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On a Conjecture of Hong and Won

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Abstract. We give an explicit counter-example to a conjecture of Kyusik Hong and Joonyeong Won about α-invariants of polarized smooth del Pezzo surfaces of degree one.

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1. Introduction

In [11], Tian defined the α-invariant of a smooth Fano variety[1] and proved

Theorem 1 ([11]). *Let X be a smooth Fano variety of dimension n such that $\alpha(X) > \frac{n}{n+1}$. Then X admits a Kähler–Einstein metric.*

In [10], Odaka and Sano proved

Theorem 2. *Let X be a smooth Fano variety of dimension n such that $\alpha(X) > \frac{n}{n+1}$. Then X is K-stable.*

Two-dimensional smooth Fano varieties are also known as smooth del Pezzo surfaces. The possible values of their α-invariants are given by

Theorem 3 ([1, Theorem 1.7]). *Let S be a smooth del Pezzo surface. Then*

$$\alpha(S) = \begin{cases} \frac{1}{3} \text{ if } S \cong \mathbb{F}_1 \text{ or } K_S^2 \in \{7, 9\}, \\ \frac{1}{2} \text{ if } S \cong \mathbb{P}^1 \times \mathbb{P}^1 \text{ or } K_S^2 \in \{5, 6\}, \\ \frac{2}{3} \text{ if } K_S^2 = 4, \\ \frac{2}{3} \text{ if } S \text{ is a cubic surface in } \mathbb{P}^3 \text{ with an Eckardt point}, \\ \frac{3}{4} \text{ if } S \text{ is a cubic surface in } \mathbb{P}^3 \text{ without Eckardt points}, \\ \frac{3}{4} \text{ if } K_S^2 = 2 \text{ and } |-K_S| \text{ has a tacnodal curve}, \\ \frac{5}{6} \text{ if } K_S^2 = 2 \text{ and } |-K_S| \text{ has no tacnodal curves}, \\ \frac{5}{6} \text{ if } K_S^2 = 1 \text{ and } |-K_S| \text{ has a cuspidal curve}, \\ 1 \text{ if } K_S^2 = 1 \text{ and } |-K_S| \text{ has no cuspidal curves}. \end{cases}$$

[1] All varieties are assumed to be algebraic, projective and defined over $\mathbb{C}$.

Let X be an arbitrary smooth algebraic variety, and let L be an ample $\mathbb{Q}$-divisor on it. Donaldson, Tian and Yau conjectured that the following conditions are equivalent:

- the pair (X, L) is K-polystable,
- the variety X admits a constant scalar curvature Kähler metric in $\mathrm{c}_1(L)$.

In [6], this conjecture has been proved in the case when X is a Fano variety and $L = -K_X$.

In [12], Tian defined a new invariant $\alpha(X, L)$ that generalizes the classical α-invariant. If X is a smooth Fano variety, then $\alpha(X) = \alpha(X, -K_X)$. By [3, Theorem A.3], one has

$$\alpha\big(X, L\big) = \sup\left\{\lambda \in \mathbb{Q}\ \middle|\ \begin{array}{l}\text{the log pair } (X, \lambda D) \text{ is log canonical}\\ \text{for every effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} L\end{array}\right\} \in \mathbb{R}_{>0}.$$

In [8], Dervan generalized Theorem 2 as follows:

Theorem 4 ([8, Theorem 1.1]). *Suppose that* $-K_X - \frac{n}{n+1}\frac{-K_X\cdot L^{n-1}}{L^n}L$ *is nef, and*

$$\alpha\big(X, L\big) > \frac{n}{n+1}\frac{-K_X \cdot L^{n-1}}{L^n}.$$

Then the pair (X, L) *is* K*-stable.*

For smooth del Pezzo surfaces, Theorem 4 gives

Theorem 5 ([2, 9]). *Let* S *be a smooth del Pezzo surface such that* $K_S^2 = 1$ *or* $K_S^2 = 2$*. Let* A *be an ample* $\mathbb{Q}$*-divisor on the surface* S *such that the divisor*

$$-K_S - \frac{2}{3}\frac{-K_S \cdot A}{A^2}A$$

is nef. Then the pair (S, A) *is* K*-stable.*

This result is closely related to

Problem 6 (cf. Theorem 3). *Let* S *be a smooth del Pezzo surface. Compute*

$$\alpha(S, A) \in \mathbb{R}_{>0}$$

for every ample $\mathbb{Q}$*-divisor* A *on the surface* S*.*

Hong and Won suggested an answer to Problem 6 for del Pezzo surfaces of degree one. This answer is given by their [9, Conjecture 4.3], which is Conjecture 11 in Section 2.

The main result of this paper is

Theorem 7 (cf. Theorem 3). *Let* S *be a smooth del Pezzo surface such that* $K_S^2 = 1$*. Let* C *be an irreducible smooth curve in* S *such that* $C^2 = -1$*. Then there is a unique curve*

$$\widetilde{C} \in \big| -2K_S - C\big|.$$

The curve $\widetilde{C}$ *is also irreducible and smooth. One has* $\widetilde{C}^2 = -1$ *and* $1 \leqslant |C \cap \widetilde{C}| \leqslant C \cdot \widetilde{C} = 3$. *Let* λ *be a rational number such that* $0 \leqslant \lambda < 1$. *Then* $-K_S + \lambda C$ *is ample and*

$$\alpha\big(S, -K_S + \lambda C\big) = \begin{cases} \min\Big(\alpha(S), \frac{2}{1+2\lambda}\Big) & \text{if } |C \cap \widetilde{C}| \geqslant 2, \\ \min\Big(\alpha(S), \frac{4}{3+3\lambda}\Big) & \text{if } |C \cap \widetilde{C}| = 1. \end{cases}$$

Theorem 7 implies that [9, Conjecture 4.3] is wrong. To be precise, this follows from

Example 8. Let S be a surface in $\mathbb{P}(1, 1, 2, 3)$ that is given by

$$w^2 = z^3 + zx^2 + y^6,$$

where x, y, z, w are coordinates such that $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 2$ and $\mathrm{wt}(w) = 3$. Then S is a smooth del Pezzo surface and $K_S^2 = 1$. Let C be the curve in X given by

$$z = w - y^3 = 0.$$

Similarly, let $\widetilde{C}$ be the curve in S that is given by $z = w + y^3 = 0$. Then $C + \widetilde{C} \sim -2K_S$. Both curves C and $\widetilde{C}$ are smooth rational curves such that $C^2 = \widetilde{C}^2 = -1$ and $|C \cap \widetilde{C}| = 1$. All singular curves in $| - K_S|$ are nodal. Then $\alpha(S) = 1$ by Theorems 3, so that

$$\alpha\big(S, -K_S + \lambda C\big) = \min\Big(1, \frac{4}{3 + 3\lambda}\Big)$$

by Theorem 7. But [9, Conjecture 4.3] says that $\alpha(S, -K_S + \lambda C) = \min(1, \frac{2}{1+2\lambda})$.

Theorem 7 has two applications. By Theorem 4, it implies

Corollary 9 ([8, Theorem 1.2]). *Let* S *be a smooth del Pezzo surface such that* $K_S^2 = 1$. *Let* C *be an irreducible smooth curve in* S *such that* $C^2 = -1$. *Fix* $\lambda \in \mathbb{Q}$ *such that*

$$3 - \sqrt{10} \leqslant \lambda \leqslant \frac{\sqrt{10} - 1}{9}.$$

Then the pair $(S, -K_S + \lambda C)$ *is* K*-stable.*

By [5, Remark 1.1.3], Theorem 7 implies

Corollary 10. *Let* S *be a smooth del Pezzo surface. Suppose that* $K_S^2 = 1$ *and* $\alpha(S) = 1$. *Let* C *be an irreducible smooth curve in* S *such that* $C^2 = -1$. *Fix* $\lambda \in \mathbb{Q}$ *such that*

$$-\frac{1}{4} \leqslant \lambda \leqslant \frac{1}{3}.$$

Then S *does not contain* $(-K_S + \lambda C)$*-polar cylinders (see [5, Definition 1.2.1]).*

Corollary 9 follows from Theorem 5. Corollary 10 follows from [5, Theorem 2.2.3].

Let us describe the structure of this paper. In Section 2, we describe [9, Conjecture 4.3]. In Section 3, we present several well-known local results about singularities of log pairs. In Section 4, we prove eight local lemmas that are crucial for the proof of Theorem 7. In Section 5, we prove Theorem 7 using Lemmas 23, 24, 25, 26, 27, 28, 29, 30.

2. Conjecture of Hong and Won

Let S be a smooth del Pezzo surface, and let A be an ample $\mathbb{Q}$-divisor on S. Put

$$\mu = \inf\Big\{\lambda \in \mathbb{Q}_{>0} \,\Big|\, \text{the } \mathbb{Q}\text{-divisor } K_S + \lambda A \text{ is pseudo-effective}\Big\} \in \mathbb{Q}_{>0}.$$

Then $K_S + \mu A$ is contained in the boundary of the Mori cone $\overline{\mathbb{NE}}(S)$ of the surface S.

Suppose that $K_S^2 = 1$. Then $\overline{\mathbb{NE}}(S)$ is polyhedral and is generated by (-1)-curves in S. By a (-1)-curve, we mean a smooth irreducible rational curve $E \subset S$ such that $E^2 = -1$.

