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Log Pairs on Hypersurfaces of Degree N in $\mathbb{P}^N$

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ABSTRACT. The objective of this paper is to study the birational structure of smooth hypersurfaces of degree N in $\mathbb{P}^N$ by examining properties of moving log pairs on them.

KEY WORDS: moving log pair, birational structure of smooth hypersurfaces, algebraic variety, canonical singularities, Kodaira dimension.

All the varieties under consideration are projective and defined over the field $\mathbb{C}$, unless otherwise stated. The basic definitions, notions and notations are contained in [1].

1. Introduction

By a *moving log pair*

$$(X, M_X) = \left(X, \sum_{i=1}^{n} b_i \mathcal{M}_i\right),$$

we mean a variety X together with a formal finite linear combination of linear systems $\mathcal{M}_i$ without fixed components such that all the coefficients b_i belong to $\mathbb{Q}_{\geq 0}$.

Discrepancy, terminality, canonicity, the Iitaka map $I(X, M_X)$, and the Kodaira dimension $\kappa(X, M_X)$ are defined for moving log pairs (X, M_X) similarly to the corresponding notions for usual log pairs (see [1, 2]).

We say that an irreducible subvariety $Y \subset X$ is a *center of canonical singularities of a moving log pair* (X, M_X) if there exist a birational morphism $f\colon W \to X$ and an f-exceptional divisor $E \subset W$ such that $a(X, M_X, E) \leq 0$ and $f(E) = Y$. The set of all centers of canonical singularities of a moving log pair (X, M_X) is denoted by $CS(X, M_X)$.

In what follows, we refer to moving log pairs briefly as log pairs.

From now on, X denotes a sufficiently general smooth hypersurface of degree N in $\mathbb{P}^N$ for $N \geq 5$. Note that we then have

$$\operatorname{Pic}(X) = -\mathbb{Z}K_X \quad \text{and} \quad -K_X \sim \mathcal{O}_{\mathbb{P}^N}(1)|_X.$$

Consider $\lambda \in \mathbb{Q}_{>0} \cup \{+\infty\}$ such that $K_X + \lambda M_X \sim_{\mathbb{Q}} 0$; for $M_X = \varnothing$, $\lambda = +\infty$.

The main result of this paper is the following theorem.

Theorem 1. *Let $\lambda = 1$. Then the log pair (X, M_X) is canonical, $\kappa(X, M_X) = 0$, and*

$$CS(X, M_X) = \begin{cases} \varnothing, \\ X \cap H & \text{for a linear space } H \text{ of dimension } N-2. \end{cases}$$

We prove Theorem 1 in Secs. 3 and 4; in Sec. 5, we derive the following important result, which is a corollary and a refinement of Theorem 1.

Theorem 2. *If $\lambda = 1$ and $CS(X, M_X) \neq \varnothing$, then the boundary M_X can be lifted from $\mathbb{P}^1$ by a rational mapping $\varphi_{\mathcal{P}}$ for some pencil $\mathcal{P}$ in $|-K_X|$ such that $CS(X, M_X) = \{Bs(\mathcal{P})\}$.*

In Sec. 6, we apply Theorems 1 and 2 to prove the following two theorems concerning log pairs with $\lambda \neq 1$.

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Theorem 3. *If $\lambda < 1$ and $\kappa(X, M_X) \neq N-1$, then the log pair (X, M_X) is not canonical, $\kappa(X, M_X) = 1$, and there exists a pencil $\mathcal{P}$ in the linear system $|-K_X|$ such that the boundary M_X can be lifted from $\mathbb{P}^1$ by the rational mapping $\varphi_{\mathcal{P}}$ coinciding with $I(X, M_X)$.*

Theorem 4. *If $\lambda > 1$, then $\kappa(X, M_X) = -\infty$ and the log pair (X, M_X) is terminal.*

The main applications of Theorems 1–4 are described in Sec. 2.

2. The birational geometry of the hypersurface X

In this section, we describe the main applications of Theorems 2–4.

