

Del Pezzo Surfaces and Local Inequalities

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Abstract I prove new local inequality for divisors on smooth surfaces, describe its applications, and compare it to a similar local inequality that is already known by experts.

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Let X be a Fano variety of dimension $n \geq 1$ with at most Kawamata log terminal singularities (see [6, Definition 6.16]). In many applications, it is useful to *measure* how singular effective \mathbb{Q} -divisors D on X can be provided that $D \sim_{\mathbb{Q}} -K_X$. Of course, this can be done in many ways depending on what I mean by *measure*. A possible *measurement* can be given by the so-called α -invariant of the Fano variety X that can be defined as

$$\alpha(X) = \sup \left\{ \lambda \in \mathbb{Q} \left| \begin{array}{l} \text{the pair } (X, \lambda D) \text{ is Kawamata log terminal} \\ \text{for every effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X. \end{array} \right. \right\} \in \mathbb{R}.$$

The invariant $\alpha(X)$ has been studied intensively by many people who used different notation for $\alpha(X)$. The notation $\alpha(X)$ is due to Tian who defined $\alpha(X)$ in a different way. However, his definition coincides with the one I just gave by [4, Theorem A.3]. The α -invariants play a very important role in Kähler geometry due to

Throughout this chapter, I assume that most of the considered varieties are algebraic, normal, and defined over complex numbers.

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Theorem 1 ([13], [7, Criterion 6.4]). *Let X be a Fano variety of dimension n that has at most quotient singularities. If $\alpha(X) > \frac{n}{n+1}$, then X admits an orbifold Kähler–Einstein metric.*

The α -invariants are usually very tricky to compute. But they are computed in many cases. For example, the α -invariants of smooth del Pezzo surfaces have been computed as follows:

Theorem 2 ([1, Theorem 1.7]). *Let S_d be a smooth del Pezzo surface of degree d . Then*

$$\begin{aligned}\alpha(S_d) &= \begin{cases} \frac{1}{3} & \text{if } d = 9, 7 \text{ or } S_d = \mathbb{F}_1, \\ \frac{1}{2} & \text{if } d = 5, 6 \text{ or } S_d = \mathbb{P}^1 \times \mathbb{P}^1, \\ \frac{2}{3} & \text{if } d = 4, \end{cases} \\ \alpha(S_3) &= \begin{cases} \frac{2}{3} & \text{if } S_3 \text{ is a cubic surface in } \mathbb{P}^3 \text{ with an Eckardt point,} \\ \frac{3}{4} & \text{if } S_3 \text{ is a cubic surface in } \mathbb{P}^3 \text{ without Eckardt points,} \end{cases} \\ \alpha(S_2) &= \begin{cases} \frac{3}{4} & \text{if } |-K_{S_2}| \text{ has a tacnodal curve,} \\ \frac{5}{6} & \text{if } |-K_{S_2}| \text{ has no tacnodal curves,} \end{cases} \\ \alpha(S_1) &= \begin{cases} \frac{5}{6} & \text{if } |-K_{S_1}| \text{ has a cuspidal curve,} \\ 1 & \text{if } |-K_{S_1}| \text{ has no cuspidal curves.} \end{cases}\end{aligned}$$

Note that $\alpha(X) < 1$ if and only if there exists an effective \mathbb{Q} -divisor D on X such that $D \sim_{\mathbb{Q}} -K_X$ and the pair (X, D) is not log canonical. Such divisors (if they exist) are called non-log canonical *special tigers* by Keel and McKernan (see [9, Definition 1.13]). They play an important role in birational geometry of X . How does one describe non-log canonical special tigers? Note that if D is a non-log canonical special tiger on X , then

$$(1 - \mu)D + \mu D'$$

is also a non-log canonical special tiger on X for any effective \mathbb{Q} -divisor D' on X such that $D' \sim_{\mathbb{Q}} -K_X$ and any sufficiently small $\mu \geq 0$. Thus, to describe non-log canonical special tigers on X , I only need to consider those of them whose supports do not contain supports of other non-log canonical special tigers. Let me call such non-log canonical special tigers *Siberian tigers*. Unfortunately, Siberian tigers are

not easy to describe in general. But sometimes it is possible. For example, Kosta proved

Lemma 3 ([11, Lemma 3.1]). *Let S be a hypersurface of degree 6 in $\mathbb{P}(1, 1, 2, 3)$ that has exactly one singular point O . Suppose that O is a Du Val singular point of type \mathbb{A}_3 . Then all Siberian tigers on X are cuspidal curves in $|-K_S|$, which implies, in particular, that*

$$\alpha(S) = \begin{cases} \frac{5}{6} & \text{if there is a cuspidal curve in } |-K_S|, \\ 1 & \text{otherwise.} \end{cases}$$

The original proof of Lemma 3 is *global* and lengthy. In [11], Kosta applied the very same *global* method to compute the α -invariants of all del Pezzo surfaces of degree 1 that has at most Du Val singularities (in most of cases her computations do not give description of Siberian tigers). Later I noticed that the nature of her *global* method is, in fact, purely *local*. Implicitly, Kosta proved

Theorem 4 ([3, Corollary 1.29]). *Let S be a surface, let P be a smooth point in S , let Δ_1 and Δ_2 be two irreducible curves on S that are both smooth at P and intersect transversally at P , and let a_1 and a_2 be non-negative rational numbers. Suppose that $\frac{2n-2}{n+1}a_1 + \frac{2}{n+1}a_2 \leq 1$ for some positive integer $n \geq 3$. Let Ω be an effective \mathbb{Q} -divisor on the surface S whose support does not contain the curves Δ_1 and Δ_2 . Suppose that the log pair $(S, a_1\Delta_1 + a_2\Delta_2 + \Omega)$ is not log canonical at P . Then $\text{mult}_P(\Omega \cdot \Delta_1) > 2a_1 - a_2$ or $\text{mult}_P(\Omega \cdot \Delta_2) > \frac{n}{n-1}a_2 - a_1$.*

Unfortunately, Theorem 4 has a very limited application scope. Together with Kosta, I generalized Theorem 4 as

Theorem 5 ([3, Theorem 1.28]). *Let S be a surface, let P be a smooth point in S , let Δ_1 and Δ_2 be two irreducible curves on S that both are smooth at P and intersect transversally at P , let a_1 and a_2 be non-negative rational numbers, and let Ω be an effective \mathbb{Q} -divisor on the surface S whose support does not contain the curves Δ_1 and Δ_2 . Suppose that the log pair $(S, a_1\Delta_1 + a_2\Delta_2 + \Omega)$ is not log canonical at P . Suppose that there are non-negative rational numbers α, β, A, B, M , and N such that $\alpha a_1 + \beta a_2 \leq 1$, $A(B-1) \geq 1$, $M \leq 1$, $N \leq 1$, $\alpha(A+M-1) \geq A^2(B+N-1)\beta$, $\alpha(1-M) + A\beta \geq A$. Suppose, in addition, that $2M + AN \leq 2$ or $\alpha(B+1-MB-N) + \beta(A+1-AN-M) \geq AB-1$. Then $\text{mult}_P(\Omega \cdot \Delta_1) > M + Aa_1 - a_2$ or $\text{mult}_P(\Omega \cdot \Delta_2) > N + Ba_2 - a_1$.*

Despite the fact that Theorem 5 looks very ugly, it is much more flexible and much more applicable than Theorem 4. By [6, Exercice 6.26], an analogue of Theorem 5 holds for surfaces with at most quotient singularities. This helped me to find in [2] many new applications of Theorem 5 that do not follow from Theorem 4.