Let Δ_A be the smallest extremal face of the Mori cone $\overline{\mathbb{NE}}(S)$ that contains $K_S + \mu A$. Let $\phi\colon S \to Z$ be the contraction given by the face Δ_A. Then

- either ϕ is a birational morphism and Z is a smooth del Pezzo surface,
- or ϕ is a conic bundle and $Z \cong \mathbb{P}^1$.

If ϕ is birational and $Z \ncong \mathbb{P}^1 \times \mathbb{P}^1$, we call A a divisor of $\mathbb{P}^2$-type. In this case, we have

$$K_S + \mu A \sim_{\mathbb{Q}} \sum_{i=1}^{8} a_i E_i,$$

where E_1, E_2, E_3, E_4, E_5, E_6, E_7, E_8 are eight disjoint (-1)-curves in our surface S, and a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 are non-negative rational numbers such that

$$1 > a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4 \geqslant a_5 \geqslant a_6 \geqslant a_7 \geqslant a_8 \geqslant 0.$$

In this case, we put $s_A = a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8$.

If our ample divisor A is not a divisor of $\mathbb{P}^2$-type, then the surface S contains a smooth irreducible rational curve C such that $C^2 = 0$ and

$$K_S + \mu A \sim_{\mathbb{Q}} \delta C + \sum_{i=1}^{7} a_i E_i,$$

where E_1, E_2, E_3, E_4, E_5, E_6, E_7 are disjoint (-1)-curves in S that are disjoint from C, and δ, a_1, a_2, a_3, a_4, a_5, a_6, a_7 are non-negative rational numbers such that

$$1 > a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4 \geqslant a_5 \geqslant a_6 \geqslant a_7 \geqslant 0.$$

In this case, let $\psi\colon S \to \overline{S}$ be the contraction of the curves E_1, E_2, E_3, E_4, E_5, E_6, E_7, and let $\eta\colon S \to \mathbb{P}^1$ be a conic bundle given by $|C|$. Then either $\overline{S} \cong \mathbb{F}_1$ or

$\overline{S} \cong \mathbb{P}^1 \times \mathbb{P}^1$. In both cases, there exists a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\ \psi\ } & \overline{S} \\ & {\scriptstyle\eta}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \mathbb{P}^1 & \end{array}$$

where π is a natural projection. Then $\delta > 0 \iff \phi$ is a conic bundle and $\phi = \eta$. Similarly, if ϕ is birational and $Z \cong \mathbb{P}^1 \times \mathbb{P}^1$, then $\delta = 0$, $a_7 > 0$, and $\phi = \psi$. Then

- we call A a divisor of $\mathbb{F}_1$-type in the case when $\overline{S} \cong \mathbb{F}_1$,
- we call A a divisor of $\mathbb{P}^1 \times \mathbb{P}^1$-type in the case when $\overline{S} \cong \mathbb{P}^1 \times \mathbb{P}^1$.

In both cases, we put $s_A = a_2 + a_3 + a_4 + a_5 + a_6 + a_7$.

In order to study $\alpha(S, A)$, we may assume that $\mu = 1$, because

$$\alpha(S, A) = \mu\alpha(S, \mu A)$$

If A is a divisor of $\mathbb{P}^2$-type, let us define a number $\alpha_c(S, A)$ as follows:

- if $s_A > 4$, we put $\alpha_c(S, A) = \frac{1}{2+a_1}$,
- if $4 \geqslant s_A > 1$, we let $\alpha_c(S, A)$ to be

$$\max\left(\frac{2}{2 + 2a_1 + s_A - a_2 - a_3}, \frac{4}{3 + 4a_1 + 2s_A - a_2 - a_3 - a_4}, \frac{3}{2 + 3a_1 + s_A}\right),$$

- if $1 \geqslant s_A$, we put $\alpha_c(S, A) = \min(\frac{2}{1+2a_1+s_A}, 1)$.

Similarly, if A is a divisor of $\mathbb{F}_1$-type, we define $\alpha_c(S, A)$ as follows:

- if $s_A > 4$, we put $\alpha_c(S, A) = \frac{1}{2+a_1+\delta}$,
- if $4 \geqslant s_A > 1$, we let $\alpha_c(S, A)$ to be

$$\max\left(\frac{2}{2 + 2a_1 + s_A - a_2 - a_3 + 2\delta}, \frac{4}{3 + 4a_1 + 2s_A - a_2 - a_3 - a_4 + 4\delta}, \frac{3}{2 + 3a_1 + s_A + 3\delta}\right),$$

- if $1 \geqslant s_A$, we put $\alpha_c(S, A) = \min(\frac{2}{1+2a_1+s_A+2\delta}, 1)$.

Finally, if A is a divisor of $\mathbb{P}^1 \times \mathbb{P}^1$-type, we define $\alpha_c(S, A)$ as follows:

- if $s_A > 4$, we put $\alpha_c(S, A) = \frac{1}{2+a_1+\delta}$,
- if $4 \geqslant s_A > 1$, we let $\alpha_c(S, A)$ to be

$$\max\left(\frac{2}{2 + s_A - a_7 - a_2 - a_3 + 2\delta}, \frac{4}{3 + 2s_A - 2a_7 - a_2 - a_3 - a_4 + 4\delta}, \frac{3}{2 + s_A - a_7 + 3\delta}\right),$$

- if $1 \geqslant s_A$, we put $\alpha_c(S, A) = \min\left(\frac{2}{1+s_A-a_7+2\delta}, 1\right)$.

The conjecture of Hong and Won is

Conjecture 11 ([9, Conjecture 4.3]). *If* $\alpha(S) = 1$*, then* $\alpha(S, A) = \alpha_c(S, A)$*.*

The main evidence for this conjecture is

Theorem 12 ([9]). *Let* D *be an effective* $\mathbb{Q}$*-divisor on the surface* S *such that* $D \sim_{\mathbb{Q}} A$*. Then the log pair* $(S, \alpha_c(S, A)D)$ *is log canonical outside of finitely many points.*

As we already mentioned in Section 1, Example 8 shows that Conjecture 11 is wrong. However, the smooth del Pezzo surface of degree one in Example 8 is rather special. Therefore, Conjecture 11 may hold for *general* smooth del Pezzo surfaces of degree one.

By [5, Remark 1.1.3], it follows from Conjecture 11 that S does not contain A-polar cylinders (see [5, Definition 1.2.1]) when $\alpha(S) = 1$ and a_1 and δ are small enough.

3. Singularities of log pairs

Let S be a smooth surface, and let D be an effective $\mathbb{Q}$-divisor on it. Write

$$D = \sum_{i=1}^{r} a_i C_i$$

where each C_i is an irreducible curve on S, and each a_i is a non-negative rational number. We assume here that all curves $C_1, \ldots, C_r$ are different.

Let $\gamma\colon \mathcal{S} \to S$ be a birational morphism such that the surface $\mathcal{S}$ is smooth as well. It is well known that the morphism γ is a composition of n blow ups of smooth points. Thus, the morphism γ contracts n irreducible curves. Denote these curves by $\Gamma_1, \ldots, \Gamma_n$. For each curve C_i, denote by $\mathcal{C}_i$ its proper transform on the surface $\mathcal{S}$. Then

$$K_{\mathcal{S}} + \sum_{i=1}^{r} a_i \mathcal{C}_i + \sum_{j=1}^{n} b_j \Gamma_j \sim_{\mathbb{Q}} \gamma^*\big(K_S + D\big)$$

for some rational numbers $b_1, \ldots, b_n$. Suppose, in addition, that the divisor

$$\sum_{i=1}^{r} \mathcal{C}_i + \sum_{j=1}^{n} \Gamma_j$$

has simple normal crossing singularities. Fix a point $P \in S$.

Definition 13. The log pair (S, D) is *log canonical* (respectively *Kawamata log terminal*) at the point P if the following two conditions are satisfied:

- $a_i \leqslant 1$ (respectively $a_i < 1$) for every C_i such that $P \in C_i$,
- $b_j \leqslant 1$ (respectively $b_j < 1$) for every Γ_j such that $\pi(\Gamma_j) = P$.

This definition does not depend on the choice of the birational morphism γ.

The log pair (S, D) is said to be *log canonical* (respectively *Kawamata log terminal*) if it is log canonical (respectively, *Kawamata log terminal*) at every point in S.

The following result follows from Definition 13. But it is very handy.

Lemma 14. *Suppose that the singularities of the pair (S, D) are not log canonical at P. Let D' be an effective $\mathbb{Q}$-divisor on S such that (S, D') is log canonical at P and $D' \sim_{\mathbb{Q}} D$. Then there exists an effective $\mathbb{Q}$-divisor D'' on the surface S such that*

$$D'' \sim_{\mathbb{Q}} D,$$

the log pair (S, D'') is not log canonical at P, and $\mathrm{Supp}(D') \not\subseteq \mathrm{Supp}(D'')$.

Proof. Let ϵ be the largest rational number such that $(1+\epsilon)D - \epsilon D'$ is effective. Then

$$(1+\epsilon)D - \epsilon D' \sim_{\mathbb{Q}} D.$$

Put $D'' = (1+\epsilon)D - \epsilon D'$. Then (S, D'') is not log canonical at P, because

$$D = \frac{1}{1+\epsilon} D'' + \frac{\epsilon}{1+\epsilon} D'.$$

Furthermore, we have $\mathrm{Supp}(D') \not\subseteq \mathrm{Supp}(D'')$ by construction. □

Let $f\colon \widetilde{S} \to S$ be a blow up of the point P. Let us denote the f-exceptional curve by F. Denote by $\widetilde{D}$ the proper transform of the divisor D via f. Put $m = \mathrm{mult}_P(D)$.

