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Factoriality of nodal three-dimensional varieties and connectedness of the locus of log canonical singularities

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Abstract. Shokurov's vanishing theorem is used for the proof of the $\mathbb{Q}$-factoriality of the following nodal threefolds: a complete intersection of hypersurfaces F and G in $\mathbb{P}^5$ of degrees n and k, $n \geqslant k$, such that G is smooth and $|\mathrm{Sing}(F \cap G)| \leqslant (n+k-2)(n-1)/5$; a double cover of a smooth hypersurface $F \subset \mathbb{P}^4$ of degree n branched over the surface cut on F by a hypersurface $G \subset \mathbb{P}^4$ of degree $2r \geqslant n$, provided that $|\mathrm{Sing}(F \cap G)| \leqslant (2r + n - 2)r/4$.

Bibliography: 71 titles.

§ 1. Introduction

Recall that a Weil divisor is a $\mathbb{Q}$-Cartier divisor if some non-zero multiple of it is a Cartier divisor and a variety [1] has $\mathbb{Q}$-factorial singularities if each Weil divisor on it is a $\mathbb{Q}$-Cartier divisor; a variety is $\mathbb{Q}$-factorial if its singularities are $\mathbb{Q}$-factorial. In particular, smooth varieties are $\mathbb{Q}$-factorial.

The birational geometry of many singular varieties depends crucially on the condition of $\mathbb{Q}$-factoriality. For example, all $\mathbb{Q}$-factorial nodal [2] (see [1]–[4]) and all $\mathbb{Q}$-factorial double covers of $\mathbb{P}^3$ branched over nodal sextic surfaces are non-rational (see [5]–[7]). Of course, both results fail without the global topological condition of $\mathbb{Q}$-factoriality.

Example 1. As is well known, a nodal quartic threefold in $\mathbb{P}^4$ has at most 45 singular points (see [8], [9]). One can show that there exist nodal quartic threefolds with an arbitrary number of singular points between 0 and 45 (see [9]), and there exists a unique (see [10]) nodal quartic threefold $\mathscr{B}_4$ with 45 singular points, which is called the Burkhardt quartic (see [11]–[14]) and can be defined by the equation

$$w^4 - w(x^3 + y^3 + z^3 + t^3) + 3xyzt = 0 \subset \mathbb{P}^4 \cong \mathrm{Proj}\,(\mathbb{C}[x, y, z, t, w]),$$

so that it is determinantal and rational. The quartic $\mathscr{B}_4$ is the unique invariant of degree 4 of the simple group $\mathrm{PSp}(4, \mathbb{Z}_3)$ of order 25920 (see [15]–[18]), and singular points of $\mathscr{B}_4$ correspond to the 45 tritangents of a smooth cubic surface, which is related to the fact that the Weil group E_6 is a non-trivial extension of the group $\mathrm{PSp}(4, \mathbb{Z}_3)$ by $\mathbb{Z}_2$. It is easy to see that the quartic $\mathscr{B}_4$ contains a plane, which is not a Cartier divisor because the plane is not cut on $\mathscr{B}_4$ by a hypersurface in $\mathbb{P}^4$. On the other hand the local class group of an ordinary double point is $\mathbb{Z}$, therefore

[1] All varieties are assumed to be projective, normal, and defined over $\mathbb{C}$.

[2] A variety is said to be *nodal* if all its singularities are isolated ordinary double points.

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no non-zero multiple of the plane lying in $\mathscr{B}_4$ is a Cartier divisor either. Hence the singularities of $\mathscr{B}_4$ are not $\mathbb{Q}$-factorial. Moreover, it follows from [17] that $\mathrm{Cl}(\mathscr{B}_4) \cong \mathbb{Z}^{16}$, whereas $\mathrm{Pic}(\mathscr{B}_4) \cong \mathbb{Z}$ by Lefschetz's theorem (see [19], [20]).

