of the curve C, $E = f^{-1}(C)$,

$$(W, \mathcal{M}_W) = (W, f^{-1}(\mathcal{M}_X)),$$

and

$$A = (f^*(\deg(C)H) - E) \cdot \mathcal{M}_W^2.$$

On one hand, $A \geq 0$ because the linear system $|f^*(\deg(C)H) - E|$ does not have base curves. On the other hand,

$$A = (\deg(C) - \deg^2(C) + 2g(C) - 2)\,\mathrm{mult}_C^2(\mathcal{M}_X) - 2\deg(C)\,\mathrm{mult}_C(\mathcal{M}_X) + 4\deg(C).$$

This implies

$$A \leq 3\deg(C) - \deg^2(C) + 2g(C) - 2 < 0.$$

We consider case d. Let $f = g \circ h$, where $g : V \to X$ and $h : W \to V$ are respective blowups of the point P and the curve $h^{-1}(C)$. We set $G = g^{-1}(P)$ and $E = h^{-1}(g^{-1}(C))$. It follows from Lemma 8.2 that $CS(X, \mathcal{M}_X)$ does not contain the point P. Hence,

$$2 > \mathrm{mult}_P(\mathcal{M}_X) \geq \mathrm{mult}_C(\mathcal{M}_X) \geq 1.$$

We consider the log pair

$$(W, \mathcal{M}_W) = (W, f^{-1}(\mathcal{M}_X))$$

and set

$$A = (f^*(4H) - E) \cdot \mathcal{M}_W^2.$$

On one hand, $A \geq 0$ because the linear system $|f^*(4H) - E - 2G|$ does not have basic curves. On the other hand,

$$A = -14\,\mathrm{mult}_C^2(\mathcal{M}_X) + (4\,\mathrm{mult}_P(\mathcal{M}_X) - 8)\,\mathrm{mult}_C(\mathcal{M}_X) + 16 - 2\,\mathrm{mult}_P^2(\mathcal{M}_X)$$

and

$$A \leq -2\,\mathrm{mult}_P^2(\mathcal{M}_X) + 4\,\mathrm{mult}_P(\mathcal{M}_X) - 6 < 0.$$

□

Proof of Theorem 8.1. We suppose that log pair (6) is not terminal. Then Lemma 8.2 implies that $CS(X, \mathcal{M}_X)$ contains the irreducible reduced curve C. Lemma 8.4 implies the existence of a plane T containing the curve C, and Lemma 8.3 implies $\deg(C) \leq 4$.

We consider a linear system of hyperplane sections $\mathcal{H}_T$ containing the plane T. We note that the rational map $\phi_{\mathcal{H}_T}$ is a restriction on X of the projection from the plane T.

We suppose that $\deg(C) = 4$. We resolve the indeterminacy of the rational map $\phi_{\mathcal{H}_T}$ using the commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, dashed, "\phi_{\mathcal{H}_T}"] & & \mathbb{P}^1 \end{tikzcd}$$

such that the threefold W contains one divisor E that dominates the curve C and f is an isomorphism outside the curve C.

We consider a general fiber D of the morphism g. Then

$$D \sim f^*(-K_X) - E - \sum_{i=1}^{k} a_i F_i,$$

and for every divisor F_i, $f(F_i)$ is a point on the curve C. We note that D is a smooth K3 surface.

We consider the log pair

$$(D, \mathcal{M}_D) = (D, f^{-1}(\mathcal{M}_X)|_D).$$

For its boundary,

$$\mathcal{M}_D \sim_{\mathbb{Q}} ((1 - \mathrm{mult}_C(\mathcal{M}_X))E + \sum_{i=1}^{k} c_i F_i)|_D,$$

where all $c_i \in \mathbb{Q}$. The movability of the log pair $(D, \mathcal{M}_D)$ implies $\mathcal{M}_D = \emptyset$. Hence, all linear systems $\mathcal{M}_i$ are composed from the pencil $\mathcal{H}_T$.

We now suppose $\deg(C) \in (1, 4)$. We take a sufficiently general divisor D from the linear system $\mathcal{H}_T$. The divisor D is a smooth K3 surface, and

$$X \cdot T = D \cdot T = C \cup \sum_{i=1}^{r} C_i,$$

where all C_i are irreducible reduced curves on D. If

$$C_i \in CS(X, \mathcal{M}_X) \quad \text{for } i = 1, \dots, r,$$

then we can use the arguments in the previous case ($\deg(C) = 4$) to complete the proof.

We now prove that the intersection form of the curves C_i on the surface D is negative definite. On the surface D,

$$(\sum_{i=1}^{r} C_i) \cdot C_j = (D|_D - C) \cdot C_j = \deg(C_j) - C \cdot C_j.$$

But, on the other hand,

$$\deg(C_j) - C \cdot C_j = \deg(C_j) - \deg(C)\deg(C_j) < 0$$

on the plane T. All curves C_j are different from C, and the surface D is smooth. Hence,

$$(C \cdot C_j)_D = (C \cdot C_j)_T.$$

The results in [1] imply that the intersection form of the curves C_i on the surface D is negative definite.

We note that

$$\mathcal{M}_X|_D \sim_{\mathbb{Q}} D|_D \sim C + \sum_{i=1}^{r} C_i.$$

The divisor

$$\mathcal{M}_X|_D - \mathrm{mult}_C(\mathcal{M}_X)C - \sum_{i=1}^{r} \mathrm{mult}_{C_i}(\mathcal{M}_X)C_i$$

is nef on the surface D and $\mathbb{Q}$-rationally equivalent to the divisor

$$(1 - \mathrm{mult}_C(\mathcal{M}_X))C + \sum_{i=1}^{r}(1 - \mathrm{mult}_{C_i}(\mathcal{M}_X))C_i.$$

Hence, on the surface D,

$$\sum_{i=1}^{r}(1 - \mathrm{mult}_{C_i}(\mathcal{M}_X))C_i \cdot C_j \geq 0 \quad \text{for } \jmath = 1, \dots, r.$$

Therefore, all $\mathrm{mult}_{C_i}(\mathcal{M}_X) \geq 1$. All curves C_i are therefore contained in $CS(X, \mathcal{M}_X)$, and, as mentioned above, all linear systems $\mathcal{M}_i$ are composed from the pencil $\mathcal{H}_T$.

To complete the proof, we must consider the case where C is a line. We blow it up via $f : W \to X$ and set $E = f^{-1}(C)$. We consider the log pair

$$(W, \mathcal{M}_W) = (W, f^{-1}(\mathcal{M}_X))$$

and the linear system $\mathcal{H}_C$ consisting of hyperplane sections of the quartic X passing through the line C. Then

$$\mathcal{M}_W + (\mathrm{mult}_C(\mathcal{M}_X) - 1)E \sim_{\mathbb{Q}} f^{-1}(\mathcal{H}_C) \sim -K_W,$$

$\dim(|-K_W|) = 2$, the linear system $|-K_W|$ is free, and a general fiber of the morphism $\phi_{|-K_W|}$ is an elliptic curve. The equality

$$\mathcal{M}_W \cdot (f^{-1}(\mathcal{H}_C)^2 = (1 - \mathrm{mult}_C(\mathcal{M}_X))$$

implies that all linear systems $f^{-1}(\mathcal{M}_i)$ are contained in fibers of the morphism $\phi_{|-K_W|}$.