Theorem 15 ([7, Exercise 6.18]). *If (S, D) is not log canonical at P, then $m > 1$.*

Let C be an irreducible curve in the surface S. Suppose that $P \in C$ and $C \not\subseteq \mathrm{Supp}(D)$. Denote by $\widetilde{C}$ the proper transform of the curve C via f. Fix $a \in \mathbb{Q}$ such that $0 \leqslant a \leqslant 1$. Then $(S, aC + D)$ is not log canonical at P if and only if the log pair

$$\left(\widetilde{S}, a\widetilde{C} + \widetilde{D} + \Big(a\,\mathrm{mult}_P\big(C\big) + m - 1\Big)F\right) \tag{1}$$

is not log canonical at some point in F. This follows from Definition 13.

Theorem 16 ([7, Exercise 6.31]). *Suppose that C is smooth at P, and $(D \cdot C)_P \leqslant 1$. Then the log pair $(S, aC + D)$ is log canonical at P.*

Corollary 17. *Suppose that the log pair (1) is not log canonical at some point in $F \setminus \widetilde{C}$. Then either $a\,\mathrm{mult}_P(C) + m > 2$ or $m > 1$ (or both).*

Let us give another application of Theorem 16.

Lemma 18. *Suppose that there is a double cover $\pi\colon S \to \mathbb{P}^2$ branched in a curve $R \subset \mathbb{P}^2$. Suppose also that (S, D) is not log canonical at P, and $D \sim_{\mathbb{Q}} \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$. Then $\pi(P) \in R$.*

Proof. The log pair $(\widetilde{S}, \widetilde{D} + (m-1)F)$ is not log canonical at some point $Q \in F$. Then

$$m + \mathrm{mult}_Q\big(\widetilde{D}\big) > 2 \tag{2}$$

by Theorem 15. Suppose that $\pi(P) \notin R$. Then there is $Z \in |\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))|$ such that

- the curve Z passes through the point P,
- the proper transform of the curve Z on the surface $\widetilde{S}$ contains Q.

Denote by $\widetilde{Z}$ the proper transform of the curve Z on the surface $\widetilde{S}$.

By Lemma 14, we may assume that the support of the $\mathbb{Q}$-divisor D does not contain at least one irreducible component of the curve Z, because (S, Z) is log canonical at P. Thus, if Z is irreducible, then $2 - m = \widetilde{Z} \cdot \widetilde{D} \geqslant \mathrm{mult}_Q(\widetilde{D})$, which contradicts (2).

We see that $Z = Z_1 + Z_2$, where Z_1 and Z_2 are irreducible smooth rational curves. We may assume that $Z_2 \not\subseteq \mathrm{Supp}(D)$. If $P \in Z_2$, then $1 = D \cdot Z_2 \geqslant m > 1$ by Theorem 15. This shows that $P \in Z_1$ and $Z_1 \subseteq \mathrm{Supp}(D)$.

Let d be the degree of the curve R. Then $Z_1^2 = Z_2^2 = \frac{2-d}{2}$ and $Z_1 \cdot Z_2 = \frac{d}{2}$.

We may assume that $C_1 = Z_1$. Put $\Delta = a_2C_2 + \cdots + a_rC_r$. Then $a_1 \leqslant \frac{2}{d}$, since

$$1 = Z_2 \cdot D = Z_2 \cdot \Big(a_1C_1 + \Delta\Big) = a_1Z_2 \cdot C_1 + Z_2 \cdot \Delta \geqslant a_1Z_2 \cdot C_1 = \frac{a_1d}{2}.$$

Denote by $\widetilde{C}_1$ the proper transform of the curve C_1 on the surface $\widetilde{S}$. Then $Q \in \widetilde{C}_1$. Denote by $\widetilde{\Delta}$ the proper transform of the $\mathbb{Q}$-divisor Δ on the surface $\widetilde{S}$. The log pair

$$\left(\widetilde{S}, a_1\widetilde{C}_1 + \widetilde{\Delta} + \Big(a_1 + \mathrm{mult}_P\left(\Delta\right) - 1\Big)F\right)$$

is not log canonical at the point Q by construction. By Theorem 16, we have

$$1 + \frac{d-2}{2}a_1 - \mathrm{mult}_P\big(\Delta\big) = \widetilde{C}_1 \cdot \widetilde{\Delta} \geqslant \big(\widetilde{C}_1 \cdot \widetilde{\Delta}\big)_Q > 1 - \Big(a_1 + \mathrm{mult}_P\left(\Delta\right) - 1\Big),$$

so that $a_1 > \frac{2}{d}$. But we already proved that $a_1 \leqslant \frac{2}{d}$. □

Fix a point $Q \in F$. Put $\widetilde{m} = \mathrm{mult}_Q(\widetilde{D})$. Let $g \colon \widehat{S} \to \widetilde{S}$ be a blow up of the point Q. Denote by $\widehat{C}$ and $\widehat{F}$ the proper transforms of the curves $\widetilde{C}$ and F via g, respectively. Similarly, let us denote by $\widehat{D}$ the proper transform of the $\mathbb{Q}$-divisor D on the surface $\widehat{S}$. Denote by G the g-exceptional curve. If the log pair (1) is not log canonical at Q, then

$$\left(\widehat{S}, a\widehat{C} + \widehat{D} + \Big(a\,\mathrm{mult}_P\left(C\right) + m - 1\Big)\widehat{F} + \Big(a\,\mathrm{mult}_P\left(C\right) + a\,\mathrm{mult}_Q\left(\widetilde{C}\right) + m + \widetilde{m} - 2\Big)G\right) \tag{3}$$

is not log canonical at some point in G.

Lemma 19. *Suppose $m \leqslant 1$, $a\,\mathrm{mult}_P(C) + m \leqslant 2$ and $a\,\mathrm{mult}_P\left(C\right) + a\,\mathrm{mult}_Q(\widetilde{C}) + 2m \leqslant 3$. Then* (3) *is log canonical at every point in $G \setminus \widehat{C}$.*

Proof. Suppose that (3) is not log canonical at some point $O \in G$ such that $O \notin \widehat{C}$. If $O \notin \widehat{F}$, then $1 \geqslant m \geqslant \widetilde{m} = \widehat{D} \cdot G \geqslant (\widehat{D} \cdot G)_O > 1$ by Theorem 16. Then $O \in \widehat{F}$. Then

$$m - \widetilde{m} = \left(\widehat{D} \cdot \widehat{F}\right)_O > 1 - \left(a\,\mathrm{mult}_P\left(C\right) + a\,\mathrm{mult}_Q\left(\widetilde{C}\right) + m + \widetilde{m} - 2\right)$$

by Theorem 16. This is impossible, since $a\,\mathrm{mult}_P(C) + a\,\mathrm{mult}_Q(\widetilde{C}) + 2m \leqslant 3$. □

Fix a point $O \in G$. Put $\widehat{m} = \mathrm{mult}_O(\widehat{D})$. Let $h\colon \overline{S} \to \widehat{S}$ be a blow up of the point O. Denote by $\overline{C}$, $\overline{F}$, $\overline{G}$ the proper transforms of the curves $\widehat{C}$, $\widehat{F}$ and G via h, respectively. Similarly, let us denote by $\overline{D}$ the proper transform of the $\mathbb{Q}$-divisor D on the surface $\overline{S}$. Let H be the h-exceptional curve. If $O = G \cap \widehat{F}$ and (3) is not log canonical at O, then

$$\Big(\overline{S}, a\overline{C} + \overline{D} + \left(2a\,\mathrm{mult}_P\left(C\right) + a\,\mathrm{mult}_Q\left(\widetilde{C}\right) + a\,\mathrm{mult}_O\left(\widehat{C}\right) + 2m + \widetilde{m} + \widehat{m} - 4\right)H$$
$$+ \left(a\,\mathrm{mult}_P\left(C\right) + m - 1\right)\overline{F} + \left(a\,\mathrm{mult}_P\left(C\right) + a\,\mathrm{mult}_Q\left(\widetilde{C}\right) + m + \widetilde{m} - 2\right)\overline{G}\Big) \tag{4}$$

is not log canonical at some point in H.

Lemma 20. *Suppose that* $O = G \cap \widehat{F}$, $m \leqslant 1$, $a\,\mathrm{mult}_P(C) + a\,\mathrm{mult}_Q(\widetilde{C}) + m + \widetilde{m} \leqslant 3$ *and*

$$2a\,\mathrm{mult}_P\left(C\right) + a\,\mathrm{mult}_Q\left(\widetilde{C}\right) + a\,\mathrm{mult}_O\left(\widehat{C}\right) + 4m \leqslant 5.$$

Then the log pair (4) *is log canonical at every point in* $H \setminus \overline{C}$.

Proof. Suppose that the pair (4) is not log canonical at some point $E \in H$ such that $E \notin \overline{C}$. If $E \notin \overline{F} \cup \overline{G}$, then $m \geqslant \widehat{m} = \overline{D} \cdot H \geqslant (\overline{D} \cdot H)_E > 1$ by Theorem 16. Then $E \in \overline{F} \cup \overline{G}$.