Example 2. Let $\pi\colon X \to \mathbb{P}^3$ be a double cover ramified in the Barth sextic surface

$$\begin{aligned}4(\tau^2x^2-y^2)(\tau^2y^2-z^2)(\tau^2z^2-x^2)-t^2(1+2\tau)(x^2+y^2+z^2-t^2)^2=0\\ \subset \mathbb{P}^3 \cong \operatorname{Proj}(\mathbb{C}[x,y,z,t]),\end{aligned}$$

where $\tau = (1+\sqrt{5})/2$. Then X is nodal and $|\mathrm{Sing}(X)| = 65$ (see [21]). One can show that a nodal sextic in $\mathbb{P}^3$ has at most 65 singular points (see [22], [23]), and there exist nodal sextics in $\mathbb{P}^3$ with an arbitrary number of singular points between 0 and 65 (see [24]), so that X has the maximum possible number of singular points. Moreover, there exists a determinantal quartic threefold $Y \subset \mathbb{P}^4$ with 42 ordinary double points such that the diagram

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, dashed, "\rho"'] & \mathbb{P}^4 \arrow[d, dashed, "\gamma"] \\ X \arrow[r, "\pi"'] & \mathbb{P}^3 \end{tikzcd}$$

is commutative (see [25], [14]), where ρ is a birational map and γ is the projection from an ordinary double point of Y. Hence X is rational because determinantal quartics are rational. The rational map ρ is a composite of the blow-up of a singular point of the quartic Y and a subsequent blow-down of the proper transforms of 24 lines on the quartic Y passing through the singular point blown up. A non-zero multiple of the image of the exceptional divisor of the blow-up of the singular point of Y cannot be a Cartier divisor on Y, so that X is not $\mathbb{Q}$-factorial. Furthermore, one can show that $\mathrm{Pic}(X) \cong \mathbb{Z}$ and $\mathrm{Cl}(X) \cong \mathbb{Z}^{14}$ (see [25]).

It is therefore natural to ask how the global topological condition of being $\mathbb{Q}$-factorial depends on the number of singular points of a nodal threefold. To illustrate the general picture we consider nodal hypersurfaces. Let V be a nodal hypersurface in $\mathbb{P}^4$ of degree n with at most ordinary double points. Then V is $\mathbb{Q}$-factorial if and only if

$$\operatorname{rk} H^2(V,\mathbb{Z}) = \operatorname{rk} H_4(V,\mathbb{Z}),$$

which always holds in the smooth case in view of the Poincaré duality. Moreover, the following important result holds (see [26]–[29]).

Proposition 3. *The hypersurface V is $\mathbb{Q}$-factorial if and only if its singular points impose independent linear conditions on global sections of the sheaf $\mathscr{O}_{\mathbb{P}^4}(2n-5)$.*

In particular, V is $\mathbb{Q}$-factorial if $|\mathrm{Sing}(V)| \leqslant 2n-4$.

Remark 4. Let X be either a nodal complete intersection of two hypersurfaces in $\mathbb{P}^5$ or a nodal double cover of a smooth hypersurface in $\mathbb{P}^4$. Then by Lefschetz's theorem for varieties with isolated singularities (see [30]) $\mathrm{Pic}(X)$ is generated either by the class of a hyperplane section of X or by the pull-back of the class of a hyperplane section. The threefold X is usually said to be factorial in the case when a similar result holds for the group $\mathrm{Cl}(X)$ However, the local class group of an isolated ordinary double point is $\mathbb{Z}$ [31], therefore the following conditions are equivalent:

- the variety V is $\mathbb{Q}$-factorial;
- the variety V is factorial;
- $\mathrm{Cl}(V) \cong \mathrm{Pic}(V)$;
- $\mathrm{Cl}(V) \cong \mathbb{Z}$;
- $\mathrm{rk}\,\mathrm{Cl}(V) = 1$.

We now consider the simplest example of a hypersurface V that is not $\mathbb{Q}$-factorial.

Example 5. Let V be the hypersurface given by the equation

$$xg(x, y, z, t, w) + yf(x, y, z, t, w) = 0 \subset \mathbb{P}^4 \cong \mathrm{Proj}\,(\mathbb{C}[x, y, z, t, w]),$$

where g and f are sufficiently general polynomials of degree $n-1$. Then V is nodal, it contains the plane $x = y = 0$, and $|\mathrm{Sing}(V)| = (n-1)^2$; in particular, V is not $\mathbb{Q}$-factorial.

As pointed out in [32], the problem of the $\mathbb{Q}$-factoriality of nodal threefolds is related to the Shokurov vanishing (see [33]–[36]). We illustrate this relation by the following example.