Remark 6. How does one apply Theorem 5? Let me say few words about this. Let S be a smooth surface, and let D be an effective \mathbb{Q} -divisor on S . The purpose of

Theorem 5 is to prove that (S, D) is log canonical provided that D satisfies some *global* numerical conditions. To do so, I assume that (S, D) is not log canonical at P and seek for a contradiction. First, I look for some nice curves that pass through P that has very small intersection with D . Suppose I found two such curves, say Δ_1 and Δ_2 , that are both irreducible and both pass through P . If Δ_1 or Δ_2 are not contained in the support of the divisor D , I can bound $\text{mult}_P(D)$ by $D \cdot \Delta_1$ or $D \cdot \Delta_2$ and, hopefully, get a contradiction with $\text{mult}_P(D) > 1$, which follows from the fact (S, D) is not log canonical at P . This shows that I should look for the curves Δ_1 and Δ_2 among the curves which are close enough to the boundary of the Mori cone $\overline{\text{NE}}(S)$. Suppose that both curves Δ_1 and Δ_2 lie in the boundary of the Mori cone $\overline{\text{NE}}(S)$. Then $\Delta_1^2 \leq 0$ and $\Delta_2^2 \leq 0$. Keeping in mind, that the curves Δ_1 and Δ_2 can, a priori, be contained in the support of the divisor D , I must put $D = a_1 \Delta_1 + a_2 \Delta_2 + \Omega$ for some non-negative rational numbers a_1 and a_2 , where Ω is an effective \mathbb{Q} -divisor on S whose support does not contain the curves Δ_1 and Δ_2 . Then I try to bound a_1 and a_2 using some *global* methods. Usually, I end up with two non-negative rational numbers α and β such that $\alpha a_1 + \beta a_2 \leq 1$. Put $M = D \cdot \Delta_1$, $N = C \cdot \Delta_2$, $A = -\Delta_1^2$, and $B = -\Delta_2^2$. Suppose that Δ_1 and Δ_2 are both smooth at P and intersect transversally at P (otherwise I need to blow up the surface S and replace the pair (S, D) by its *log pull back*). If I am lucky, then $A(B - 1) \geq 1$, $M \leq 1$, $N \leq 1$, $\alpha(A + M - 1) \geq A^2(B + N - 1)\beta$, $\alpha(1 - M) + \beta \geq A$, and either $2M + AN \leq 2$ or $\alpha(B + 1 - MB - N) + \beta(A + 1 - AN - M) \geq AB - 1$ (or both), which implies that

$$M + Aa_1 - a_2 \geq M + Aa_1 - a_2 \Delta_1 \cdot \Delta_2 = \Omega \cdot \Delta_1 \geq \text{mult}_P(\Omega \cdot \Delta_1) > M + Aa_1 - a_2$$

or

$$N + Ba_2 - a_1 \geq N + Ba_2 - a_1 \Delta_1 \cdot \Delta_2 = \Omega \cdot \Delta_2 \geq \text{mult}_P(\Omega \cdot \Delta_2) > N + Ba_2 - a_1$$

by Theorem 5. This is the contradiction I was looking for.

Unfortunately, the hypotheses of Theorem 5 are not easy to verify in general. Moreover, the proof of Theorem 5 is very lengthy. It seems likely that Theorem 5 is a special case or, perhaps, a corollary of a more general statement that looks better and has a shorter proof. Ideally, the proof of such generalization, if it exists, should be inductive like the proof of

Theorem 7 ([6, Exercise 6.31]). *Let S be a surface, let P be a smooth point in S , let Δ be an irreducible curve on S that is smooth at P , let a be a non-negative rational number such that $a \leq 1$, and let Ω be an effective \mathbb{Q} -divisor on the surface S whose support does not contain the curve Δ . Suppose that the log pair $(S, a\Delta + \Omega)$ is not log canonical at P . Then $\text{mult}_P(\Omega \cdot \Delta) > 1$.*

Proof. Put $m = \text{mult}(\Omega)$. If $m > 1$, then I am done, since $\text{mult}_P(\Omega \cdot \Delta) \geq m$. In particular, I may assume that the log pair $(S, a\Delta + \Omega)$ is log canonical in a punctured neighborhood of the point P . Since the log pair $(S, a\Delta + \Omega)$ is not log

canonical at P , there exists a birational morphism $h: \hat{S} \rightarrow S$ that is a composition of $r \geq 1$ blow ups of smooth points dominating P , and there exists an h -exceptional divisor, say E_r , such that $e_r > 1$, where e_r is a rational number determined by

$$K_{\hat{S}} + a\hat{\Delta} + \hat{\Omega} + \sum_{i=1}^r e_i E_i \sim_{\mathbb{Q}} h^*(K_S + a\Delta + \Omega),$$

where each e_i is a rational number, each E_i is an h -exceptional divisor, $\hat{\Omega}$ is a proper transform on \hat{S} of the divisor Ω , and $\hat{\Delta}$ is a proper transform on \hat{S} of the curve Δ .

Let $f: \tilde{S} \rightarrow S$ be the blow up of the point P , let $\tilde{\Omega}$ be the proper transform of the divisor Ω on the surface \tilde{S} , let E be the f -exceptional curve, and let $\tilde{\Delta}$ be the proper transform of the curve Δ on the surface \tilde{S} . Then the log pair $(\tilde{S}, a\tilde{\Delta} + (a + m - 1)E + \tilde{\Omega})$ is not log canonical at some point $Q \in E$.

Let me prove the inequality $\text{mult}_P(\Omega \cdot \Delta) > 1$ by induction on r . If $r = 1$, then $a + m - 1 > 1$, which implies that $m > 2 - a \geq 1$. This implies that $\text{mult}_P(\Omega \cdot \Delta) > 1$ if $r = 1$. Thus, I may assume that $r \geq 2$. Since

$$\text{mult}_P(\Omega \cdot \Delta) \geq m + \text{mult}_Q(\tilde{\Omega} \cdot \tilde{\Delta}),$$

it is enough to prove that $m + \text{mult}_Q(\tilde{\Omega} \cdot \tilde{\Delta}) > 1$. Moreover, I may assume that $m \leq 1$, since $\text{mult}_P(\Omega \cdot \Delta) \geq m$. Then the log pair $(\tilde{S}, a\tilde{\Delta} + (a + m - 1)E + \tilde{\Omega})$ is log canonical at a punctured neighborhood of the point $Q \in E$, since $a + m - 1 \leq 2$.

If $Q \notin \tilde{\Delta}$, then the log pair $(\tilde{S}, (a + m - 1)E + \tilde{\Omega})$ is not log canonical at the point Q , which implies that

$$m = \tilde{\Omega} \cdot E \geq \text{mult}_Q(\tilde{\Omega} \cdot E) > 1$$

by induction. The latter implies that $Q = \tilde{\Delta} \cap E$, since $m \leq 1$. Then

$$a + m - 1 + \text{mult}_Q(\tilde{\Omega} \cdot \tilde{\Delta}) = \text{mult}_Q\left(\left((a + m - 1)E + \tilde{\Omega}\right) \cdot \tilde{\Delta}\right) > 1$$

by induction. This implies that $\text{mult}_Q(\tilde{\Omega} \cdot \tilde{\Delta}) > 2 - a - m$. Then $m + \text{mult}_Q(\tilde{\Omega} \cdot \tilde{\Delta}) > 2 - a \geq 1$ as required. \square

Recently, I jointly with Park and Won proved that all Siberian tigers on smooth cubic surfaces are just anticanonical curves that have non-log canonical singularities (see [5, Theorem 1.12]). This follows from

Theorem 8 ([5, Corollary 1.13]). *Let S be a smooth cubic surface in \mathbb{P}^3 , let P be a point in S , let T_P be the unique hyperplane section of the surface S that is singular at P , let D be any effective \mathbb{Q} -divisor on the surface S such that $D \sim_{\mathbb{Q}} -K_S$. Then*

(S, D) is log canonical at P provided that $\text{Supp}(D)$ does not contain at least one irreducible component of $\text{Supp}(T_P)$.