If $E \in \overline{G}$, then $E \notin \overline{F}$, so that Theorem 16 gives

$$\begin{aligned}\widetilde{m} - \widehat{m} &= \left(\overline{D} \cdot \overline{F}\right)_E \\ &> 1 - \left(2a\,\mathrm{mult}_P\left(C\right) + a\,\mathrm{mult}_Q\left(\widetilde{C}\right) + a\,\mathrm{mult}_O\left(\widehat{C}\right) + 2m + \widetilde{m} + \widehat{m} - 4\right),\end{aligned}$$

which is impossible, since $2a\,\mathrm{mult}_P(C) + a\,\mathrm{mult}_Q(\widetilde{C}) + a\,\mathrm{mult}_O(\widehat{C}) + 4m \leqslant 5$ by assumption. Similarly, if $E \in \overline{F}$, then $E \notin \overline{G}$, so that Theorem 16 gives

$$\begin{aligned}m - \widetilde{m} - \widehat{m} &= \left(\overline{D} \cdot \overline{F}\right)_E \\ &> 1 - \left(2a\,\mathrm{mult}_P\left(C\right) + a\,\mathrm{mult}_Q\left(\widetilde{C}\right) + a\,\mathrm{mult}_O\left(\widehat{C}\right) + 2m + \widetilde{m} + \widehat{m} - 4\right),\end{aligned}$$

which is impossible, since $2a\,\mathrm{mult}_P(C) + a\,\mathrm{mult}_Q(\widetilde{C}) + a\,\mathrm{mult}_O(\widehat{C}) + 4m \leqslant 5$. □

Let Z be an irreducible curve in S such that $P \in Z$. Suppose also that $Z \not\subseteq \mathrm{Supp}(D)$. Denote its proper transforms on the surfaces $\widetilde{S}$ and $\widehat{S}$ by the symbols $\widetilde{Z}$ and $\widehat{Z}$, respectively. Fix $b \in \mathbb{Q}$ such that $0 \leqslant b \leqslant 1$. If $(S, aC + bZ + D)$ is not log canonical at P, then

$$\left(\widetilde{S}, a\widetilde{C} + b\widetilde{Z} + \widetilde{D} + \Big(a\,\mathrm{mult}_P\left(C\right) + b\,\mathrm{mult}_P\left(Z\right) + m - 1\Big)F\right) \tag{5}$$

is not log canonical at some point in F.

Lemma 21. *Suppose that $m \leqslant 1$ and*

$$a\,\mathrm{mult}_P\left(C\right) + b\,\mathrm{mult}_P\left(Z\right) + m \leqslant 2.$$

Then (5) *is log canonical at every point in $Q \in F \setminus (\widetilde{C} \cup \widetilde{Z})$.*

Proof. Suppose that (5) is not log canonical at some point $Q \in F$ such that $Q \notin \widetilde{C} \cup \widetilde{Z}$. Then $m = \widetilde{D} \cdot F \geqslant (\widetilde{D} \cdot F)_Q > 1$ by Theorem 16. But $m \leqslant 1$ by assumption. □

If the log pair (5) is not log canonical at Q, then the log pair

$$\Big(\widehat{S}, a\widehat{C} + b\widehat{Z} + \widehat{D} + \Big(a\,\mathrm{mult}_P\left(C\right) + b\,\mathrm{mult}_P\left(Z\right) + m - 1\Big)F \tag{6}$$
$$+ \Big(a\,\mathrm{mult}_P\left(C\right) + a\,\mathrm{mult}_Q\left(\widetilde{C}\right) + b\,\mathrm{mult}_P\left(Z\right) + b\,\mathrm{mult}_Q\left(\widetilde{Z}\right) + m + \widetilde{m} - 2\Big)G\Big)$$

is not log canonical at some point in G.

Lemma 22. *Suppose that $m \leqslant 1$, $a\,\mathrm{mult}_P(C) + b\,\mathrm{mult}_P(Z) + m \leqslant 2$ and*

$$a\,\mathrm{mult}_P\left(C\right) + a\,\mathrm{mult}_Q\left(\widetilde{C}\right) + b\,\mathrm{mult}_P\left(Z\right) + b\,\mathrm{mult}_Q\left(\widetilde{Z}\right) + 2m \leqslant 3.$$

Then the log pair (6) *is log canonical at every point in $G \setminus (\widehat{C} \cup \widehat{Z})$.*

Proof. We may assume that the log pair (6) is not log canonical at O and $O \notin \widehat{C} \cup \widehat{Z}$. If $O \notin \widehat{F}$, then $m \geqslant \widetilde{m} = \widehat{D} \cdot G \geqslant (\widehat{D} \cdot G)_O > 1$ by Theorem 16, so that $O \in \widehat{F}$. Then

$$\begin{aligned} m - \widetilde{m} &= \big(\widehat{D} \cdot \widehat{F}\big)_O \\ &> 1 - \Big(a\,\mathrm{mult}_P\left(C\right) + a\,\mathrm{mult}_Q\left(\widetilde{C}\right) + b\,\mathrm{mult}_P\left(Z\right) + b\,\mathrm{mult}_Q\left(\widetilde{Z}\right) + m + \widetilde{m} - 2\Big), \end{aligned}$$

by Theorem 16, so that

$$a\,\mathrm{mult}_P\left(C\right) + a\,\mathrm{mult}_Q(\widetilde{C}) + b\,\mathrm{mult}_P(Z) + b\,\mathrm{mult}_Q(\widetilde{Z}) + 2m > 3.$$ □

4. Eight local lemmas

Let us use notations and assumptions of Section 3. Fix $x \in \mathbb{Q}$ such that $0 \leqslant x \leqslant 1$. Put

$$\mathrm{lct}_P\big(S, C\big) = \sup\Big\{\lambda \in \mathbb{Q} \,\Big|\, \text{the log pair } \big(S, \lambda C\big) \text{ is log canonical at } P\Big\} \in \mathbb{Q}_{>0}.$$

Lemma 23. *Suppose that C has an ordinary node or an ordinary cusp at P, $a \leqslant \frac{x}{2}$ and*

$$\big(D \cdot C\big)_P \leqslant \frac{4}{3} + \frac{x}{6} - a.$$

Then the log pair $(S, aC + D)$ is log canonical at P.

Proof. We have $2m \leqslant \mathrm{mult}_P(D)\,\mathrm{mult}_P(C) \leqslant (D{\cdot}C)_P \leqslant \frac{4}{3}+\frac{x}{6}-a$, so that $2m+a \leqslant \frac{4}{3}+\frac{x}{6}$. Then $m \leqslant \frac{3}{4}$ and $m+2a = m+\frac{a}{2}+\frac{3a}{2} \leqslant \frac{\frac{4}{3}+\frac{x}{6}}{2}+\frac{3a}{2} \leqslant \frac{\frac{4}{3}+\frac{x}{6}}{2}+\frac{3x}{4} = \frac{2}{3}+\frac{5}{6}x \leqslant \frac{3}{2}$.

Suppose that $(S, aC + D)$ is not log canonical at P. Let us seek for a contradiction. We may assume that (1) is not log canonical at Q. Then $Q \in \widetilde{C}$ by Corollary 17. Then

$$\big(\widetilde{D} \cdot \widetilde{C}\big)_O > 1 - \big(2a + m - 1\big)\big(\widetilde{C} \cdot F\big)_O \geqslant 1 - 2\big(2a + m - 1\big) = 3 - 4a - 2m.$$

On the other hand, we have $\frac{4}{3} + \frac{x}{6} - a \geqslant (D \cdot C)_P \geqslant 2m + (\widetilde{D} \cdot \widetilde{C})_O$, so that $a > \frac{5}{9} - \frac{x}{18}$. Then $\frac{x}{2} \geqslant a > \frac{5}{9} - \frac{x}{18}$, so that $x > 1$. But $x \leqslant 1$ by assumption. □

Lemma 24. *Suppose that C has an ordinary node or an ordinary cusp at P, and*

$$\big(D \cdot C\big)_P \leqslant \mathrm{lct}_P(S, C) + \frac{x}{2}.$$

Suppose also that $a \leqslant \mathrm{lct}_P(S, C) - \frac{x}{2}$. Then $(S, aC + D)$ is log canonical at P.

Proof. We have $2m \leqslant (D \cdot C)_P$. This gives $2m + a \leqslant 1 + \frac{x}{2}$. Thus, we have $m \leqslant \frac{1+\frac{x}{2}}{2} \leqslant \frac{3}{4}$. Similarly, we get $m+2a = m+\frac{a}{2}+\frac{3a}{2} \leqslant \frac{1+\frac{x}{2}}{2}+\frac{3a}{2} \leqslant \frac{1+\frac{x}{2}}{2}+\frac{3}{2}(1-\frac{x}{2}) = 2 - \frac{x}{2} \leqslant 2$.