Proposition 6. *Let $\mathscr{H}$ be the linear system of hypersurfaces of degree $k < n/2$ in $\mathbb{P}^4$ passing through the singular points of V and let $\widehat{\mathscr{H}} = \mathscr{H}|_V$. Suppose that $\dim(\mathrm{Bs}(\widehat{\mathscr{H}})) = 0$. Then V is $\mathbb{Q}$-factorial.*

Proof. Let P be an arbitrary singular point of V. It follows from Proposition 3 that for the proof of the proposition we must find a hypersurface in $\mathbb{P}^4$ of degree $2n-5$ passing through all the points in $\mathrm{Sing}(V) \setminus P$, but not passing through P.

Assume first that $\dim(\mathrm{Bs}(\mathscr{H})) = 0$. Let Λ be the base locus of $\mathscr{H}$. Then $\mathrm{Sing}(V) \subseteq \Lambda$. Consider sufficiently general divisors $H_1, \dots, H_s$ in $\mathscr{H}$ for $s \gg 0$, let $X = \mathbb{P}^4$ and

$$B_X = \frac{4}{s}\sum_{i=1}^{s} H_i.$$

Let $\mathrm{Sing}(V) \setminus P = \{P_1, \dots, P_r\}$, where the P_i are points of X. We consider the blow-up $f\colon V \to X$ of all points in $\mathrm{Sing}(V) \setminus P$. Then

$$K_V + \left(B_V + \sum_{i=1}^{r}(\mathrm{mult}_{P_i}(B_X) - 4)E_i\right) + f^*(H) = f^*((4k-4)H) - \sum_{i=1}^{r} E_i,$$

where $E_i = f^{-1}(P_i)$, $B_V = f^{-1}(B_X)$, and H is a hyperplane in $\mathbb{P}^4$.

Let $\overline{P} = f^{-1}(P)$ and let

$$\widehat{B}_V = B_V + \sum_{i=1}^{r}(\mathrm{mult}_{P_i}(B_X) - 4)E_i.$$

Then the divisor $\widehat{B}_V$ is effective because $\mathrm{mult}_{P_i}(B_X) \geqslant 4$ for each i. Moreover, $\mathrm{mult}_P(B_X) \geqslant 4$, therefore $\overline{P}$ is an isolated centre of log canonical singularities of the log pair $(V, \widehat{B}_V)$. On the other hand the map

$$H^0\Big(\mathscr{O}_V\Big(f^*((4k-4)H) - \sum_{i=1}^{r} E_i\Big)\Big) \\ \to H^0\Big(\mathscr{O}_{\mathscr{L}(V,\widehat{B}_V)} \otimes \mathscr{O}_V\Big(f^*((4k-4)H) - \sum_{i=1}^{r} E_i\Big)\Big)$$

is surjective by Shokurov's vanishing theorem (see Theorem 23), where $\mathscr{L}(V, \widehat{B}_V)$ is the subscheme of log canonical singularities of the log pair $(V, \widehat{B}_V)$. However, in the neighbourhood of the point $\overline{P}$ the support of the subscheme $\mathscr{L}(V, \widehat{B}_V)$ contains only the point $\overline{P}$, therefore there exists an effective divisor

$$D \in \Big|f^*((4k-4)H) - \sum_{i=1}^{r} E_i\Big|$$

not passing through the point $\overline{P}$. Therefore, the divisor $f(D)$ is a hypersurface of degree $4k-4$ in $\mathbb{P}^4$ passing through all points in the set $\mathrm{Sing}(V) \setminus P$, but not passing through P. By assumption $4k-4 \leqslant 2n-5$, so that there exists a hypersurface of degree $2n-5$ in $\mathbb{P}^4$ containing the set $\mathrm{Sing}(V) \setminus P$ and not passing through P.

In the general case we can apply the previous arguments to the linear system $\widehat{\mathscr{H}}$ instead of $\mathscr{H}$ setting $X = V$, and then use the projective normality of the hypersurface V.

Corollary 7. *Let* $\mathrm{Sing}(V) \subset \mathbb{P}^4$ *be a set-theoretical intersection of hypersurfaces of degree* $k < n/2$. *Then the hypersurface* V *is* $\mathbb{Q}$*-factorial.*

As shown in [37], if $\mathrm{Sing}(V) < (n-1)^2$, then each smooth surface in V is a Cartier divisor . It is natural to expect that V is $\mathbb{Q}$-factorial for $|\mathrm{Sing}(V)| < (n-1)^2$, which is proved however only for $n \leqslant 4$ (see [38], [39]). The arguments used in the proof of Proposition 6, and the properties of linear systems on rational surfaces enabled us to prove in [32] that the hypersurface V is $\mathbb{Q}$-factorial in the case when $|\mathrm{Sing}(V)| \leqslant (n-1)^2/4$.

