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TWO RATIONAL NODAL QUARTIC 3-FOLDS

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Abstract

We prove that the quartic 3-folds defined by

$$\sum_{i=0}^{5} x_i = \sum_{i=0}^{5} x_i^4 - t\left(\sum_{i=0}^{5} x_i^2\right)^2 = 0$$

in $\mathbb{P}^5$ are rational for $t = \frac{1}{6}$ and $t = \frac{7}{10}$.

1. Introduction

Consider the six-dimensional permutation representation $\mathbb{W}$ of the group $\mathfrak{S}_6$. Choose coordinates $x_0, \ldots, x_5$ in $\mathbb{W}$ so that they are permuted by $\mathfrak{S}_6$. Then, $x_0, \ldots, x_5$ also serve as homogeneous coordinates in the projective space $\mathbb{P}^5 = \mathbb{P}(\mathbb{W})$.

Let us identify $\mathbb{P}^4$ with a hyperplane

$$x_0 + x_1 + x_2 + x_3 + x_4 + x_5 = 0$$

in $\mathbb{P}^5$. Denote by X_t the quartic 3-fold in $\mathbb{P}^4$ that is given by the equation

$$x_0^4 + x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 = t\left(x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2\right)^2, \tag{1.1}$$

where t is an element of the ground field, which we will always assume to be the field $\mathbb{C}$ of complex numbers. Every $\mathfrak{S}_6$-invariant quartic in $\mathbb{P}(\mathbb{W})$ is one of the quartics X_t. Moreover, every quartic 3-fold with a faithful $\mathfrak{S}_6$-action is isomorphic to some X_t. All quartics X_t are singular. Indeed,

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denote by Σ_{30} the $\mathfrak{S}_6$-orbit of the point $[1:1:\omega:\omega:\omega^2:\omega^2]$, where $\omega = e^{\frac{2\pi i}{3}}$. Then, $|\Sigma_{30}| = 30$, and X_t is singular at every point of Σ_{30} for every $t \in \mathbb{C}$ (see, for example, [12, Theorem 4.1]).

The possible singularities of the quartic 3-fold X_t have been described by van der Geer in [12, Theorem 4.1]. To recall his description, denote by $\mathcal{L}_{15}$ the $\mathfrak{S}_6$-orbit of the line that passes through the points $[1:0:-1:1:0:-1]$ and $[0:1:-1:0:1:-1]$, and denote by Σ_6, Σ_{10} and Σ_{15} the $\mathfrak{S}_6$-orbits of the points $[-5:1:1:1:1:1]$, $[-1:-1:-1:1:1:1]$ and $[1:-1:0:0:0:0]$, respectively. Then, the curve $\mathcal{L}_{15}$ is a union of 15 lines, while $|\Sigma_6| = 6$, $|\Sigma_{10}| = 10$ and $|\Sigma_{15}| = 15$. Moreover, one has

$$\mathrm{Sing}(X_t) = \begin{cases} \mathcal{L}_{15} \text{ if } t = \dfrac{1}{4}, \\ \Sigma_{30} \cup \Sigma_{15} \text{ if } t = \dfrac{1}{2}, \\ \Sigma_{30} \cup \Sigma_{10} \text{ if } t = \dfrac{1}{6}, \\ \Sigma_{30} \cup \Sigma_{6} \text{ if } t = \dfrac{7}{10}, \\ \Sigma_{30} \text{ otherwise.} \end{cases}$$

Furthermore, if $t \neq \frac{1}{4}$, then all singular points of the quartic 3-fold X_t are isolated ordinary double points (nodes).

The 3-fold $X_{\frac{1}{2}}$ is classical. It is the so-called *Burkhardt quartic*. In [3], Burkhardt discovered that the subset $\Sigma_{30} \cup \Sigma_{15}$ is invariant under the action of the simple group $\mathrm{PSp}_4(F_3)$ of order 25 920. In [7], Coble proved that $\Sigma_{30} \cup \Sigma_{15}$ is the singular locus of the 3-fold $X_{\frac{1}{2}}$, and proved that $X_{\frac{1}{2}}$ is also $\mathrm{PSp}_4(F_3)$-invariant. Later Todd proved in [22] that $X_{\frac{1}{2}}$ is rational. In [15], de Jong, Shepherd-Barron and Van de Ven proved that $X_{\frac{1}{2}}$ is the unique quartic 3-fold in $\mathbb{P}^4$ with 45 singular points.

The quartic 3-fold $X_{\frac{1}{4}}$ is also classical. It is known as the *Igusa quartic* from its modular interpretation as the Satake compactification of the moduli space of Abelian surfaces with level 2 structure (see [12]). The projectively dual variety of the quartic 3-fold $X_{\frac{1}{4}}$ is the so-called *Segre cubic*. Since the Segre cubic is rational, $X_{\frac{1}{4}}$ is rational as well.

During *Kul!fest* conference dedicated to the 60th anniversary of Viktor Kulikov which was held in Moscow in December 2012, Alexei Bondal and Yuri Prokhorov posed

PROBLEM 1.1 *Determine all $t \in \mathbb{C}$ such that X_t is rational.*

Since X_t is singular, we cannot apply Iskovskikh and Manin's theorem from [14] to X_t. Similarly, we cannot apply Mella's [18, Theorem 2] to X_t either, because the quartic 3-fold X_t is not $\mathbb{Q}$-factorial by Beauville [1, Lemma 2]. Nevertheless, Beauville proved

THEOREM 1.2 [(1)]. *If $t \notin \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{7}{10}\right\}$, then X_t is non-rational.*

Both $X_{\frac{1}{2}}$ and $X_{\frac{1}{4}}$ are rational. The goal of this paper is to prove

THEOREM 1.3 *The quartic 3-folds $X_{\frac{1}{6}}$ and $X_{\frac{7}{10}}$ are also rational.*

Surprisingly, the proof of Theorem 1.3 goes back to two classical papers of Todd. Namely, we will construct an explicit $\mathfrak{A}_6$-birational map $\mathbb{P}^3 \to X_{\frac{7}{10}}$ that is a special case of Todd's construction from Todd [20]. Similarly, we will construct an explicit $\mathfrak{S}_5$-birational map $\mathbb{P}^3 \to X_{\frac{1}{6}}$ that is a degeneration of Todd's construction from [21]. We emphasize that our proof is self-contained, that is, it does not rely on the results proved in [20] and [21], but recovers the necessary facts in our particular situation using additional symmetries arising from group actions.

Remark 1.4 Todd proved in [22] that the Burkhardt quartic $X_{\frac{1}{2}}$ is determinantal (see also [19, Section 5.1]). The constructions of our birational maps $\mathbb{P}^3 \to X_{\frac{7}{10}}$ and $\mathbb{P}^3 \to X_{\frac{1}{6}}$ imply that both $X_{\frac{7}{10}}$ and $X_{\frac{1}{6}}$ are determinantal (see [19, Example 6.4.2] and [19, Example 6.2.1]). Yuri Prokhorov pointed out that the quartic 3-fold

$$\det \begin{pmatrix} y_0 & y_1 & y_2 & y_3 \\ y_4 & y_0 & y_3 & y_4 \\ y_2 & y_1 & y_1 & y_0 \\ y_0 & y_3 & y_2 & y_4 \end{pmatrix} = 0$$

in $\mathbb{P}^4$ with homogeneous coordinates $y_0, \ldots, y_4$ has exactly 45 singular points. Thus, it is isomorphic to the Burkhardt quartic $X_{\frac{1}{2}}$ by de Jong [15]. It would be interesting to find similar determinantal equations of the 3-folds $X_{\frac{7}{10}}$ and $X_{\frac{1}{6}}$.

The plan of the paper is as follows. In Section 2, we recall some preliminary results on representations of a central extension of the group $\mathfrak{S}_6$, and some of its subgroups. In Section 3, we collect results concerning a certain action of the group $\mathfrak{A}_5$ on $\mathbb{P}^3$, and study $\mathfrak{A}_5$-invariant quartic surfaces; the reason we pay so much attention to this group is that it is contained both in $\mathfrak{A}_6$ and in $\mathfrak{S}_5$, and thus the information about its properties simplifies the study of the latter two groups. In Section 4, we collect auxiliary results about the groups $\mathfrak{S}_6$, $\mathfrak{A}_6$ and $\mathfrak{S}_5$, in particular about their actions on curves and their five-dimensional irreducible representations. In Section 5, we construct an $\mathfrak{A}_6$-equivariant birational map $\mathbb{P}^3 \to X_{\frac{7}{10}}$. Finally, in Section 6, we construct an $\mathfrak{S}_5$-equivariant birational map $\mathbb{P}^3 \to X_{\frac{1}{6}}$ and make some concluding remarks.

Throughout the paper, we denote a cyclic group of order n by $\boldsymbol{\mu}_n$, and we denote a dihedral group of order $2n$ by D_{2n}. In particular, one has $\mathrm{D}_{12} \cong \mathfrak{S}_3 \times \boldsymbol{\mu}_2$. By F_{36} we denote a group isomorphic to $(\boldsymbol{\mu}_3 \times \boldsymbol{\mu}_3) \rtimes \boldsymbol{\mu}_4$, and by F_{20} we denote a group isomorphic to $\boldsymbol{\mu}_5 \rtimes \boldsymbol{\mu}_4$.

2. Representation theory

Recall that the permutation group $\mathfrak{S}_6$ has two central extensions $2^+\mathfrak{S}_6$ and $2^-\mathfrak{S}_6$ by the group $\boldsymbol{\mu}_2$ with the central subgroup contained in the commutator subgroup (see [8, p. xxiii] for details). We denote the first of them (i.e. the one where the preimages of a transposition in $\mathfrak{S}_6$ under the natural projection have order two) by $2.\mathfrak{S}_6$ to simplify notation. Similarly, for any group Γ we denote by $2.\Gamma$ a non-split central extension of Γ by the group $\boldsymbol{\mu}_2$.

We start with recalling some facts about four- and five-dimensional representations of the group $2.\mathfrak{S}_6$ we will be working with. A reader who is not interested in details here can skip to Corollary 2.1, or even to Section 4 where we reformulate everything in geometric language. Also, we will see in Section 4 that our further constructions do not depend much on the choice of representations, and all computations one makes for one of them actually apply to all others.

Let $\mathbb{I}$ and $\mathbb{J}$ be the trivial and the non-trivial one-dimensional representations of the group $\mathfrak{S}_6$, respectively. Consider the six-dimensional permutation representation $\mathbb{W}$ of $\mathfrak{S}_6$. One has

$$\mathbb{W} \cong \mathbb{I} \oplus \mathbb{W}_5 \otimes \mathbb{J}$$

for some irreducible representation $\mathbb{W}_5$ of $\mathfrak{S}_6$. We can regard $\mathbb{I}$, $\mathbb{J}$ and $\mathbb{W}_5$ as representations of the group $2.\mathfrak{S}_6$. Recall that there is a double cover

$$\nabla\colon \mathrm{SL}_4(\mathbb{C}) \to \mathrm{SO}_6(\mathbb{C}),$$

see, for example, [**10**, Exercise 20.39]. Taking the embedding of the group $2.\mathfrak{S}_6$ into $\mathrm{SO}_6(\mathbb{C})$ via the representation $\mathbb{I} \oplus \mathbb{W}_5$ and considering its preimage with respect to ∇, we produce an embedding of the group $2.\mathfrak{S}_6$ into $\mathrm{SL}_4(\mathbb{C})$. This embedding gives rise to two four-dimensional representations of $2.\mathfrak{S}_6$ that differ by a tensor product with $\mathbb{J}$. We fix one of these two representations $\mathbb{U}_4$. Note that

$$\mathbb{I} \oplus \mathbb{W}_5 \cong \Lambda^2\left(\mathbb{U}_4\right).$$

Recall that there are coordinates $x_0,\ldots,x_5$ in $\mathbb{W}$ that are permuted by the group $\mathfrak{S}_6$. We will refer to a subgroup of $2.\mathfrak{S}_6$ fixing one of the corresponding points as a *standard* subgroup $2.\mathfrak{S}_5$; we denote any such subgroup by $2.\mathfrak{S}_5^{\mathrm{st}}$. A subgroup of $2.\mathfrak{S}_6$ that is isomorphic to $2.\mathfrak{S}_5$, but is not conjugate to a standard $2.\mathfrak{S}_5$, will be called a *non-standard* subgroup $2.\mathfrak{S}_5$; we denote any such subgroup by $2.\mathfrak{S}_5^{nst}$. These agree with standard and non-standard subgroups of $\mathfrak{S}_6$ isomorphic to $\mathfrak{S}_5$, although outer automorphisms of $\mathfrak{S}_6$ do not lift to $2.\mathfrak{S}_6$. Any subgroup of $2.\mathfrak{S}_6$ that is isomorphic to $2.\mathfrak{A}_5$, $2.\mathfrak{S}_4$ or $2.\mathfrak{A}_4$ and is contained in $2.\mathfrak{S}_5^{st}$ is denoted by $2.\mathfrak{A}_5^{st}$, $2.\mathfrak{S}_{4st}$ or $2.\,\mathfrak{A}_4^{st}$, respectively. Similarly, any subgroup of $2.\mathfrak{S}_6$ that is isomorphic to $2.\mathfrak{A}_5$, $2.\mathfrak{S}_4$ or $2.\mathfrak{A}_4$ and is contained in $2.\mathfrak{S}_5^{nst}$ is denoted by $2.\mathfrak{A}_5^{nst}$, $2.\mathfrak{S}_4^{nst}$ or $2.\mathfrak{A}_4^{nst}$, respectively.