Suppose that $(S, aC + D)$ is not log canonical at P. Let us seek for a contradiction. We may assume that the pair (1) is not log canonical at Q. Then $Q \in \widetilde{C}$ by Corollary 17. We may assume that (3) is not log canonical at O. Then $O \in \widehat{C}$ by Lemma 19, since

$$3a + 2m \leqslant 2a + 1 + \frac{x}{2} \leqslant 2 - x + 1 + \frac{x}{2} = 3 - \frac{x}{2} \leqslant 3,$$

because $2m + a \leqslant 1 + \frac{x}{2}$ and $a \leqslant 1 - \frac{x}{2}$. If $O \notin \widehat{F}$, then Theorem 16 gives

$$1 + \frac{x}{2} - a \geqslant \big(D \cdot C\big)_P - 2m - \widetilde{m} \geqslant \big(\widehat{D} \cdot \widehat{C}\big)_O > 1 - \big(3a + m + \widetilde{m} - 2\big),$$

which implies that $2a+\frac{x}{2} > 2+m$. But $2a+\frac{x}{2} \leqslant 2-\frac{x}{2}$, because $a \leqslant \mathrm{lct}_P(S, C)-\frac{x}{2} \leqslant 1 - \frac{x}{2}$. This shows that $O = G \cap \widehat{F} \cap \widehat{C}$. In particular, the curve C has an ordinary cusp at P. By assumption, we have $a \leqslant \frac{5}{6} - \frac{x}{2}$ and $2m + a \leqslant \frac{5}{6} + \frac{x}{2}$. This gives $6a + 4m \leqslant 5 - x \leqslant 5$.

Put $E = H \cap \overline{C}$. Then (4) is not log canonical at E by Lemma 20. Then

$$\left(\overline{D}\cdot\overline{C}\right)_E > 1-\left(6a+2m+\widetilde{m}+\widehat{m}-4\right) = 5-6a-2m-\widetilde{m}-\widehat{m}$$

by Theorem 16. Thus, we have $\frac{5}{6}+\frac{x}{2}-a \geqslant (D\cdot C)_P \geqslant 2m+\widetilde{m}+\widehat{m}+(\overline{D}\cdot\overline{C})_E > 5-6a$. This gives $a > \frac{5}{6}-\frac{x}{10}$. But $a \leqslant \frac{5}{6}-\frac{x}{2}$, which is absurd. □

Lemma 25. *Suppose that C is smooth at P, $a \leqslant \frac{1}{3}+\frac{x}{2}$, $m+a \leqslant 1+\frac{x}{2}$ and*

$$\left(D\cdot C\right)_P \leqslant 1-\frac{x}{2}+a.$$

Then the log pair $(S, aC+D)$ is log canonical at P.

Proof. We have $m \leqslant (D\cdot C)_P$, so that $m-a \leqslant 1-\frac{x}{2}$. Then $m \leqslant 1$, since $m+a \leqslant 1+\frac{x}{2}$.

Suppose that $(S, aC+D)$ is not log canonical at P. Let us seek for a contradiction. We may assume that the pair (1) is not log canonical at Q. Then $Q \in \widetilde{C}$ by Corollary 17. We may assume that (3) is not log canonical at O. Then $O \in \widehat{C}$ by Lemmas 19. Then

$$\left(\widehat{D}\cdot\widehat{C}\right)_O > 1-\left(2a+m+\widetilde{m}-2\right) = 3-2a-m-\widetilde{m}$$

by Theorem 16. Then $1-\frac{x}{2}+a \geqslant (D\cdot C)_P \geqslant m+(\widetilde{D}\cdot\widetilde{C})_Q \geqslant m+\widetilde{m}+(\widehat{D}\cdot\widehat{C})_O > 3-2a$. This give $a > \frac{2}{3}+\frac{x}{6}$, which is impossible, since $a \leqslant \frac{1}{3}+\frac{x}{2}$ and $x \leqslant 1$. □

Lemma 26. *Suppose that C is smooth at P, $a \leqslant \frac{8}{9}-\frac{x}{18}$, $m+a \leqslant \frac{4}{3}+\frac{x}{6}$ and*

$$\left(D\cdot C\right)_P \leqslant \frac{x}{2}+a.$$

Then the log pair $(S, aC+D)$ is log canonical at P.

Proof. We have $m \leqslant (D\cdot C)_P$, so that $m-a \leqslant \frac{x}{2}$. Then $m \leqslant \frac{2}{3}+\frac{x}{3} \leqslant 1$, since $m+a \leqslant \frac{4}{3}+\frac{x}{6}$.

Suppose that $(S, aC+D)$ is not log canonical at P. Let us seek for a contradiction. We may assume that the pair (1) is not log canonical at Q. Then $Q \in \widetilde{C}$ by Corollary 17. We may assume that (3) is not log canonical at O. Then $O \in \widehat{C}$ by Lemmas 19. Then

$$\left(\widehat{D}\cdot\widehat{C}\right)_O > 1-\left(2a+m+\widetilde{m}-2\right) = 3-2a-m-\widetilde{m}$$

by Theorem 16. Then $\frac{x}{2}+a \geqslant (D\cdot C)_P \geqslant m+(\widetilde{D}\cdot\widetilde{C})_Q \geqslant m+\widetilde{m}+(\widehat{D}\cdot\widehat{C})_O > 3-2a$. This gives $a > 1-\frac{x}{6}$, which is impossible, since $a \leqslant \frac{8}{9}-\frac{x}{18}$ and $x \leqslant 1$. □

Lemma 27. *Suppose that C has an ordinary node or an ordinary cusp at P, $a \leqslant \frac{1+x}{3}$ and*

$$\left(D\cdot C\right)_P \leqslant 2-2a.$$

Then the log pair $(S, aC+D)$ is log canonical at P.

Proof. We have $2m \leqslant (D \cdot C)_P \leqslant 2 - 2a$. This gives $m + a \leqslant 1$, so that we have $m \leqslant 1$. Then $m + 2a \leqslant 1 + a \leqslant 1 + \frac{1+x}{3} = \frac{4+x}{3} \leqslant \frac{5}{3}$ and $3a + 2m \leqslant 2 + a \leqslant 2 + \frac{1+x}{3} = \frac{7+x}{3} \leqslant \frac{8}{3}$.

Suppose that $(S, aC + D)$ is not log canonical at P. Let us seek for a contradiction. We may assume that the pair (1) is not log canonical at Q. Then $Q \in \widetilde{C}$ by Corollary 17. We may assume that (3) is not log canonical at O. Then $O \in \widehat{C}$ by Lemma 19.

If $O \notin \widehat{F}$, then $(\widehat{D} \cdot \widehat{C})_O > 3 - 3a - m - \widetilde{m}$ by Theorem 16, so that

$$2 - 2a \geqslant \big(D \cdot C\big)_P \geqslant 2m + \big(\widetilde{D} \cdot \widetilde{C}\big)_Q \geqslant 2m + \widetilde{m} + \big(\widehat{D} \cdot \widehat{C}\big)_O > 3 - 3a,$$

which is absurd. This shows that $O = G \cap \widehat{F} \cap \widehat{C}$. Then

$$\big(\widehat{D} \cdot \widehat{C}\big)_O > 1 - \big(2a + m - 1\big) - \big(3a + m + \widetilde{m} - 2\big) = 4 - 5a - 2m - \widetilde{m}$$

by Theorem 16. Then $2 - 2a \geqslant (D \cdot C)_P \geqslant 2m + \widetilde{m} + (\widehat{D} \cdot \widehat{C})_O > 4 - 5a$, so that $a > \frac{2}{3}$. But $a \leqslant \frac{1+x}{3} \leqslant \frac{2}{3}$ by assumption. This is a contradiction. □

Lemma 28. *Suppose that C has an ordinary node or an ordinary cusp at P, $a \leqslant \frac{2}{3}$ and*

$$\big(D \cdot C\big)_P \leqslant \frac{4}{3} + \frac{2x}{3} - 2a.$$

Then the log pair $(S, aC + D)$ is log canonical at P.

Proof. We have $2m \leqslant (D \cdot C)_P$, so that $m + a \leqslant \frac{2}{3} + \frac{x}{3} \leqslant 1$. Then $m \leqslant 1$ and $m + 2a \leqslant \frac{5}{3}$. Similarly, we see that $3a + 2m \leqslant \frac{4}{3} + \frac{2x}{3} + a \leqslant \frac{4}{3} + \frac{2x}{3} + \frac{2}{3} = 2 + \frac{2x}{3} \leqslant \frac{8}{3} < 3$.

Suppose that $(S, aC + D)$ is not log canonical at P. Let us seek for a contradiction. We may assume that the pair (1) is not log canonical at Q. Then $Q \in \widetilde{C}$ by Corollary 17. We may assume that (3) is not log canonical at O. Then $O \in \widehat{C}$ by Lemma 19.

If $O \notin \widehat{F}$, then $\frac{4}{3} + \frac{2x}{3} - 2a \geqslant (D \cdot C)_P \geqslant 2m + \widetilde{m} + (\widehat{D} \cdot \widehat{C})_O > m + 3 - 3a$ by Theorem 16. Therefore, if $O \notin \widehat{F}$, then $a > \frac{5}{3} - \frac{2x}{3} \geqslant 1$. But $a \leqslant \frac{2}{3}$. This shows that $O = G \cap \widehat{F} \cap \widehat{C}$. Then $(\widehat{D} \cdot \widehat{C})_O > 1 - (2a + m - 1) - (3a + m + \widetilde{m} - 2) = 4 - 5a - 2m - \widetilde{m}$ by Theorem 16. Then $\frac{4}{3} + \frac{2x}{3} - 2a \geqslant (D \cdot C)_P \geqslant 2m + \widetilde{m} + (\widehat{D} \cdot \widehat{C})_O > 4 - 5a$, which gives $a > \frac{2}{3}$. □

Lemma 29. *Suppose that C and Z are smooth at P, $(C \cdot Z)_P \leqslant 2$, and $a + b + m \leqslant 1 + \frac{x}{2}$. Suppose also that $a \leqslant \frac{1+x}{3}$, $b \leqslant \frac{1+x}{3}$, $(D \cdot C)_P \leqslant 1 + a - 2b$ and $(D \cdot Z)_P \leqslant 1 + b - 2a$. Then the log pair $(S, aC + bZ + D)$ is log canonical at P.*

Proof. We have $m \leqslant (D \cdot C)_P \leqslant 1 + a - 2b$ and $m \leqslant (D \cdot Z)_P \leqslant 1 + b - 2a$. Then $m + \frac{a+b}{2} \leqslant 1$.