The main result of the present paper is as follows.

Theorem 8. *The following nodal threefolds* X *are* $\mathbb{Q}$*-factorial*:
- *X is the complete intersection of hypersurfaces F and G of degrees n and k, respectively, in $\mathbb{P}^5$ such that G is smooth, $n \geqslant k$, and $|\mathrm{Sing}(X)| \leqslant (n+k-2)(n-1)/5$;*
- *there exists a double cover $\eta\colon X \to F$ of a smooth hypersurface F of degree $n \geqslant 2$ in $\mathbb{P}^4$ ramified in a surface $S \subset F$ cut out on F by a hypersurface $G \subset \mathbb{P}^4$ of degree $2r \geqslant n$ such that the number of singular points of S is at most $(2r+n-2)r/4$.*

Nodal threefolds arise in a natural way in many problems of algebraic geometry.

Example 9. Let Y be a general divisor of bidegree $(2,3)$ in $\mathbb{P}^1 \times \mathbb{P}^3$ given by a bihomogeneous equation

$$f_3(x,y,z,w)s^2 + g_3(x,y,z,w)st + h_3(x,y,z,w)t^2 = 0$$

in bihomogeneous coordinates $(s:t;x:y:z:w)$ on $\mathbb{P}^1 \times \mathbb{P}^3$ (see [40]), where f_3, g_3, and h_3 are sufficiently general homogeneous polynomials of degree 3. Let $\xi\colon Y \to \mathbb{P}^3$ be the natural projection. Then Y contains precisely 27 smooth rational curves $C_1, C_2, \dots, C_{27}$ such that $-K_Y \cdot C_i = 0$ because the system of equations

$$f_3(x,y,z,w) = g_3(x,y,z,w) = h_3(x,y,z,w) = 0$$

has precisely 27 solutions. The projection ξ has degree 2 outside the 27 curves C_i, and

$$X = \operatorname{Proj}\biggl(\bigoplus_{n\geqslant 0} H^0(Y, \mathscr{O}_V(-nK_Y))\biggr)$$

is a double cover of $\mathbb{P}^3$ branched over the nodal surface

$$g_3^2(x,y,z,w) - 4f_3(x,y,z,w)h_3(x,y,z,w) = 0,$$

so that the threefold X is nodal with precisely 27 ordinary double points that are the images of the smooth rational curves C_i contracted by the morphism

$$\varphi_{|-nK_Y|}\colon Y \to X$$

for some integer $n \gg 0$. The threefold X is not $\mathbb{Q}$-factorial, and it is well known that X is not rational (see [41]–[43]).

We point out, however that the geometry of nodal threefolds can be more complicated than that of smooth ones, as seen in the following examples:

- each surface on a smooth hypersurface in $\mathbb{P}^4$ is a complete intersection by Lefschetz's theorem, which is no longer true in the nodal case (see Example 2);
- the birational automorphisms of a smooth quadric threefold form a finite group (see [1]), which does not hold in the nodal case (see [2], [4]);
- smooth cubic threefolds are not rational (see [44]), while nodal ones are rational.

An isolated ordinary double point has two small resolutions, which are birational via an ordinary flop (see [27], [45]). Therefore, a nodal threefold with k singular points has precisely 2^k small resolutions, which must all be non-projective in the $\mathbb{Q}$-factorial case because each exceptional curve in that case must be homologous to zero. It is therefore natural to expect that a singular nodal threefold is $\mathbb{Q}$-factorial if and only if all its small resolutions are non-projective. The following example, which is due to Wotzlaw, shows that this is not true.