We note that the previous arguments imply that for every irreducible reduced curve C in $CS(X, \mathcal{M}_X)$, we have $\mathrm{mult}\, C(\mathcal{M}_X) = 1$. Together with Lemma 8.2, this implies the canonicity of log pair (6). □

We now state analogues of Theorems 7.2 and 7.3 for a quartic without proof.

Theorem 8.5. *Let $\lambda > 1$. Then $\kappa(X, \mathcal{M}_X) = -\infty$, and log pair* (6) *is terminal.*

Theorem 8.6. *Let $\lambda < 1$. Then for log pair* (6), *we have three possibilities:*

1. $\kappa(X, \mathcal{M}_X) = 1$, *all linear systems $\mathcal{M}_i$ are composed from one pencil $\mathcal{P}$ in $|-K_X|$, and $I(X, \mathcal{M}_X) = \psi_{\mathcal{P}}$;*
2. $\kappa(X, \mathcal{M}_X) = 2$, *for the blowup $f : W \to X$ of some line on the quartic X all linear systems $f^{-1}(\mathcal{M}_i)$ are contained in the fibers of the elliptic fibration $\psi_{|-K_W|} : W \to \mathbb{P}^2$, and $I(X, \mathcal{M}_X) = \psi_{|-K_W|} \circ f^{-1}$, and*
3. $\kappa(X, \mathcal{M}_X) = 3$.

Moreover, in cases a *and* b, *log pair* (6) *is not canonical.*

Results generalizing classical theorems on the birational rigidity of a smooth quartic threefold follow from Theorems 8.1, 8.5, and 8.6.

Corollary 8.7. *A quartic X cannot be birationally transformed into a conic bundle and a fibration on rational surfaces.*

Corollary 8.8. *Let a quartic X be birationally isomorphic to an elliptic fibration $\tau : Y \to Z$ via a birational map ρ. Then the map $\tau \circ \rho$ is a projection from some line on X.*

Corollary 8.9. *Let X be birationally isomorphic to a fibration on surfaces with Kodaira dimension zero $\tau : Y \to Z$ via a birational map ρ. Then there is a pencil in the linear system $|-K_X|$ such that $\tau \circ \rho = \phi_{\mathcal{P}}$.*

Corollary 8.10. $\mathrm{Bir}(X) = \mathrm{Aut}(X)$, *and the quartic X is not birationally isomorphic to a Fano threefold with canonical singularities different from X.*

We omit the proofs of Corollaries 8.7–8.10 because they are similar to the proofs of Lemma 5.13 and Corollaries 5.14 and 5.15.

9. Double Cover of a Quadric

We now consider the main problem for a double cover of a smooth quadric $\theta : X \to Q$ ramified in a smooth surface S such that S can be obtained as the intersection of the quadric $Q \subset \mathbb{P}^4$ and a quartic. We note that $\mathrm{Pic}(X) = \mathbb{Z}$ and

$$-K_X \sim \theta^*(\mathcal{O}_{\mathbb{P}^4}(1)|_Q).$$

The variety X is not birationally isomorphic to a Mori fibration not isomorphic to X (see [3, 7]). Pencils in the linear system $|-K_X|$ give K3 fibrations that are birationally isomorphic to X. Every line on the quadric Q induces a birational transformation of the variety X into an elliptic fibration.

We describe differences between the variety X and the varieties considered in Secs. 7 and 8. First, $\mathrm{Bir}(X) \neq \mathrm{Aut}(X)$ (see [7]). Second, X can be birationally transformed into Fano threefolds with canonical singularities nonisomorphic to X. We describe the last transformations in detail because they have not been previously mentioned in the literature.

We consider the curve $C \in X$ such that $-K_X \cdot C = 1$. The curve C is smooth and rational. We call such curves "lines." We note that the variety X contains a one-dimensional family of such "lines" (see [7]).

Let $f : W \to X$ be a blowup of the curve C and $E = f^{-1}(C)$. We consider the curve $C_1 \subset W$ such that in the case where $\theta(C) \not\subset S$,

$$\theta^{-1}(\theta(C)) = C \cup C_1,$$

and in the case where $\theta(C) \subset S$, C_1 is an exceptional section of the ruled surface E (we see later that $E \cong \mathbb{F}_5$).

Lemma 9.1. *There is an antiflip $\rho : W \dashrightarrow \widehat{W}$ in the curve C_1, and singularities of the threefold $\widehat{W}$ consist of exactly one terminal point of type $\frac{1}{2}(1,1,1)$.*

Proof. First, we prove that $E \cong \mathbb{F}_5$ in the case where $\theta(C) \subset S$ (see [7]). Let

$$N_{X/C} \cong \mathcal{O}_C(m) \oplus \mathcal{O}_C(n) \quad \text{for } m \geq n.$$

Then

$$m + n = \deg(N_{X/C}) = c_1(X) \cdot C - c_1(C) = -1,$$

and the exact sequence

$$0 \to N_{\theta^{-1}(S)/C} \to N_{X/C} \to N_{X/\theta^{-1}(S)}|_C \to 0$$

implies $n \geq \deg(N_{\theta^{-1}(S)/C}) = -3$. Therefore, $m - n$ is one of the numbers 1, 3, or 5.

We note that the linear system $|-K_W|$ consists of proper transfoms of hyperplane sections of the quadric Q containing the line $\theta(C)$. Therefore, a general surface D of the linear system $|-K_W|$ is a smooth K3 surface, and the morphism $f|_D$ contracts exactly four curves into four simply double points of the surface $f(D)$.

It follows from the last remark that

$$D|_E = C_1 + \alpha L,$$

where L is a fiber of the ruled surface E and $\alpha \geq 4$. Therefore,

$$D^2 = C_1^2 + 2\alpha$$

on the surface E. But on the threefold X,

$$D^2 \cdot E = (f^*(-K_X) - E)^2 \cdot E = 3.$$

Therefore, $\alpha = 4$ and $E \cong \mathbb{F}_5$.

We now construct the birational map ρ step by step.

Step 1. We blow up a curve C_1 via $g : V \to W$ and set $G = g^{-1}(C_1)$. We show that $G \cong \mathbb{F}_1$. The linear system $|-K_V|$ consists of proper transforms of hyperplane sections of the quadric Q passing through the line $\theta(C)$. This implies the freeness of the linear system $|-K_V|$ and the existence of the elliptic fibration $\phi_{|-K_V|} : V \to \mathbb{P}^2$ with a section G. Hence, $G \cong \mathbb{F}_1$.

Step 2. We make a flop in the exceptional section C_2 of the ruled surface G. For the existence of such a flop, we must show that

$$N_{V/C_2} \cong \mathcal{O}_{C_2}(-1) \oplus \mathcal{O}_{C_2}(-1).$$

Indeed, if $r : Y \to V$ is a blowup of the curve C_2 and $R = r^{-1}(C_2)$, then $R \cong \mathbb{P}^1 \times \mathbb{P}^1$ up to the above isomorphism, and there is an analytic contraction $\hat{r} : Y \to \widehat{V}$ of the divisor R to the curve $\widehat{C}_2$ different from r. It is easy to see that $\hat{r} \circ r^{-1}$ is a flop in the curve C_2.

Let

$$N_{V/C_2} \cong \mathcal{O}_{C_2}(m) \oplus \mathcal{O}_{C_2}(n) \quad \text{for } m \geq n.$$

We note that

$$m + n = \deg(N_{V/C_2}) = c_1(V) \cdot C_2 - c_1(C_2) = -2$$

and the exact sequence

$$0 \to N_{G/C_2} \to N_{V/C_2} \to N_{V/C_2}|_{C_2} \to 0$$

implies $n \geq \deg(N_{G/C_2}) = -1$. Therefore, $n = m = -1$.