Suppose that $(S, aC + bZ + D)$ is not log canonical at P. Let us seek for a contradiction. We may assume that (5) is not log canonical at Q. Then $Q \in \widetilde{C} \cup \widetilde{Z}$ by Lemma 21. Without loss of generality, we may assume that $\widetilde{C}$ contains Q. Then

$\widetilde{Z}$ also contains Q. Indeed, if $Q \notin \widetilde{Z}$, then $1+a-2b \geqslant (D\cdot C)_P \geqslant m + (\widetilde{D}\cdot\widetilde{C})_Q > 2-a-b$ by Theorem 16. But $1+b-2a \geqslant 0$. Thus, we have $Q = G\cap\widetilde{C}\cap\widetilde{Z}$, so that $(C\cdot Z)_P = 2$.

We may assume that (6) is not log canonical at O. Then $O \in \widehat{C}\cup\widehat{Z}$ by Lemma 22. In particular, we have $O \notin \widehat{F}$. Without loss of generality, we may assume that $O \in \widehat{C}$. By Theorem 16, we have $1+a-2b-m-\widetilde{m} \geqslant (\widehat{D}\cdot\widehat{C})_O > 1-(2a+2b+m+\widetilde{m}-2)$. This gives $a > \frac{2}{3}$, which is impossible, since $a \leqslant 1+\frac{x}{2} \leqslant \frac{2}{3}$. □

Lemma 30. *Suppose that C and Z are smooth at P, $(C\cdot Z)_P \leqslant 2$, and $a+b+m \leqslant \frac{4}{3}+\frac{x}{6}$. Suppose also that $a \leqslant \frac{2}{3}$, $b \leqslant \frac{2}{3}$, $(D\cdot C)_P \leqslant \frac{2+x}{3}+a-2b$ and $(D\cdot Z)_P \leqslant \frac{2+x}{3}+b-2a$. Then the log pair $(S, aC+bZ+D)$ is log canonical at P.*

Proof. We have $m \leqslant (D\cdot C)_P \leqslant \frac{2+x}{3}+a-2b$ and we have $m \leqslant (D\cdot Z)_P \leqslant \frac{2+x}{3}+b-2a$. Then $m+\frac{a+b}{2} \leqslant \frac{2+x}{3} \leqslant 1$, $m+a+b \leqslant \frac{4}{3}+\frac{x}{6} \leqslant \frac{3}{2}$ and $2a-b \leqslant 1$.

Suppose that $(S, aC+bZ+D)$ is not log canonical at P. Let us seek for a contradiction. We may assume that (5) is not log canonical at Q. Then $Q \in \widetilde{C}\cup\widetilde{Z}$ by Lemma 21. Without loss of generality, we may assume that Q is contained in $\widetilde{C}$. Then $Q \in \widetilde{C}\cap\widetilde{Z}$. Indeed, if $\widetilde{Z}$ does not contain Q, then $\frac{2+x}{3}+a-2b \geqslant m+\big(\widetilde{D}\cdot\widetilde{C}\big)_Q > 2-a-b$ by Theorem 16. The later inequality immediately leads to a contradiction, since $2a-b \leqslant 1$.

We may assume that (6) is not log canonical at O. Then $O \in \widehat{C}\cup\widehat{Z}$ by Lemmas 22. In particular, we have $O \notin \widehat{F}$. Without loss of generality, we may assume that $O \in \widehat{C}$. Then $\frac{2+x}{3}+a-2b-m-\widetilde{m} \geqslant (\widehat{D}\cdot\widehat{C})_O > 1-(2a+2b+m+\widetilde{m}-2)$ by Theorem 16. This gives $a > \frac{7-x}{9}$, which is impossible, since $a \leqslant \frac{2}{3}$. □

5. The proof of the main result

Let S be a smooth del Pezzo surface such that $K_S^2 = 1$. Then $|-2K_S|$ is base point free. It is well known that the linear system $|-2K_S|$ gives a double cover $S \to \mathbb{P}(1,1,2)$. This double cover induces an involution $\tau \in \mathrm{Aut}(S)$.

Let C be an irreducible curve in S such that $C^2 = -1$. Then $-K_S\cdot C = 1$ and $C \cong \mathbb{P}^1$. Put $\widetilde{C} = \tau(C)$. Then $\widetilde{C}^2 = K_S\cdot\widetilde{C} = -1$ and $\widetilde{C} \cong \mathbb{P}^1$. Moreover, we have $C+\widetilde{C} \sim -2K_S$. Furthermore, the irreducible curve $\widetilde{C}$ is uniquely determined by this rational equivalence. Since $C\cdot(C+\widetilde{C}) = -2K_S\cdot C = 2$ and $C^2 = -1$, we have $C\cdot\widetilde{C} = 3$, so that $1 \leqslant |C\cap\widetilde{C}| \leqslant 3$.

Fix $\lambda \in \mathbb{Q}$. Then $-K_S+\lambda C$ is ample $\iff -\frac{1}{3} < \lambda < 1$. Indeed, we have

$$-K_S+\lambda C \sim_{\mathbb{Q}} \frac{1}{2}\big(C+\widetilde{C}\big)+\lambda C = \Big(\frac{1}{2}+\lambda\Big)C+\frac{1}{2}\widetilde{C} \sim_{\mathbb{Q}} \big(1+2\lambda\big)\Big(-K_S-\frac{\lambda}{1+2\lambda}\widetilde{C}\Big). \tag{7}$$

One the other hand, we have $(-K_S+\lambda C)\cdot C = 1-\lambda$ and $(-K_S+\lambda C)\cdot\widetilde{C} = 1-3\lambda$.

Note that Theorem 7 and (7) imply

Corollary 31. *Suppose that* $-\frac{1}{3}<\lambda<1$*. If* $|C\cap\widetilde{C}|\geqslant 2$*, then*

$$\alpha\big(S,-K_S+\lambda C\big)=\begin{cases}\min\left(\frac{\alpha(X)}{1+2\lambda},2\right) \textit{ if } -\frac{1}{3}<\lambda<0,\\ \min\left(\alpha(X),\frac{2}{1+2\lambda}\right) \textit{ if } 0\leqslant\lambda<1.\end{cases}$$

Similarly, if $|C\cap\widetilde{C}|=1$*, then*

$$\alpha\big(S,-K_S+\lambda C\big)=\begin{cases}\min\left(\frac{\alpha(X)}{1+2\lambda},\frac{4}{3+3\lambda}\right) \textit{ if } -\frac{1}{3}<\lambda<0,\\ \min\left(\alpha(X),\frac{4}{3+3\lambda}\right) \textit{ if } 0\leqslant\lambda<1.\end{cases}$$

Now let us prove Theorem 7. Suppose that $0\leqslant\lambda<1$. Put

$$\mu=\begin{cases}\min\left(\alpha(S),\frac{2}{1+2\lambda}\right) \text{ when } |C\cap\widetilde{C}|\geqslant 2,\\ \min\left(\alpha(S),\frac{4}{3+3\lambda}\right) \text{ when } |C\cap\widetilde{C}|=1.\end{cases}\tag{8}$$

Lemma 32. *One has* $\alpha(S,-K_S+\lambda C)\leqslant\mu$*.*

Proof. Since we have $(\frac{1}{2}+\lambda)C+\frac{1}{2}\widetilde{C}\sim_{\mathbb{Q}}-K_S+\lambda C$, we see that $\alpha(S,-K_S+\lambda C)\leqslant\frac{2}{1+2\lambda}$. Similarly, we see that $\alpha(S,-K_S+\lambda C)\leqslant\alpha(S)$. If $|C\cap\widetilde{C}|=1$, then the log pair

$$\left(S,\frac{2+4\lambda}{3+3\lambda}C+\frac{2}{4+3\lambda}\widetilde{C}\right)$$

is not Kawamata log terminal at the point $C\cap\widetilde{C}$, so that $\alpha(S,-K_S+\lambda C)\leqslant\frac{4}{3+3\lambda}$. □

Thus, to complete the proof of Theorem 7, we have to show that $\alpha(S,-K_S+\lambda C)\geqslant\mu$. Suppose that $\alpha(S,-K_S+\lambda C)<\mu$. Let us seek for a contradiction.

Since $\alpha(S,-K_S+\lambda C)<\mu$, there exists an effective $\mathbb{Q}$-divisor D on S such that

$$D\sim_{\mathbb{Q}}-K_S+\lambda C,$$

and $(S,\mu D)$ is not log canonical at some point $P\in S$.

By Lemma 14 and (7), we may assume that $\mathrm{Supp}(D)$ does not contain C or $\widetilde{C}$. Indeed, one can check that the log pair $(S,\mu(\frac{1}{2}+\lambda)C+\frac{\mu}{2}\widetilde{C})$ is log canonical at P.