Siberian tigers on smooth del Pezzo surfaces of degree 1 and 2 are also just anti-canonical curves that have non-log canonical singularities (see [5, Theorem 1.12]). This follows easily from the proofs of [1, Lemmas 3.1 and 3.5]. Surprisingly, smooth del Pezzo surfaces of degree 4 contains much more Siberian tigers.

Example 9. Let S be a smooth complete intersection of two quadric hypersurfaces in \mathbb{P}^4 , let L be a line on S , and let P_0 be a point in L such that L is the only line in S that passes through P_0 . Then there exists exactly five conics in S that pass through P_0 . Let me denote them by $C_1^0, C_2^0, C_3^0, C_4^0$, and C_5^0 . Then

$$\frac{\sum_{i=1}^5 C_i^0}{3} + \frac{2}{3}L \sim_{\mathbb{Q}} -K_S,$$

is a Siberian tiger. Let Z be a general smooth rational cubic curve in S such that $Z + L$ is cut out by a hyperplane section and $P \in Z$. Then $Z \cap L$ consists of a point P and another point which I denote by Q . Let $f: \tilde{S} \rightarrow S$ be a blow up of the point Q , and let E be its exceptional curve. Denote by \tilde{L} and \tilde{Z} the proper transforms of the curves L and Z on the surface \tilde{S} , respectively. Then $\tilde{Z} \cap \tilde{L} = \emptyset$. Let $g: \hat{S} \rightarrow \tilde{S}$ be the blow up of the point $\tilde{Z} \cap E$, and let F be its exceptional curve. Denote by \hat{E}, \hat{L} and \hat{Z} the proper transforms of the curves E, \tilde{L} and \tilde{Z} on the surface \hat{S} , respectively. Then \hat{S} is a minimal resolution of a singular del Pezzo surface of degree 2, and $|-K_{\hat{S}}|$ gives a morphism $\hat{S} \rightarrow \mathbb{P}^2$ that is a double cover away from the curves \hat{E} and \hat{L} . This double cover induces an involution $\tau \in \text{Bir}(S)$. Put $C_i^1 = \tau(C_i^0)$ for every i . Then $C_1^1, C_2^1, C_3^1, C_4^1$ and C_5^1 are curves of degree 5 that all intersect exactly in one point in L . Denote this point by P_1 . Iterate this construction k times. This gives me five irreducible curves $C_1^k, C_2^k, C_3^k, C_4^k$ and C_5^k that intersect exactly in one point P_k . Then

$$\frac{\sum_{i=1}^5 C_i^k}{a_{2k+1} + a_{2k+3}} + \frac{4a_{2k+1} - a_{2k+3}}{a_{2k+1} + a_{2k+3}}L \sim_{\mathbb{Q}} -K_S, \quad (1)$$

where a_i is the i -th Fibonacci number. Moreover, each curve C_i^k is a curve of degree a_{2k+3} . Furthermore, the log canonical threshold of the divisor (1) is

$$\frac{a_{2k+3}(a_{2k+1} + a_{2k+3})}{1 + a_{2k+3}(a_{2k+1} + a_{2k+3})} < 1,$$

which easily implies that the divisor (1) is a Siberian tiger.

Quite surprisingly, Theorem 8 has other applications as well. For example, it follows from [10, Corollary 2.12], [5, Lemma 1.10] and Theorem 8 that every cubic cone in \mathbb{A}^4 having unique singular point does not admit non-trivial regular \mathbb{G}_a -actions (cf. [8, Question 2.22]).

The crucial part in the proof of Theorem 8 is played by two sibling lemmas. The first one is

Lemma 11 ([5, Lemma 4.8]). *Let S be a smooth cubic surface in \mathbb{P}^3 , let P be a point in S , let T_P be the unique hyperplane section of the surface S that is singular at P , let D be any effective \mathbb{Q} -divisor on the surface S such that $D \sim_{\mathbb{Q}} -K_S$. Suppose that T_P consists of three lines such that one of them does not pass through P . Then (S, D) is log canonical at P .*

Its younger sister is

Lemma 12 ([5, Lemma 4.9]). *Let S be a smooth cubic surface in \mathbb{P}^3 , let P be a point in S , let T_P be the unique hyperplane section of the surface S that is singular at P , let D be any effective \mathbb{Q} -divisor on the surface S such that $D \sim_{\mathbb{Q}} -K_S$. Suppose that T_P consists of a line and a conic intersecting transversally. Then (S, D) is log canonical at P .*

The proofs of Lemmas 11 and 12 we found in [5] are *global*. In fact, they resemble the proofs of classical results by Segre and Manin on cubic surfaces (see [6, Theorems 2.1 and 2.2]). Once the paper [5] has been written, I asked myself a question: can I prove Lemmas 11 and 12 using just *local* technique? To answer this question, let me sketch their *global* proofs first.

Global proof of Lemma 11. Let me use the notation and assumptions of Lemma 11. I write $T_P = L + M + N$, where L , M , and N are lines on the cubic surface S . Without loss of generality, I may assume that the line N does not pass through the point P . Let D be any effective \mathbb{Q} -divisor on the surface S such that $D \sim_{\mathbb{Q}} -K_S$. I must show that (S, D) is log canonical at P . Suppose that the log pair (S, D) is not log canonical at the point P . Let me seek for a contradiction.

Put $D = aL + bM + cN + \Omega$, where a , b , and c are non-negative rational numbers and Ω is an effective \mathbb{Q} -divisor on S whose support contains none of the lines L , M and N . Put $m = \text{mult}_P(\Omega)$. Then $a \leq 1$, $b \leq 1$ and $c \leq 1$. Moreover, the pair (S, D) is log canonical outside finitely many points. This follows from [6, Lemma 5.3.6] and is very easy to prove (see, for example, [5, Lemma 4.1] or the proof of [1, Lemma 3.4]).

Since (S, D) is not log canonical at the point P , I have

$$m + a + b = \text{mult}_P(D) > 1$$

by [6, Exercice 6.18] (this also follows from Theorem 7). In particular, the rational number a must be positive, since otherwise I would have

$$1 = L \cdot D \geq \text{mult}_P(D) > 1.$$

Similarly, the rational number b must be positive as well.

The inequality $m + a + b > 1$ is very handy. However, a stronger inequality $m + a + b > c + 1$ holds. Indeed, there exists a non-negative rational number μ

such that the divisor $(1 + \mu)D - \mu T_P$ is effective and its support does not contain at least one components of T_P . Now to obtain $m + a + b > c + 1$, it is enough to apply [6, Exercice 6.18] to the divisor $(1 + \mu)D - \mu T_P$, since $(S, (1 + \mu)D - \mu T_P)$ is not log canonical at P .