Step 3. We now contract the surface $\widehat{G} = \hat{r} \circ r^{-1}(G)$ to the point. The morphism $\hat{r}|_{r^{-1}(G)}$ contracts the exceptional section $R \cap r^{-1}(G)$ of the ruled surface $r^{-1}(G) \cong \mathbb{F}_1$. Hence, $\widehat{G} \cong \mathbb{P}^2$ and

$$N_{\widehat{V}/\widehat{G}} \cong \mathcal{O}_{\widehat{G}}(-2).$$

Therefore, there is an analytic contraction $\hat{g} : \widehat{V} \to \widehat{W}$ of the surface $\widehat{G}$ to the singular terminal point O of type $\frac{1}{2}(1,1,1)$. We set

$$\rho = \hat{g} \circ \hat{r} \circ r^{-1} \circ g^{-1}$$

and $\widehat{C}_1 = \hat{g}(\widehat{C}_2)$.

We construct the commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "r"'] \arrow[dr, "\hat{r}"] & \\
V \arrow[d, "g"'] & & \widehat{V} \arrow[d, "\hat{g}"] \\
W \arrow[rr, dashed, "\rho"] & & \widehat{W}.
\end{tikzcd}$$

It is easy to verify that the map ρ is an antiflip of the threefold W in the curve C_1 (i.e., ρ^{-1} is a flip of the threefold $\widehat{W}$ in the curve $\widehat{C}_1$). $\square$

We now show that the threefold obtained in Lemma 9.1 is projective.

Lemma 9.2. *In Lemma* 9.1, *the threefold* $\widehat{W}$ *is projective and* $\mathbb{Q}$*-factorial.*

Proof. The map ρ is a flip for the log terminal log pair

$$(W, (1+\epsilon)|-K_W|) \quad \text{for } 1 \gg \epsilon > 0$$

because

$$K_W + (1+\epsilon)|-K_W|$$

has a negative intersection only with the curve C_1. $\square$

Lemma 9.3. *In Lemma* 9.1, $-K^3_{\widehat{W}} = \frac{1}{2}$ *and* $\mathrm{Bs}(|-K_{\widehat{W}}|) = O$. *In particular, the divisor* $-K_{\widehat{W}}$ *is nef and big.*

Proof. We use the notation in the proof of Lemma 9.1. The flop $\hat{r} \circ r^{-1}$ "changes" the curve in the fibers of $\phi_{|-K_V|}$. Therefore, the linear system $|-K_{\widehat{V}}|$ is free, and the morphism $\phi|-K_{\widehat{V}}|$ is an elliptic fibration.

It follows from the construction of the map ρ that

$$\mathrm{Bs}(|-K_{\widehat{W}}|) = O$$

and

$$0 = -K^3_{\widehat{V}} = (\hat{g}^*(-K_{\widehat{W}}) - \frac{1}{2}\widehat{G})^3 = -K^3_{\widehat{W}} - \frac{1}{8}\widehat{G}^3 = -K^3_{\widehat{W}} - \frac{1}{2}.$$

$\square$

It follows directly from Lemma 9.3 that the linear system $|-nK_{\widehat{W}}|$ is free for $n \gg 0$ and gives the birational morphism

$$\phi_{|-nK_{\widehat{W}}|} : \widehat{W} \to X_C,$$

where the threefold X_C is normal. By construction, X_C is a Fano threefold with canonical singularities, and $-K^3_{X_C} = \frac{1}{2}$.

What more can be said about the singularities of X_C? It was shown in [7] that on the threefold X, every "line" intersects a finite number of "lines." Therefore, the fibration $\phi_{|-K_V|}$ has a finite number of reducible fibers in the case where $\theta(C) \not\subset S$. In the case where $\theta(C) \subset S$, the fibration $\phi_{|-K_V|}$ has a one-dimensional family of reducible fibers, whose images on the quadric Q are lines tangent to the surface S in the points of the curve $\theta(C)$. Therefore, in the case where $\theta(C) \not\subset S$, the threefold X_C has terminal singularities, and in the case where $\theta(C) \subset S$, the threefold X_C has canonical singularities along an irreducible curve.

We have shown that for every "line" C on the threefold X, there is a nontrivial birational map

$$\psi_C = \phi_{|-nK_{\widehat{W}}|} \circ \rho$$

from X to a Fano threefold with canonical singularities X_C.

We fix the log pair

$$(X, \mathcal{M}_X) = (X, \sum_{i=1}^{N} b_i \mathcal{M}_i). \tag{7}$$

For $\mathcal{M}_X \neq \emptyset$, we consider $\lambda \in \mathbb{Q}_{>0}$ such that

$$K_X + \lambda \mathcal{M}_X \sim_{\mathbb{Q}} 0.$$

For $\mathcal{M}_X = \emptyset$, we formally set $\lambda = +\infty$.

We recall that we introduced the notion of a maximal log pair on a del Pezzo surface with the Picard group $\mathbb{Z}$ in Sec. 5. This notion is still valid in our current situation.

We now state the main result in this section.

Theorem 9.4. *Let* $\lambda = 1$ *and log pair* (7) *be maximal. Then* $\kappa(X, \mathcal{M}_X) = 0$, *and log pair* (7) *is canonical. If it is not terminal, then one of the following holds:*

1. *all linear systems* $\mathcal{M}_i$ *are composed from one pencil* $\mathcal{P}$ *in* $|-K_X|$;
2. *if in the commutative diagram*

$$\begin{tikzcd} & W \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, dashed, "\psi"] & & Y \end{tikzcd}$$

the threefold W *is smooth,* f *is birational, and the map* ψ *is a composition of the morphism* θ *and a projection from some line in the quadric* Q, *then* $f^{-1}(\mathcal{M}_X)$ *lies in the fibers of* g; *or*

3. *there is a "line" C on the threefold X such that the log pair*

$$(X_C, \mathcal{M}_{X_C}) = (X_C, \psi_C(\mathcal{M}_X))$$

is semiterminal.

Before proving Theorem 9.4, we state two more theorems.

Theorem 9.5. *Let $\lambda < 1$ and log pair (7) be maximal. Then $\kappa(X, \mathcal{M}_X) = -\infty$, and log pair (7) is terminal.*

Theorem 9.6. *Let $\lambda > 1$ and log pair (7) be maximal. Then one of the following holds:*

1. *$\kappa(X, \mathcal{M}_X) = 1$, log pair (7) is not canonical, all linear systems $\mathcal{M}_i$ are composed from one pencil $\mathcal{P}$ in $|-K_X|$, and $I(X, \mathcal{M}_X) = \phi_{\mathcal{P}}$;*
2. *$\kappa(X, \mathcal{M}_X) = 2$, log pair (7) is not canonical, and there is a commutative diagram*

$$\begin{tikzcd} & W \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, dashed, "\psi"] & & Y \end{tikzcd}$$

such that the threefold W is smooth, f is birational, the map ψ is a composition of the morphism θ and a projection from some line in the quadric Q, $f^{-1}(\mathcal{M}_X)$ lies in the fibers of g, and $I(X, \mathcal{M}_X) = \psi$; or

3. *$\kappa(X, \mathcal{M}_X) = 3$.*

We omit proofs of Theorems 9.5 and 9.6 because they are very similar to the proofs of Theorems 5.11, 5.12, 7.2, 7.3, 8.5, and 8.6. We split the proof of Theorem 9.4 into several lemmas.