The values of characters of important representations of the group $2.\mathfrak{S}_6$, and the information about some of its subgroups are presented in Table 1, cf. [**8**, p. 5]. The first two columns of

Table 1. Characters and subgroups of the group $2.\mathfrak{S}_6$.

ord	Type	$\mathbb{W}$	$\mathbb{W}_5$	$\mathbb{U}_4$	$2.\mathfrak{S}_6$	$2.\mathfrak{A}_6$	$2.\mathfrak{S}_5^{nst}$	$2.\mathfrak{A}_{5st}$	$2.\mathfrak{A}_5^{nst}$	$2.\mathfrak{S}_4^{nst}$	$2.\mathfrak{A}_4^{nst}$	$2.F_{36}$	$2.F_{20}$	$2.\mathrm{D}_{12}^{nst}$
1	id	6	5	4	1	1	1	1	1	1	1	1	1	1
2	z	6	5	-4	1	1	1	1	1	1	1	1	1	1
2	[2]	4	-3	0	30	0	0	0	0	0	0	0	0	0
4	[2, 2]	2	1	0	90	90	30	30	30	6	6	18	10	6
4	[2, 2, 2]	0	1	0	30	0	20	0	0	12	0	0	0	8
6	[3]	3	2	2	40	40	0	20	0	0	0	4	0	0
3	[3]	3	2	-2	40	40	0	20	0	0	0	4	0	0
6	[3, 2]	1	0	0	120	0	0	0	0	0	0	0	0	0
6	[3, 2]	1	0	0	120	0	0	0	0	0	0	0	0	0
6	[3, 3]	0	-1	-1	40	40	20	0	20	8	8	4	0	2
3	[3, 3]	0	-1	1	40	40	20	0	20	8	8	4	0	2
8	[4]	2	-1	0	180	0	60	0	0	12	0	0	20	0
8	[4, 2]	0	-1	0	180	180	0	0	0	0	0	36	0	0
10	[5]	1	0	1	144	144	24	24	24	0	0	0	4	0
5	[5]	1	0	-1	144	144	24	24	24	0	0	0	4	0
12	[6]	0	1	$\sqrt{-3}$	120	0	20	0	0	0	0	0	0	2
12	[6]	0	1	$-\sqrt{-3}$	120	0	20	0	0	0	0	0	0	2

Table 1 describes conjugacy classes of elements of the group $2.\mathfrak{S}_6$. The first column lists the orders of the elements in the corresponding conjugacy class, and the second column, except for the entries in the second and the third rows, gives a cycle type of the image of an element under projection to $\mathfrak{S}_6$ (for example, $[3, 2]$ denotes a product of two disjoint cycles of lengths 3 and 2). By id we denote the identity element of $2.\mathfrak{S}_6$, and z denotes the unique non-trivial central element of $2.\mathfrak{S}_6$. Note that the preimages of some of conjugacy classes in $\mathfrak{S}_6$ split into a union of two conjugacy classes in $2.\mathfrak{S}_6$. The next three columns list the values of the characters of the representations $\mathbb{W}$, $\mathbb{W}_5$ and $\mathbb{U}_4$ of $2.\mathfrak{S}_6$. Note that there is no real ambiguity in the choice of $\sqrt{-3}$ since we did not specify any way to distinguish the two conjugacy classes in $2.\mathfrak{S}_6$ whose elements are projected to cycles of length 6 in $\mathfrak{S}_6$ up to this point (note that the two ways to choose a sign here is exactly a tensor multiplication of the representation with $\mathbb{J}$, that is, the choice between two homomorphisms of $2.\mathfrak{S}_6$ to $\mathrm{SL}_4(\mathbb{C})$ having the same image). The remaining columns list the numbers of elements from each of the conjugacy classes of $2.\mathfrak{S}_6$ in subgroups of certain types. By $2.F_{36}$ (respectively, by $2.F_{20}$, by $2.\mathrm{D}_{12}^{nst}$) we denote a subgroup of $2.\mathfrak{S}_6$ (respectively, of $2.\mathfrak{S}_6$, or of $2.\mathfrak{S}_5^{nst}$) isomorphic to a central extension of F_{36} (respectively, of F_{20}, or of D_{12}) by $\boldsymbol{\mu}_2$. A subgroup $2.F_{20}$ is actually contained in a subgroup $2.\mathfrak{S}_5 st$ and in a subgroup $2.\mathfrak{S}_5^{nst}$. Note that the intersection of a conjugacy class in a group with a subgroup may (and often does) split into several conjugacy classes in this subgroup.

It is immediate to see from Table 1 that $\mathbb{U}_4$ is an irreducible representation of the group $2.\mathfrak{S}_6$. Using the information provided by Table 1, we immediately obtain the following results.

Corollary 2.1 *Let Γ be a subgroup of $2.\mathfrak{S}_6$. After restriction to the subgroup Γ the $2.\mathfrak{S}_6$-representation $\mathbb{U}_4$*

- *remains irreducible, if Γ is one of the subgroups $2.\mathfrak{A}_6$, $2.\mathfrak{S}_5^{nst}$, $2.\mathfrak{A}_5^{nst}$, $2.\mathfrak{S}_4^{nst}$, $2.F_{36}$ or $2.F_{20}$;*
- *splits into a sum of two non-isomorphic irreducible two-dimensional representations, if Γ is one of the subgroups $2.\mathfrak{A}_5^{st}$, $2.\mathfrak{A}_4^{nst}$ or $2.\mathrm{D}_{12}^{nst}$.*

Proof. Compute inner products of the corresponding characters with themselves, and keep in mind that neither of the groups $2.\mathfrak{A}_5^{st}$, $2.\mathfrak{A}_4^{nst}$ and $2.\mathrm{D}_{12}^{nst}$ has an irreducible three-dimensional representation with a non-trivial action of the central subgroup. □

Remark 2.2 By Corollary 2.1(i), the $2.\mathfrak{S}_5^{nst}$-representation $\mathbb{U}_4$ is irreducible. One can check that it is not induced from any proper subgroup of $2.\mathfrak{S}_5^{nst}$, that is, it defines a primitive subgroup isomorphic to $2.\mathfrak{S}_5$ in $\mathrm{GL}_4(\mathbb{C})$. Note that this subgroup is not present in the list given in [**9**, Section 8.5]. It is still listed by some other classical surveys, see, for example, [**2**, Section 119].

Corollary 2.3 *Let Γ be a subgroup of $\mathfrak{S}_6$. After restriction to the subgroup Γ, the $\mathfrak{S}_6$-representation $\mathbb{W}_5$*

(i) *remains irreducible, if Γ is one of the subgroups $\mathfrak{A}_6$, $\mathfrak{S}_5^{nst}$ or $\mathfrak{A}_5^{nst}$;*
(ii) *splits into a sum of the trivial and an irreducible four-dimensional representation if Γ is a subgroup $\mathfrak{A}_5^{st}$;*
(iii) *splits into a sum of the trivial and two different irreducible two-dimensional representations if Γ is a subgroup D_{12}^{nst}.*

In the sequel, we will denote the restrictions of the $2.\mathfrak{S}_6$-representation $\mathbb{U}_4$ and of the $\mathfrak{S}_6$-representation $\mathbb{W}_5$ to various subgroups by the same symbols for simplicity. The next two corollaries are implied by direct computations (we used GAP software [**11**] to perform them).

COROLLARY 2.4 *The following assertions hold:*

(i) *the* $\mathfrak{A}_6$*-representation* $\mathrm{Sym}^2(\mathbb{U}_4^\vee)$ *does not contain one-dimensional subrepresentations*;
(ii) *the* $\mathfrak{A}_6$*-representation* $\mathrm{Sym}^4(\mathbb{U}_4^\vee)$ *does not contain one-dimensional subrepresentations*;
(iii) *the* $\mathfrak{A}_5^{nst}$*-representation* $\mathrm{Sym}^2(\mathbb{U}_4^\vee)$ *splits into a sum of two different irreducible three-dimensional representations and one irreducible four-dimensional representation*;
(iv) *the* $2.\mathfrak{A}_5^{nst}$*-representation* $\mathrm{Sym}^3(\mathbb{U}_4^\vee)$ *does not contain one-dimensional subrepresentations*;
(v) *the* $\mathfrak{A}_5^{nst}$*-representation* $\mathrm{Sym}^4(\mathbb{U}_4^\vee)$ *has a unique two-dimensional subrepresentation, and this subrepresentation splits into a sum of two trivial representations of* $\mathfrak{A}_5^{nst}$.

Recall that all representations of a symmetric group are self-dual. Therefore, to study invariant hypersurfaces in $\mathbb{P}(\mathbb{W}_5)$, we will use the following result.

COROLLARY 2.5 *Let* Γ *be one of the groups* $\mathfrak{S}_6$, $\mathfrak{A}_6$ *or* $\mathfrak{S}_5^{nst}$*. Then,*

(i) *the* Γ*-representation* $\mathrm{Sym}^2(\mathbb{W}_5)$ *has a unique one-dimensional subrepresentation*;
(ii) *the* Γ*-representation* $\mathrm{Sym}^4(\mathbb{W}_5)$ *has a unique two-dimensional subrepresentation, and this subrepresentation splits into a sum of two trivial representations of* Γ.

We conclude this section by recalling some information about several subgroups of $2.\mathfrak{S}_6$ that are smaller than those listed in Table 1. Namely, we list in Table 2 orders, types and numbers of elements in certain subgroups of $2.\mathfrak{A}_5^{nst}$. We keep the notation used in Table 1. By $2.\mathfrak{S}_3'$ we denote a subgroup of $2.\mathfrak{A}_5^{nst}$ isomorphic to $2.\mathfrak{S}_3$. Note that the preimage in $2.\mathfrak{S}_6$ of any subgroup $\boldsymbol{\mu}_5 \subset \mathfrak{S}_6$ is isomorphic to $\boldsymbol{\mu}_{10}$.

Looking at Table 2 (and keeping in mind character values provided by Table 1) we immediately obtain the following.

COROLLARY 2.6 *Let* Γ *be a subgroup of* $2.\mathfrak{A}_5^{nst} \subset 2.\mathfrak{S}_6$*. After restriction to* Γ *the* $2.\mathfrak{S}_6$*-representation* $\mathbb{U}_4$

(i) *splits into a sum of two non-isomorphic irreducible two-dimensional representations if* Γ *is a subgroup* $2.\mathrm{D}_{10}$;
(ii) *splits into a sum of an irreducible two-dimensional representation and two non-isomorphic one-dimensional representations if* Γ *is a subgroup* $2.\mathfrak{S}_3'$;
(iii) *splits into a sum of two isomorphic irreducible two-dimensional representations if* Γ *is a subgroup* $2.(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)$;
(iv) *splits into a sum of four pairwise non-isomorphic one-dimensional representations if* Γ *is a subgroup* $\boldsymbol{\mu}_{10}$.

Table 2. Subgroups of $2.\mathfrak{A}_5^{nst}$.

ord	type	$2.\mathrm{D}_{10}$	$2.\mathfrak{S}_3'$	$2.(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)$	$\boldsymbol{\mu}_{10}$
1	id	1	1	1	1
2	z	1	1	1	1
4	[2, 2]	10	6	6	0
6	[3, 3]	0	2	0	0
3	[3, 3]	0	2	0	0
10	[5]	4	0	0	4
5	[5]	4	0	0	4

3. Icosahedral group in 3 dimensions

In this section, we consider the action of the group $\mathfrak{A}_5$ on the projective space $\mathbb{P}^3$ arising from a non-standard embedding of $\mathfrak{A}_5 \hookrightarrow \mathfrak{S}_6$. Namely, we identify $\mathbb{P}^3$ with the projectivization $\mathbb{P}(\mathbb{U}_4)$, where $\mathbb{U}_4$ is the restriction of the four-dimensional irreducible representation of the group $2.\mathfrak{S}_6$ introduced in Section 2 to a subgroup $2.\mathfrak{A}_5^{nst}$ (which we will refer to just as $2.\mathfrak{A}_5$ in this section). Recall from Corollary 2.1(i) that $\mathbb{U}_4$ is an irreducible representation of $2.\mathfrak{A}_5$. Moreover, this is the unique faithful four-dimensional irreducible representation of the group $2.\mathfrak{A}_5$ (see e.g. [**8**, p. 2]).

Remark 3.1 (see e.g. [**8**, p. 2]). Let Γ be a proper subgroup of $\mathfrak{A}_5$ such that the index of Γ is at most 15. Then, Γ is isomorphic either to $\mathfrak{A}_4$, or to D_{10}, or to $\mathfrak{S}_3$, or to $\boldsymbol{\mu}_5$, or to $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$. In particular, if $\mathfrak{A}_5$ acts transitively on the set of $r < 15$ elements, then $r \in \{5, 6, 10, 12\}$.

Lemma 3.2 *Let Ω be an $\mathfrak{A}_5$-orbit of length $r \leq 15$ in $\mathbb{P}^3$. Then, either $r = 10$, or $r = 12$. Moreover, $\mathbb{P}^3$ contains exactly two $\mathfrak{A}_5$-orbits of length 10 and exactly two $\mathfrak{A}_5$-orbits of length 12.*

Proof. By Remark 3.1 one has $r \in \{1, 5, 6, 10, 12, 15\}$. The case $r = 1$ is impossible since $\mathbb{U}_4$ is an irreducible $2.\mathfrak{A}_5$-representation. Restricting $\mathbb{U}_4$ to subgroups of $2.\mathfrak{A}_5$ isomorphic to $2.\mathfrak{A}_4$, $2.\mathrm{D}_{10}$ and $2.(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2)$, and applying Corollaries 2.1(ii) and 2.6(i), (iii), we see that $r \notin \{5, 6, 15\}$.

Restricting $\mathbb{U}_4$ to a subgroup of $2.\mathfrak{A}_5$ isomorphic to $2.\mathfrak{S}_3$, applying Corollary 2.6(ii) and keeping in mind that there are 10 subgroups isomorphic to $\mathfrak{S}_3$ in $\mathfrak{A}_5$, we see that $\mathbb{P}^3$ contains exactly two $\mathfrak{A}_5$-orbits of length 10.

Finally, restricting $\mathbb{U}_4$ to a subgroup of $2.\mathfrak{A}_5$ isomorphic to $\boldsymbol{\mu}_{10}$, applying Corollary 2.6(iv) and keeping in mind that there are six subgroups isomorphic to $\boldsymbol{\mu}_5$ in $\mathfrak{A}_5$, we see that $\mathbb{P}^3$ contains exactly two $\mathfrak{A}_5$-orbits of length 12. ∎

Lemma 3.3 *There are no $\mathfrak{A}_5$-invariant surfaces of degree at most three in $\mathbb{P}^3$.*

Proof. Apply Corollary 2.4(iii), (iv). ∎

By Corollary 2.1(ii), the subgroup $\mathfrak{A}_4 \subset \mathfrak{A}_5$ leaves invariant two disjoint lines in $\mathbb{P}^3$, say L_1 and L_1'. Let $L_1, \ldots, L_5$ be the $\mathfrak{A}_5$-orbit of the line L_1, and let $L_1', \ldots, L_5'$ be the $\mathfrak{A}_5$-orbit of the line L_1'.

Lemma 3.4 *The lines $L_1, \ldots, L_5$ (respectively, the lines $L_1', \ldots, L_5'$) are pairwise disjoint.*

Proof. Suppose that some of the lines $L_1, \ldots, L_5$ have a common point. Since the action of $\mathfrak{A}_5$ on the set $\{L_1, \ldots, L_5\}$ is doubly transitive, this implies that every two of the lines $L_1, \ldots, L_5$ have a common point. Therefore, either all lines $L_1, \ldots, L_5$ are coplanar, or all of them pass through one point. Both of these cases are impossible since the $2.\mathfrak{A}_5$-representation $\mathbb{U}_4$ is irreducible by Corollary 2.1(i). Therefore, the lines $L_1, \ldots, L_5$ are pairwise disjoint. The same argument applies to the lines $L_1', \ldots, L_5'$. ∎

Corollary 3.5 *Any $\mathfrak{A}_5$-orbit contained in the union $L_1 \cup \ldots \cup L_5$ has length at least 20.*

Proof. Corollary 2.1(ii) implies that the stabilizer $\Gamma \cong \mathfrak{A}_4$ of the line L_1 acts on L_1 faithfully. Therefore, the length of any Γ-orbit contained in L_1 is at least four. Thus, the required assertion follows from Lemma 3.4. ∎

We are going to describe the configuration formed by the lines $L_1, \ldots, L_5$, $L_1', \ldots, L_5'$.