Recall that X is a sufficiently general smooth hypersurface of degree N in $\mathbb{P}^N$ for $N \geq 5$.

Theorem 5. *The hypersurface X is birationally nonisomorphic to a fibration into varieties of Kodaira dimension $-\infty$.*

We assume that all fibrations have connected fibers, they are not birational, and their bases are not points.

Proof. Suppose that there exists a birational surgery ρ of the hypersurface X into a fibration $\tau\colon Y \to Z$ such that the Kodaira dimension of its general fiber equals $-\infty$. We put $\mathcal{H} = |\tau^*(H)|$ for a "sufficiently large" very ample divisor H on the variety Z. Take $\mu \in \mathbb{Q}_{>0}$ such that the log pair $(X, M_X) = (X, \mu\rho^{-1}(\mathcal{H}))$ satisfies the relation

$$K_X + M_X \sim_{\mathbb{Q}} 0.$$

The log pair (X, M_X) constructed is not terminal, because otherwise, for small $\alpha \in \mathbb{Q}_{>0}$, we have

$$N - 1 = \kappa(X, (1+\alpha)M_X) = -\infty.$$

The required assertion now follows from Theorem 2. $\square$

Theorem 6. *The hypersurface X is not birationally isomorphic to any Fano variety with canonical singularities other than X and $\mathrm{Bir}(X) = \mathrm{Aut}(X)$.*

Proof. Suppose that there exists a birational map $\rho\colon X \dashrightarrow Y$ such that Y is a Fano variety with canonical singularities. Let us show that ρ is then an isomorphism.

Take $n \in \mathbb{Z}_{\gg 0}$ and consider the log pair

$$(Y, M_Y) = \Big(Y, \frac{1}{n}|-nK_Y|\Big).$$

For $M_X = \rho^{-1}(M_Y)$, we have $\kappa(X, M_X) = 0$. Take $\lambda \in \mathbb{Q} \cap (0, 1]$ such that

$$K_X + \lambda M_X \sim_{\mathbb{Q}} 0.$$

Theorem 2 readily implies that the log pair $(X, \lambda M_X)$ is terminal.

Suppose that $\lambda < 1$. Consider $\delta \in \mathbb{Q} \cap (\lambda, 1)$ such that the log pair $(X, \delta M_X)$ is terminal. We have

$$N - 1 = \kappa(X, \delta M_X) \leq \kappa(X, M_X) = 0.$$

Thus $\lambda = 1$.

Let us resolve the indeterminacies of the rational map ρ by means of the commutative diagram

$$\begin{array}{ccc} & W & \\ {}^{f}\swarrow & & \searrow^{g} \\ X & \overset{\rho}{\dashrightarrow} & Y, \end{array}$$

where W is a smooth variety. Then

$$\sum_{j=1}^{k} a(X, M_X, F_j)F_j \sim_{\mathbb{Q}} g^*(K_Y + M_Y) + \sum_{i=1}^{l} a(Y, M_Y, G_i)G_i,$$

where G_i and F_j are exceptional divisors for the morphisms g and f, respectively. Lemma 2.19 from [3] implies that

$$a(X, M_X, E) = a(Y, M_Y, E)$$

for all divisors E on the variety W. In particular, the log pair (Y, M_Y) is terminal, and there exists $\zeta \in \mathbb{Q}_{>1}$ such that both log pairs $(X, \zeta M_X)$ and $(Y, \zeta M_Y)$ are canonical models. Since the canonical model is unique, the map ρ is an isomorphism. □

Theorem 7. *All fibrations into varieties birationally isomorphic to X and having Kodaira dimension zero are birationally equivalent to a fibration into hypersurfaces of degree N in $\mathbb{P}^{N-1}$ associated to a pencil of hyperplane sections of X.*

Proof. Suppose that there exists a birational surgery ρ of the hypersurface X into a fibration $\tau: Y \to Z$ such that the Kodaira dimension of its general fiber equals 0. We must show that $\tau \circ \rho = \varphi_{\mathcal{P}}$ for some pencil $\mathcal{P}$ in $|-K_X|$.