Lemma 9.7. *In Theorem 9.4, $CS(X, \mathcal{M}_X)$ does not contain points.*

Proof. To prove this assertion, we change the words "hyperplane section" in Lemma 8.2 to "a general element of the linear system $|-K_X|$." ☐

Lemma 9.8. *Let $CS(X, \mathcal{M}_X)$ in Theorem 9.4 contain an irreducible reduced curve C. Then* $\deg(\theta(C)) \leq 4$.

Proof. We consider a general element H of the linear system $|-K_X|$. Then

$$4 = H \cdot \mathcal{M}_X^2 \geq \mathrm{mult}_C(\mathcal{M}_X^2) H \cdot C \geq \deg(C)$$

because

$$\mathrm{mult}_C(\mathcal{M}_X) \geq 1.$$

☐

Lemma 9.9. *In Lemma 9.8, the curve $\theta(C)$ is a plane.*

Proof. We can assume that θ_C is an isomorphism. If the curve $\theta(C)$ is not a plane, then it should be one of the following curves:

a. a smooth rational curve of degree 3 not lying on the surface S,
b. a smooth rational curve of degree 3 lying on the surface S,
c. a smooth rational curve of degree 4 not lying on the surface S,
d. a smooth rational curve of degree 4 lying on the surface S, or
e. a smooth rational curve of degree 4 contained in some hyperplane.

First, we show that cases a, b, c, and d are impossible. Let $f : W \to X$ be a blowup of the curve C and $E = f^{-1}(C)$. We show that the divisor $f^*(-2K_X) - E$ is nef.

In cases a and c,

$$\theta^{-1}(\theta(C)) = C \cup \tilde{C}.$$

It is easy to see that

$$\mathrm{Bs}(|f^*(-2K_X) - E|) = f^{-1}(\tilde{C}).$$

Therefore, the divisor $f^*(-2K_X)-E$ has a nonnegative intersection with all curves on the threefold W except possibly $f^{-1}(\tilde{C})$, but

$$(f^*(-2K_X)-E)\cdot f^{-1}(\tilde{C})=0.$$

In cases b and d,

$$\mathrm{Bs}(|f^*(-2K_X)-E|)\subset E.$$

Let s_∞ be an exceptional section of the ruled surface $f|_E: E\to C$. The nefness of the divisor $f^*(-2K_X)-E$ follows from the inequality

$$(f^*(-2K_X)-E)|_E\cdot s_\infty\geq 0.$$

We can easily show that $E^3=2-\deg(\theta(C))$ and

$$(f^*(-2K_X)-E)|_E\cdot s_\infty=2\deg(\theta(C))+\frac{s_\infty^2+2-\deg(\theta(c))}{2}.$$

Therefore, we must show that $s_\infty^2\geq -2-3\deg(\theta(C))$. Let

$$N_{X/C}\cong\mathcal{O}_C(m)\oplus\mathcal{O}_C(n)\quad\text{for } m\geq n.$$

Then

$$m+n=\deg(N_{X/C})=c_1(X)\cdot C-c_1(C)=\deg(\theta(C))-2,$$

and the exact sequence

$$0\to N_{\theta^{-1}(S)/C}\to N_{X/C}\to N_{X/\theta^{-1}(S)}|_C\to 0$$

implies $n\geq\deg(N_{\theta^{-1}(S)/C})=-2-\deg(\theta(C))$. Therefore,

$$s_\infty^2=n-m=2n+2-\deg(\theta(C))\geq -2-3\deg(\theta(C)).$$

The divisor $f^*(-2K_X)-E$ is therefore nef. We consider the log pair

$$(W,\mathcal{M}_W)=(W,f^{-1}(\mathcal{M}_X)).$$

Then

$$A=(f^*(2H)-E)\cdot\mathcal{M}_W^2\geq 0.$$

On the other hand,

$$A=(f^*(-2K_X)-E)\cdot(f^*(-K_X)-\mathrm{mult}_C(\mathcal{M}_X))^2.$$

Therefore,

$$A=8-\mathrm{mult}_C(\mathcal{M}_X)((2+\deg(\theta(C)))\,\mathrm{mult}_C(\mathcal{M}_X)+2\deg(\theta(C)))<0$$

because $\mathrm{mult}_C(\mathcal{M}_X)\geq 1$.

We now consider case e. We note that $\mathcal{M}_X^2=C$ and $\mathrm{mult}_C(\mathcal{M}_X)=1$. Hence, the surface H intersects the curve C in four different points: x_1, x_2, x_3, and x_4. Let $g: V\to H$ be their blowup and $E_i=g^{-1}(x_i)$, $i=1,\dots,4$. Then

$$\Big(\sum_{i=1}^{N}b_i g^{-1}(\mathcal{M}_i|_H)\Big)^2\sim_{\mathbb{Q}} g^*(H|_H)-\sum_{i=1}^{4}E_i.$$

The points $\theta(x_1)$, $\theta(x_2)$, $\theta(x_3)$, and $\theta(x_4)$ are contained in one plane, but they are not contained in one line. This implies that the linear system

$$|g^*(H|_H)-\sum_{i=1}^{4}E_i|$$

contains exactly one effective divisor D. On the other hand, the linear system $|nD|$ does not have fixed components for $n\gg 0$ and $D^2=0$. Therefore, for $n\gg 0$, the linear system $|nD|$ is free, and

$$\phi_{|nD|}(V)=\mathbb{P}^1.$$

Hence, for $k\in(1,n]$, the fibration $\phi_{|nD|}$ has the multiple fiber kD. Therefore, the arithmetic genus of the curve D should be equal to 1, but we can see that it is equal to 4. ☐

Lemma 9.10. *If the curve $\theta(C)$ in Lemma 9.7 is conic, then log pair (7) is canonical in a general point of the curve C and all linear systems $\mathcal{M}_i$ are composed from one pencil in the linear system $|-K_X|$.*

Proof. We consider a pencil $\mathcal{H}_C$ in the linear system $|-K_X|$ that contains surfaces containing the curve C. We note that the pencil $\mathcal{H}_C$ does not have basic components.

We have three cases:

a. $-K_X \cdot C = 4$, $\mathcal{M}_X^2 = C$, and $\theta|_C$ is a double cover of a conic $\theta(C) \subset Q$;
b. $-K_X \cdot C = 2$, $\theta|_C$ is an isomorphism to a conic $\theta(C) \subset Q$, and $\theta(C) \not\subset S$; and
c. $-K_X \cdot C = 2$, and $\theta|_C$ is an isomorphism to a conic $\theta(C) \subset S$.

First, we consider case a. We resolve the indeterminacy of the map $\phi_{\mathcal{H}_C}$ using the commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, dashed, "\phi_{\mathcal{H}_C}"] & & \mathbb{P}^1. \end{tikzcd}$$

We can assume that the threefold W is smooth and contains one f-exceptional divisor E lying over a general point of the curve C and that f is an isomorphism outside of C.