Definition 3.6 Let $T_1, \ldots, T_5$, $T'_1, \ldots, T'_5$ be different lines in a projective space. We say that they form a *double five configuration* if the following conditions hold:

- the lines $T_1, \ldots, T_5$ (respectively, the lines $T'_1, \ldots, T'_5$) are pairwise disjoint;
- for every i the lines T_i and T'_i are disjoint;
- for every $i \neq j$, the line T_i meets the line T'_j.

Lemma 3.7 *The lines $L_1, \ldots, L_5$, $L'_1, \ldots, L'_5$ form a double five configuration. Moreover, the only line in $\mathbb{P}^3$ that intersects all lines of $L_1, \ldots, L_5$, but L_i is the line L'_i, and the only line in $\mathbb{P}^3$ that intersects all lines of $L'_1, \ldots, L'_5$, but L'_i is the line L_i.*

Proof. For any i, the lines L_i and L'_i are disjoint by construction. The lines $L_1, \ldots, L_5$ (respectively, the lines $L'_1, \ldots, L'_5$) are pairwise disjoint by Lemma 3.4.

Since any three pairwise skew lines in $\mathbb{P}^3$ are contained in a smooth quadric surface, and an intersection of two different quadric surfaces in $\mathbb{P}^3$ cannot contain three pairwise skew lines, we see that for any three indices $1 \leq i < j < k \leq 5$ there is a unique quadric surface Q_{ijk} in $\mathbb{P}^3$ passing through the lines L_i, L_j and L_k. Moreover, the quadric Q_{ijk} is smooth. Note also that the quadric Q_{ijk} is not $\mathfrak{A}_5$-invariant by Lemma 3.3. This implies that all five lines $L_1, \ldots, L_5$ are not contained in a quadric.

Therefore, we may assume that the quadric Q_{123} does not contain the line L_4. It is well known that in this case either there is a unique line L meeting all four lines $L_1, \ldots, L_4$, or there are exactly two lines L and L' meeting $L_1, \ldots, L_4$. In the latter case, the stabilizer $\Gamma \subset \mathfrak{A}_5$ of the quadruple $L_1, \ldots, L_4$ (that is, the stabilizer of the line L_5) preserves the lines L_5, L and L'. However, the lines L and L' are different from L_5 since L_5 meets neither of the lines $L_1, \ldots, L_4$; moreover, the group $\Gamma \cong \mathfrak{A}_4$ fixes both L and L'. But Γ cannot fix three different lines in $\mathbb{P}^3$ by Corollary 2.1(ii). The contradiction shows that there is a unique line L meeting $L_1, \ldots, L_4$. Again we see that $L \neq L_5$, so that $L = L'_5$ by Corollary 2.1(ii).

Since the group $\mathfrak{A}_5$ permutes the lines $L_1, \ldots, L_5$ transitively, we conclude that the only line in $\mathbb{P}^3$ that intersects all lines of $L_1, \ldots, L_5$ except L_i is the line L'_i. Similarly, we see that the only line in $\mathbb{P}^3$ that intersects all lines of $L'_1, \ldots, L'_5$ except L'_i is the line L_i. In particular, the lines $L_1, \ldots, L_5$, $L'_1, \ldots, L'_5$ form a double five configuration. □

Lemma 3.8 *Every $\mathfrak{A}_5$-invariant curve of degree at most three in $\mathbb{P}^3$ is a twisted cubic. Moreover, there are exactly two $\mathfrak{A}_5$-invariant twisted cubic curves in $\mathbb{P}^3$.*

Proof. Let C be an $\mathfrak{A}_5$-invariant curve of degree at most three in $\mathbb{P}^3$. Since the $2.\mathfrak{A}_5$-representation $\mathbb{U}_4$ is irreducible, we conclude that C is a twisted cubic.

By Corollary 2.4(iii), one has

$$\mathrm{Sym}^2\big(\mathbb{U}_4\big) \cong \mathbb{V}_3 \oplus \mathbb{V}'_3 \oplus \mathbb{V}_4, \tag{3.1}$$

where $\mathbb{V}_3$, $\mathbb{V}'_3$ and $\mathbb{V}_4$, are irreducible representations of the group $\mathfrak{A}_5$ of dimensions 3, 3 and 4, respectively. Note that $\mathbb{V}_3$ and $\mathbb{V}'_3$ are not isomorphic.

Denote by $\mathcal{Q}$ and $\mathcal{Q}'$ the linear systems of quadrics in $\mathbb{P}^3$ that correspond to $\mathbb{V}_3$ and $\mathbb{V}'_3$, respectively. Since $\mathbb{P}^3$ does not contain $\mathfrak{A}_5$-orbits of lengths less than or equal to eight by Lemma 3.2, we see that the base loci of $\mathcal{Q}$ and $\mathcal{Q}'$ contain $\mathfrak{A}_5$-invariant curves $\mathscr{C}^1$ and $\mathscr{C}^2$, respectively. The degrees of these curves must be <4, so that they are twisted cubic curves. This also implies that the base loci of $\mathcal{Q}$ and $\mathcal{Q}'$ are exactly the curves $\mathscr{C}^1$ and $\mathscr{C}^2$, respectively.

Now take an arbitrary $\mathfrak{A}_5$-invariant twisted cubic curve C in $\mathbb{P}^3$. The quadrics in $\mathbb{P}^3$ passing through C define a three-dimensional $\mathfrak{A}_5$-subrepresentation in $\mathrm{Sym}^2(\mathbb{U}_4)$. Moreover, different $\mathfrak{A}_5$-invariant twisted cubics give different $\mathfrak{A}_5$-subrepresentations of $\mathrm{Sym}^2(\mathbb{U}_4)$. Thus, Equation (3.1) implies that C coincides either with $\mathscr{C}^1$ or with $\mathscr{C}^2$. □

Keeping in mind Lemma 3.8, we will denote the two $\mathfrak{A}_5$-invariant twisted cubic curves in $\mathbb{P}^3$ by $\mathscr{C}^1$ and $\mathscr{C}^2$ throughout this section.

Remark 3.9 The curves $\mathscr{C}^1$ and $\mathscr{C}^2$ are disjoint. Indeed, otherwise, their intersection would contain at least 12 points, which is impossible, since a twisted cubic curve is an intersection of quadrics.

The lines in $\mathbb{P}^3$ that are tangent to the curves $\mathscr{C}^1$ and $\mathscr{C}^2$ sweep out quartic surfaces $\mathcal{S}^1$ and $\mathcal{S}^2$, respectively. These surfaces are $\mathfrak{A}_5$-invariant. The singular loci of $\mathcal{S}^1$ and $\mathcal{S}^2$ are the curves $\mathscr{C}^1$ and $\mathscr{C}^2$, respectively. In particular, the surfaces $\mathcal{S}^1$ and $\mathcal{S}^2$ are different. Their singularities along these curves are locally isomorphic to a product of $\mathbb{A}^1$ and an ordinary cusp.

Denote by $\mathcal{P}$ the pencil of quartics in $\mathbb{P}^3$ generated by $\mathcal{S}^1$ and $\mathcal{S}^2$.

Lemma 3.10 *All $\mathfrak{A}_5$-invariant quartic surfaces in $\mathbb{P}^3$ are contained in the pencil $\mathcal{P}$.*

Proof. This follows from Corollary 2.4(v). □

We proceed by describing the base locus of the pencil $\mathcal{P}$. This was done in [**4**, Remark 2.6], but we reproduce the details here for the convenience of the reader.

Lemma 3.11 *The base locus of the pencil $\mathcal{P}$ is an irreducible curve B of degree* 16. *It has* 24 *singular points, these points are in a union of two $\mathfrak{A}_5$-orbits of length* 12, *and each of them is an ordinary cusp of the curve B. The curve B contains a unique $\mathfrak{A}_5$-orbit of length* 20.

Proof. Denote by B the base curve of the pencil $\mathcal{P}$. Let us show that the curves $\mathscr{C}^1$ and $\mathscr{C}^2$ are not contained in B. Since $\mathscr{C}^1$ is projectively normal, there is an exact sequence of $2.\mathfrak{A}_5$-representations

$$0 \to H^0\big(\mathcal{I}_{\mathscr{C}^1} \otimes \mathcal{O}_{\mathbb{P}^3}(4)\big) \to H^0\big(\mathcal{O}_{\mathbb{P}^3}(4)\big) \to H^0\big(\mathcal{O}_{\mathscr{C}^1} \otimes \mathcal{O}_{\mathbb{P}^3}(4)\big) \to 0,$$

where $\mathcal{I}_{\mathscr{C}^1}$ is the ideal sheaf of $\mathscr{C}^1$. The $2.\mathfrak{A}_5$-representation $H^0(\mathcal{O}_{\mathscr{C}^1} \otimes \mathcal{O}_{\mathbb{P}^3}(4))$ contains a one-dimensional subrepresentation corresponding to the unique $\mathfrak{A}_5$-orbit of length 12 in $\mathscr{C}^1 \cong \mathbb{P}^1$. This shows that $\mathcal{P}$ contains a surface that does not pass through $\mathscr{C}^1$, so that $\mathscr{C}^1$ is not contained in B. Similarly, we see that $\mathscr{C}^2$ is not contained in B.

Let $\rho\colon \hat{\mathcal{S}}^1 \to \mathcal{S}^1$ be the normalization of the surface $\mathcal{S}^1$, and let $\hat{\mathscr{C}}^1$ be the preimage of the curve $\mathscr{C}^1$ via ρ. Then, the action of the group $\mathfrak{A}_5$ lifts to $\hat{\mathcal{S}}^1$. One has $\hat{\mathcal{S}}^1 \cong \mathbb{P}^1 \times \mathbb{P}^1$, and $\rho^*(\mathcal{O}_{\mathcal{S}^1} \otimes \mathcal{O}_{\mathbb{P}^3}(1))$ is a divisor of bi-degree $(1, 2)$. This shows that $\hat{\mathscr{C}}^1$ is of bi-degree $(1, 1)$. Thus, the action of $\mathfrak{A}_5$ on $\hat{\mathcal{S}}$ is diagonal by Cheltsov and Shramov [**6**, Lemma 6.4.3(i)].

Denote by $\hat{B}$ be the preimage of the curve B via ρ. Then, $\hat{B}$ is a divisor of bi-degree $(4, 8)$. Hence, the curve $\hat{B}$ is irreducible by Cheltsov and Shramov [**6**, Lemma 6.4.4(i)], so that the curve B is irreducible as well.

Note that the curve $\hat{B}$ is singular. Indeed, the intersection $\mathcal{S}^1 \cap \mathscr{C}^2$ is an $\mathfrak{A}_5$-orbit Σ_{12} of length 12, because $\mathscr{C}^2$ is not contained in $\mathcal{S}^1$. Similarly, we see that the intersection $\mathcal{S}^2 \cap \mathscr{C}^1$ is also an $\mathfrak{A}_5$-orbit Σ'_{12} of length 12. These $\mathfrak{A}_5$-orbits Σ_{12} and Σ'_{12} are different by Remark 3.9. Since B is the scheme theoretic intersection of the surfaces $\mathcal{S}^1$ and $\mathcal{S}^2$, it must be singular at every point of

$\Sigma_{12} \cup \Sigma'_{12}$. Denote by $\hat{\Sigma}_{12}$ and $\hat{\Sigma}'_{12}$ the preimages via ρ of the $\mathfrak{A}_5$-orbits Σ_{12} and Σ'_{12}, respectively. Then, $\hat{B}$ is singular in every point of $\hat{\Sigma}'_{12}$.

The curve $\hat{B}$ is smooth away of $\hat{\Sigma}'_{12}$, because its arithmetic genus is 21, and the surface $\hat{\mathcal{S}}^1$ does not contain $\mathfrak{A}_5$-orbits of length <12. However, we have

$$\hat{B} \bigcap \hat{\mathscr{C}}^1 = \hat{\Sigma}_{12},$$

because $\hat{B} \cdot \mathscr{C}^1 = 12$ and $\hat{\Sigma}_{12} \subset \hat{B}$. This shows that B is an irreducible curve whose only singularities are the points of $\Sigma_{12} \cup \Sigma'_{12}$, and each such point is an ordinary cusp of the curve B. In particular, the genus of the normalization of the curve B is 9. By Cheltsov and Shramov [18, Lemma 5.1.5], this implies that B contains a unique $\mathfrak{A}_5$-orbit of length 20. □

The following classification of $\mathfrak{A}_5$-invariant quartic surfaces in $\mathbb{P}^3$ was obtained in [17, Theorem 2.4].

LEMMA 3.12 *The pencil $\mathcal{P}$ contains two surfaces $\mathcal{R}^1$ and $\mathcal{R}^2$ with ordinary double singularities, such that the singular loci of $\mathcal{R}^1$ and $\mathcal{R}^2$ are $\mathfrak{A}_5$-orbits of length* 10. *Every surface in $\mathbb{P}^3$ is different from $\mathcal{S}^1$, $\mathcal{S}^2$, $\mathcal{R}^1$ and $\mathcal{R}^2$ is smooth.*

Proof. Let S be a surface in $\mathcal{P}$ that is different from $\mathcal{S}^1$ and $\mathcal{S}^2$. It follows from Lemma 3.3 that S is irreducible. Assume that S is singular.

We claim that S has isolated singularities. Indeed, suppose that S is singular along some $\mathfrak{A}_5$-invariant curve Z. Taking a general plane section of S, we see that the degree of Z is at most three. Thus, one has either $Z = \mathscr{C}^1$ or $Z = \mathscr{C}^2$ by Lemma 3.8. Since neither of these curves is contained in the base locus of $\mathcal{P}$ by Lemma 3.11, this would imply that either $S = \mathcal{S}^1$ or $S = \mathcal{S}^2$. The latter is not the case by assumption.

We see that the singularities of S are isolated. Hence, S contains at most two non-Du Val singular points by [23, Theorem 1] applied to the minimal resolution of singularities of the surface S. Since the set of all non-Du Val singular points of the surface S must be $\mathfrak{A}_5$-invariant, we see that S has none of them by Lemma 3.2. Thus, all singularities of S are Du Val.