Since a, b, c do not exceed 1 and $(S, L + M + N)$ is log canonical, $\Omega \neq 0$. Let me write $\Omega = \sum_{i=1}^r e_i C_i$, where every e_i is a positive rational number, and every C_i is an irreducible reduced curve of degree $d_i > 0$ on the surface S . Then

$$a + b + c + \sum_{i=1}^r e_i d_i = 3,$$

since $-K_S \cdot D = 3$.

Let $f: \tilde{S} \rightarrow S$ be a blow up of the point P , and let E be the exceptional divisor of f . Denote by \tilde{L}, \tilde{M} and \tilde{N} the proper transforms on \tilde{S} of the lines L, M and N , respectively. For each i , denote by \tilde{C}_i the proper transform of the curve C_i on the surface \tilde{S} . Then

$$K_{\tilde{S}} + a\tilde{L} + b\tilde{M} + c\tilde{N} + (a + b + m - 1)E + \sum_{i=1}^r e_i \tilde{C}_i \sim_{\mathbb{Q}} f^*(K_S + D),$$

which implies that the log pair $(\tilde{S}, a\tilde{L} + b\tilde{M} + c\tilde{N} + (a + b + m - 1)E + \sum_{i=1}^r e_i \tilde{C}_i)$ is not log canonical at some point $Q \in E$.

I claim that either $Q \in \tilde{L} \cap E$ or $Q \in \tilde{M} \cap E$. Indeed, it follows from

$$\begin{cases} 1 = D \cdot L = (aL + bM + cN + \Omega) \cdot L = -a + b + c + \Omega \cdot L \geq -a + b + c + m, \\ 1 = D \cdot M = (aL + bM + cN + \Omega) \cdot M = a - b + c + \Omega \cdot M \geq a - b + c + m, \\ 1 = D \cdot N = (aL + bM + cN + \Omega) \cdot N = a + b - c + \Omega \cdot N \geq a + b - c, \end{cases}$$

that $m \leq 1 - c$ and $a + b + m - 1 \leq 1$, because $a \leq 1$ and $b \leq 1$. On the other hand, if $Q \notin \tilde{L} \cup \tilde{M}$, then the log pair $(\tilde{S}, (a + b + m - 1)E + \sum_{i=1}^r e_i \tilde{C}_i)$ is not log canonical at Q , which implies that

$$m = \left(\sum_{i=1}^r e_i \tilde{C}_i \right) \cdot E > 1$$

by Theorem 7. This shows that either $Q \in \tilde{L} \cap E$ or $Q \in \tilde{M} \cap E$, since $m \leq 1 - c \leq 1$. Without loss of generality, I may assume that $Q = \tilde{L} \cap E$.

Let $\rho: S \dashrightarrow \mathbb{P}^2$ be the linear projection from the point P . Then ρ is a generically two-to-one rational map. Thus the map ρ induces an involution $\tau \in \text{Bir}(S)$ known as the Geiser involution (see [6, Sect. 2.14]). The involution τ is biregular outside $P \cup N$, $\tau(L) = L$ and $\tau(M) = M$.

For each i , denote by \hat{d}_i the degree of the curve $\tau(C_i)$. Put $\hat{\Omega} = \sum_{i=1}^r e_i \tau(C_i)$. Then

$$aL + bM + (a + b + m - 1)N + \hat{\Omega} \sim_{\mathbb{Q}} -K_S,$$

and $(S, aL + bM + (a + b + m - 1)M + \hat{\Omega})$ is not log canonical at the point $L \cap N$. Thus, I can replace the original effective \mathbb{Q} -divisor D by the divisor

$$aL + bM + (a + b + m - 1)N + \hat{\Omega} \sim_{\mathbb{Q}} -K_S$$

that has the same properties as D . Moreover, I have

$$\sum_{i=1}^r e_i \hat{d}_i < \sum_{i=1}^r e_i d_i,$$

since $m + a + b > c + 1$. Iterating this process, I obtain a contradiction after finitely many steps. \square

Global proof of Lemma 12. Let me use the notations and assumptions of Lemma 12. I write $T_P = L + C$, where L is a line, and C is a conic. Let D be any effective \mathbb{Q} -divisor on the surface S such that $D \sim_{\mathbb{Q}} -K_S$. I must show that the log pair (S, D) is log canonical at P . Suppose that (S, D) is not log canonical at the point P . Let me seek for a contradiction.

Let me write $D = nL + kC + \Omega$, where n and k are non-negative rational numbers and Ω is an effective \mathbb{Q} -divisor on S whose support contains none of the curves L and C . Put $m = \text{mult}_P(\Omega)$. Then $2n + m \leq 2$ and $2k + m \leq 1 + n$, since

$$\begin{cases} 1 = D \cdot L = (nL + kC + \Omega) \cdot L = -n + 2k + \Omega \cdot L \geq -n + 2k + m, \\ 2 = D \cdot C = (nL + kC + \Omega) \cdot C = 2n + \Omega \cdot C \geq 2n + m. \end{cases}$$

Arguing as in the proof [1, Lemma 3.4], I see that the log pair (S, D) is log canonical outside finitely many points (this follows, for example, from [6, Lemma 5.3.6]). In particular, both rational numbers n and k do not exceed 1. On the other hand, it follows from [6, Exercice 6.18] that

$$m + n + k = \text{mult}_P(D) > 1,$$

because the log pair (S, D) is not log canonical at the point P . The later implies that $n > 0$, since $1 = L \cdot D \geq \text{mult}_P(D)$ if $n = 0$.

I claim that $n > k$ and $m + n > 1$. Indeed, there exists a non-negative rational number μ such that the divisor $(1 + \mu)D - \mu T_P$ is effective and its support does not contain at least one components of T_P . Then $(S, (1 + \mu)D - \mu T_P)$ is not log

canonical at P . If $n \leq k$, then the support of $(1 + \mu)D - \mu T_P$ does not contain L , which is impossible, since

$$\text{mult}_P((1 + \mu)D - \mu T_P) > 1$$

and $1 = L \cdot ((1 + \mu)D - \mu T_P)$. Thus, I proved that $n > k$. Now I can apply [6, Exercice 6.18] to the divisor $(1 + \mu)D - \mu T_P$ and obtain $m + n > 1$.

Let $f: \tilde{S} \rightarrow S$ be the blow up of the point P , let $\tilde{\Omega}$ be the proper transform of the divisor Ω on the surface \tilde{S} , let \tilde{L} be the proper transform of the line L on the surface \tilde{S} , let \tilde{C} be the proper transform of the conic C on the surface \tilde{S} , and let E be the f -exceptional curve. Then

$$K_{\tilde{S}} + n\tilde{L} + k\tilde{C} + \tilde{\Omega} + (n + k + m - 1)E \sim_{\mathbb{Q}} f^*(K_S + D) \sim_{\mathbb{Q}} 0,$$

which implies that the log pair $(\tilde{S}, n\tilde{L} + k\tilde{C} + (n + k + m - 1)E + \tilde{\Omega})$ is not log canonical at some point $Q \in E$. On the other hand, I must have $n + k + m - 1 \leq 1$, because $2n + m \leq 2$, $2k + m \leq 1 + n$ and $n \leq 1$.