A general fiber D of g is a smooth K3 surface, and

$$D \sim f^*(-K_X) - E - \sum_{i=1}^{k} a_i F_i,$$

where $f(F_i)$ is a point for every divisor F_i. We consider the log pair

$$(D, \mathcal{M}_D) = (D, f^{-1}(\mathcal{M}_X)|_D).$$

Then

$$\mathcal{M}_D \sim_{\mathbb{Q}} ((1 - \mathrm{mult}_C(\mathcal{M}_X))E + \sum_{i=1}^{k} c_i F_i)|_D,$$

and all $c_i \in \mathbb{Q}$. Hence, $\mathrm{mult}_C(\mathcal{M}_X) = 1$ and $\mathcal{M}_D = \emptyset$. In particular, log pair (7) is canonical in a general point of the curve C. The condition $\mathcal{M}_D = \emptyset$ implies that all linear systems $\mathcal{M}_i$ are composed from one pencil $\mathcal{H}_C$.

We consider case b. Let

$$\theta^{-1}(\theta(C)) = C \cup \tilde{C}.$$

As in the proofs of Theorems 7.1 and 8.1, it is sufficient to show that $CS(X, \mathcal{M}_X)$ contains the curve $\tilde{C}$. We take a sufficiently general divisor D in the pencil $\mathcal{H}_C$. The divisor D is a smooth K3 surface, $\tilde{C} \subset D$, and

$$\mathcal{M}_X|_D = \mathrm{mult}_C(\mathcal{M}_X)C + \mathrm{mult}_{\tilde{C}}(\mathcal{M}_X)\tilde{C} + R,$$

where R is an effective divisor on D, whose support does not contain the curves C and $\tilde{C}$. On the surface D,

$$C^2 = \tilde{C}^2 = -2 \qquad \text{and} \qquad C \cdot \tilde{C} = 4.$$

Therefore,

$$2 = \mathcal{M}_X|_D \cdot \tilde{C} = 4\,\mathrm{mult}_C(\mathcal{M}_X) - 2\,\mathrm{mult}_{\tilde{C}}(\mathcal{M}_X) + R \cdot \tilde{C}.$$

Therefore, $\mathrm{mult}_{\tilde{C}}(\mathcal{M}_X) \geq 1$, and $CS(X, \mathcal{M}_X)$ contains the curve $\tilde{C}$. This implies that all linear systems $\mathcal{M}_i$ are composed from one pencil $\mathcal{H}_C$ and log pair (7) is canonical in general points of the curves C and $\tilde{C}$.

We now consider case c. Let $f : W \to X$ be a blowup of the curve C and $f^{-1}(C) = E$. Then the base locus of the linear system $f^{-1}(\mathcal{H}_C)$ consists of a section of the ruled surface $f_E : E \to C$. This section is denoted by $\tilde{C}$. We consider the log pair

$$(W, \mathcal{M}_W) = (W, f^{-1}(\mathcal{M}_X))$$

and a sufficiently general divisor D in the linear system $f^{-1}(\mathcal{H}_C)$. Then D is a smooth K3 surface, and

$$\mathcal{M}_W|_D = \mathrm{mult}_{\tilde{C}}(f^{-1}(\mathcal{M}_X)\tilde{C} + R,$$

where the support of the effective divisor R does not contain the curve $\tilde{C}$. On the other hand, on the surface D,

$$f^{-1}(\mathcal{M}_X)|_D \sim_{\mathbb{Q}} \tilde{C} + (1 - \mathrm{mult}_C(\mathcal{M}_X))E|_D,$$

and $\widetilde{C}^2 = -2$. Therefore,

$$\mathrm{mult}_{\widetilde{C}}(\mathcal{M}_W) = \mathrm{mult}_C(\mathcal{M}_X) = 1.$$

As in case a, we now find that all linear systems $\mathcal{M}_i$ are composed from one pencil $\mathcal{H}_C$ and log pair (7) is canonical in a general point of the curve C. □

Lemma 9.11. *In Lemma* 9.7, *let* $-K_X \cdot C = 2$ *and* $\theta(C)$ *be a line. Let* $\mathcal{H}_C$ *be a linear system of surfaces in* $|-K_X|$ *containing the curve* C. *We consider the commutative diagram*

$$\begin{tikzcd} & W \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, dashed, "\phi_{\mathcal{H}_C}"] & & \mathbb{P}^2, \end{tikzcd}$$

which resolves the indeterminacy of the rational map $\phi_{\mathcal{H}_C}$. *Then all linear systems* $f^{-1}(\mathcal{M}_i)$ *are contained in the fibers of the elliptic fibration* g.

Proof. We note that $\theta|_C$ is a double cover. We suppose that the curve C is smooth and f is a blowup of the curve C. We set $E = f^{-1}(C)$ and consider the log pair

$$(W, \mathcal{M}_W) = (W, f^{-1}(\mathcal{M}_X)).$$

The claim then follows from the equality

$$\mathcal{M}_W \cdot (f^{-1}(\mathcal{H}_C))^2 = 1 - \mathrm{mult}_C(\mathcal{M}_X).$$

We note that the previous arguments used only the following properties of f and C:

1. The two-dimensional linear system $f^{-1}(\mathcal{H}_C)$ is free.
2. The divisor E is not contained in the fibers of the fibration g.
3. All f-exceptional divisors except E are contained in fibers of g.

But all these properties hold for an arbitrary resolution of the indeterminacy of the map $\phi_{\mathcal{H}_C}$ without assuming smoothness of C. □

Lemma 9.12. *In Theorem* 9.4, *log pair* (7) *is canonical.*

Proof. We suppose that log pair (7) is not canonical. Then Lemmas 9.7–9.11 imply that log pair (7) is not canonical in a general point of an irreducible reduced curve C such that $K_X \cdot C = 1$. In particular, $\theta(C)$ is a line, and

$$\mathrm{mult}_C(\mathcal{M}_X) > 1.$$

We use the notation in Lemma 9.1. We suppose that the line $\theta(C)$ is not contained in the surface S. Then we can find $\mu \in \mathrm{Bir}(V)$ such that μ is a reflection in the fibers of an elliptic fibration $\phi_{|-K_V}$ with respect to the section G (see [7]). As shown in [7], μ is an isomorphism in codimension one, and its action on $\mathrm{Pic}(V)$ can be given by the relations

$$\begin{aligned} &\mu^*(K_V) = K_V, \\ &\mu^*(G) = G, \\ &\mu^*(g^{-1}(E)) = -8K_V - g^{-1}(E) + 2G. \end{aligned}$$

This implies that

$$\lambda(\mu) = \frac{1}{9 - 8\,\mathrm{mult}_C(\mathcal{M}_X)} > 1,$$

which contradicts the maximality of log pair (7).

We now suppose that $\theta(C) \subset S$. Then the boundary $\mathcal{M}_Y$ of the log pair

$$(W, \mathcal{M}_W) = (W, f^{-1}(\mathcal{M}_X))$$

has a negative intersection with the inverse images on W of all lines on Q tangent to the surface S in the points of the curve $\theta(C)$. This contradicts the movability of the log pair $(W, \mathcal{M}_W)$ because such lines span the surface on Q. $\square$

Proof of Theorem 9.4. Lemmas 9.7–9.12 imply that log pair (7) is canonical and $CS(X, \mathcal{M}_X)$ contains an irreducible reduced curve C such that $K_X \cdot C = 1$.

We use the notation in Lemma 9.1. We suppose that the log pair

$$(W, \mathcal{M}_W) = (W, f^{-1}(\mathcal{M}_X))$$

is terminal. We want to show that in this case, the log pair

$$(X_C, \mathcal{M}_{X_C}) = (X_C, \psi_C(\mathcal{M}_X))$$

is semiterminal.