By Cheltsov and Shramov [6, Lemma 6.7.3(iii)], the surface S has only ordinary double singularities, the set $\mathrm{Sing}(S)$ consists of one $\mathfrak{A}_5$-orbit, and

$$|\mathrm{Sing}(S)| \in \{5, 6, 10, 12, 15\}.$$

Since $\mathbb{P}^3$ does not contain $\mathfrak{A}_5$-orbits of lengths 5, 6 and 15 by Lemma 3.2, we see that $\mathrm{Sing}(S)$ is either an $\mathfrak{A}_5$-orbit of length 10 or an $\mathfrak{A}_5$-orbit of length 12.

Suppose that the singular locus of S is an $\mathfrak{A}_5$-orbit Σ_{12} of length 12. Then, S does not contain other $\mathfrak{A}_5$-orbits of length 12 by Cheltsov and Shramov [6, Lemma 6.7.3(iv)]. Since $\mathscr{C}^1$ is not contained in the base locus of $\mathcal{P}$ by Lemma 3.11, and $\mathscr{C}^1$ is contained in $\mathcal{S}^1$, we see that $\mathscr{C}^1 \subset S$. Since

$$S \cdot \mathscr{C}^1 = 12$$

and Σ_{12} is the only $\mathfrak{A}_5$-orbit of length at most 12 in $\mathscr{C}^1 \cong \mathbb{P}^1$, we have $S \cap \mathscr{C}^1 = \Sigma_{12}$. Thus,

$$12 = S \cdot \mathscr{C}^1 \geq \sum_{P \in \Sigma_{12}} \mathrm{mult}_P(S) = 2|\Sigma_{12}| = 24,$$

which is absurd.

Therefore, we see that the singular locus of S is an $\mathfrak{A}_5$-orbit Σ_{12} of length 10. Vice versa, if an $\mathfrak{A}_5$-invariant quartic surface passes through an $\mathfrak{A}_5$-orbit of length 10, then it is singular by Cheltsov and Shramov [6, Lemma 6.7.1(ii)]. We know from Lemma 3.2 that there are exactly two $\mathfrak{A}_5$-orbits of length 10 in $\mathbb{P}^3$, and it follows from Lemma 3.11 that they are not contained in the base locus of $\mathcal{P}$. Thus, there are two surfaces $\mathcal{R}^1$ and $\mathcal{R}^2$ that are singular exactly at the points of these two $\mathfrak{A}_5$-orbits, respectively. The above argument shows that every surface in $\mathcal{P}$ except $\mathcal{S}^1$, $\mathcal{S}^2$, $\mathcal{R}^1$ and $\mathcal{R}^2$ is smooth. ☐

Keeping in mind Lemma 3.12, we will denote by $\mathcal{R}^1$ and $\mathcal{R}^2$ the two nodal surfaces contained in the pencil $\mathcal{P}$ until the end of this section.

Lemma 3.13 *There is a unique $\mathfrak{A}_5$-invariant quartic surface in $\mathbb{P}^3$ that contains the lines $L_1, \ldots, L_5$ (respectively, the lines $L_1', \ldots, L_5'$). Moreover, this surface is smooth, and it does not contain the lines $L_1', \ldots, L_5'$ (respectively, $L_1, \ldots, L_5$).*

Proof. Put $\mathcal{L} = \sum_{i=1}^5 L_i$ and $\mathcal{L}' = \sum_{i=1}^5 L_i'$. Corollary 2.1(ii) implies that the stabilizer in $\mathfrak{A}_5$ of a general point of L_1 is trivial. Therefore, there exists a surface $S \in \mathcal{P}$ that contains all lines $L_1, \ldots, L_5$. By Lemma 3.11 such surface S is unique.

We claim that $S \neq \mathcal{S}^1$. Indeed, all lines contained in $\mathcal{S}^1$ are tangent to the curve $\mathscr{C}^1$, and there are no $\mathfrak{A}_5$-orbits of length five in $\mathscr{C}^1 \cong \mathbb{P}^1$. Similarly, one has $S \neq \mathcal{S}^2$.

We claim that S is not one of the two nodal surfaces $\mathcal{R}^1$ and $\mathcal{R}^2$ contained in the pencil $\mathcal{P}$. Indeed, suppose that $S = \mathcal{R}^1$. Since the singular locus of $\mathcal{R}^1$ is an $\mathfrak{A}_5$-orbit of length 10 by Lemma 3.12, we see that the lines $L_1, \ldots, L_5$ are contained in the smooth locus of $\mathcal{R}^1$ by Corollary 3.5. However, one has $\mathcal{L}^2 = -10$ by Lemma 3.4. This means that $\mathrm{rk}\,\mathrm{Pic}\,(S)^{\mathfrak{A}_5} \geq 2$, which is impossible by Cheltsov and Shramov [6, Lemma 6.7.3(i), (ii)].

We see that the surface S is different from $\mathcal{R}^1$. The same argument shows that S is different from $\mathcal{R}^2$. Hence, S is smooth by Lemma 3.12.

Let us show that S does not contain the lines $L_1', \ldots, L_5'$. Suppose that it does. By Lemma 3.4 one has

$$\mathcal{L} \cdot \mathcal{L} = \mathcal{L}' \cdot \mathcal{L}' = -10.$$

By Cheltsov and Shramov [6, Lemma 6.7.1(i)], we have $\mathrm{rk}\,\mathrm{Pic}\,(S)^{\mathfrak{A}_5} = 2$. Let Π_S be the class of a plane section of S. Then, the determinant of the matrix

$$\begin{pmatrix} \mathcal{L} \cdot \mathcal{L} & \mathcal{L} \cdot \mathcal{L}' & \Pi_S \cdot \mathcal{L} \\ \mathcal{L} \cdot \mathcal{L}' & \mathcal{L}' \cdot \mathcal{L}' & \Pi_S \cdot \mathcal{L}' \\ \Pi_S \cdot \mathcal{L} & \Pi_S \cdot \mathcal{L}' & \Pi_S \cdot \Pi_S \end{pmatrix} = \begin{pmatrix} -10 & 20 & 5 \\ 20 & -10 & 5 \\ 5 & 5 & 4 \end{pmatrix}$$

must vanish. This is a contradiction, because it is equal to 300.

Applying the same argument, we see that the lines $L_1', \ldots, L_5'$ are contained in a unique $\mathfrak{A}_5$-invariant quartic surface, this surface is smooth and does not contain the lines $L_1, \ldots, L_5$. ☐

REMARK 3.14 One can use the properties of the pencil $\mathcal{P}$ to give an alternative proof of Lemma 3.7. Namely, we know from Lemma 3.13 that there are two (different) smooth $\mathfrak{A}_5$-invariant quartic surfaces S and S' containing the lines $L_1,\ldots,L_5$ and $L_1',\ldots,L_5'$, respectively. By Lemma 3.11, the base locus of the pencil $\mathcal{P}$ is an irreducible curve B that contains a unique $\mathfrak{A}_5$-orbit Σ of length 20. By Corollary 3.5, this implies that Σ is contained in the union $L_1 \cup \ldots \cup L_5$, because

$$B \cdot (L_1 + \ldots + L_5) = 20$$

on the surface S. Similarly, we see that Σ is contained in $L_1' \cup \ldots \cup L_5'$. These facts together with Lemma 3.4 easily imply that the lines $L_1,\ldots,L_5$ and $L_1',\ldots,L_5'$ form a double five configuration.

Now we will obtain some restrictions on low degree $\mathfrak{A}_5$-invariant curves in $\mathbb{P}^3$.

LEMMA 3.15 *Let C be an irreducible $\mathfrak{A}_5$-invariant curve in $\mathbb{P}^3$ of degree $d \leq 10$. Denote by g the genus of the normalization of the curve C. Then,*

$$g \leq \frac{d^2}{8} + 1 - |\mathrm{Sing}(C)|.$$

Proof. Since $\mathbb{U}_4$ is an irreducible $2.\mathfrak{A}_5$-representation, the curve C is not contained in a plane in $\mathbb{P}^3$. This implies that a stabilizer in $\mathfrak{A}_5$ of a general point of the curve C is trivial. In particular, the $\mathfrak{A}_5$-orbit of a general point of C has length $|\mathfrak{A}_5| = 60$.

Let S be a surface in the pencil $\mathcal{P}$ that passes through a general point of C. Then, the curve C is contained in S, because otherwise one would have

$$60 \leq |S \bigcap C| \leq S \cdot C = 4d \leq 40,$$

which is absurd. Since the assertion of the lemma clearly holds for the twisted cubic curves $\mathscr{C}^1$ and $\mathscr{C}^2$, we may assume that C is different from these two curves.

Suppose that $S = \mathcal{S}^1$. Let us use the notation of the proof of Lemma 3.11. Denote by $\hat{C}$ the preimage of the curve C via ρ. Then, $\hat{C}$ is a divisor of bi-degree (a, b) for some non-negative integers a and b such that $d = 2a + b$. However, one has

$$|\hat{C} \bigcap \hat{\mathscr{C}}^1| \leq \hat{C} \cdot \hat{\mathscr{C}}^1 = a + b \leq 2a + b = d \leq 10,$$

which is impossible, since the curve $\hat{\mathscr{C}}^1 \cong \mathscr{C}^1 \cong \mathbb{P}^1$ does not contain $\mathfrak{A}_5$-orbits of length <12.

We see that $S \neq \mathcal{S}^1$. Similarly, we see that $S \neq \mathcal{S}^2$. By Lemma 3.12, either S is a smooth quartic $K3$ surface, or S is one of the surfaces $\mathcal{R}^1$ and $\mathcal{R}^2$. Denote by Π_S a plane section of S. Then,

$$\det\begin{pmatrix} \Pi_S^2 & \Pi_S \cdot C \\ \Pi_S \cdot C & C^2 \end{pmatrix} = \det\begin{pmatrix} 4 & d \\ d & C^2 \end{pmatrix} = 4C^2 - d^2 \leq 0$$

by the Hodge index theorem.

Suppose that C is contained in the smooth locus of the surface S. Denote by $p_a(C)$ the arithmetic genus of the curve C. Then,

$$C^2 = 2p_a(C) - 2,$$

by the adjunction formula. Thus, we get

$$p_a(C) \leq \frac{d^2}{8} + 1.$$

Since $g \leq p_a(C) - |\mathrm{Sing}(C)|$, this implies the assertion of the lemma.

To complete the proof, we may assume that C contains a singular point of the surface S. By Lemma 3.12, this means either $S = \mathcal{R}^1$ or $S = \mathcal{R}^2$. The singularities of the surface S are ordinary double points, and its singular locus is an $\mathfrak{A}_5$-orbit of length 10. In particular, the curve C contains the whole singular locus of S. By Cheltsov and Shramov [6, Lemma 6.7.3(i), (ii)], one has $\mathrm{Pic}(S)^{\mathfrak{A}_5} \cong \mathbb{Z}$. Since $\Pi_S^2 = 4$ and the self-intersection of any Cartier divisor on the surface S is even, we see that the group $\mathrm{Pic}(S)^{\mathfrak{A}_5}$ is generated by Π_S.

Suppose that C is a Cartier divisor on S. Then, either $C \sim \Pi_S$ or $C \sim 2\Pi_S$, because $d \leq 10$. Since the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^3}(n)) \to H^0(\mathcal{O}_S(n\Pi_S))$$

is an isomorphism for $n \leq 3$, we conclude that there is an $\mathfrak{A}_5$-invariant quadric in $\mathbb{P}^3$. This is not the case by Lemma 3.3.

Therefore, we see that C is not a Cartier divisor on S. Since S has only ordinary double points, the divisor $2C$ is Cartier. Thus,

$$2C \sim l\Pi_S,$$

for some odd positive integer l. Since

$$2d = 2C \cdot \Pi_S = l\Pi_S \cdot \Pi_S = 4l,$$

we see that $l = \frac{d}{2}$. In particular, d is even and $l \leq 5$.

Let $\theta\colon \tilde{S} \to S$ be the minimal resolution of singularities of the surface S. Denote by $\tilde{C}$ the proper transform of the curve C on the surface $\tilde{S}$, and denote by $\Theta_1, \ldots, \Theta_{10}$ the exceptional curves of θ. Then,

$$2\tilde{C} \sim \theta^*(l\Pi_S) - m\sum_{i=1}^{10}\Theta_i,$$

for some positive integer m. Moreover, m is odd, because C is not a Cartier divisor. We have

$$4\tilde{C}^2 = \Pi_S^2 l^2 - 20m^2 = 4l^2 - 20m^2,$$

which implies that $\tilde{C}^2 = l^2 - 5m^2$. Since $\tilde{C}^2$ is even, m is odd and $l \leq 5$, we see that either $l = 3$ or $l = 5$.

Denote by $p_a(\tilde{C})$ the arithmetic genus of the curve $\tilde{C}$. Then,

$$l^2 - 5m^2 = \tilde{C}^2 = 2p_a(\tilde{C}) - 2,$$

by the adjunction formula. In particular, we have

$$25 - 5m^2 \geq l^2 - 5m^2 \geq -2,$$

so that $l \in \{3, 5\}$ and $m = 1$. The latter means that C is smooth at every point of $\mathrm{Sing}(S)$, so that

$$|\mathrm{Sing}(\tilde{C})| = |\mathrm{Sing}(C)|.$$

If $l = \frac{d}{2} = 3$, then $p_a(\tilde{C}) = 3$. This gives

$$g \leq p_a(\tilde{C}) - |\mathrm{Sing}(\tilde{C})| = 3 - |\mathrm{Sing}(C)| \leq \frac{d^2}{8} + 1 - |\mathrm{Sing}(C)|.$$

Similarly, if $l = \frac{d}{2} = 5$, then $p_a(\tilde{C}) = 11$. This gives

$$g \leq p_a(\tilde{C}) - |\mathrm{Sing}(\tilde{C})| = 11 - |\mathrm{Sing}(C)| \leq \frac{d^2}{8} + 1 - |\mathrm{Sing}(C)|.$$

□

Recall from [6, Lemma 5.4.1] that there exists a unique smooth irreducible curve of genus 4 with a faithful action of the group $\mathfrak{A}_5$. This curve is known as Bring's curve. Its canonical model is a complete intersection of a quadric and a cubic in a three-dimensional projective space. However, this sextic curve does not appear in our $\mathbb{P}^3 = \mathbb{P}(\mathbb{U}_4)$.