I claim that $Q \in \tilde{L}$. Indeed, if $Q \in \tilde{C}$, then the log pair $(\tilde{S}, k\tilde{C} + (n + k + m - 1)E + \tilde{\Omega})$ is not log canonical at Q , which implies that $k > n$, since

$$1 - n + k = (\tilde{\Omega} + (n + k + m - 1)E) \cdot \tilde{C} > 1,$$

by Theorem 7. Since I proved already that $n > k$, the curve \tilde{C} does not contain Q . Thus, if $Q \notin \tilde{L}$, then $Q \notin \tilde{L} \cup \tilde{C}$, which contradicts [5, Lemma 3.2], since

$$n\tilde{L} + k\tilde{C} + \tilde{\Omega} + (n + k + m - 1)E \sim_{\mathbb{Q}} -K_{\tilde{S}}.$$

Since n and k do not exceed 1 and the log pair $(S, L + C)$ is log canonical, the effective \mathbb{Q} -divisor Ω cannot be the zero-divisor. Let r be the number of the irreducible components of the support of the \mathbb{Q} -divisor Ω . Let me write $\Omega = \sum_{i=1}^r e_i C_i$, where every e_i is a positive rational number, and every C_i is an irreducible reduced curve of degree $d_i > 0$ on the surface S . Then

$$n + 2k + \sum_{i=1}^r a_i d_i = 3,$$

since $-K_S \cdot D = 3$.

Let $\rho: S \dashrightarrow \mathbb{P}^2$ be the linear projection from the point P . Then ρ is a generically 2-to-1 rational map. Thus the map ρ induces a birational involution τ of the cubic surface S . This involution is also known as the Geiser involution (cf. the proof of Lemma 11). The involution τ is biregular outside of the conic C , and $\tau(L) = L$.

For every i , put $\hat{C}_i = \tau(C_i)$, and denote by \hat{d}_i the degree of the curve \hat{C}_i . Put $\hat{\Omega} = \sum_{i=1}^r e_i \hat{C}_i$. Then

$$nL + (n + k + m - 1)C + \hat{\Omega} \sim_{\mathbb{Q}} -K_S,$$

and $(S, nL + (n + k + m - 1)C + \hat{\Omega})$ is not log canonical at the point $L \cap C$ that is different from P . Thus, I can replace the original effective \mathbb{Q} -divisor D by $nL + (n + k + m - 1)C + \hat{\Omega}$ that has the same properties as D . Moreover, since $m + n > 1$, the inequality

$$\sum_{i=1}^r e_i \hat{d}_i < \sum_{i=1}^r e_i d_i$$

holds. Iterating this process, I obtain a contradiction in a finite number of steps as in the proof of Lemma 11. \square

It came as a surprise that Theorem 5 can be used to replace the *global* proof of Lemma 12 by its *local* counterpart. Let me show how to do this.

Local proof of Lemma 12. Let me use the assumptions and notation of Lemma 12. I write $T_P = L + C$, where L is a line, and C is a conic. Let D be any effective \mathbb{Q} -divisor on the surface S such that $D \sim_{\mathbb{Q}} -K_S$. I must show that the log pair (S, D) is log canonical at P . Suppose that (S, D) is not log canonical at the point P . Let me seek for a contradiction.

Put $D = nL + kC + \Omega$, where n and k are non-negative rational numbers and Ω is an effective \mathbb{Q} -divisor on S whose support contains none of the curves L and C . Put $m = \text{mult}_P(\Omega)$. Then

$$m + n + k = \text{mult}_P(D) > 1,$$

since (S, D) is not log canonical at P . The latter implies that $n > 0$, since $1 = L \cdot D \geq \text{mult}_P(D)$ if $n = 0$.

Replacing D by an effective \mathbb{Q} -divisor $(1 + \mu)D - \mu T_P$ for an appropriate $\mu \geq 0$, I may assume that $k = 0$. Then $2 = C \cdot D = 2n + \Omega \cdot C \geq 2n + m$. Moreover, the log pair (S, D) is log canonical outside finitely many points. The latter follows, for example, from [6, Lemma 5.3.6] and is very easy to prove (cf. the proof of [1, Lemma 3.4]).

Let $f: \tilde{S} \rightarrow S$ be the blow up of the point P , let $\tilde{\Omega}$ be the proper transform of the divisor Ω on the surface \tilde{S} , let \tilde{L} be the proper transform of the line L on the surface \tilde{S} , and let E be the f -exceptional curve. Then

$$K_{\tilde{S}} + n\tilde{L} + \tilde{\Omega} + (n + m - 1)E \sim_{\mathbb{Q}} f^*(K_S + D) \sim_{\mathbb{Q}} 0,$$

which implies that $(\tilde{S}, n\tilde{L} + (n + m - 1)E + \tilde{\Omega})$ is not log canonical at some point $Q \in E$. Arguing as in the proof of [1, Lemma 3.5], I get $Q = \tilde{L} \cap E$. Now I can apply Theorem 5 to the log pair $(\tilde{S}, n\tilde{L} + (n + m - 1)E + \tilde{\Omega})$ at the point Q .

Put $\Delta_1 = E$, $\Delta_2 = \tilde{L}$, $M = 1$, $A = 1$, $N = 0$, $B = 2$, and $\alpha = \beta = 1$. Check that all hypotheses of Theorem 5 are satisfied. By Theorem 5, I have

$$m = \text{mult}_Q(\tilde{\Omega} \cdot E) > 1 + (n + m - 1) - n = m$$

or $1 + n - m = \text{mult}_Q(\tilde{\Omega} \cdot \tilde{L}) > 2n - (n + m - 1) = 1 + n - m$, which is absurd. \square

I tried to apply Theorem 5 to find a *local* proof of Lemma 11 as well. But I failed. This is not surprising. Let me explain why. The proof of Theorem 5 is *asymmetric* with respect to the curves Δ_1 and Δ_2 . The *global* proof of Lemma 12 is also *asymmetric* with respect to the curves L and C . The proof of Theorem 5 is based on uniquely determined iterations of blow ups: I must keep blowing up the point of the proper transform of the curve Δ_2 that dominates the point P . The *global* proof of Lemma 12 is based on uniquely determined composition of Geiser involutions. So, Lemma 12 can be considered as a *global* wrap up of a purely *local* special case of Theorem 5, where the line L plays the role of the curve Δ_2 in Theorem 5. On the other hand, Lemma 11 is *symmetric* with respect to the lines L and M . Moreover, its proof is not deterministic at all, since the composition of Geiser involutions in the proof of Lemma 11 is not uniquely determined by the initial data, i.e., every time I apply Geiser involution, I have exactly two possible candidates for the next one: either I can use the Geiser involution induced by the projection from $L \cap N$ or I can use the Geiser involution induced by the projection from $M \cap N$. So, there is a little hope that Theorem 5 can be used to replace the usage of Geiser involutions in the proof of Lemma 11. Of course, there is a chance that the proof of Lemma 11 cannot be *localized* like the proof of Lemma 12. Fortunately, this is not the case. Indeed, instead of using Geiser involutions in the *global* proof of Lemma 11, I can use

Theorem 13. *Let S be a surface, let P be a smooth point in S , let Δ_1 and Δ_2 be two irreducible curves on S that both are smooth at P and intersect transversally at P , let a_1 and a_2 be non-negative rational numbers, and let Ω be an effective \mathbb{Q} -divisor on the surface S whose support does not contain the curves Δ_1 and Δ_2 . Suppose that the log pair $(S, a_1\Delta_1 + a_2\Delta_2 + \Omega)$ is not log canonical at P . Put $m = \text{mult}_P(\Omega)$. Suppose that $m \leq 1$. Then $\text{mult}_P(\Omega \cdot \Delta_1) > 2(1 - a_2)$ or $\text{mult}_P(\Omega \cdot \Delta_2) > 2(1 - a_1)$.*

Proof. I may assume that $a_1 \leq 1$ and $a_2 \leq 1$. Then the log pair $(S, a_1\Delta_1 + a_2\Delta_2 + \Omega)$ is log canonical in a punctured neighborhood of the point P , because $m \leq 1$.