We recall the construction of the birational map ψ_C. First, we blow up the curve C via $f : W \to X$, and let C_1 denote the unique base curve of the linear system $|-K_W|$. We then make the antiflip $\rho : W \dashrightarrow \widehat{W}$ in the curve C_1, and using the freeness of the linear system $|-nK_{\widehat{W}}|$ for $n \gg 0$, we set

$$\psi_C = \phi_{|-nK_{\widehat{W}}|} \circ \rho.$$

We take $\zeta \in \mathbb{Q}_{>1}$ such that the log pair $(W, \zeta\mathcal{M}_W)$ is still terminal. Then ρ is the log flip of the log pair $(W, \zeta\mathcal{M}_W)$. In particular, the log pair

$$(\widehat{W}, \zeta\mathcal{M}_{\widehat{W}}) = (\widehat{W}, \zeta\rho \circ f^{-1}(\mathcal{M}_X))$$

is terminal. On the threefold $\widehat{W}$, the relation

$$K_{\widehat{W}} + \mathcal{M}_{\widehat{W}} \sim_{\mathbb{Q}} 0$$

holds. Therefore, the morphism $\phi_{|-nK_{\widehat{W}}|}$ is is crepant for the log pair $(\widehat{W}, \zeta\mathcal{M}_{\widehat{W}})$. This implies the canonicity of the log pair $(X_C, \zeta\mathcal{M}_{X_C})$. Hence, the log pair $(X_C, \mathcal{M}_{X_C})$ is semiterminal.

We suppose that the log pair $(W, \mathcal{M}_W)$ is not terminal. Using Lemmas 9.7–9.11, we find that $CS(W, \mathcal{M}_W)$ contains a smooth irrreducible reduced curve T such that $f(T)$ is a "line" on the threefold X. We have four cases:

a. $T = C_1$,
b. $\theta \circ f(T) \cap \theta(C) = \emptyset$,
c. $f(T) = C$, and
d. $\theta \circ f(T) \cap \theta(C) \neq \emptyset$.

The proof of Lemma 9.11 implies that in case a, all linear systems $(f \circ g)^{-1}(|M_i)$ are contained in the fibers of the elliptic fibration $\phi_{|-K_V}$.

In cases b and c, we consider a sufficiently general divisor D_W in the linear system $|-K_W|$. It is a smooth K3 surface, and

$$\mathcal{M}_W|_{D_W} \sim_{\mathbb{Q}} C_1 + F,$$

where F is an elliptic curve such that $F \cdot C_2 = 1$. On the other hand,

$$\mathcal{M}_W|_{D_W} = \mathrm{mult}_{C_1}(\mathcal{M}_W)C_1 + R$$

for some effective divisor R such that the support of R does not contain C_1. Moreover, $\mathrm{mult}_{C_1}(\mathcal{M}_W) > 0$ because

$$\mathcal{M}_W|_{D_W} \cdot C_1 = -1.$$

In case b,

$$\emptyset \neq T \cap D_W \subset E,$$

and we can assume that R does not contain the points $T \cap D_W$. Taking the intersection of the divisor $\mathcal{M}_W|_{D_W}$ and a curve from the linear system $|F|$ passing through any point of $T \cdot D_W$, we obtain a contradiction.

In case c, $T \subset E$. Considering the intersection of the divisor $\mathcal{M}_W|_{D_W}$ and a fiber of the ruled surface E, we obtain $E \subset \mathcal{M}_W$. This contradicts the movability of the log pair $(W, \mathcal{M}_W)$.

To complete the proof, it remains to consider case d. Let $\mathcal{H}_T$ be a pencil consisting of surfaces in the linear system $|-K_W|$ passing through the curve T. We note that the pencil $\mathcal{H}_T$ consists of the inverse images on W of hyperplane sections of the quadric Q passing through the lines $\theta \circ f(C)$ and $\theta \circ f(T)$.

Let $h : U \to W$ be a blowup of the curve T. On the threefold U, the base locus of the pencil $|-K_U|$ consists of two smooth irreducible curves. One of these curves is $h^{-1}(C_1)$; let T_1 denote the other one. It is easy to verify that a general element D_U of the pencil $|-K_U|$ is a smooth K3 surface. On D_U,

$$h^{-1}(\mathcal{M}_X)|_{D_U} \sim_{\mathbb{Q}} h^{-1}(C_1) + T_1$$

and

$$h^{-1}(C_1)^2 = T_1^2 = (h^{-1}(C_1) + T_1)^2 = -2.$$

Hence,

$$\mathrm{mult}_{C_1}(\mathcal{M}_W) = \mathrm{mult}_{T_1}(\mathcal{M}_W) = 1.$$

As in the proof of Lemma 9.10, we can now show that all linear systems $\mathcal{M}_i$ are composed from one pencil $f(\mathcal{H}_T)$. □

Now, we state four corollaries of Theorems 9.4–9.6 describing birational transformations of the threefold X into Fano fibrations, elliptic fibrations, and fibrations on surfaces with Kodaira dimension zero. We omit their proofs because they are similar to the proofs of Lemma 5.13 and Corollaries 5.14 and 5.15.

Corollary 9.13. *The threefold X is not birationally isomorphic to a conic bundle and a fibration on rational surfaces.*

Corollary 9.14. *If the variety X is birationally isomorphic to an elliptic fibration $\tau : Y \dashrightarrow Z$ via a birational map ρ, then up to the action of* $\mathrm{Bir}(X)$*, the map $\tau \circ \rho$ is a composition of a double cover θ and a projection from some line on Q.*

Corollary 9.15. *If X is birationally isomorphic to a fibration on surfaces with Kodaira dimension zero $\tau : Y \dashrightarrow Z$ via a birational map ρ, then there is a pencil $\mathcal{P}$ in the linear system $|-K_X|$ such that $\tau \circ \rho = \phi_{\mathcal{P}}$ up to the action of* $\mathrm{Bir}(X)$.

Corollary 9.16. *If X is birationally isomorphic to a Fano threefold Y with canonical singularities, then either $Y \cong X$ or $Y \cong X_C$ for some "line" C on X.*

10. Threefold Conic Bundles

In this section, we obtain a three-dimensional generalization of the results of Sec. 6.

We consider the threefold Mori fibration $\pi : X \to S$ with $\dim(X/S) = 1$ and the log pair

$$(X, \mathcal{M}_X) = \left(X, \sum_{i=1}^{N} b_i \mathcal{M}_i\right). \tag{8}$$

We fix $\lambda \in \mathbb{Q}_{>0} \cup \{+\infty\}$ such that

$$K_X + \lambda \mathcal{M}_X \sim_{\mathbb{Q}} \pi^*(L),$$

where L is a $\mathbb{Q}$-Cartier divisor on the surface S. Moreover, if $\mathcal{M}_X$ is contained in the fibers of π, then we formally set $\gamma = +\infty$ and $L = f(\mathcal{M}_X)$.

We want to study the properties of log pair (8) with its dependence on λ. In general, this problem seems unsolvable. The results of Sec. 6 suggest that we should consider a "very degenerate" fibration π.

We now violate our agreement on the movability of log pairs. On the surface S, we fix the "classical" (immovable) log pair

$$\left(S, \frac{1}{4} D_S\right), \tag{9}$$

where D_S is a degeneration divisor of π.

In the rest of this section, all log pairs on surfaces are immovable. We trust that this will not create confusion.

We impose the following two conditions on the fibration π:

1. log pair (9) has canonical singularities and
2. $\kappa(X, \frac{1}{4}D_S) = 2$.

We note that if π is standard (see [16, 17]), then log pair (9) has canonical singularities.