LEMMA 3.16 *Let C be a smooth irreducible $\mathfrak{A}_5$-invariant curve in $\mathbb{P}^3$ of degree $d \leq 6$ and genus* g. *Then, $g \neq 4$.*

Proof. Suppose that $g = 4$. Denote by Π_C the plane section of the curve C. Then,

$$h^0(\mathcal{O}_C(\Pi_C)) = d - 3 + h^0(\mathcal{O}_C(K_C - \Pi_C))$$

by the Riemann–Roch theorem. Since C is not contained in a plane, this implies that $\Pi_C \sim K_C$. Therefore, the projective space $\mathbb{P}^3$ is identified with a projectivization of an $\mathfrak{A}_5$-representation $H^0(\mathcal{O}_C(K_C))^\vee$, that is, of a representation of the group $2.\mathfrak{A}_5$ where the center of $2.\mathfrak{A}_5$ acts trivially. The latter is not the case by construction of $\mathbb{U}_4$. □

LEMMA 3.17 *Let C be an irreducible smooth $\mathfrak{A}_5$-invariant curve in $\mathbb{P}^3$ of degree $d = 10$ and genus g. Then, $g \neq 10$.*

Proof. Suppose that $g = 10$. By Lemma 3.11, the base locus of the pencil $\mathcal{P}$ is an irreducible curve B of degree 16. In particular, there exists a surface $S \in \mathcal{P}$ that does not contain C. Thus, the intersection $S \cap C$ is an $\mathfrak{A}_5$-invariant set that consists of

$$C \cdot S = 4d = 40$$

points (counted with multiplicities). However, by Cheltsov and Shramov [6, Lemma 5.1.5], any $\mathfrak{A}_5$-orbit in C has length 12, 30 or 60. $\square$

4. Large subgroups of $\mathfrak{S}_6$

In this section, we collect some auxiliary results about the groups $\mathfrak{S}_6$, $\mathfrak{A}_6$ and $\mathfrak{S}_5$. We start with recalling some general properties of the group $\mathfrak{A}_6$.

Remark 4.1 (see, for example, [8, p. 4]). Let Γ be a proper subgroup of $\mathfrak{A}_6$ such that the index of Γ is at most 15. Then, Γ is isomorphic either to $\mathfrak{A}_5$, or to F_{36}, or to $\mathfrak{S}_4$. In particular, if $\mathfrak{A}_6$ acts transitively on the set of $r < 15$ elements, then either $r = 6$ or $r = 10$.

We will need the following result about possible actions of the group $\mathfrak{A}_6$ on curves of small genera (cf. [5, Theorem 2.18] and [6, Lemma 5.1.5]).

Lemma 4.2 *Suppose that C is a smooth irreducible curve of genus $g \leq 15$ with a non-trivial action of the group $\mathfrak{A}_6$. Then, $g = 10$.*

Proof. Let $\Omega \subset C$ be an $\mathfrak{A}_6$-orbit. Then, a stabilizer of a point in Ω is a cyclic subgroup of $\mathfrak{A}_6$, which implies that

$$|\Omega| \in \{72, 90, 120, 180, 360\}.$$

From the classification of finite subgroups of $\mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}_2(\mathbb{C})$, we know that $g \neq 0$. Also, it follows from the non-solvability of the group $\mathfrak{A}_6$ that $g \neq 1$.

Put $\bar{C} = C/\mathfrak{A}_6$. Then, $\bar{C}$ is a smooth curve. Let $\bar{g}$ be the genus of the curve $\bar{C}$. The Riemann–Hurwitz formula gives

$$2g - 2 = 360(2\bar{g} - 2) + 180a_{180} + 240a_{120} + 270a_{90} + 288a_{72},$$

where a_k is the number of $\mathfrak{A}_6$-orbits in C of length k.

Since $a_k \geq 0$ and $2 \leq g \leq 15$, one has $\bar{g} = 0$. Thus, we obtain

$$2g - 2 = -720 + 180a_{180} + 240a_{120} + 270a_{90} + 288a_{72}.$$

Going through the values $2 \leq g \leq 15$, and solving this equation case by case, we see that the only possibility is $g = 10$. $\square$

We proceed by recalling some general properties of the group $\mathfrak{S}_5$.

Remark 4.3 (see e.g. [12, p. 2]). Let Γ be a proper subgroup of $\mathfrak{S}_5$ such that the index of Γ is <12. Then, Γ is isomorphic either to $\mathfrak{A}_5$, or to $\mathfrak{S}_4$, or to F_{20}, or to $\mathfrak{A}_4$, or to D_{12}. In particular, if $\mathfrak{S}_5$ acts transitively on the set of $r < 12$ elements, then $r \in \{2, 5, 6, 10\}$.

Lemma 4.4 *The group $\mathfrak{S}_5$ cannot act faithfully on a smooth irreducible curve of genus* 5.

Proof. Suppose that C is a curve of genus 5 with a faithful action of $\mathfrak{S}_5$. Considering the action of the subgroup $\mathfrak{A}_5 \subset \mathfrak{S}_5$ on C and applying [6, Lemma 5.4.3], we see that C is hyperelliptic. This gives a natural homomorphism

$$\theta\colon \mathfrak{S}_5 \to \mathrm{Aut}\big(\mathbb{P}^1\big) \cong \mathrm{PGL}_2\big(\mathbb{C}\big)$$

whose kernel is either trivial or isomorphic to $\boldsymbol{\mu}_2$. Thus, θ is injective, which gives a contradiction. $\square$

Now we will prove some auxiliary facts about actions of the groups $\mathfrak{S}_6$, $\mathfrak{A}_6$ and $\mathfrak{S}_5$ on the four-dimensional projective space.

Remark 4.5 The group $\mathfrak{S}_6$ has exactly four irreducible five-dimensional representations (see e.g. [8, p. 5]). Starting from one of them, one more can be obtained by a twist by an outer automorphism of $\mathfrak{S}_6$, and two remaining ones are obtained from these two by a tensor product with the sign representation. Although these four representations are not isomorphic, the images of $\mathfrak{S}_6$ in $\mathrm{PGL}_5(\mathbb{C})$ under them are the same. Every irreducible five-dimensional representation of $\mathfrak{S}_6$ restricts to an irreducible representation of the subgroup $\mathfrak{A}_6 \subset \mathfrak{S}_6$, and restricts to an irreducible representation of *some* of the subgroups $\mathfrak{S}_5 \subset \mathfrak{S}_6$. The group $\mathfrak{A}_6$ has exactly two irreducible five-dimensional representations, each of them arising this way (see e.g. [8, p. 5]). Similarly, the group $\mathfrak{S}_5$ has exactly two irreducible five-dimensional representations, each of them arising this way (see e.g. [8, p. 2]). Note also that every five-dimensional representation of a group $\mathfrak{A}_6$ or $\mathfrak{S}_5$ that does not contain one-dimensional subrepresentations is irreducible.

Let $\mathbb{V}_5$ be an irreducible five-dimensional representation of the group $\mathfrak{S}_6$. Put $\mathbb{P}^4 = \mathbb{P}(\mathbb{V}_5)$. Keeping in mind Remark 4.5, we see that the image of the corresponding homomorphism $\mathfrak{S}_6$ to $\mathrm{PGL}_5(\mathbb{C})$ is the same for any choice of $\mathbb{V}_5$, and thus the $\mathfrak{S}_6$-orbits and $\mathfrak{S}_6$-invariant hypersurfaces in $\mathbb{P}^4$ do not depend on $\mathbb{V}_5$ either.

Remark 4.5 implies that there are six linear forms $x_0, \ldots, x_5$ on $\mathbb{P}^4$ that are permuted by the group $\mathfrak{S}_6$ (cf. Sections 1 and 2). Indeed, up to a twist by an outer automorphoism of $\mathfrak{S}_6$ and a tensor product with the sign representation, $\mathbb{V}_5$ is a subrepresentation of the six-dimensional representation $\mathbb{W}$ of $\mathfrak{S}_6$, so that one can take restrictions of the natural coordinates in $\mathbb{W}$ to be these linear forms. Let Q be the three-dimensional quadric in $\mathbb{P}^4$ given by equation

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0. \tag{4.1}$$

The quadric Q is smooth and $\mathfrak{S}_6$-invariant. Note also that Equation (1.1) makes sense in our $\mathbb{P}^4$.

We will use the notation introduced above until the end of the paper.

Lemma 4.6 *Let Γ be either the group $\mathfrak{S}_6$, or its subgroup $\mathfrak{A}_6$, or a subgroup $\mathfrak{S}_5$ of $\mathfrak{S}_6$ such that $\mathbb{V}_5$ is an irreducible representation of Γ. Then, the only Γ-invariant quadric 3-fold in $\mathbb{P}^4$ is the quadric Q. Similarly, every (reduced) Γ-invariant quartic 3-fold in $\mathbb{P}^4$ is given by Equation (1.1) for some $t \in \mathbb{C}$.*

Proof. Apply Corollary 2.5. $\square$

By a small abuse of notation, we will refer to the points in $\mathbb{P}^4$ using x_i as if they were homogeneous coordinates, that is, a point in $\mathbb{P}^4$ will be encoded by a ratio of six linear forms x_i. As in

Section 1, let Σ_6 and Σ_{10} be the $\mathfrak{S}_6$-orbits of the points $[-5:1:1:1:1:1]$ and $[-1:-1:-1:1:1:1]$, respectively. Looking at Equation (4.1), we obtain Corollary 4.7.

COROLLARY 4.7 *The quadric Q does not contain the $\mathfrak{S}_6$-orbits Σ_6 and Σ_{10}.*

Now we will have a look at the action of the group $\mathfrak{A}_6$ on $\mathbb{P}^4$. Note that $\mathbb{V}_5$ is an irreducible $\mathfrak{A}_6$-representation by Remark 4.5.

LEMMA 4.8 *There are no $\mathfrak{A}_6$-orbits of length < 6 in $\mathbb{P}^4$. Moreover, the only $\mathfrak{A}_6$-orbit of length six in $\mathbb{P}^4$ is Σ_6.*

Proof. The only subgroup of $\mathfrak{A}_6$ of index < 6 is $\mathfrak{A}_6$ itself (cf. Remark 4.1), so that the first assertion of the lemma follows from irreducibility of the $\mathfrak{A}_6$-representation $\mathbb{V}_5$. Also, the only subgroups of $\mathfrak{A}_6$ of index six are $\mathfrak{A}_5^{st}$ and $\mathfrak{A}_5^{nst}$, so that the second assertion of the lemma also follows from Corollary 2.3. □

LEMMA 4.9 *Let X be an $\mathfrak{A}_6$-invariant quartic 3-fold in $\mathbb{P}^4$ that contains an $\mathfrak{A}_6$-orbit of length at most six. Then, $X = X_{\frac{7}{10}}$.*

Proof. By Lemma 4.6, one has $X = X_t$ for some $t \in \mathbb{C}$, and by Lemma 4.8 the $\mathfrak{A}_6$-orbit Σ_6 is contained in X_t. Since Σ_6 is not contained in the quadric Q by Corollary 4.7, we see that there is a unique $t \in \mathbb{C}$ such that Σ_6 is contained in a quartic given by Equation (1.1). Therefore, we conclude that $t = \frac{7}{10}$. □

Now we will make a couple of observations about the action of the group $\mathfrak{S}_5$ on $\mathbb{P}^4$. We choose $\mathfrak{S}_5$ to be a subgroup of $\mathfrak{S}_6$ such that $\mathbb{V}_5$ is an irreducible $\mathfrak{S}_5$-representation (cf. Remark 4.5 and Corollary 2.3).

LEMMA 4.10 *Let $P \in \mathbb{P}^4$ be a point such that its stabilizer in $\mathfrak{S}_5$ contains a subgroup isomorphic to D_{12}. Then, the $\mathfrak{S}_5$-orbit of P is Σ_{10}.*

Proof. By Corollary 2.3(iii), the point in $\mathbb{P}^4$ fixed by a subgroup $\mathrm{D}_{12} \subset \mathfrak{S}_5$ is unique. However, it is straightforward to check that a stabilizer in $\mathfrak{S}_5$ of a point of Σ_{10} contains a subgroup isomorphic to D_{12}. It remains to notice that the latter stabilizer is actually isomorphic to D_{12}, since the only subgroups of $\mathfrak{S}_5$ that contain D_{12} are D_{12} and $\mathfrak{S}_5$ itself, while $\mathfrak{S}_5$ has no fixed points on $\mathbb{P}^4$. □

LEMMA 4.11 *Let X be an $\mathfrak{S}_5$-invariant quartic 3-fold in $\mathbb{P}^4$ that contains Σ_{10}. Then, $X = X_{\frac{1}{6}}$.*

Proof. By Lemma 4.6, one has $X = X_t$ for some $t \in \mathbb{C}$. Since Σ_{10} is not contained in the quadric Q by Corollary 4.7, we see that there is a unique $t \in \mathbb{C}$ such that Σ_{10} is contained in a quartic given by Equation (1.1). Therefore, we conclude that $t = \frac{1}{6}$. □

5. Rationality of the quartic 3-fold $X_{\frac{7}{10}}$

In this section, we will construct an explicit $\mathfrak{A}_6$-equivariant birational map $\mathbb{P}^3 \dashrightarrow X_{\frac{7}{10}}$. Implicitly, the construction of this map first appeared in the proof of Cheltsov and Shramov [5, Theorem 1.20]. Here we will present a much simplified proof of its existence.

We identify $\mathbb{P}^3$ with the projectivization $\mathbb{P}(\mathbb{U}_4)$, where $\mathbb{U}_4$ is the restriction of the four-dimensional irreducible representation of the group $2.\mathfrak{S}_6$ introduced in Section 2 to the subgroup $2.\mathfrak{A}_6$. By Corollary 2.1(i), the $2.\mathfrak{A}_6$-representation $\mathbb{U}_4$ is irreducible.

LEMMA 5.1 *There are no $\mathfrak{A}_6$-invariant surfaces of odd degree in $\mathbb{P}^3$, and no $\mathfrak{A}_6$-invariant pencils of surfaces of odd degree in $\mathbb{P}^3$. Moreover, there are no $\mathfrak{A}_6$-invariant quadric and quartic surfaces in $\mathbb{P}^3$.*

Proof. Recall that the only one-dimensional representation of the group $2.\mathfrak{A}_6$ is the trivial representation. Therefore, any $\mathfrak{A}_6$-invariant surface of odd degree d in $\mathbb{P}^3$ gives rise to a trivial $2.\mathfrak{A}_6$-subrepresentation in $R_d = \mathrm{Sym}^d(\mathbb{U}_4)$. However, the non-trivial central element z of $2.\mathfrak{A}_6$ acts on R_d by a scalar matrix with diagonal entries equal to -1, which shows that R_d does not contain trivial $2.\mathfrak{A}_6$-representations. Also, since the only two-dimensional representation of $2.\mathfrak{A}_6$ is the sum of two trivial representations, this implies that there are no $\mathfrak{A}_6$-invariant pencils of surfaces of odd degree in $\mathbb{P}^3$.

The last assertion of the lemma follows from Corollary 2.4(i), (ii). □

LEMMA 5.2 *Let Ω be an $\mathfrak{A}_6$-orbit in $\mathbb{P}^3$. Then, $|\Omega| \geq 16$.*

Proof. Lemma 5.1 implies that there are no $\mathfrak{A}_6$-orbits of odd length in $\mathbb{P}^3$. Thus, if Ω is an $\mathfrak{A}_6$-orbit in $\mathbb{P}^3$ of length at most 15, then by Remark 4.1 a stabilizer of its general point is isomorphic either to $\mathfrak{A}_5$ or to F_{36}. Both of these cases are impossible by Corollary 2.1. □

Actually, the minimal degree of an $\mathfrak{A}_6$-invariant surface in $\mathbb{P}^3$ equals 8 (see [**5**, Lemma 3.7]), and the minimal length of an $\mathfrak{A}_6$-orbit in $\mathbb{P}^3$ equals to 36 (see [**5**, Lemma 3.8]), but we will not need this here.