Since the log pair $(S, a_1\Delta_1 + a_2\Delta_2 + \Omega)$ is not log canonical at P , there exists a birational morphism $h: \hat{S} \rightarrow S$ that is a composition of $r \geq 1$ blow ups of smooth points dominating P , and there exists an h -exceptional divisor, say E_r , such that $e_r > 1$, where e_r is a rational number determined by

$$K_{\hat{S}} + a_1\hat{\Delta}_1 + a_2\hat{\Delta}_2 + \hat{\Omega} + \sum_{i=1}^r e_i E_i \sim_{\mathbb{Q}} h^*(K_S + a_1\Delta_1 + a_2\Delta_2 + \Omega),$$

where e_i is a rational number, each E_i is an h -exceptional divisor, $\hat{\Omega}$ is a proper transform on \hat{S} of the divisor Ω , $\hat{\Delta}_1$ and $\hat{\Delta}_2$, are proper transforms on \hat{S} of the curves Δ_1 and Δ_2 , respectively.

Let $f: \tilde{S} \rightarrow S$ be the blow up of the point P , let $\tilde{\Omega}$ be the proper transform of the divisor Ω on the surface \tilde{S} , let E be the f -exceptional curve, let $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$ be the proper transforms of the curves Δ_1 and Δ_2 on the surface \tilde{S} , respectively. Then

$$K_{\tilde{S}} + a_1 \tilde{\Delta}_1 + a_2 \tilde{\Delta}_2 + (a_1 + a_2 + m - 1)E + \tilde{\Omega} \sim_{\mathbb{Q}} f^*(K_S + a_1 \Delta_1 + a_2 \Delta_2 + \Omega).$$

which implies that the log pair $(\tilde{S}, a_1 \tilde{\Delta}_1 + a_2 \tilde{\Delta}_2 + (a_1 + a_2 + m - 1)E + \tilde{\Omega})$ is not log canonical at some point $Q \in E$.

If $r = 1$, then $a_1 + a_2 + m - 1 > 1$, which implies that $m > 2 - a_1 - a_2$. On the other hand, if $m > 2 - a_1 - a_2$, then either $m > 2(1 - a_1)$ or $m > 2(1 - a_2)$, because otherwise I would have $2m \leq 4 - 2(a_1 + a_2)$, which contradicts to $m > 2 - a_1 - a_2$. Thus, if $r = 1$, then $\text{mult}_P(\Omega \cdot \Delta_1) > 2(1 - a_2)$ or $\text{mult}_P(\Omega \cdot \Delta_2) > 2(1 - a_1)$.

Let me prove the required assertion by induction on r . The case $r = 1$ is done. Thus, I may assume that $r \geq 2$. If $Q \neq E \cap \tilde{\Delta}_1$ and $Q \neq E \cap \tilde{\Delta}_2$, then it follows from Theorem 7 that $m = \tilde{\Omega} \cdot E > 1$, which is impossible, since $m \leq 1$ by assumption. Thus, either $Q = E \cap \tilde{\Delta}_1$ or $Q = E \cap \tilde{\Delta}_2$. Without loss of generality, I may assume that $Q = E \cap \tilde{\Delta}_1$.

By induction, I can apply the required assertion to $(\tilde{S}, a_1 \tilde{\Delta}_1 + (a_1 + a_2 + m - 1)E + \tilde{\Omega})$ at the point Q . This implies that either

$$\text{mult}_Q(\tilde{\Omega} \cdot \tilde{\Delta}_1) > 2(1 - (a_1 + a_2 + m - 1)) = 4 - 2a_1 - 2a_2 - 2m$$

or $\text{mult}_Q(\tilde{\Omega} \cdot E) > 2(1 - a_1)$. In the latter case, I have

$$\text{mult}_P(\Omega \cdot \Delta_2) \geq m > 2(1 - a_1),$$

since $m = \text{mult}_Q(\tilde{\Omega} \cdot E) > 2(1 - a_1)$, which is exactly what I want. Thus, to complete the proof, I may assume that $\text{mult}_Q(\tilde{\Omega} \cdot \tilde{\Delta}_1) > 4 - 2a_1 - 2a_2 - 2m$.

If $\text{mult}_P(\Omega \cdot \Delta_2) > 2(1 - a_1)$, then I am done. Thus, to complete the proof, I may assume that $\text{mult}_P(\Omega \cdot \Delta_2) \leq 2(1 - a_1)$. This gives me $m \leq 2(1 - a_1)$, since $\text{mult}_P(\Omega \cdot \Delta_2) \geq m$. Then

$$\begin{aligned} \text{mult}_P(\Omega \cdot \Delta_1) &\geq m + \text{mult}_Q(\tilde{\Omega} \cdot \tilde{\Delta}_1) > m + 4 - 2a_1 - 2a_2 - 2m \\ &= 4 - 2a_1 - 2a_2 - m > 2(1 - a_2), \end{aligned}$$

because $m \leq 2(1 - a_1)$. This completes the proof. \square

Let me show how to prove Lemma 11 using Theorem 13. This is very easy.

Local proof of Lemma 11. Let me use the assumptions and notation of Lemma 11. I write $T_P = L + M + N$, where L , M , and N are lines on the cubic surface S .

Without loss of generality, I may assume that the line N does not pass through the point P . Let D be any effective \mathbb{Q} -divisor on the surface S such that $D \sim_{\mathbb{Q}} -K_S$. I must show that the log pair (S, D) is log canonical at P . Suppose that the log pair (S, D) is not log canonical at P . Let me seek for a contradiction.

The log pair (S, D) is log canonical in a punctured neighborhood of the point P (use [6, Lemma 5.3.6] or the proof of [1, Lemma 3.4]). Put $D = aL + bM + cN + \Omega$, where a, b , and c are non-negative rational numbers and Ω is an effective \mathbb{Q} -divisor on S whose support contains none of the lines L, M , and N . Put $m = \text{mult}_P(\Omega)$.

Since $(S, L + M + N)$ is log canonical, $D \neq L + M + N$. Then there exists a non-negative rational number μ such that the divisor $(1 + \mu)D - \mu T_P$ is effective and its support does not contain at least one components of $T_P = L + M + N$. Thus, replacing D by $(1 + \mu)D - \mu T_P$, I can assume that at least one number among a, b , and c is zero. On the other hand, I know that

$$\text{mult}_P(D) = m + a + b > 1,$$

because the log pair (S, D) is not log canonical at P . Thus, if $a = 0$, then

$$1 = L \cdot D \geq \text{mult}_P(L) \text{mult}_P(D) = \text{mult}_P(D) = m + b > 1,$$

which is absurd. This shows that $a > 0$. Similarly, $b > 0$. Therefore, $c = 0$. Then

$$1 = N \cdot D = N \cdot (aL + bM + \Omega) = a + b + N \cdot \Omega \geq a + b,$$

which implies that $a + b \leq 1$. On the other hand, I know that

$$\begin{cases} 1 = L \cdot (aL + bM + \Omega) = -a + b + L \cdot \Omega \geq -a + b + m, \\ 1 = M \cdot (aL + bM + \Omega) = a - b + M \cdot \Omega \geq a - b + m, \end{cases}$$

which implies that $m \leq 1$. Thus, I can apply Theorem 13 to $(S, aL + bM + \Omega)$. This gives either

$$1 + a - b = \text{mult}_P(\Omega \cdot L) > 2(1 - b)$$

or $1 - a + b = \text{mult}_P(\Omega \cdot M) > 2(1 - a)$. Then either $1 + a - b > 2 - 2b$ or $1 - a + b > 2 - 2a$. In both cases, $a + b > 1$, which is not the case (I proved this earlier). \square

I was very surprised to find out that Theorem 13 has many other applications as well. Let me show how to use Theorem 13 to give a short proof of Lemma 3.