The main result in this section is the following theorem.

Theorem 10.1. *Let $\lambda = 1$. Then there is a commutative diagram*

$$\begin{array}{ccc} X & \overset{\rho}{\dashrightarrow} & Y \\ \pi \downarrow & & \downarrow \tau \\ S & \overset{\mu}{\dashrightarrow} & Z \end{array}$$

such that

1. *the maps ρ and μ are birational,*
2. *τ is a Mori fibration,*
3. *the log pair*
$$(Y, \mathcal{M}_Y) = (Y, \rho(\mathcal{M}_X))$$
is canonical, and
4. *the relation*
$$K_Y + \mathcal{M}_Y \sim_{\mathbb{Q}} \tau^*(H)$$
holds, where H is nef and a big $\mathbb{Q}$-Cartier divisor on the surface Z.

In particular, $\kappa(X, \mathcal{M}_X) = 2$ and $I(X, \mathcal{M}_X) = \pi$.

The proof of Theorem 10.1 is similar to the proof of Theorem 6.3. But there are two principal differences. First, we can birationally transform not only the variety X but also the surface S. Second, the induction in the proof of Theorem 6.3 uses the fact that the base of a conic bundle is a curve.

We omit the proof of the following two theorems, which can be deduced from Theorem 10.1 as Lemma 6.5 is deduced from Theorem 6.3.

Theorem 10.2. *Let $\lambda > 1$. Then $\kappa(X, \mathcal{M}_X) = -\infty$, and there is a commutative diagram*

$$\begin{array}{ccc} X & \overset{\rho}{\dashrightarrow} & Y \\ \pi \downarrow & & \downarrow \tau \\ S & \overset{\mu}{\dashrightarrow} & Z \end{array}$$

such that ρ and μ are birational maps, τ is a Mori fibration, and the log pair

$$(Y, \mathcal{M}_Y) = (Y, \rho(\mathcal{M}_X))$$

is terminal.

Theorem 10.3. *If $\lambda < 1$, then $\kappa(X, \mathcal{M}_X) = 3$.*

We split the proof of Theorem 10.1 into several lemmas. In the rest of this section, we assume that $\lambda = 1$. The results in [13] imply the existence of a commutative diagram

$$\begin{array}{ccc} X & \overset{\psi}{\dashrightarrow} & \widehat{X} \\ \pi \downarrow & & \downarrow \hat{\pi} \\ S & \overset{\sigma}{\leftarrow} & \widehat{S} \end{array}$$

such that the map ψ and the morphism σ are birational, $\hat{\pi}$ is a Mori fibration, and the log pair

$$(\widehat{X}, \mathcal{M}_{\widehat{X}}) = (\widehat{X}, \psi(\mathcal{M}_X))$$

is canonical.

We note that the singularities of the surface $\widehat{S}$ should be log terminal.

Let $D_{\widehat{S}}$ be a degeneration divisor of the conic bundle $\hat{\pi}$.

Lemma 10.4. *The following equality holds:*

$$\kappa\left(K_{\widehat{S}}+\frac{1}{4}D_{\widehat{S}}\right)=2.$$

We note that Lemma 10.4 does not imply that

$$\kappa\left(\widehat{S},\frac{1}{4}D_{\widehat{S}}\right)=2,$$

because the singularities of the log pair $(\widehat{S},\frac{1}{4}D_{\widehat{S}})$ can be noncanonical a priori.

Proof of Lemma 10.4. We can see that the divisor

$$D_{\widehat{S}}-\sigma^{-1}(D_S)$$

is effective. The canonicity of the log pair $(S,\frac{1}{4}D_S)$ implies the effectiveness of some multiple of the divisor

$$K_{\widehat{S}}+\frac{1}{4}\sigma^{-1}(D_S)-\sigma^*(K_S+\frac{1}{4}D_S).$$

Therefore, for $n\gg 0$,

$$h^0(n(K_{\widehat{S}}+\frac{1}{4}D_{\widehat{S}}))\geq h^0(n(K_{\widehat{S}}+\frac{1}{4}\sigma^{-1}(D_S)))\geq h^0(n(K_S+\frac{1}{4}D_S))).$$

□

We consider the relation

$$K_{\widehat{X}}+\mathcal{M}_{\widehat{X}}\sim_{\mathbb{Q}}\hat{\pi}^*(\widehat{L}).$$

What can be said about the $\mathbb{Q}$-Cartier divisor $\widehat{L}$ on the surface $\widehat{S}$?

Lemma 10.5. *The following relation holds:*

$$\widehat{L}\sim_{\mathbb{Q}}K_{\widehat{S}}+\frac{1}{4}D_{\widehat{S}}+\frac{1}{4}\hat{\pi}_*(\mathcal{M}^2_{\widehat{X}}).$$

Proof. It is well known (see [16, 17]) that

$$-\hat{\pi}_*(K^2_{\widehat{X}})\sim_{\mathbb{Q}}K_{\widehat{S}}+\frac{1}{4}D_{\widehat{S}}.$$

The claim follows from

$$\hat{\pi}^*(\widehat{L})^2-2K_{\widehat{X}}\cdot\hat{\pi}^*(\widehat{L})\sim_{\mathbb{Q}}K^2_{\widehat{X}}.$$

□

Proof of Theorem 10.1. We show how to construct the desired commutative diagram by applying the LMMP to the log pair $(\widehat{X},\mathcal{M}_{\widehat{X}})$. Lemma 10.5 implies that the divisor $\widehat{L}$ can be considered the log canonical divisor of the log pair

$$(\widehat{S},\frac{1}{4}D_{\widehat{S}}+\frac{1}{4}\hat{\pi}_*(\mathcal{M}^2_{\widehat{X}})).$$

Unfortunately, we do not know the singularities of this log pair. Nevertheless, it follows from [12] that a boundary $\mathcal{B}_{\widehat{S}}$ on the surface $\widehat{S}$ exists such that

$$\mathcal{B}_{\widehat{S}}\sim_{\mathbb{Q}}\frac{1}{4}D_{\widehat{S}}+\frac{1}{4}\hat{\pi}_*(\mathcal{M}^2_{\widehat{X}})$$

and the log pair $(\widehat{S},\mathcal{B}_{\widehat{S}})$ has log terminal singularities. Then

$$K_{\widehat{X}}+\mathcal{M}_{\widehat{X}}\sim_{\mathbb{Q}}\hat{\pi}^*(K_{\widehat{S}}+\mathcal{B}_{\widehat{S}}).$$

If the divisor $K_{\widehat{S}}+\mathcal{B}_{\widehat{S}}$ is nef, then the log abundance implies that it is big, and we have the claim.

We suppose that the divisor $K_{\widehat{S}}+\mathcal{B}_{\widehat{S}}$ is not nef. Then the LMMP implies the existence of a birational morphism $q:\widehat{S}\to\widetilde{S}$ contracting one irreducible reduced curve C on the surface $\widehat{S}$ such that

$$(K_{\widehat{S}}+\mathcal{B}_{\widehat{S}})\cdot C<0.$$

We set $\mathcal{M}_{\widetilde{S}} = q(\mathcal{B}_{\widehat{S}})$. Then the log pair $(\widetilde{S}, \mathcal{M}_{\widetilde{S}})$ is log terminal, and

$$K_{\widehat{S}} + \mathcal{B}_{\widehat{S}} \sim_{\mathbb{Q}} q^*(K_{\widetilde{S}} + \mathcal{M}_{\widetilde{S}}) + aC$$

for $a \in \mathbb{Q}_{>0}$.