Consider the complete linear system $\mathcal{H} = |\tau^*(H)|$ for a "sufficiently large" very ample divisor H on the variety Z. The log pairs $(X, M_X) = (X, \rho^{-1}(\mathcal{H}))$ satisfy the equality

$$\kappa(X, M_X) = \dim(Z).$$

It remains to apply Theorems 2–4. □

3. Proof of Theorem 1, part I

We use the notation of Sec. 1 and assume that $\lambda = 1$. The main result this section is the following theorem.

Theorem 8. *The set $CS(X, M_X)$ contains no points.*

Suppose that $CS(X, M_X)$ contains a smooth point O.

We abandon our convention that all log pairs under consideration are moving. Hopefully, this will cause no confusion, because it will always be clear whether or not a log pair is moving.

We need the following version of Theorem 3.1 from [4].

Lemma 1. *If a moving log pair (H, M_H) is not log canonical at a smooth point O on the surface, then* $\mathrm{mult}_P(M_H^2) > 4$.

The following result is usually called the Iskovskikh–Pukhlikov inequality.

Lemma 2. *The inequality* $\mathrm{mult}_O(M_X^2) > 4$ *holds.*

Proof. Let H be a sufficiently general very ample divisor on X containing the point O. Then

$$\mathrm{mult}_O(M_X^2) = \mathrm{mult}_O\big((M_X|_H)^2\big)$$

and $O \in LCS(X, H + M_X)$, where LCS denotes the set of centers of log canonical singularities (see [4]). Shokurov's connectedness theorem (see [4]) implies that $O \in LCS(H, M_X|_H)$.

Repeating the construction described above, we can assume that X is two-dimensional and the log pair (X, M_X) is not log canonical at the point O. Now the required assertion follows from Lemma 1. □

Proof of Theorem 8. It follows from the results obtained in [5] and Lemma 2. □

4. Proof of Theorem 1, part II

In this section, we complete the proof of Theorem 1. We use the notation of Sec. 1 and assume that $\lambda = 1$.

By virtue of the results of the preceding section, we can assume that $CS(X, M_X)$ contains a variety S of nonzero dimension. The results obtained in [6] imply that $\mathrm{mult}_S(M_X) = 1$; in particular, the log pair (X, M_X) is canonical.

Lemma 3. *The equality* $\dim(S) = N - 3$ *holds.*

Proof. Let $f\colon W \to X$ be a blow-up of a general point of the variety S. We can assume that the variety W is quasi-projective. Then

$$a(X, M_X, E) = N - 2 - \dim(S) - \mathrm{mult}_S(M_X) = N - 3 - \dim(S).$$

If the assertion of the lemma does not hold, then there exists a variety $T \subset E$ such that the morphism $f|_T\colon T \to S$ is surjective and

$$T \in CS(W, f^{-1}(M_X) - a(X, M_X, E)E);$$

in particular,

$$\mathrm{mult}_S(M_X) \geq \mathrm{mult}_T\big(f^{-1}(M_X)\big) > 1. \quad \square$$

Thus we can assume that $\dim(S) = N - 3$.

Lemma 4. *The inequality* $\deg(S) \leq N$ *holds.*

Proof. We have $N = (-K_X)^{N-3} \cdot M_X^2 \geq \mathrm{mult}_S(M_X^2)\deg(S)$. $\square$

We can assume that $\deg(S) \neq 1$. The proof of the following lemma is the main technical difficulty in this section.

Lemma 5. *The variety S lies in a linear space of dimension* $N - 2$.

Proof. Considering intersections with sufficiently general hyperplane sections, we can assume that X is a hypersurface of degree N in $\mathbb{P}^4$ and contains a curve S such that

$$\mathrm{mult}_S(M_X) = 1 \quad \text{and} \quad M_X \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}^4}(1)|_X.$$

We must show that the curve S is contained in a plane. We assume the converse and obtain a contradiction by using a trick from [6].