Proof of Lemma 3. Let me use the assumptions and notation of Lemma 3. Every cuspidal curve in $|-K_S|$ is a Siberian tigers, since all curves in $|-K_S|$ are

irreducible. Let D be a Siberian tiger. I must prove that D is a cuspidal curve in $|-K_S|$.

The pair (S, D) is not log canonical at some point $P \in S$. Let C be a curve in $|-K_S|$ that contains P . If P is the base locus of the pencil $|-K_S|$, then (S, C) is log canonical at P , because every curve in the pencil $|-K_S|$ is smooth at its unique base point. Moreover, if $P = O$, then (S, C) is also log canonical at P by [12, Theorem 3.3]. In the latter case, the curve C has an ordinary double point at P by [12, Theorem 3.3], which also follows from Kodaira's table of singular fibers of elliptic fibration. Furthermore, if C is singular at P and (S, C) is not log canonical at P , then C has an ordinary cusp at P .

If $D = C$ and C is a cuspidal curve, then I am done. Thus, I may assume that this is not the case. Let me seek for a contradiction.

I claim that $C \not\subseteq \text{Supp}(D)$. Indeed, if C is cuspidal curve, then $C \not\subseteq \text{Supp}(D)$, since D is a Siberian tiger. If (S, C) is log canonical, put $D = aC + \Omega$, where a is a non-negative rational number, and Ω is an effective \mathbb{Q} -divisor on S whose support does not contain the curve C . Then $a < 1$, since $D \sim_{\mathbb{Q}} C$ and $D \neq C$. Then

$$\frac{1}{1-a}D - \frac{a}{1-a}C = \frac{1}{1-a}(aC + \Omega) - \frac{a}{1-a}C = \frac{1}{1-a}\Omega \sim_{\mathbb{Q}} -K_S$$

and the log pair $(S, \frac{1}{1-a}\Omega)$ is not log canonical at P , because (S, C) is log canonical at P , and (S, D) is not log canonical at P . Since D is a Siberian tiger, I see that $a = 0$, i.e., $C \not\subseteq \text{Supp}(D)$.

If $P \neq O$, then

$$1 = C \cdot D \geq \text{mult}_P(D),$$

which is impossible by [6, Exercise 6.18], since the log pair (S, D) is not log canonical at the point P . Thus, I see that $P = O$.

Let $f: \tilde{S} \rightarrow S$ be a minimal resolution of singularities of the surface S . Then there are three f -exceptional curves, say E_1, E_2 , and E_3 , such that $E_1^2 = E_2^2 = E_3^2 = -2$. I may assume that $E_1 \cdot E_3 = 0$ and $E_1 \cdot E_2 = E_2 \cdot E_3 = 1$. Let \tilde{C} be the proper transform of the curve C on the surface \tilde{S} . Then $\tilde{C} \sim_{\mathbb{Q}} f^*(C) - E_1 - E_2 - E_3$.

Let \tilde{D} be the proper transform of the \mathbb{Q} -divisor D on the surface \tilde{S} . Then

$$\tilde{D} \sim_{\mathbb{Q}} f^*(D) - a_1 E_1 - a_2 E_2 - a_3 E_3$$

for some non-negative rational numbers a_1, a_2 and a_3 . Then

$$\begin{cases} 1 - a_1 - a_3 = \tilde{D} \cdot \tilde{C} \geq 0, \\ 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq 0, \\ 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq 0, \\ 2a_3 - a_2 = \tilde{D} \cdot E_3 \geq 0, \end{cases}$$

which gives $1 \geq a_1 + a_3$, $2a_1 \geq a_2$, $3a_2 \geq 2a_3$, $2a_3 \geq a_2$, $3a_2 \geq 2a_1$, $a_1 \leq \frac{3}{4}$, $a_2 \leq 1$, $a_3 \leq \frac{3}{4}$. On the other hand, I have

$$K_{\tilde{S}} + \tilde{D} + \sum_{i=1}^3 a_i E_i \sim_{\mathbb{Q}} f^*(K_S + D) \sim_{\mathbb{Q}} 0,$$

which implies that $(\tilde{S}, \tilde{D} + a_1 E_1 + a_2 E_2 + a_3 E_3)$ is not log canonical at some point $Q \in E_1 \cup E_2 \cup E_3$.

Suppose that $Q \in E_1$ and $Q \notin E_2$. Then $(\tilde{S}, \tilde{D} + a_1 E_1)$ is not log canonical at Q . Then $2a_1 - a_2 = \tilde{D} \cdot E_1 > 1$ by Theorem 7. Therefore, I have

$$1 \geq \frac{4}{3}a_1 \geq 2a_1 - \frac{2}{3}a_1 \geq 2a_1 - a_2 > 1,$$

which is absurd. Thus, if $Q \in E_1$, then $Q = E_1 \cap E_2$. Similarly, I see that if $Q \in E_3$, then $Q = E_3 \cap E_2$.

Suppose that $Q \in E_2$ and $Q \notin E_1 \cup E_3$. Then $(\tilde{S}, \tilde{D} + a_2 E_2)$ is not log canonical at Q . Then $2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 > 1$ by Theorem 7. Therefore, I have

$$1 \geq a_2 = 2a_2 - \frac{a_2}{2} - \frac{a_2}{2} \geq 2a_2 - a_1 - a_3 > 1,$$

which is absurd. Thus, I proved that either $Q = E_1 \cap E_2$ or $Q = E_3 \cap E_2$. Without loss of generality, I may assume that $Q = E_1 \cap E_2$.

The log pair $(\tilde{S}, \tilde{D} + a_1 E_1 + a_2 E_2)$ is not log canonical at Q . Put $m = \text{mult}_Q(\tilde{D})$. Then

$$\begin{cases} 2a_1 - a_2 = \tilde{D} \cdot E_1 \geq m, \\ 2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 \geq m, \\ 2a_3 - a_2 = \tilde{D} \cdot E_3 \geq 0, \end{cases}$$

which implies that $a_1 + a_3 \geq 2m$. Since I already proved that $a_1 + a_3 \leq 1$, $m \leq \frac{1}{2}$. Applying Theorem 13 to the log pair $(\tilde{S}, \tilde{D} + a_1 E_1 + a_2 E_2)$ at the point Q , I see that $\tilde{D} \cdot E_1 > 2(1 - a_2)$ or $\tilde{D} \cdot E_2 > 2(1 - a_1)$. In the former case, one has

$$2a_1 - a_2 = \tilde{D} \cdot E_1 > 2(1 - a_2),$$

which implies that $2 \geq 2a_1 + 2a_3 \geq 2a_1 + a_2 > 2$, since $1 \geq a_1 + a_3$ and $2a_3 \geq a_2$. Thus, I proved that

$$2a_2 - a_1 - a_3 = \tilde{D} \cdot E_2 > 2(1 - a_1),$$

which implies that $2a_2 + a_1 > 2 + a_3$. Then $2a_2 + 1 - a_3 > 2a_2 + a_1 > 2 + a_3$, since $a_1 + a_3 \leq 1$. The last inequality implies that $2a_2 > 1 + 2a_3$. Since I already proved that $2a_3 \geq a_2$, I conclude that $2a_2 > 1 + a_2$, which is impossible, since $a_1 \leq 1$. The obtained contradiction completes the proof. \square