Example 10. Let $\mathscr{I}_5$ be the quintic hypersurface

$$x_5 - 6x_5^3 \sum_{i=0}^{6} x_i - 27x_5\biggl(\biggl(\sum_{i=0}^{5} x_i\biggr)^2 - 4\sum_{i=0}^{5}\sum_{j=i+1}^{5} x_i x_j\biggr) - 648x_0x_1x_2x_3x_4 = 0$$

in $\mathbb{P}^5 \cong \operatorname{Proj}(\mathbb{C}[x_0,x_1,x_2,x_3,x_4,x_5])$. Then $\mathscr{I}_5$ is invariant under the standard action of the Weil group E_6 on $\mathbb{P}^5$ by reflections; moreover, $\mathscr{I}_5$ is the unique invariant of degree 5 of E_6 under such action (see [16], § 6; [46]).

The singularities of the quintic $\mathscr{I}_5$ consist of 120 lines L_i intersecting at 36 points O_k, $i=1,\dots,120$ and $k=1,\dots,36$, and the projectivization of the tangent cone to $\mathscr{I}_5$ at each point O_k is isomorphic to the so-called Segre cubic (see [47], [16], [46]), while at each point of the set

$$\bigcup_{i=1}^{120} L_i \setminus \bigcup_{k=1}^{36} O_k$$

the quintic $\mathscr{I}_5$ is locally isomorphic to the product $\mathbb{C}\times\mathbb{A}$, where $\mathbb{A}$ is a neighbourhood of a three-dimensional ordinary double point.

Let H_α be a hyperplane section of the quintic $\mathscr{I}_5$ corresponding to a general point $\alpha\in(\mathbb{P}^5)^*$, and let T_β be a hyperplane section of $\mathscr{I}_5$ corresponding to a general point $\beta\in(\mathscr{I}_5)^*\subset(\mathbb{P}^5)^*$. In particular, T_β is tangent to $\mathscr{I}_5$ at some point $P\in\mathscr{I}_5$. There exist therefore a five-dimensional family of hyperplane sections H_α of $\mathscr{I}_5$ and a four-dimensional family of tangent hyperplane sections H_β. It follows from [16] or from explicit computer-based calculations (see [48], [49]) that both families are versal. By construction H_α is a nodal hypersurface in $\mathbb{P}^4$ of degree 5 with 120 ordinary double points $Q_i = L_i\cap H_\alpha$, and T_β is a nodal hypersurface of degree 5 with 121 ordinary double points $P_i = L_i\cap T_\beta$ and P. It follows by Lefschetz's theorem that

$$\operatorname{rk}\operatorname{Pic}(H_\alpha)=\operatorname{rk}\operatorname{Pic}(T_\beta)=1,$$

but it follows from [50] that

$$\operatorname{rk}\operatorname{Cl}(H_\alpha)=\operatorname{rk}\operatorname{Cl}(T_\beta)=25,$$

so that H_α and T_β are not $\mathbb{Q}$-factorial.

Let $\pi\colon \widehat{T}_\beta\to T_\beta$ be a small resolution and let C_i and C be curves on $\widehat{T}_\beta$ contracted to the points P_i and P, respectively. Then

$$\mathscr{N}_{C/\widehat{T}_\beta}\cong\mathscr{N}_{C_i/\widehat{T}_\beta}\cong\mathscr{O}_{\mathbb{P}^1}(-1)\oplus\mathscr{O}_{\mathbb{P}^1}(-1),$$

where $C\cong C_i\cong\mathbb{P}^1$.

Let $\psi\colon \overline{H}_\alpha\to H_\alpha$ be a small resolution and $\tau\colon\widehat{T}_\beta\to\overline{T}_\beta$ a small contraction of a smooth rational curve C into an ordinary double point $\overline{P}\in\overline{T}_\beta$. Then $\overline{P}$ is the unique singular point of $\overline{T}_\beta$, and the five-dimensional family of smooth threefolds $\overline{H}_\alpha$ is a smooth deformation of the threefold $\overline{T}_\beta$. Therefore, there exists an exact sequence (see [27])

$$0\to H_3(\widehat{T}_\beta,\mathbb{Z})\to H_3(\overline{T}_\beta,\mathbb{Z})\to H_2(C,\mathbb{Z})\to H_2(\widehat{T}_\beta,\mathbb{Z})\to H_2(\overline{T}_\beta,\mathbb{Z})\to 0$$

and an isomorphism $H_2(\overline{T}_\beta,\mathbb{Z})\cong H_2(\overline{H}_\alpha,\mathbb{Z})$; but

$$h_2(\widehat{T}_\beta,\mathbb{Z})=\operatorname{rk}\operatorname{Cl}(T_\beta)=\operatorname{rk}\operatorname{Cl}(H_\alpha)=h_2(\overline{H}_\alpha,\mathbb{Z}),$$

so that the natural map $H_2(C,\mathbb{Z}) \to H_2(\widehat{T}_\beta,\mathbb{Z})$ takes the entire homology group $H_2(C,\mathbb{Z})$ to zero. Hence the curve C on the smooth threefold $\widehat{T}_\beta$ is homologous to zero, and therefore $\widehat{T}_\beta$ is not projective.