We now apply the LMMP over the surface $\widetilde{S}$ to the log pair $(\widehat{X}, \mathcal{M}_{\widehat{X}})$. We obtain the commutative diagram

$$\begin{array}{ccc} \widehat{X} & \overset{p}{\dashrightarrow} & \widetilde{X} \\ \hat{\pi}\downarrow & & \downarrow\tilde{\pi} \\ \widehat{S} & \overset{q}{\rightarrow} & \widetilde{S} \end{array}$$

such that the map p is birational. We consider the log pair

$$(\widetilde{X}, \mathcal{M}_{\widetilde{X}}) = (\widetilde{X}, p \circ \psi(\mathcal{M}_X)).$$

Let $G = \hat{\pi}^{-1}(C)$. Then we have two possibilities:

a. the map p is a composition of log flips and a contraction of the direct image of G, or
b. the map p is a composition of log flips, and the divisor $K_{\widetilde{X}} + \mathcal{M}_{\widetilde{X}}$ is $\tilde{\pi}$-nef.

In both cases, the threefold $\widetilde{X}$ has terminal $\mathbb{Q}$-factorial singularities, and

$$K_{\widetilde{X}} + \mathcal{M}_{\widetilde{X}} \sim_{\mathbb{Q}} \tilde{\pi}^*(K_{\widetilde{S}} + \mathcal{M}_{\widetilde{S}}) + ap(G).$$

In case a, $\tilde{\pi}$ is a Mori fibration, and

$$K_{\widetilde{X}} + \mathcal{M}_{\widetilde{X}} \sim_{\mathbb{Q}} \tilde{\pi}^*(K_{\widetilde{S}} + \mathcal{M}_{\widetilde{S}}).$$

We want to show that case b is impossible. We consider a very ample divisor H on $\widetilde{S}$ such that the divisor

$$K_{\widetilde{S}} + \mathcal{M}_{\widetilde{S}} + H$$

is ample. The log abundance for the log pair

$$(\widetilde{X}, \mathcal{M}_{\widetilde{X}} + |\tilde{\pi}^*(H)|)$$

implies that the linear system

$$|\tilde{\pi}^*(n(K_{\widetilde{S}} + \mathcal{M}_{\widetilde{S}} + H)) + anp(G)|$$

is free for $n \gg 0$. This contradicts the inequality $a > 0$.

Thus, we have shown that the conic bundle $\tilde{\pi} : \widetilde{X} \to \widetilde{S}$ is a Mori fibration,

$$K_{\widetilde{X}} + \mathcal{M}_{\widetilde{X}} \sim_{\mathbb{Q}} \tilde{\pi}^*(K_{\widetilde{S}} + \mathcal{M}_{\widetilde{S}}),$$

and

$$\kappa(K_{\widetilde{S}} + \mathcal{M}_{\widetilde{S}}) = \kappa(K_{\widehat{S}} + \mathcal{B}_{\widehat{S}}) = 2.$$

Repeating our construction at most $\mathrm{rk}(\mathrm{Pic}(\widehat{S}))$ times, we obtain the claim. □

The following corollary is deduced from Theorems 10.1–10.3 in the same way as Theorem 6.6 is deduced from Theorems 6.3, Definition 6.4, and Lemma 6.5. We therefore omit its proof.

Corollary 10.6. *The threefold X cannot be birationally transformed into a Fano threefold with canonical singularities, a fibration on rational surfaces, an elliptic fibration, or a fibration on surfaces with Kodaira dimension zero nonequivalent to a conic bundle π.*

A trivial example of a direct product of a rational curve and an elliptic surface shows that the conditions of Corollary 10.6 cannot be weakened.

We now give an example of a conic bundle with a base $\mathbb{P}^2$ satisfying our conditions.

Example 10.7. We consider the fourfold

$$V = \mathrm{Proj}(\mathcal{O}_{\mathbb{P}^2}(-5) \oplus \mathcal{O}_{\mathbb{P}^2}(-5) \oplus \mathcal{O}_{\mathbb{P}^2})$$

together with the natural projection $f : V \to \mathbb{P}^2$. Let X be a general divisor in the linear system

$$|\mathcal{O}_{V/\mathbb{P}^2}(2) + f^*(\mathcal{O}_{\mathbb{P}^2}(11))|.$$

Then the induced morphism $f|_X : X \to \mathbb{P}^2$ is a conic bundle. We can easily prove (see [2]) that

1. the threefold X is smooth,
2. $\mathrm{Pic}(X/\mathbb{P}^2) = \mathbb{Z}$,
3. the degeneration divisor D_S of $f|_X$ has only double points, and
4. $\deg(D_S) = 13$.

Therefore, X satisfies all assumptions in this section.

REFERENCES

1. M. Artin, "On isolated rational singularities of surfaces," *Amer. J. Math.*, **88**, 129–136 (1966).
2. A. Beauville, "Variétiés de Prym et Jacobiennes intermédiaires," *Ann. Sci. Ecole Norm. Sup.*, **10**, 309–399 (1977).
3. A. Corti, "Factorizing birational maps of threefolds after Sarkisov," *J. Alg. Geom.*, **4**, 223–254 (1995).
4. I. Dolgachev, "Rational surfaces with a pencil of elliptic curves," *Izv. Akad. Nauk SSSR, Ser. Mat.*, **30**, 1073–1100 (1966).
5. V. A. Iskovskikh, "Rational surfaces with a pencil of rational curves," *Mat. Sb.*, **74**, 608–638 (1967).
6. V. A. Iskovskikh, "Anticanonical models of three-dimensional algebraic manifolds," In: *Sovr. Problemy Matematiki*, Vol. 12 (1978), pp. 59–157.
7. V. A. Iskovskikh, "Birational automorphisms of three-dimensional algebraic manifolds," In: *Sovr. Problemy Matematiki*, Vol. 12 (1978), pp. 159–236.
8. V. A. Iskovskikh and Yu. I. Manin, "Three-dimensional quartics and counterexamples to the Lüroth problem," *Mat. Sb.*, **86**, 140–166 (1971).
9. Yu. I. Manin, "Rational surfaces over perfect fields," *Publ. Math. IHES*, **30**, 55–114 (1966).
10. Yu. I. Manin, "Rational surfaces over perfect fields: II," *Mat. Sb.*, **72**, 161–192 (1967).
11. Y. Kawamata, K. Matsuda, and K. Matsuki, "Introduction to the minimal model problem," *Adv. Stud. Pure Math.*, **10**, 283–360 (1987).
12. S. Keel, K. Matsuki, and J. McKernan, "Log abundance theorem for threefolds," *Duke Math. J.*, **75**, 99–119 (1994).
13. J. Kollár et al., "Flips and abundance for algebraic threefolds," *Astérique*, **211** (1992).
14. J. Kollár, *Rational Curves on Algebraic Varieties*, Springer-Verlag (1996).
15. A. V. Pukhlikov, "Essentials of the method of maximal singularities," Preprint (1996).
16. V. S. Sarkisov, "Birational automorphisms of fibers on conics," *Izv. Akad. Nauk SSSR, Ser. Mat.*, **44**, 918–945 (1980).
17. V. G. Sarkisov, "On the structure of fibers on conics," *Izv. Akad. Nauk SSSR, Ser. Mat.*, **46**, 371–408 (1982).
18. V. V. Shokurov, "3-Fold log models," *J. Math. Sci.*, **81**, 2667–2699 (1996).