Let $\mathcal{C}$ be a curve in the pencil $|-K_S|$ that passes through P. Then $\mathcal{C}+\lambda C\sim-K_S+\lambda C$. Moreover, the curve $\mathcal{C}$ is irreducible, and the log pair $(S,\mu\mathcal{C}+\mu\lambda C)$ is log canonical at P. Thus, we may assume that $\mathrm{Supp}(D)$ does not contain C or $\mathcal{C}$ by Lemma 14.

Lemma 33. *The curve* $\mathcal{C}$ *is smooth at the point* P*.*

Proof. Suppose that $\mathcal{C}$ is singular at P. If $\mathcal{C} \not\subseteq \mathrm{Supp}(D)$, then Theorem 15 gives

$$1+\lambda = \mathcal{C}\cdot\Big(-K_S+\lambda C\Big) = \mathcal{C}\cdot D \geqslant \mathrm{mult}_P\big(\mathcal{C}\big)\,\mathrm{mult}_P\big(D\big) \geqslant 2\,\mathrm{mult}_P\big(D\big) > \frac{2}{\mu},$$

which is impossible by (8). Thus, we have $\mathcal{C} \subseteq \mathrm{Supp}(D)$. Then $C \not\subseteq \mathrm{Supp}(D)$.

Write $D = \epsilon\mathcal{C}+\Delta$, where ϵ is a positive rational number, and Δ is an effective $\mathbb{Q}$-divisor on the surface S whose support does not contain the curves $\mathcal{C}$ and C. Then

$$1-\lambda = C\cdot\Big(-K_S+\lambda C\Big) = C\cdot D = C\cdot\Big(\epsilon\mathcal{C}+\Delta\Big) = \epsilon + C\cdot\Delta \geqslant \epsilon,$$

so that $\epsilon \leqslant 1-\lambda$. Similarly, we have

$$1+\lambda-\epsilon = \mathcal{C}\cdot\Delta \geqslant \big(\mathcal{C}\cdot\Delta\big)_P. \tag{9}$$

We claim that $\lambda \leqslant \frac{1}{2}$. Indeed, suppose that $\lambda > \frac{1}{2}$. Then it follows from (9) that

$$\big(\Delta\cdot\mathcal{C}\big)_P \leqslant 1+\lambda-\epsilon = \frac{1+2\lambda}{2}\left(\frac{4}{3}+\frac{\frac{4-4\lambda}{1+2\lambda}}{6}-\frac{2}{1+2\lambda}\epsilon\right).$$

Thus, we can apply Lemma 23 to the log pair $(S, \frac{2}{1+2\lambda}D)$ with $x = \frac{4-4\lambda}{1+2\lambda}$ and $a = \frac{2}{1+2\lambda}\epsilon$. This implies that $(S, \frac{2}{1+2\lambda}D)$ is log canonical at P, which is impossible, because $\mu \leqslant \frac{2}{1+2\lambda}$.

If $\mathcal{C}$ has a node at P, then we can apply Lemma 24 to (S,D) with $x = 2\lambda$ and $a = \epsilon$. This implies that (S,D) is log canonical, which is absurd, since $\mu \leqslant 1$.

Therefore, the curve $\mathcal{C}$ has an ordinary cusp at P and $\lambda \leqslant \frac{1}{2}$. Then $\mu \leqslant \alpha(S) = \frac{5}{6}$. Thus, we can apply Lemma 23 to the log pair $(S, \frac{5}{6}D)$ with $x = \frac{5}{3}\lambda$ and $a = \frac{5}{6}\epsilon$, since

$$\big(\Delta\cdot\mathcal{C}\big)_P \leqslant \frac{6}{5}\left(\frac{5}{6}+\frac{5}{6}\lambda-\frac{5}{6}\epsilon\right).$$

This implies that $(S, \frac{5}{6}D)$ is log canonical at P, which is impossible, since $\mu \leqslant \frac{5}{6}$. □

The next step in the proof of Theorem 7 is

Lemma 34. *The point P is not contained in the curve C.*

Proof. Suppose that $P \in C$. Let us seek for a contradiction. If $C \not\subseteq \mathrm{Supp}(D)$, then

$$1-\lambda = C\cdot\Big(-K_S+\lambda C\Big) = C\cdot D \geqslant \mathrm{mult}_P\big(C\big)\,\mathrm{mult}_P\big(D\big) \geqslant \mathrm{mult}_P\big(D\big) > \frac{1}{\mu}$$

by Theorem 15. But (8) implies that $\mu > \frac{1}{1-\lambda}$, which is impossible, because $\mu \leqslant 1$. Therefore, we must have $C \subseteq \mathrm{Supp}(D)$. Then $\mathcal{C} \not\subseteq \mathrm{Supp}(D)$ and also $\widetilde{C} \not\subseteq \mathrm{Supp}(D)$.

Write $D = \epsilon C + \Delta$, where ϵ is a positive rational number, and Δ is an effective divisor whose support does not contain $\mathcal{C}$, C and $\widetilde{C}$. Then $1+\lambda-\epsilon =$

$\mathcal{C} \cdot \Delta \geqslant \mathrm{mult}_P(\Delta)$. Similarly, we have $1 + 3\lambda - 3\epsilon = \widetilde{C} \cdot \Delta \geqslant 0$. Finally, we have $1 - \lambda + \epsilon = C \cdot \Delta \geqslant (C \cdot \Delta)_P$.

If $\lambda \leqslant \frac{1}{2}$, we can apply Lemma 25 to the log pair (S, D) with $x = 2\lambda$ and $a = \epsilon$. This implies that (S, D) is log canonical, which is impossible since $\mu \leqslant 1$.

Therefore, we have $\lambda > \frac{1}{2}$. Since $\epsilon \leqslant \frac{1}{3} + \lambda$, we have $\frac{2}{1+2\lambda}\epsilon \leqslant \frac{2}{1+2\lambda}(\frac{1}{3} + \lambda) = \frac{8}{9} - \frac{\frac{4-4\lambda}{1+2\lambda}}{18}$. Since $\epsilon + \mathrm{mult}_P(\Delta) \leqslant 1 + \lambda$, we have $\frac{2}{1+2\lambda}\epsilon + \frac{2}{1+2\lambda}\mathrm{mult}_P(\Delta) \leqslant \frac{2}{1+2\lambda}(1 + \lambda) = \frac{4}{3} + \frac{\frac{4-4\lambda}{1+2\lambda}}{6}$. But

$$\big(\Delta \cdot C\big)_P \leqslant 1 - \lambda + \epsilon = \frac{1+2\lambda}{2}\left(\frac{\frac{4-4\lambda}{1+2\lambda}}{2} + \frac{2}{1+2\lambda}\epsilon\right).$$

Thus, we can apply Lemma 26 to the log pair $(S, \frac{2}{1+2\lambda}D)$ with $x = \frac{4-4\lambda}{1+2\lambda}$ and $a = \frac{2}{1+2\lambda}\epsilon$. This implies that $(S, \frac{2}{1+2\lambda}D)$ is log canonical at P, which is impossible, since $\mu \leqslant \frac{2}{1+2\lambda}$. □

Let $h\colon S \to \overline{S}$ be the contraction of the curve C. Put $\overline{D} = h(D)$. Then $\overline{D} \sim_{\mathbb{Q}} -K_{\overline{S}}$. Moreover, it follows from Lemma 34 that $(\overline{S}, \mu\overline{D})$ is not log canonical at the point $h(P)$.

By construction, the surface $\overline{S}$ is a smooth del Pezzo surface such that $K_{\overline{S}}^2 = K_S^2 + 1 = 2$. Then $|-K_{\overline{S}}|$ gives a double cover $\pi\colon \overline{S} \to \mathbb{P}^2$ branched in a smooth quartic curve $R_4 \subset \mathbb{P}^2$. By Lemma 18, there exists a unique curve $\overline{Z} \in |-K_{\overline{S}}|$ such that $\overline{Z}$ is singular at $h(P)$. Moreover, the log pair $(\overline{S}, \overline{Z})$ is not log canonical at the point $h(P)$ by [4, Theorem 1.12]. Note that $\pi(\overline{Z})$ is the line in $\mathbb{P}^2$ that is tangent to the curve R_4 at the point $\pi \circ h(P)$.

Let Z be the proper transform of the curve $\overline{Z}$ on the surface S. Then $h(C) \notin \overline{Z}$. Indeed, if $h(C)$ is contained in $\overline{Z}$, then $Z \sim -K_S$, which is impossible by Lemma 33. Thus, we see that $C \cap Z = \varnothing$. Then $Z \sim -K_S + C$.

Lemma 35. *The curve Z is reducible.*

Proof. Suppose that Z is irreducible. Then Z has an ordinary node or ordinary cusp at P. In fact, if $Z \not\subseteq \mathrm{Supp}(D)$, then $2 = Z \cdot D > \frac{2}{\mu}$ by Theorem 15, which contradicts to (8). Therefore, we have $Z \subseteq \mathrm{Supp}(D)$. Put $\widetilde{Z} = \tau(Z)$. Then $Z + \widetilde{Z} \sim -4K_S$ and

$$\frac{3\lambda+1}{4}Z + \frac{1-\lambda}{4}\widetilde{Z} \sim_{\mathbb{Q}} \frac{1-\lambda}{4}\Big(Z + \widetilde{Z}\Big) + \lambda Z \sim_{\mathbb{Q}} -K_S + \lambda C.$$

Furthermore, one can show (using Definition 13) that the log pair

$$\left(S, \mu\frac{3\lambda+1}{4}Z + \mu\frac{1-\lambda}{4}\widetilde{Z}\right)$$

is log canonical at P. Hence, we may assume that $\widetilde{Z} \not\subseteq \mathrm{Supp}(D)$ by Lemma 14.