We consider now two examples inspired by [51] and [4].

Example 11. Let $\pi\colon X \to \mathbb{P}^3$ be the double cover ramified along a nodal hypersurface $S \subset \mathbb{P}^3$ of degree 6 such that X can be defined by an equation

$$y^2 + g_3^2(x_0,x_1,x_2,x_3) = h_1(x_0,x_1,x_2,x_3)f_5(x_0,x_1,x_2,x_3)$$

in $\mathbb{P}(1^4,3) \cong \operatorname{Proj}(\mathbb{C}[x_0,x_1,x_2,x_3,y])$, where g_3, h_1, and f_5 are general polynomials of degrees 3, 1, and 5, respectively, defined over $\mathbb{R}$. Then X is not $\mathbb{Q}$-factorial over $\mathbb{C}$ because the divisor $h_1 = 0$ on X splits into the union of two non $\mathbb{Q}$-Cartier divisors conjugate by means of $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$ and given by the equation

$$\bigl(y+\sqrt{-1}\,g_3(x_0,x_1,x_2,x_3)\bigr)\bigl(y-\sqrt{-1}\,g_3(x_0,x_1,x_2,x_3)\bigr)=0.$$

The surface $S \subset \operatorname{Proj}(\mathbb{C}[x_0,x_1,x_2,x_3])$ has 15 ordinary double points, which are defined on X by the equations

$$h_1(x,y,z,w) = g_3(x,y,z,w) = f_5(x,y,z,w) = 0.$$

Introducing new variables α and β of weight 2 by the formulae

$$\begin{aligned}\alpha &= \frac{y+\sqrt{-1}\,g_3(x_0,x_1,x_2,x_3)}{h_1(x_0,x_1,x_2,x_3)} = \frac{f_5(x_0,x_1,x_2,x_3)}{y-\sqrt{-1}\,g_3(x_0,x_1,x_2,x_3)}\,,\\ \beta &= \frac{y-\sqrt{-1}\,g_3(x_0,x_1,x_2,x_3)}{h_1(x_0,x_1,x_2,x_3)} = \frac{f_5(x_0,x_1,x_2,x_3)}{y+\sqrt{-1}\,g_3(x_0,x_1,x_2,x_3)}\,,\end{aligned}$$

we can *unproject* $X \subset \mathbb{P}(1^4,3)$ in the sense of [52] into two complete intersections

$$\begin{aligned}\widehat{V} &= \left\{\begin{aligned}&\alpha h_1(x_0,x_1,x_2,x_3) = y+\sqrt{-1}\,g_3(x_0,x_1,x_2,x_3)\\ &\alpha(y-\sqrt{-1}\,g_3(x_0,x_1,x_2,x_3)) = f_5(x_0,x_1,x_2,x_3)\end{aligned}\right\} \subset \mathbb{P}(1^4,3,2),\\ \overline{V} &= \left\{\begin{aligned}&\beta h_1(x_0,x_1,x_2,x_3) = y-\sqrt{-1}\,g_3(x_0,x_1,x_2,x_3)\\ &\beta(y+\sqrt{-1}\,g_3(x_0,x_1,x_2,x_3)) = f_5(x_0,x_1,x_2,x_3)\end{aligned}\right\} \subset \mathbb{P}(1^4,3,2),\end{aligned}$$

which are not defined over $\mathbb{R}$. Eliminating the variable u we obtain the isomorphisms

$$\begin{aligned}\widehat{V} &\cong \{\alpha^2 h_1 - 2\sqrt{-1}\,\alpha g_3 - f_5 = 0\} \subset \mathbb{P}(1^4,2),\\ \overline{V} &\cong \{\beta^2 h_1 + 2\sqrt{-1}\,\beta g_3 - f_5 = 0\} \subset \mathbb{P}(1^4,2).\end{aligned}$$