LEMMA 5.3 (cf. [**20**, Lemma 4.26]) *Let C be a smooth irreducible $\mathfrak{A}_6$-invariant curve of degree* 9 *and genus g in $\mathbb{P}^3$. Then, $g \neq 10$.*

Proof. Suppose that $g = 10$. Then, it follows from Hartshorne [**13**, Example 6.4.3] that either C is contained in a unique quadric surface in $\mathbb{P}^3$, or C is a complete intersection of two cubic surfaces in $\mathbb{P}^3$. The former case is impossible, since there are no $\mathfrak{A}_6$-invariant quadrics in $\mathbb{P}^3$ by Lemma 5.1. The latter case is impossible, because there are no $\mathfrak{A}_6$-invariant pencils of cubic surfaces in $\mathbb{P}^3$ by Lemma 5.1. □

Recall that the group $\mathfrak{A}_6$ contains six standard subgroups isomorphic to $\mathfrak{A}_5$ and six non-standard subgroups isomorphic to $\mathfrak{A}_5$ (see the conventions made in Section 2). Denote the former ones by $H_1', \ldots, H_6'$, and denote the latter ones by $H_1, \ldots, H_6$. By Corollary 2.1(ii), each group H_i' leaves invariant two lines L_i^1 and L_i^2 in $\mathbb{P}^3$. Note that each group H_i permutes transitively the lines $L_1^1, \ldots, L_6^1$ (respectively, $L_1^2, \ldots, L_6^2$).

Put $\mathcal{L}^1 = L_1^1 + \ldots + L_6^1$ and $\mathcal{L}^2 = L_1^2 + \ldots + L_6^2$. Then, the curves $\mathcal{L}^1$ and $\mathcal{L}^2$ are $\mathfrak{A}_6$-invariant, and the curve $\mathcal{L}^1 + \mathcal{L}^2$ is $\mathfrak{S}_6$-invariant.

LEMMA 5.4 *The lines $L_1^1, \ldots, L_6^1$ (respectively, the lines $L_1^2, \ldots, L_6^2$) are pairwise disjoint. Moreover, the curves $\mathcal{L}^1$ and $\mathcal{L}^2$ are disjoint.*

Proof. We use an argument similar to one in the proof of Lemma 3.4. Suppose that some of the lines $L_1^1, \ldots, L_6^1$ have a common point. Since the action of $\mathfrak{A}_6$ on the set $\{L_1^1, \ldots, L_6^1\}$ is doubly transitive, this implies that any two of the lines $L_1^1, \ldots, L_6^1$ have a common point. Therefore, either all lines $L_1^1, \ldots, L_6^1$ are coplanar, or all of them pass through one point. Both of these cases are impossible since $\mathbb{U}_4$ is an irreducible $2.\mathfrak{A}_6$-representation (see Corollary 2.1(i)). Therefore, the lines $L_1^1, \ldots, L_6^1$ are pairwise disjoint. The same argument applies to the lines $L_1^2, \ldots, L_6^2$.

Suppose that some of the lines $L_1^1, \ldots, L_6^1$, say, L_1^1, intersects some of the lines $L_1^2, \ldots, L_6^2$. Since the lines L_1^1 and L_2^1 are disjoint by construction, we may assume that L_1^1 intersects L_2^2. Since the stabilizer $H_1' \subset \mathfrak{A}_6$ of L_1^1 acts transitively on the lines $L_2^2, \ldots, L_6^2$, we conclude that all five lines $L_2^2, \ldots, L_6^2$ intersect L_1^1. Therefore, the line L_1^1 contains a subset of at most five points that is invariant with respect to the group $H_1' \cong \mathfrak{A}_5$, which is a contradiction. Thus, $\mathcal{L}^1$ and $\mathcal{L}^2$ are disjoint. □

LEMMA 5.5 *Let C be an $\mathfrak{A}_6$-invariant curve in $\mathbb{P}^3$ of degree $d \leq 10$. Then, either $C = \mathcal{L}^1$ or $C = \mathcal{L}^2$.*

Proof. Suppose first that C is reducible. We may assume that $\mathfrak{A}_6$ permutes the irreducible components of C transitively. Thus, C has either 6 or 10 irreducible components by Remark 4.1, and these irreducible components are lines. By Remark 4.1 and Corollary 2.1, the latter case is impossible, and in the former case one has either $C = \mathcal{L}^1$ or $C = \mathcal{L}^2$.

Therefore, we assume that the curve C is irreducible. Let g be the genus of the normalization of the curve C. We have

$$g \leq \frac{d^2}{8} + 1 - |\mathrm{Sing}(C)| \leq 13 - |\mathrm{Sing}(C)| \tag{5.1}$$

by Lemma 3.15. This implies that the curve C is smooth, because $\mathbb{P}^3$ does not contain $\mathfrak{A}_6$-orbits of length <16 by Lemma 5.2.

If $d \leq 8$, then Equation (5.1) gives $g \leq 9$. This is impossible by Lemma 4.2.

If $d = 9$, then Equation (5.1) gives $g \leq 11$, so that $g = 10$ by Lemma 4.2. This is impossible by Lemma 5.3.

Therefore, we see that $d = 10$. Thus, Equation (5.1) gives $g \leq 13$, so that $g = 10$ by Lemma 4.2. The latter is impossible by Lemma 3.17. □

Denote by $\mathcal{M}$ the linear system on $\mathbb{P}^3$ consisting of all quartic surfaces passing through the lines $L_1^1, \ldots, L_6^1$. Then, $\mathcal{M}$ is not empty. In fact, its dimension is at least four by parameter count. Moreover, the linear system $\mathcal{M}$ does not have base components by Lemma 5.1.

LEMMA 5.6 *The base locus of $\mathcal{M}$ does not contain curves except the lines $L_1^1, \ldots, L_6^1$. Moreover, a general surface in $\mathcal{M}$ is smooth at a general point of each of the lines $L_1^1, \ldots, L_6^1$.*

Proof. Denote by Z the union of the curves that are contained in the base locus of $\mathcal{M}$ and are different from the lines $L_1^1, \ldots, L_6^1$. Then, Z is a (possibly empty) $\mathfrak{A}_6$-invariant curve. Denote its degree by d. Pick two general surfaces M_1 and M_2 in $\mathcal{M}$. Then,

$$M_1 \cdot M_2 = Z + m\mathcal{L}^1 + \Delta,$$

where m is a positive integer, and Δ is an effective one-cycle on $\mathbb{P}^3$ that contains none of the lines $L_1^1, \ldots, L_6^1$. Note that Δ may contain irreducible components of the curve Z. Let Π be a plane in $\mathbb{P}^3$. Then,

$$16 = \Pi \cdot M_1 \cdot M_2 = \Pi \cdot Z + m\Pi \cdot \mathcal{L}^1 + \Pi \cdot \Delta = d + 6m + \Pi \cdot \Delta \leq d + 6m,$$

which implies that $m \leq 2$ and $d \leq 10$. By Lemma 5.5, we have $d = 0$, so that Z is empty. Since

$$2 \geq m \geq \mathrm{mult}_{L_i^1}(M_1)\mathrm{mult}_{L_i^1}(M_2),$$

we see that a general surface in $\mathcal{M}$ is smooth at a general point of L_i^1. □

Let $\alpha\colon U \to \mathbb{P}^3$ be a blow up along the lines $L_1^1,\dots, L_6^1$. Then, the action of $\mathfrak{A}_6$ lifts to U. Denote by $E_1,\dots, E_6$ the α-exceptional surfaces that are mapped to $L_1^1,\dots, L_6^1$, respectively. Denoting by Π a plane in $\mathbb{P}^3$, we compute

$$(-K_U)^3 = \left(4\alpha^*\Pi - \sum_{i=1}^{6} E_i\right)^3 = 64\left(\alpha^*\Pi\right)^3 + 12\sum_{i=1}^{6}\alpha^*\Pi\cdot E_i^2 - \sum_{i=1}^{6} E_i^3 = 64 - 72 + 12 = 4.$$

Lemma 5.7 *The action of the stabilizer $H_i' \cong \mathfrak{A}_5$ in $\mathfrak{A}_6$ of the line L_i^1 on the surface $E_i \cong \mathbb{P}^1\times\mathbb{P}^1$ is twisted diagonal, that is, E_i is identified with $\mathbb{P}(\mathbb{U}_2)\times\mathbb{P}(\mathbb{U}_2')$, where $\mathbb{U}_2$ and $\mathbb{U}_2'$ are different two-dimensional irreducible representations of the group $2.\mathfrak{A}_5$.*

Proof. This follows from Corollary 2.1(ii). □

Let us denote by $\mathcal{M}_U$ the proper transform of the linear system $\mathcal{M}$ on the 3-fold U. Then, $\mathcal{M}_U \sim -K_U$ by Lemma 5.6.

Lemma 5.8 *The linear system $\mathcal{M}_U$ is base point free.*

Proof. Let us first show that $\mathcal{M}_U$ is free from base curves. Suppose that the base locus of the linear system $\mathcal{M}_U$ contains some curves. Then, each of these curves is contained in some of the α-exceptional surfaces by Lemma 5.6. Denote by Z the union of all such curves that are contained in E_1. Then, Z is an H_1'-invariant curve. For some non-negative integers a and b, one has

$$Z \sim as + bl,$$

where s is a section of the natural projection $E_1 \to L_1^1$ such that $s^2 = 0$ on E_1, and l is a fiber of this projection. However, we have

$$\mathcal{M}_U|_{E_1} \sim -K_U|_{E_1} \sim s + 3l.$$

This gives $a \leq 1$ and $b \leq 3$. Since the action of H_1' on the surface E_1 is twisted diagonal by Lemma 5.7, the latter is impossible by Cheltsov and Shramov [6, Lemma 6.4.2(i)] and [6, Lemma 6.4.11(o)].

We see that $\mathcal{M}_U$ is free from base curves. Since $\mathcal{M}_U \sim -K_U$, the linear system $\mathcal{M}_U$ cannot have more than $-K_U^3 = 4$ base points. By Lemma 5.2, this implies that $\mathcal{M}_U$ is base point free. □

Corollary 5.9 *The base locus of the linear system $\mathcal{M}$ consists of the lines $L_1^1,\dots,L_6^1$.*

By Lemma 5.8, the divisor $-K_U$ is nef. Since $-K_U^3 = 4$, it is also big. Thus, we have

$$h^1\left(\mathcal{O}_U(-K_U)\right) = h^2\left(\mathcal{O}_U(-K_U)\right) = 0$$

by the Kawamata–Viehweg vanishing theorem (see [16]). Hence, the Riemann–Roch formula gives

$$h^0(\mathcal{O}_U(-K_U)) = 5. \tag{5.2}$$

In particular, we see that $|-K_U| = \mathcal{M}_U$.

Lemma 5.10 *The $\mathfrak{A}_6$-representation $H^0(\mathcal{O}_U(-K_U))$ is irreducible.*

Proof. By Lemma 5.1, there are no $\mathfrak{A}_6$-invariant quartic surfaces in $\mathbb{P}^3$. This implies that $H^0(\mathcal{O}_U(-K_U))$ does not contain one-dimensional subrepresentations. Hence, it is irreducible by Remark 4.5. □

Lemma 5.8, together with Equation (5.2), implies that there is an $\mathfrak{A}_6$-equivariant commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \mathbb{P}^3 \arrow[rr, dashed, "\phi"'] & & \mathbb{P}^4 \end{tikzcd} \tag{5.3}$$

where ϕ is the rational map given by $\mathcal{M}$, and β is a morphism given by the anticanonical linear system $|-K_U|$. By Lemma 5.10, the projective space $\mathbb{P}^4$ in Equation (5.3) is a projectivization of an irreducible $\mathfrak{A}_6$-representation.

Recall from Lemma 3.8 that $\mathbb{P}^3$ contains exactly two H_1-invariant twisted cubic curves $\mathscr{C}_1^1$ and $\mathscr{C}_1^2$.

Lemma 5.11 *The curve $\mathcal{L}^1$ intersects exactly one curve among $\mathscr{C}_1^1$ and $\mathscr{C}_1^2$. Moreover, each line among $L_1^1, \ldots, L_6^1$ contains two points of this intersection. Similarly, the curve $\mathcal{L}^2$ intersects exactly one curve among $\mathscr{C}_1^1$ and $\mathscr{C}_1^2$, and this curve is different from the one that intersects $\mathcal{L}^1$.*

Proof. By Remark 3.1, the stabilizer in H_1 of the curve L_1^1 is isomorphic to D_{10}, and thus it has an orbit of length 2 on L_1^1. Thus, the curve $\mathcal{L}^1$ contains an H_1-orbit Σ_{12}^1 of length 12 by Lemma 3.2. Similarly, the curve $\mathcal{L}^2$ contains an H_1-orbit Σ_{12}^2 of length 12. By Lemma 5.4, one has $\Sigma_{12}^1 \neq \Sigma_{12}^2$. Moreover, Σ_{12}^1 and Σ_{12}^2 are the only H_1-orbits in $\mathbb{P}^3$ of length 12 by Lemma 3.2. Since $\mathscr{C}_1^1$ and $\mathscr{C}_1^2$ are disjoint by Remark 3.9, and each of them contains an H_1-orbit of length 12, we see that either $\Sigma_{12}^1 \subset \mathscr{C}_1^1$ and $\Sigma_{12}^2 \subset \mathscr{C}_1^2$, or $\Sigma_{12}^2 \subset \mathscr{C}_1^1$ and $\Sigma_{12}^1 \subset \mathscr{C}_1^2$. Since a line cannot have more than two common points with a twisted cubic, this also implies the last assertion of the lemma. □

Without loss of generality, we may assume that the curve $\mathcal{L}^1$ intersects $\mathscr{C}_1^1$, and the curve $\mathcal{L}^2$ intersects $\mathscr{C}_1^2$. Let $\mathscr{C}_1^1, \ldots, \mathscr{C}_6^1$ be the $\mathfrak{A}_6$-orbit of the curve $\mathscr{C}_1^1$, and let $\mathscr{C}_1^2, \ldots, \mathscr{C}_6^2$ be the $\mathfrak{A}_6$-orbit of the curve $\mathscr{C}_1^2$. By Lemma 3.8, the curves $\mathscr{C}_i^1$ and $\mathscr{C}_i^2$ are the only twisted cubic curves in $\mathbb{P}^3$ that are H_i-invariant. By Lemma 5.11, we have

Corollary 5.12 *Every twisted cubic curve $\mathscr{C}_i^1$ intersects each line among $L_1^1, \ldots, L_6^1$ by two points. Similarly, every twisted cubic curve $\mathscr{C}_i^2$ intersects each line among $L_1^2, \ldots, L_6^2$ by two points.*

Denote by $\widetilde{\mathscr{C}}_1^1, \ldots, \widetilde{\mathscr{C}}_6^1$ the proper transforms on U of the curves $\mathscr{C}_1^1, \ldots, \mathscr{C}_6^1$, respectively.