Similarly, I can use Theorem 13 instead of Theorem 5 in the *local* proof of Lemma 12 (I leave the details to the reader). Theorem 13 has a nice and clean inductive proof like Theorem 7 has. So, what if Theorem 13 is the desired generalization of Theorem 5? This may seem unlikely keeping in mind how both theorems look like. However, Theorem 13 does generalize Theorem 4, which is the ancestor and a special case of Theorem 5. The latter follows from

Remark 14. Let S be a surface, let Δ_1 and Δ_2 be two irreducible curves on S that are both smooth at P and intersect transversally at P . Take an effective \mathbb{Q} -divisor $a_1\Delta_1 + a_2\Delta_2 + \Omega$, where a_1 and a_2 are non-negative rational numbers, and Ω is an effective \mathbb{Q} -divisor on the surface S whose support does not contain the curves Δ_1 and Δ_2 . Put $m = \text{mult}_P(\Omega)$. Let n be a positive integer such that $n \geq 3$. Theorem 4 asserts that $\text{mult}_P(\Omega \cdot \Delta_1) > 2a_1 - a_2$ or

$$\text{mult}_P(\Omega \cdot \Delta_2) > \frac{n}{n-1}a_2 - a_1$$

provided that $\frac{2n-2}{n+1}a_1 + \frac{2}{n+1}a_2 \leq 1$ and the log pair $(S, a_1\Delta_1 + a_2\Delta_2 + \Omega)$ is not log canonical at P . On the other hand, $\text{mult}_P(\Omega \cdot \Delta_1) \geq m$ and $\text{mult}_P(\Omega \cdot \Delta_2) \geq m$. Thus, Theorem 4 asserts something non-obvious only if

$$\begin{cases} 2a_1 - a_2 \geq m, \\ \frac{n}{n-1}a_2 - a_1 \geq m, \\ \frac{2n-2}{n+1}a_1 + \frac{2}{n+1}a_2 \leq 1. \end{cases} \quad (15)$$

Note that (15) implies that $a_1 \leq \frac{1}{2}$, $a_2 \leq 1$, and $m \leq 1$. Thus, if (15) holds, then I can apply Theorem 13 to the log pair $(S, a_1\Delta_1 + a_2\Delta_2 + \Omega)$ to get $\text{mult}_P(\Omega \cdot \Delta_1) > 2(1 - a_2)$ or $\text{mult}_P(\Omega \cdot \Delta_2) > 2(1 - a_1)$. On the other hand, if (15) holds, then $2(1 - a_2) \geq 2a_1 - a_2$ and

$$2(1 - a_1) \geq \frac{2n-2}{n+1}a_1 + \frac{2}{n+1}a_2.$$

Nevertheless, Theorem 13 is not a generalization of Theorem 5, i.e., I cannot use Theorem 13 instead of Theorem 5 in general. I checked this in many cases considered in [2]. To convince the reader, let me give

Example 16. Put $S = \mathbb{P}^2$. Take some integers $m \geq 2$ and $k \geq 2$. Put $r = km(m-1)$. Let C be a curve in S that is given by $z^{r-1}y = x^r$, where $[x : y : z]$ are

projective coordinates on S . Put $\Omega = \lambda C$ for some positive rational number λ . Let Δ_1 be a line in S that is given by $x = 0$, and let Δ_2 be a line in S that is given by $y = 0$. Put $a_1 = \frac{1}{m}$ and $a_2 = 1 - \frac{1}{m}$. Let P be the intersection point $\Delta_1 \cap \Delta_2$. Then $(S, a_1 \Delta_1 + a_2 \Delta_2 + \Omega)$ is log canonical P if and only if $\lambda \leq \frac{1}{m} + \frac{1}{km^2}$. Take any $\lambda > \frac{1}{m} + \frac{1}{km^2}$ such that $\lambda < \frac{k}{km-1}$. Then $\text{mult}_P(\Omega) = \lambda < \frac{2}{m} \leq 1$ and

$$\text{mult}_P(\Omega \cdot \Delta_1) = \lambda < \frac{k}{km-1} < \frac{2}{m} = 2(1-a_2),$$

which implies that

$$k(m-1) + \frac{m-1}{m} > km(m-1)\lambda = \text{mult}_P(\Omega \cdot \Delta_2) > 2(1-a_1) = \frac{2m-2}{m}$$

by Theorem 13. Taking λ close enough to $\frac{1}{m} + \frac{1}{km^2}$, I can get $\text{mult}_P(\Omega \cdot \Delta_2)$ as close to $k(m-1) + \frac{m-1}{m}$ as I want. Thus, the inequality $\text{mult}_P(\Omega \cdot \Delta_2) > \frac{2m-2}{m}$ provided by Theorem 13 is not very good when $k \gg 0$. Now let me apply Theorem 5 to the log pair $(S, a_1 \Delta_1 + a_2 \Delta_2 + \Omega)$ to get much better estimate for $\text{mult}_P(\Omega \cdot \Delta_2)$. Put $\alpha = 1$, $\beta = 1$, $M = 1$, $B = km$, $A = \frac{1}{km-1}$, and $N = 0$. Then

$$\left\{ \begin{array}{l} 1 = \alpha a_1 + \beta a_2 \leq 1, \\ 1 = A(B-1) \geq 1, \\ 1 = M \leq 1, \\ 0 = N \leq 1, \\ \frac{1}{km-1} = \alpha(A+M-1) \geq A^2(B+N-1)\beta = \frac{1}{km-1}, \\ \frac{1}{km-1} = \alpha(1-M) + A\beta \geq A = \frac{1}{km-1}, \\ 2 = 2M + AN \leq 2. \end{array} \right.$$

By Theorem 5, $\text{mult}_P(\Omega \cdot \Delta_1) > M + Aa_1 - a_2$ or $\text{mult}_P(\Omega \cdot \Delta_2) > N + Ba_2 - a_1$. Since $\text{mult}_P(\Omega \cdot \Delta_1) = \lambda < \frac{k}{km-1} = M + Aa_1 - a_2$, it follows from Theorem 5 that

$$\text{mult}_P(\Omega \cdot \Delta_2) > N + Ba_2 - a_1 = k(m-1) - \frac{1}{m}.$$

For $k \gg 0$, the latter inequality is much stronger than $\text{mult}_P(\Omega \cdot \Delta_2) > \frac{2m-2}{m}$ given by Theorem 13. Moreover, I can always choose λ close enough to $\frac{1}{m} + \frac{1}{km^2}$ so that the multiplicity $\text{mult}_P(\Omega \cdot \Delta_2) = km(m-1)\lambda$ is as close to $k(m-1) + \frac{m-1}{m}$ as I want. This shows that the inequality $\text{mult}_P(\Omega \cdot \Delta_2) > k(m-1) - \frac{1}{m}$ provided by Theorem 5 is almost sharp.

I have a strong feeling that Theorems 5 and 13 are special cases of some more general result that is not yet found. Perhaps, it can be found by analyzing the proofs of Theorems 5 and 13.

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