The maps $\widehat{\rho}\colon X \dashrightarrow \widehat{V}$ and $\overline{\rho}\colon X \dashrightarrow \overline{V}$ fit in a commutative diagram

$$\begin{tikzcd} & \widehat{Y} \arrow[dl, "\widehat{\psi}"'] \arrow[dr, "\widehat{\varphi}"] & & \overline{Y} \arrow[dl, "\overline{\varphi}"'] \arrow[dr, "\overline{\psi}"] & \\ \widehat{V} & & X \arrow[ll, dashed, "\overline{\rho}"'] \arrow[rr, dashed, "\widehat{\rho}"] & & \overline{V} \end{tikzcd}$$

with birational morphisms $\widehat{\varphi}$, $\widehat{\psi}$, $\overline{\varphi}$, and $\overline{\psi}$ such that $\widehat{\psi}$ and $\overline{\psi}$ are extremal contractions in the sense of [53], while $\widehat{\varphi}$ and $\overline{\varphi}$ are flopping contractions.

It is easy to verify that the weighted hypersurfaces $\widehat{V}$ and $\overline{V}$ are quasismooth (see [54]) and $\mathbb{Q}$-factorial, with Picard group $\mathbb{Z}$ (see [55], [56], [57], [58]). In fact, the weighted hypersurfaces $\widehat{V}$ and $\overline{V}$ are projectively isomorphic in $\mathbb{P}(1^4, 2)$ by the natural action of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}_2$. Thus,

$$\mathrm{Pic}(\widehat{Y}) \cong \mathrm{Pic}(\overline{Y}) \cong \mathbb{Z} \oplus \mathbb{Z},$$

which shows that $\widehat{Y}$ and $\overline{Y}$ are $\mathbb{Q}$-factorial and $\mathrm{Cl}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$.

By construction the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-invariant part of the group $\mathrm{Cl}(X)$ is $\mathbb{Z}$, so that the threefold X is $\mathbb{Q}$-factorial over $\mathbb{R}$ and therefore not rational over $\mathbb{R}$ (see [7]), but one can also show that X is not rational over $\mathbb{C}$ either [55]. Moreover, the biregular involution of X interchanging the fibres of π induces a non-biregular birational involution $\tau \in \mathrm{Bir}(\widehat{V})$, which is regularized by $\widehat{\rho}$ (see [59]).

Example 12. Let $V \subset \mathbb{P}^4$ be a general hypersurface of degree 4 with precisely one ordinary double point O. Then V is $\mathbb{Q}$-factorial and $\mathrm{Pic}(V) \cong \mathbb{Z}$. It is easy to see that V can be described by an equation

$$t^2 f_2(x,y,z,w) + t f_3(x,y,z,w) + f_4(x,y,z,w) = 0 \subset \mathbb{P}^4 = \mathrm{Proj}\,(\mathbb{C}[x,y,z,w,t]),$$

where $O = (0:0:0:0:1)$. The threefold V is known to be non-rational, but $\mathrm{Bir}(V) \neq \mathrm{Aut}(V)$ since the projection $\varphi\colon V \dashrightarrow \mathbb{P}^3$ from the singular point O has degree 2 at a generic point of V and induces a non-biregular involution $\tau \in \mathrm{Bir}(V)$.

Let $f\colon Y \to V$ be the blow-up of the point O. Then the linear system $|-nK_Y|$ has no base points for some $n \gg 0$ and defines a birational morphism

$$g = \varphi_{|-nK_Y|}\colon Y \to X$$

contracting each curve $C_i \subset Y$ such that $f(C_i)$ is a line on the quartic threefold V passing through the point O. The singularities of X are canonical Gorenstein. [3] We obtain next a double cover $\pi\colon X \to \mathbb{P}^3$ ramified along the surface $S \subset \mathbb{P}^3$ given by the equation

$$f_3^2(x,y,z,w) - 4 f_2(x,y,z,w) f_4(x,y,z,w) = 0.$$

Each line $f(C_i)$ corresponds to an intersection point of three surfaces

$$f_2(x,y,z,w) = f_3(x,y,z,w) = f_4(x,y,z,w) = 0 \subset \mathbb{P}^3 = \mathrm{Proj}\,(\mathbb{C}[x,y,z,w]),$$

which gives one 24 distinct smooth rational curves $C_1, C_2, \ldots, C_{24}$ on Y such that

$$\mathscr{N}_{C_i/Y} \cong \mathscr{O}_{C_i}(-1) \oplus \mathscr{O}_{C_i}(-1),$$

and therefore g is a standard flopping contraction mapping each curve C_i into an ordinary double point of the threefold X. In particular, the sextic S has precisely 24 ordinary double points. However, X is not $\mathbb{Q}$-factorial and $\mathrm{Cl}(X) = \mathbb{Z} \oplus \mathbb{Z}$.