Lemma 5.13 *One has $-K_U \cdot \widetilde{\mathscr{C}}_1^1 = \ldots = -K_U \cdot \widetilde{\mathscr{C}}_6^1 = 0$.*

Proof. This follows from Corollary 5.12. □

We see that each curve $\widetilde{\mathscr{C}}_i^1$ is contracted by β to a point. Since the $\mathfrak{A}_6$-orbit of $\widetilde{\mathscr{C}}_1^1$ consists of six curves, we also obtain the following.

Corollary 5.14 *The image of the morphism β contains an $\mathfrak{A}_6$-orbit of length at most six.*

Since $-K_U^3 = 4$, the image of β is either an $\mathfrak{A}_6$-invariant quartic 3-fold or an $\mathfrak{A}_6$-invariant quadric 3-fold. Using results of [**24**], one can show that the latter case is impossible. However, this immediately follows from Corollary 5.14. Indeed, an $\mathfrak{A}_6$-orbit of length at most six cannot be contained in the $\mathfrak{A}_6$-invariant quadric by Corollary 4.7 and Lemma 4.8.

Corollary 5.15 *The morphism β is birational onto its image, and its image is a quartic* 3*-fold.*

Now Lemma 4.9 implies that the image of β is the quartic $X_{\frac{7}{10}}$. This proves

Corollary 5.16 *The* 3*-fold $X_{\frac{7}{10}}$ is rational.*

Let us conclude this section by recalling two related results proved in [**5**, Section 4]. The commutative diagram (5.3) can be extended to an $\mathfrak{A}_6$-equivariant commutative diagram

$$\begin{tikzcd}
U \arrow[rrr, dashed, "\rho"] \arrow[dr, "\gamma"] \arrow[dd, "\alpha"'] & & & U \arrow[dl, "\gamma"'] \arrow[dd, "\alpha"] \\
& X_{\frac{7}{10}} \arrow[r, "\sigma"] & X_{\frac{7}{10}} & \\
\mathbb{P}^3 \arrow[ur, dashed, "\phi"'] \arrow[rrr, dashed, "\psi"'] & & & \mathbb{P}^3 \arrow[ul, dashed, "\phi"]
\end{tikzcd} \tag{5.4}$$

Here σ is an automorphism of the quartic 3-fold $X_{\frac{7}{10}}$ given by an odd permutation in $\mathfrak{S}_6$ acting on $\mathbb{P}^4$, cf. Remark 4.5. The birational map ρ is a composition of Atiyah flops in 36 curves contracted by γ, and the birational map ψ is not regular.

The diagram (5.4) is a so-called $\mathfrak{A}_6$-Sarkisov link. The subgroup $\mathfrak{A}_6 \subset \mathrm{Aut}(\mathbb{P}^3)$ together with $\psi \in \mathrm{Bir}^{\mathfrak{A}_6}(\mathbb{P}^3)$ generates a subgroup isomorphic to $\mathfrak{S}_6$. Moreover, the subgroup

$$\mathrm{Aut}^{\mathfrak{A}_6}\left(\mathbb{P}^3\right) \subset \mathrm{Bir}^{\mathfrak{A}_6}\left(\mathbb{P}^3\right)$$

is also isomorphic to $\mathfrak{S}_6$. By Cheltsov and Shramov [**5**, Theorem 1.24], the whole group $\mathrm{Bir}^{\mathfrak{A}_6}(\mathbb{P}^3)$ is a free product of these two copies of $\mathfrak{S}_6$ amalgamated over the original $\mathfrak{A}_6$.

6. Rationality of the quartic 3-fold $X_{\frac{1}{6}}$

In this section, we will construct an explicit $\mathfrak{S}_5$-equivariant birational map $\mathbb{P}^3 \to X_{\frac{1}{6}}$. We identify $\mathbb{P}^3$ with the projectivization $\mathbb{P}(\mathbb{U}_4)$, where $\mathbb{U}_4$ is the restriction of the four-dimensional irreducible representation of the group $2.\,\mathfrak{S}_6$ introduced in Section 2 to a subgroup $2.\,\mathfrak{S}_5^{nst}$, and denote the latter subgroup simply by $2.\,\mathfrak{S}_5$. By Corollary 2.1(i), the $2.\,\mathfrak{S}_5$-representation $\mathbb{U}_4$ is irreducible.

Lemma 6.1 *Let Ω be an $\mathfrak{S}_5$-orbit in $\mathbb{P}^3$. Then, $|\Omega| \geq 12$.*

Proof. Apply Remark 4.3 together with Corollary 2.1. □

Lemma 6.2 *Let C be an $\mathfrak{S}_5$-invariant curve in $\mathbb{P}^3$ of degree d. Then, $d \geq 6$.*

Proof. Suppose that $d \leq 5$. To start with, assume that C is reducible and denote by r the number of its irreducible components. We may assume that $\mathfrak{S}_5$ permutes the irreducible components of C transitively. Thus, either $r = 2$ or $r = 5$ by Remark 4.3. If $r = 5$, the irreducible components of C are lines, so that this case is impossible by Remark 4.3 and Corollary 2.1(i). Hence, we have $r = 2$, and the stabilizer of each of the two irreducible components C_1 and C_2 of C is the subgroup $\mathfrak{A}_5 \subset \mathfrak{S}_5$. Moreover, in this case one has

$$\deg(C_1) = \deg(C_2) \leq 2,$$

which is impossible by Lemma 3.8.

Therefore, we assume that the curve C is irreducible. Let g be the genus of the normalization of the curve C. Then,

$$g \leq \frac{d^2}{8} + 1 - |\mathrm{Sing}(C)|$$

by Lemma 3.15, so that $g \leq 4 - |\mathrm{Sing}(C)|$. This implies that C is smooth, because there are no $\mathfrak{S}_5$-orbits of length <12 by Lemma 6.1.

Since $\mathfrak{S}_5$ does not act faithfully on $\mathbb{P}^1$, we see that $g \neq 0$. Thus, either $g = 4$ or $g = 5$ by Cheltsov and Shramov [**6**, Lemma 5.1.5]. The former case is impossible by Lemma 3.16, while the latter case is impossible by Lemma 4.4. □

Recall from Section 3 that the subgroup $\mathfrak{A}_4 \subset \mathfrak{A}_5 \subset \mathfrak{S}_5$ fixes two disjoint lines L_1 and L_1'. As before, we consider the $\mathfrak{A}_5$-orbit $L_1, \ldots, L_5$ of the line L_1 and the $\mathfrak{A}_5$-orbit $L_1', \ldots, L_5'$ of the line L_1'. By Lemma 3.7, the lines $L_1, \ldots, L_5$, $L_1', \ldots, L_5'$ form a double five configuration (see Definition 3.6). Corollary 2.1(i) implies that the $\mathfrak{S}_5$-orbit of the line L_1 is $L_1, \ldots, L_5$, $L_1', \ldots, L_5'$.

Remark 6.3 Any subgroup $F_{20} \subset \mathfrak{S}_5$ permutes the 10 lines $L_1, \ldots, L_5$, $L_1', \ldots, L_5'$ transitively. Indeed, let $c \in F_{20}$ be an element of order five. Then, c is not contained in a stabilizer of the line L_1, so that the orbit of L_1 with respect to the group $\Gamma \cong \boldsymbol{\mu}_5$ generated by c is $L_1, \ldots, L_5$. Similarly, the Γ-orbit of the line L_1' is $L_1', \ldots, L_5'$. Also, the group F_{20} is not contained in $\mathfrak{A}_5$, so that the F_{20}-orbit of L_1 contains some of the lines $L_1', \ldots, L_5'$, and thus contains all the 10 lines $L_1, \ldots, L_5$, $L_1', \ldots, L_5'$.

Let $\mathcal{M}$ be the linear system on $\mathbb{P}^3$ consisting of all quartic surfaces passing through all lines $L_1, \ldots, L_5$ and $L_1', \ldots, L_5'$. Then, $\mathcal{M}$ is not empty. In fact, Lemma 3.7 and parameter count imply that its dimension is at least four. Moreover, the linear system $\mathcal{M}$ does not have base components by Lemma 3.3.

Lemma 6.4 *The base locus of $\mathcal{M}$ does not contain curves that are different from the lines $L_1, \ldots, L_5$, $L_1', \ldots, L_5'$. Moreover, a general surface in $\mathcal{M}$ is smooth in a general point of each of these lines. Furthermore, two general surfaces in $\mathcal{M}$ intersect transversally at a general point of each of these lines.*

Proof. Denote by Z the union of all curves that are contained in the base locus of the linear system $\mathcal{M}$ and are different from the lines $L_1, \ldots, L_5$, $L_1', \ldots, L_5'$. Then, Z is a (possibly empty) $\mathfrak{S}_5$-invariant curve. Denote its degree by d. Pick two general surfaces M_1 and M_2 in $\mathcal{M}$. Then,

$$M_1 \cdot M_2 = Z + m\sum_{i=1}^{5} L_i + m\sum_{i=1}^{5} L_i' + \Delta,$$

where m is a positive integer, and Δ is an effective one-cycle on $\mathbb{P}^3$ that contains none of the lines $L_1, \ldots, L_5$ and $L_1', \ldots, L_5'$. Note that Δ may contain irreducible components of the curve Z. Note also that $\Delta \neq 0$, because $\mathcal{M}$ is not a pencil.

Let Π be a plane in $\mathbb{P}^3$. Then,

$$16 = \Pi \cdot Z + m\sum_{i=1}^{5} \Pi \cdot L_i + m\sum_{i=1}^{5} \Pi \cdot L_i' + \Pi \cdot \Delta = d + 10m + \Pi \cdot \Delta > d + 10m,$$

which implies that $m = 1$ and $d \leq 5$. By Lemma 6.2, we have $d = 0$, so that Z is empty. Since

$$1 \geq m \geq \mathrm{mult}_{L_i}(M_1 \cdot M_2) \geq \mathrm{mult}_{L_i}(M_1)\mathrm{mult}_{L_i}(M_2),$$

we see that a general surface in $\mathcal{M}$ is smooth at a general point of L_i, and two general surfaces in $\mathcal{M}$ intersect transversally at a general point of L_i. Similarly, we see that a general surface in $\mathcal{M}$ is smooth at a general point of L_i', and two general surfaces in $\mathcal{M}$ intersect transversally at a general point of L_i'. $\square$

Let $g\colon W \to \mathbb{P}^3$ be a blow up along the lines $L_1, \ldots, L_5$, and let $g'\colon W' \to \mathbb{P}^3$ be a blow up along the lines $L_1', \ldots, L_5'$. Denote by $\widetilde{L}_1', \ldots, \widetilde{L}_5'$ (respectively, $\widetilde{L}_1, \ldots, \widetilde{L}_5$) the proper transforms of the lines $L_1', \ldots, L_5'$ on the 3-fold W (respectively, on the 3-fold W'). Let $h\colon V \to W$ be a blow up along the curves $\widetilde{L}_1', \ldots, \widetilde{L}_5'$, and let $h'\colon V' \to W'$ be a blow up along the curves $\widetilde{L}_1, \ldots, \widetilde{L}_5$. Finally, let $\alpha\colon U \to \mathbb{P}^3$ be a blow up of the (singular) curve that is a union of all lines $L_1, \ldots, L_5$, $L_1', \ldots, L_5'$. Then, U has 20 nodes by Lemma 3.7, and there exists a commutative diagram

$$\begin{tikzcd}
& V \arrow[dl, "h"'] \arrow[dr, "\upsilon"] \arrow[rr, dashed, "\tau"] & & V' \arrow[dl, "\upsilon'"'] \arrow[dr, "h'"] & \\
W \arrow[drr, "g"'] & & U \arrow[d, "\alpha"] & & W' \arrow[dll, "g'"] \\
& & \mathbb{P}^3 & &
\end{tikzcd}$$

where υ and υ' are small resolutions of singularities of the 3-fold U, and τ is a composition of 20 Atiyah flops.

REMARK 6.5 By construction, the action of group $\mathfrak{A}_5$ lifts to the 3-folds W, W', V, V' and U. Similarly, the action of the group $\mathfrak{S}_5$ lifts to the 3-fold U, but this action does not lift to W and W'. On the 3-folds V and V', the group $\mathfrak{S}_5$ acts biregularly outside of the curves flopped by τ and τ^{-1}, respectively.

Denote by $E_1, \ldots, E_5$ the g-exceptional surfaces that are mapped to $L_1, \ldots, L_5$, respectively. Similarly, denote by $E_1', \ldots, E_5'$ the g'-exceptional surfaces that are mapped to $L_1', \ldots, L_5'$, respectively. Then, all surfaces $E_1, \ldots, E_5$, $E_1', \ldots, E_5'$ are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

Denote by $\hat{E}_1', \ldots, \hat{E}_5'$ the h-exceptional surfaces that are mapped to the curves $\widetilde{L}_1', \ldots, \widetilde{L}_5'$, respectively. Similarly, denote by $\check{E}_1, \ldots, \check{E}_5$ the h'-exceptional surfaces that are mapped to the curves $\widetilde{L}_1, \ldots, \widetilde{L}_5$, respectively. Denote by $\hat{E}_1, \ldots, \hat{E}_5$ the proper transforms on V of the surfaces $E_1, \ldots, E_5$, respectively. Finally, denote by $\check{E}_1', \ldots, \check{E}_5'$ the proper transforms on V' of the surfaces $E_1',\dots,E_5'$, respectively. Then, τ maps the surfaces $\hat{E}_1,\dots,\hat{E}_5$ to the surfaces $\check{E}_1,\dots,\check{E}_5$, respectively, and it maps the surfaces $\hat{E}_1',\dots,\hat{E}_5'$ to the surfaces $\check{E}_1',\dots,\check{E}_5'$, respectively. Denoting by Π a plane in $\mathbb{P}^3$, we compute

$$(-K_W)^3 = \left(4g^*(\Pi) - \sum_{i=1}^{5} E_i\right)^3 = 64\big(g^*(\Pi)\big)^3 + 12\sum_{i=1}^{5} g^*(\Pi)\cdot E_i^2 - \sum_{i=1}^{5} E_i^3 = 64 - 60 + 10 = 14,$$

and

$$(-K_V)^3 = \left(-h^*(K_W) - \sum_{i=1}^{5} \hat{E}_i'\right)^3 = \big(-h^*(K_W)\big)^3 - 3\sum_{i=1}^{5} h^*(K_W)\cdot \hat{E}_i'^2 - \sum_{i=1}^{5} \hat{E}_i'^3 = 14 - 10 = 4.$$

Denote by $\mathcal{M}_W$, $\mathcal{M}_V$, $\mathcal{M}_{W'}$, $\mathcal{M}_{V'}$ and $\mathcal{M}_U$ the proper transforms of the linear system $\mathcal{M}$ on the 3-folds W, V, W', V' and U, respectively. Then, it follows from Lemma 6.4 that

$$\mathcal{M}_W \sim -K_W, \quad \mathcal{M}_V \sim -K_V, \quad \mathcal{M}_{W'} \sim -K_{W'}, \quad \mathcal{M}_{V'} \sim -K_{V'}$$

and $\mathcal{M}_U \sim -K_U$.