Consider a sufficiently general cone R_S over a curve S. We have

$$R_S \cdot X = S \cup \widetilde{S} \quad \text{and} \quad \deg(\widetilde{S}) = (N-1)\deg(S).$$

The generality of the cone R_S implies that

$$\widetilde{S} \not\subset \bigcup_{i=1}^{n} Bs(\mathcal{M}_i)$$

and the curves S and $\widetilde{S}$ have $(N-1)\deg(S)$ different intersection points (see [6]).

On the other hand,

$$(N-1)\deg(S) = \deg(\widetilde{S}) = \deg(M_X|_{\widetilde{S}}) \geq (N-1)\deg(S)\,\mathrm{mult}_S(M_X) = (N-1)\deg(S).$$

Therefore, the curve $\widetilde{S}$ only intersects the boundary M_X at points of $S \cap \widetilde{S}$.

Note that the proof of the last inequality does not use the assumption that the boundary M_X is moving. In particular, there exist no hyperplanes tangent to the hypersurface X along the curve S. This implies that the general secant of the curve S intersects X at precisely N points, because otherwise, it should be contained in X and must coincide with the curve S.

Consider the divisor

$$D = \sum_{i=1}^{n} b_i M_i,$$

where M_i is a sufficiently general divisor from the linear system $\mathcal{M}_i$. By assumption,

$$\mathrm{mult}_S(D) = 1.$$

Let us take two sufficiently general points P_S and P_D on the curve S and in the divisor D, respectively, and consider the straight line L through the points P_S and P_D and a sufficiently general point P on this line. Let $R_{S,P}$ be the cone over the curve S with vertex at the point P, and let

$$R_{S,P} \cdot X = S \cup \widetilde{S}_P.$$

As shown above, the curve $\widetilde{S}_P$ either is contained in the divisor D or intersects it only at points of $S \cap \widetilde{S}_P$. By construction, $P_D \in \widetilde{S}_P \cap D$ and $P_D \notin S$. Therefore, $\widetilde{S}_P \subset D$; in particular, $L \cap X \subset D$. Since the last condition is closed, we can assume that the point P_D belongs to the curve S but does not coincide with the point P_S. This implies that the general secant of the curve S intersects

$$\bigcup_{i=1}^{n} Bs(\mathcal{M}_i)$$

at N different points. On the other hand, the intersection points of the last set with the general hyperplane must be in general position, because this set contains the curve S. □

Proof of Theorem 1. By virtue of Theorem 8 and Lemmas 3 and 5, we can assume that $CS(X, M_X)$ contains a variety S of dimension $N-3$ lying in a linear space T of dimension $N-2$. Lemma 4 and the generality of the hypersurface X allow us to assume that $\deg(S) \in (1, N)$.

Consider the pencil $\mathcal{H}_T$ on X consisting of the varieties cut out by the hyperplanes that contain the linear space T. We have

$$X \cdot T = S \cup \sum_{i=1}^{r} S_i,$$

where S_i are irreducible reduced varieties on the hypersurface X (the reducedness of all the varieties S_i was implicitly obtained in the proof of Lemma 5). It is sufficient to show that all S_i are contained in $CS(X, M_X)$.

As in the proof of Lemma 5, considering intersections with sufficiently general hyperplane sections, we can assume that

$$\dim(X) = 3, \quad \deg(X) = N, \quad \dim(T) = 2, \quad \text{and} \quad \dim(S) = \dim(S_i) = 1 \quad \text{for } i = 1, \dots, r.$$

Under these assumptions, it suffices to show that

$$\mathrm{mult}_{S_i}(M_X) \geq 1$$

for all S_i.