Write $D = \epsilon Z + \Delta$, where ϵ is a positive rational number, and Δ is an effective $\mathbb{Q}$-divisor on the surface S whose support does not contain Z and $\widetilde{Z}$. Then $2+4\lambda-6\epsilon = \widetilde{Z}\cdot\Delta \geqslant 0$. Thus, we have $\epsilon \leqslant \frac{1+2\lambda}{3}$. Finally, we have

$$2-2\epsilon = Z\cdot\Delta \geqslant \big(Z\cdot\Delta\big)_P.$$

Therefore, if $\lambda \leqslant \frac{1}{2}$, then we can apply Lemma 27 to (S, D) with $x = 2\lambda$ and $a=\epsilon$. This implies that (S, D) is log canonical at P. But $\mu \leqslant 1$. Thus, we have $\lambda > \frac{1}{2}$.

We have $\mu \leqslant \frac{2}{1+2\lambda}$. Then $(S, \frac{2}{1+2\lambda}D)$ is not log canonical at P. We have $\frac{2}{1+2\lambda}\epsilon \leqslant \frac{2}{3}$. Thus, we can apply Lemma 28 to $(S, \frac{2}{1+2\lambda}D)$ with $x = \frac{4-4\lambda}{1+2\lambda}$ and $a = \frac{2}{1+2\lambda}\epsilon$, because

$$\big(\Delta\cdot Z\big)_P \leqslant \frac{1+2\lambda}{2}\left(\frac{4}{3}+\frac{2\frac{4-4\lambda}{1+2\lambda}}{3}-2\frac{2}{1+2\lambda}\epsilon\right) = 2-2\epsilon.$$

This implies that $(S, \frac{2}{1+2\lambda}D)$ is log canonical at P, which is absurd, since $\mu \leqslant \frac{2}{1+2\lambda}$. □

Since Z is reducible, $Z = Z_1 + Z_2$, where Z_1 and Z_2 are smooth irreducible curves. Then $Z_1^2 = Z_2^2 = -1$ and $Z_1\cdot Z_2 = 2$. Moreover, we have $P \in Z_1\cap Z_2$ and $(Z_1\cdot Z_2)_P \leqslant 2$. Furthermore, we have $Z_1\cap C = \varnothing$ and $Z_2 \cap C = \varnothing$.

We have $Z_1 \subseteq \mathrm{Supp}(D)$ and $Z_2 \subseteq \mathrm{Supp}(D)$. Indeed, if $Z_1 \not\subseteq \mathrm{Supp}(D)$, then

$$1 = Z_1\cdot\Big(-K_S+\lambda C\Big) = Z_1\cdot D \geqslant \mathrm{mult}_P\big(Z_1\big)\,\mathrm{mult}_P\big(D\big) \geqslant \mathrm{mult}_P\big(D\big) > \frac{1}{\mu} \geqslant 1$$

by Theorem 15. This shows that $Z_1 \subseteq \mathrm{Supp}(D)$. Similarly, we have $Z_2 \subseteq \mathrm{Supp}(D)$. But

$$\big(1-\lambda\big)\mathcal{C} + \lambda\big(Z_1+Z_2\big) \sim_{\mathbb{Q}} -K_S+\lambda C.$$

On the other hand, the log pair $(S, \mu(1-\lambda)\mathcal{C}+\mu\lambda(Z_1+Z_2))$ is log canonical at P. Therefore, we may assume that $\mathcal{C} \not\subseteq \mathrm{Supp}(D)$ by Lemma 14.

Put $\widetilde{Z}_1 = \tau(Z_1)$ and put $\widetilde{Z}_2 = \tau(Z_2)$. Then $Z_1+\widetilde{Z}_1 \sim -2K_S$ and $Z_2+\widetilde{Z}_2 \sim -2K_S$. This gives $\mathcal{C}\cdot Z_1 = \mathcal{C}\cdot Z_2 = 1$, $Z_1\cdot\widetilde{Z}_1 = Z_2\cdot\widetilde{Z}_2 = 3$, $Z_1\cdot\widetilde{Z}_2 = Z_2\cdot\widetilde{Z}_1 = 0$, $\widetilde{Z}_1\cdot C = \widetilde{Z}_2\cdot C = 2$. Moreover, we have $Z_1+Z_2 \sim -K_S + C$. Then

$$\frac{1+\lambda}{2}Z_1 + \lambda Z_2 + \frac{1-\lambda}{2}\widetilde{Z}_1 \sim_{\mathbb{Q}} \frac{1-\lambda}{2}\Big(Z_1+\widetilde{Z}_1\Big) + \lambda\Big(Z_1+Z_2\Big) \sim_{\mathbb{Q}} -K_S+\lambda C$$

Note that $P \notin \widetilde{Z}_1$, because $P \in Z_2$ and $\widetilde{Z}_1\cdot Z_2 = 0$. Using this, we see that the log pair

$$\left(S, \mu\frac{1+\lambda}{2}Z_1 + \mu\lambda Z_2 + \mu\frac{1-\lambda}{2}\widetilde{Z}_1\right)$$

is log canonical at the point P. Hence, we may assume that $\widetilde{Z}_1 \not\subseteq \mathrm{Supp}(D)$ by Lemma 14. Similarly, we may assume that $\widetilde{Z}_2 \not\subseteq \mathrm{Supp}(D)$ using Lemma 14 one more time.

Now let us write $D = \epsilon_1 Z_1 + \epsilon_2 Z_2 + \Delta$, where ϵ_1 and ϵ_2 are positive rational numbers, and Δ is an effective divisor whose support does not contain Z_1 and Z_2. Then

$$1 + \lambda - \epsilon_1 - \epsilon_2 = \mathcal{C} \cdot \Delta \geqslant \mathrm{mult}_P\big(\Delta\big).$$

This gives $\epsilon_1 + \epsilon_2 + \mathrm{mult}_P(\Delta) \leqslant 1 + \lambda$. We also have $\epsilon_1 \leqslant \frac{1+2\lambda}{3}$, since

$$1 + 2\lambda - 3\epsilon_1 = \widetilde{Z}_1 \cdot \Delta \geqslant 0.$$

Similarly, see that $\epsilon_2 \leqslant \frac{1+2\lambda}{3}$. Moreover, we have

$$1 + \epsilon_1 - 2\epsilon_2 = Z_1 \cdot \Delta \geqslant \big(Z_1 \cdot \Delta\big)_P.$$

Finally, we have

$$1 + \epsilon_2 - 2\epsilon_1 = Z_2 \cdot \Delta \geqslant \big(Z_2 \cdot \Delta\big)_P.$$

Thus, if $\lambda \leqslant \frac{1}{2}$, then we can apply Lemma 29 to (S, D) with $x = 2\lambda$, $a = \epsilon_1$ and $b = \epsilon_1$. This implies that (S, D) is log canonical at P, which is absurd. Hence, we have $\lambda > \frac{1}{2}$.

Since $\lambda > \frac{1}{2}$, we have $\mu \leqslant \frac{2}{1+2\lambda}$. Then the log pair $(S, \frac{2}{1+2\lambda}D)$ is not log canonical at P. On the other hand, we have $\frac{2}{1+2\lambda}\epsilon_1 \leqslant \frac{2}{3}$ and $\frac{2}{1+2\lambda}\epsilon_2 \leqslant \frac{2}{3}$. We also have

$$\begin{aligned}\frac{2}{1+2\lambda}\epsilon_1 + \frac{2}{1+2\lambda}\epsilon_2 + \frac{2}{1+2\lambda}\,\mathrm{mult}_P(\Delta) &\leqslant \frac{2}{1+2\lambda}\big(1+\lambda\big) \\ &= \frac{2}{1+2\lambda} + \lambda\frac{2}{1+2\lambda} = \frac{4}{3} + \frac{\frac{4-4\lambda}{1+2\lambda}}{6},\end{aligned}$$

Moreover, we have

$$\big(\Delta \cdot Z_1\big)_P \leqslant 1 + \epsilon_1 - 2\epsilon_2 = \frac{1+2\lambda}{2}\left(\frac{2}{3} + \frac{\frac{4-4\lambda}{1+2\lambda}}{3} + \frac{2}{1+2\lambda}\epsilon_1 - 2\frac{2}{1+2\lambda}\epsilon_2\right).$$

Furthermore, we also have

$$\big(\Delta \cdot Z_2\big)_P \leqslant 1 + \epsilon_1 - 2\epsilon_2 = \frac{1+2\lambda}{2}\left(\frac{2}{3} + \frac{\frac{4-4\lambda}{1+2\lambda}}{3} + \frac{2}{1+2\lambda}\epsilon_2 - 2\frac{2}{1+2\lambda}\epsilon_1\right).$$

Thus, we can apply Lemma 30 to $(S, \frac{2}{1+2\lambda}D)$ with $x = \frac{4-4\lambda}{1+2\lambda}$, $a = \frac{2}{1+2\lambda}\epsilon_1$ and $b = \frac{2}{1+2\lambda}\epsilon_2$. This implies that $(S, \frac{2}{1+2\lambda}D)$ is log canonical at P, which is absurd.

The obtained contradiction completes the proof of Theorem 7.

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