LEMMA 6.6 *The base locus of the linear system $\mathcal{M}_W$ does not contain curves that are different from the curves $\widetilde{L}_1',\dots,\widetilde{L}_5'$. Similarly, the base locus of $\mathcal{M}_{W'}$ does not contain curves that are different from the curves $\widetilde{L}_1,\dots,\widetilde{L}_5$.*

Proof. It is enough to prove the first assertion of the lemma. Suppose that the base locus of the linear system $\mathcal{M}_W$ contains an irreducible curve Z that is different from the curves $\widetilde{L}_1'$, …, $\widetilde{L}_5'$. Then, Z is contained in one of the surfaces $E_1,\dots,E_5$ by Lemma 6.4.

By Lemma 6.4, the curve Z is a fiber of some of the natural projections $E_i \to L_i$, because otherwise two general surfaces in $\mathcal{M}_W$ would be tangent in a general point of L_i. In particular, the only curves in the base locus of the linear system $\mathcal{M}_W$ are $\widetilde{L}_i'$ and possibly some fibers of the projections $E_i \to L_i$. This shows that $-K_W$ is nef. Indeed, $-K_W$ has positive intersections with the fibers of the projections $E_i \to L_i$, it has trivial intersection with all curves $\widetilde{L}_1',\dots,\widetilde{L}_5'$, and $-K_W \sim \mathcal{M}_W$ has non-negative intersection with any other curve.

Let $Z_1 = Z, Z_2,\dots,Z_r$ be the $\mathfrak{A}_5$-orbit of the curve Z. Then, $r \geq 20$ by Corollary 3.5. Pick two general surfaces M_1 and M_2 in the linear system $\mathcal{M}_W$. By Lemma 6.4, one has

$$M_1 \cdot M_2 = \sum_{i=1}^{5} \widetilde{L}_i' + m\sum_{i=1}^{r} Z_i + \Delta$$

for some positive integer m and some effective one-cycle Δ whose support contains none of the curves $\widetilde{L}_1',\dots,\widetilde{L}_5'$ and $Z_1,\dots,Z_r$. Hence,

$$\begin{aligned} 14 = -K_W^3 = -K_W\cdot M_1\cdot M_2 &= -K_W\cdot\left(\sum_{i=1}^{5}\widetilde{L}_i' + m\sum_{i=1}^{r} Z_i + \Delta\right) \\ &= -5K_W\cdot\widetilde{L}_1' - mrK_W\cdot Z - K_W\cdot\Delta = mr - K_W\cdot\Delta \geq mr \geq r \geq 20, \end{aligned}$$

which is absurd. □

LEMMA 6.7 *The linear system $\mathcal{M}_V$ is base point free.*

Proof. It is enough to show that $\mathcal{M}_V$ is free from base curves. Indeed, if the base locus of the linear system $\mathcal{M}_V$ does not contain base curves, then $\mathcal{M}_V$ cannot have more than $-K_V^3 = 4$ base points, because $\mathcal{M}_V \sim -K_V$. However, V does not contain $\mathfrak{S}_5$-orbits of length <12, because there are no $\mathfrak{S}_5$-orbits of such length on $\mathbb{P}^3$ by Lemma 6.1.

Suppose that the base locus of the linear system $\mathcal{M}_V$ contains an irreducible curve Z. If Z is not contained in any of the surfaces $\hat{E}_1',\ldots,\hat{E}_5'$, then the curve $h(Z)$ is a base curve of the linear system $\mathcal{M}_W$ and $h(Z)$ is different from the curves $\widetilde{L}_1',\ldots,\widetilde{L}_5'$. This is impossible by Lemma 6.6. Similarly, if Z is not contained in any of the surfaces $\hat{E}_1,\ldots,\hat{E}_5$, then the curve $h'\circ\tau(Z)$ is a base curve of the linear system $\mathcal{M}_{W'}$ that is different from the curves $\widetilde{L}_1,\ldots,\widetilde{L}_5$. This is again impossible by Lemma 6.6. Thus, Z is contained in one of the surfaces $\hat{E}_1,\ldots,\hat{E}_5$, and in one of the surfaces $\hat{E}_1',\ldots,\hat{E}_5'$. In particular, the curves flopped by τ are not contained in the base locus of $\mathcal{M}_V$.

Without loss of generality, we may assume that $Z = \hat{E}_1 \cap \hat{E}_2'$. Let C be the curve flopped by τ that is contained in $\hat{E}_1$ and intersects $\hat{E}_2'$. Then, C intersects Z by one point. However, we have $-K_V \cdot C = 0$. Since $\mathcal{M}_V \sim -K_V$, this implies that C is disjoint from a general surface in $\mathcal{M}_V$. This is impossible, because $C \cap Z \neq \varnothing$, while Z is contained in the base locus of the linear system $\mathcal{M}_V$. □

COROLLARY 6.8 *The linear systems $\mathcal{M}_{V'}$ and $\mathcal{M}_U$ are also base point free.*

Proof. Recall that $\mathcal{M}_V \sim -K_V$. Thus, the general surface of $\mathcal{M}_V$ is disjoint from all curves flopped by τ, because $\mathcal{M}_V$ is base point free by Lemma 6.7. □

COROLLARY 6.9 *The base locus of $\mathcal{M}$ consists of the lines $L_1,\ldots,L_5$, $L_1',\ldots,L_5'$.*

By Lemma 6.7 and Corollary 6.8, the divisors $-K_V$, $-K_{V'}$ and $-K_U$ are nef. Since

$$-K_V^3 = -K_{V'}^3 = -K_U^3 = 4,$$

these divisors are also big. Thus, the Kawamata–Viehweg vanishing theorem and the Riemann–Roch formula give

$$h^0(\mathcal{O}_V(-K_V)) = h^0(\mathcal{O}_{V'}(-K_{V'})) = h^0(\mathcal{O}_U(-K_U)) = 4. \tag{6.1}$$

In particular, one has $|-K_V| = \mathcal{M}_V$, $|-K_V'| = \mathcal{M}_{V'}$ and $|-K_U| = \mathcal{M}_U$.

LEMMA 6.10 *The $\mathfrak{S}_5$-representation $H^0(\mathcal{O}_U(-K_U))$ is irreducible.*

Proof. By Lemma 3.13, there are no $\mathfrak{S}_5$-invariant quartic surfaces in $\mathbb{P}^3$ that pass through the 10 lines $L_1,\ldots,L_5$, $L_1',\ldots,L_5'$. This implies that $H^0(\mathcal{O}_U(-K_U))$ does not contain one-dimensional subrepresentations. Hence, it is irreducible by Remark 4.5. □

Lemma 6.7 together with Equation (6.1) implies that there is an $\mathfrak{S}_5$-equivariant commutative diagram

$$\begin{tikzcd} & U \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \mathbb{P}^3 \arrow[rr, dashed, "\phi"'] & & \mathbb{P}^4 \end{tikzcd} \tag{6.2}$$

where ϕ is the rational map given by $\mathcal{M}$, and β is a morphism given by the anticanonical linear system $|-K_U|$. By Lemma 6.10, the projective space $\mathbb{P}^4$ in Equation (6.2) is a projectivization of an irreducible $\mathfrak{S}_5$-representation.

For $1 \leq i < j \leq 5$, let Λ_{ij} be the intersection line of the plane spanned by L_i and L_j' with the plane spanned by L_i' and L_j. Note that the stabilizer of Λ_{ij} in $\mathfrak{S}_5$ contains a subgroup isomorphic to D_{12}. Actually, this implies that the stabilizer of Λ_{ij} in $\mathfrak{S}_5$ is isomorphic to D_{12}, since D_{12} is a maximal proper subgroup in $\mathfrak{S}_5$ (see Remark 4.3) and there are no $\mathfrak{S}_5$-invariant lines in $\mathbb{P}^3$ by Corollary 2.1(i). Denote by $\hat{\Lambda}_{ij}$ the proper transform of the line Λ_{ij} on the 3-fold V, and denote by $\overline{\Lambda}_{ij}$ its proper transform on U. Then,

$$-K_V \cdot \hat{\Lambda}_{ij} = 0.$$

Since υ is a small birational morphism, we also obtain $-K_U \cdot \overline{\Lambda}_{ij} = 0$.

We see that each curve $\overline{\Lambda}_{ij}$ is contracted by β to a point. Note that the stabilizer of Λ_{ij} in $\mathfrak{S}_5$ is isomorphic to D_{12}. Since $-K_U^3 = 4$, the image of β is either an $\mathfrak{S}_5$-invariant quartic 3-fold or an $\mathfrak{S}_5$-invariant quadric 3-fold. Applying Corollary 4.7 together with Lemma 4.10, we obtain the following.

Corollary 6.11 *The morphism β is birational on its image, and its image is a quartic 3-fold.*

Now Lemmas 4.10 and 4.11 imply that the image of β is the quartic $X_{\frac{1}{6}}$. This proves

Corollary 6.12 *The 3-fold $X_{\frac{1}{6}}$ is rational.*

Ten curves $\overline{\Lambda}_{ij}$ are mapped by γ to 10 singular points of the 3-fold $X_{\frac{1}{6}}$. Twenty singular points of U are mapped by γ to another 20 singular points of $X_{\frac{1}{6}}$. Let us describe the curves in U that are contracted by γ to the remaining 10 singular points of the 3-fold $X_{\frac{1}{6}}$. To do this, we need

Lemma 6.13 *Let ℓ_1, ℓ_2, ℓ_3 and ℓ_4 be pairwise skew lines in $\mathbb{P}^3$. Suppose that there is a unique line $\ell \subset \mathbb{P}^3$ that intersects ℓ_1, ℓ_2, ℓ_3 and ℓ_4. Let $\pi\colon Y \to \mathbb{P}^3$ be a blow up of the line ℓ, and $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ be the exceptional divisor of π. Denote by $\tilde{\ell}_1$, $\tilde{\ell}_2$, $\tilde{\ell}_3$ and $\tilde{\ell}_4$ the proper transforms on Y of the lines ℓ_1, ℓ_2, ℓ_3 and ℓ_4, respectively. Then, there exists a unique curve $C \subset E$ of bi-degree $(1,1)$ that intersects the curves $\tilde{\ell}_1$, $\tilde{\ell}_2$, $\tilde{\ell}_3$ and $\tilde{\ell}_4$.*

Proof. The lines ℓ_1, ℓ_2 and ℓ_3 are contained in a unique quadric surface $S \subset \mathbb{P}^3$. Note that S is smooth, because ℓ_1, ℓ_2 and ℓ_3 are disjoint. Furthermore, the line ℓ is contained in S, because ℓ intersects the lines ℓ_1, ℓ_2 and ℓ_3 by assumption. Moreover, the line ℓ_4 is tangent to S, since otherwise there would be either two or infinitely many lines in $\mathbb{P}^3$ that intersect ℓ_1, ℓ_2, ℓ_3 and ℓ_4. Denote by $\tilde{S}$ the proper transform on Y of the quadric surface S. Then, $\tilde{S}$ contains the curves $\tilde{\ell}_1$, $\tilde{\ell}_2$ and $\tilde{\ell}_3$. Moreover, $\tilde{S}$ intersects the curve $\tilde{\ell}_4$. Thus, $\tilde{S}|_E$ is the required curve C. $\square$

By Lemmas 3.7 and 6.13, each surface $E_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ contains a unique smooth rational curve C_i of bi-degree $(1,1)$ that passes through all four points of the intersection of E_i with the curves $\widetilde{L}_1', \ldots, \widetilde{L}_5'$ (recall that $E_i \cap \widetilde{L}_i' = \varnothing$). Similarly, each surface $E_i' \cong \mathbb{P}^1 \times \mathbb{P}^1$ contains a unique smooth rational curve C_i' of bi-degree $(1,1)$ that passes through all four points of the intersection of E_i' with the curves $\widetilde{L}_1, \ldots, \widetilde{L}_5$. Denote by $\hat{C}_1, \ldots, \hat{C}_5$ the proper transforms on the 3-fold V of the curves $C_1, \ldots, C_5$, respectively. Similarly, denote by $\check{C}_1', \ldots, \check{C}_5'$ the proper transforms on the 3-fold V' of the curves $C_1', \ldots, C_5'$, respectively. Then,

$$-K_V \cdot \hat{C}_i = -K_{V'} \cdot \check{C}_i' = 0.$$

This implies that the proper transforms of the curves $\hat{C}_1, \ldots, \hat{C}_5$ on the 3-fold V' are (-2)-curves on the surfaces $\check{E}_1, \ldots, \check{E}_5$, respectively. Similarly, the proper transforms of the curves $\check{C}_1', \ldots, \check{C}_5'$ on the 3-fold V are (-2)-curves on the surfaces $\hat{E}_1', \ldots, \hat{E}_5'$, respectively. Thus, all surfaces $\hat{E}_1', \ldots, \hat{E}_5'$, $\check{E}_1, \ldots, \check{E}_5$ are isomorphic to the Hirzebruch surface $\mathbb{F}_2$.

Denote by $\overline{C}_1, \ldots, \overline{C}_5$, $\overline{C}_1', \ldots, \overline{C}_5'$ the images of the curves $\hat{C}_1, \ldots, \hat{C}_5$, $\check{C}_1', \ldots, \check{C}_5'$ on the 3-fold U, respectively. Then,

$$-K_U \cdot \overline{C}_i = -K_U \cdot \overline{C}_i' = 0,$$

because $-K_V \cdot \hat{C}_i = -K_{V'} \cdot \check{C}_i' = 0$, and υ and υ' are small birational morphisms. Thus, the 10 curves $\overline{C}_1, \ldots, \overline{C}_5, \overline{C}_1', \ldots, \overline{C}_5'$ are contracted by the morphism β to 10 singular points of $X_{\frac{1}{6}}$.

It would be interesting to extend the commutative diagram (6.2) to an $\mathfrak{S}_5$-Sarkisov link similar to the $\mathfrak{A}_6$-Sarkisov link (5.4).

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