Consider a smooth surface D from the pencil $\mathcal{H}_T$. Let us show that the intersection form of the curves S_i is negative definite on D. First, on the surface D, we have

$$\left(\sum_{i=1}^{r} S_i\right) \cdot S_j = (D|_D - S) \cdot S_j = \deg(S_j) - S \cdot S_j.$$

Secondly, on the plane T,

$$\deg(S_j) - S \cdot S_j = \deg(S_j) - \deg(S)\deg(S_j) < 0.$$

Thirdly, $(S \cdot S_j)_D = (S \cdot S_j)_T$, because all the curves S_j differ from S and the surface D is smooth. The results obtained in [7] imply the negative definiteness of the intersection form of S_i on D.

The divisor

$$M_X|_D - S - \sum_{i=1}^{r} \mathrm{mult}_{S_i}(M_X) S_i$$

is numerically effective on the surface D. On the other hand,

$$M_X|_D - S - \sum_{i=1}^{r} \mathrm{mult}_{S_i}(M_X) S_i \sim_{\mathbb{Q}} \sum_{i=1}^{r} (1 - \mathrm{mult}_{S_i}(M_X)) S_i,$$

and

$$\sum_{i=1}^{r} (1 - \mathrm{mult}_{S_i}(M_X)) S_i \cdot S_j \geq 0 \qquad \text{for } j = 1, \dots, r$$

on D. The fact that the intersection form of the curves S_i on D is negative definite implies that $\mathrm{mult}_{S_i}(M_X) \geq 1$ for all S_i. □

5. Log pairs with Kodaira dimension zero

In this section, we show how Theorem 2 is derived from Theorem 1.

Proof of Theorem 2. Suppose that the variety S is the union of all elements $CS(X, M_X)$. Theorem 1 implies that the dimension of S equals $N-3$ and S is contained in a linear space T of dimension $N-2$.

Consider the pencil $\mathcal{H}_T$ on X consisting of the hyperplane sections of X that contain the variety S. Let us resolve the indeterminacies of the rational map $\varphi_{\mathcal{H}_T}$ by means of the morphism $f\colon W \to X$, where W is a smooth variety; over a general point of each irreducible component of the variety S, exactly one divisor lies, and f is an isomorphism outside S. We put

$$g = \varphi_{\mathcal{H}_T} \circ f \quad \text{and} \quad E = f^{-1}(S).$$

Let D be the general fiber D of the morphism g. Then

$$D \sim f^*(-K_X) - E - \sum_{i=1}^{k} a_i F_i,$$

where all a_i belong to $\mathbb{N}$ and $\dim(f(F_i)) \leq N-4$ for all the divisors F_i. We have

$$f^{-1}(M_X)|_D \sim_{\mathbb{Q}} \sum_{i=1}^{k} c_i F_i|_D,$$

where all c_i belong to $\mathbb{Q}$, which implies that $f^{-1}(M_X)$ lies in fibers of the morphism g. □

6. Log pairs of nonzero Kodaira dimension

In this section, we derive Theorems 3 and 4 from Theorems 1 and 2.

Proof of Theorem 3. We assume that $\kappa(X, M_X) \neq N-1$. Theorem 1 implies that the log pair $(X, \lambda M_X)$ is canonical and $\kappa(X, \lambda M_X) = 0$. Therefore,

$$\kappa(X, M_X) \geq \kappa(X, \lambda M_X) \geq 0.$$

Suppose that the log pair $(X, \lambda M_X)$ is terminal. Take $\delta \in \mathbb{Q} \cap (\lambda, 1)$ such that the log pair $(X, \delta M_X)$ is also terminal. We have

$$N-1 = \kappa(X, \delta M_X) \leq \kappa(X, M_X) < N-1.$$

Therefore, $CS(X, \lambda M_X) \neq \varnothing$, and the required assertion readily follows from Theorems 1 and 2. □

Proof of Theorem 4. By Theorem 1, the log pair $(X, \lambda M_X)$ is canonical. Therefore, the log pair (X, M_X) is terminal, and $\kappa(X, M_X) = -\infty$. □

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