[3] Canonical Gorenstein singularities are rational Gorenstein singularities (see [60]).

We set $\rho = g \circ f^{-1}$. Then the involution $\gamma = \rho \circ \tau \circ \rho^{-1}$ is biregular on X and interchanges the fibres of the double cover π. Thus, the map ρ is a regularization of the birational non-biregular involution τ in the sense of [59], while the commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "f"'] \arrow[dr, "g"] & & & Y \arrow[dl, "g"'] \arrow[dr, "f"] & \\
V \arrow[rr, dashed, "\rho"] & & X \arrow[r, "\gamma"'] & X & & V \arrow[ll, dashed, "\rho"']
\end{tikzcd}$$

is a decomposition of $\tau \in \mathrm{Bir}(V)$ into a sequence of so-called Sarkisov links (see [53], [55], [61]).

Assume now that $f_2(x, y, z, w)$ and $f_4(x, y, z, w)$ are defined over $\mathbb{Q}$, and let

$$f_3(x, y, z, w) = \sqrt{2}\, g_3(x, y, z, w),$$

where $g_3(x, y, z, t)$ is also defined over $\mathbb{Q}$. Then V is defined over the field $\mathbb{Q}(\sqrt{2})$, but not over $\mathbb{Q}$, and V is not invariant under the action of $\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$. However, the sextic $S \subset \mathbb{P}^3$ has the equation

$$2g_3^2(x, y, z, w) - 4f_2(x, y, z, w)f_4(x, y, z, w) = 0 \subset \mathbb{P}^3 = \mathrm{Proj}\,(\mathbb{Q}[x, y, z, w]),$$

therefore X is also defined over $\mathbb{Q}$. Moreover, the $\mathrm{Gal}(\mathbb{Q}(\sqrt{2}\,)/\mathbb{Q})$-invariant part of $\mathrm{Cl}(X)$ is $\mathbb{Z}$, so that X is $\mathbb{Q}$-factorial and non-rational over $\mathbb{Q}$.

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§ 2. Preliminaries

The following result is well known ([29], [26], [62], [27], [28]).

Theorem 13. *Let W be a smooth fourfold and $Y \subset W$ an ample reduced and irreducible divisor such that all singularities of Y are nodal and*

$$h^2(\Omega^1_W) = h^3(\Omega^1_W \otimes \mathscr{O}_W(-Y)) = h^1(\mathscr{O}_W) = h^2(\mathscr{O}_W) = 0.$$

Let $\widetilde{Y}$ be a small resolution of Y. Then $h^1(\mathscr{O}_{\widetilde{Y}}) = h^2(\mathscr{O}_{\widetilde{Y}}) = 0$, $h^1(\Omega^1_{\widetilde{Y}}) = h^1(\Omega^1_W) + \delta$ and

$$\begin{aligned} h^2(\Omega^1_{\widetilde{Y}}) = h^0(K_W \otimes \mathscr{O}_W(2Y)) + h^3(\mathscr{O}_W) - h^0(K_W \otimes \mathscr{O}_W(Y)) \\ - h^3(\Omega^1_W) - h^4(\Omega^1_W \otimes \mathscr{O}_W(-Y)) - |\mathrm{Sing}(Y)| + \delta, \end{aligned}$$

where δ is the number of dependent conditions that vanishing at the nodes of Y imposes on global sections of the line bundle $K_W \otimes \mathscr{O}_W(2Y)$.

The following result is a consequence of Theorem 13 in [29].

Corollary 14. *Let W be a smooth fourfold and Y a reduced and irreducible divisor on W with nodal singularities. Let*

$$h^2(\Omega^1_W) = h^1(\mathscr{O}_W) = h^2(\mathscr{O}_W) = 0,$$

and assume that singular points of Y impose independent linear conditions on global sections of the line bundle $K_W \otimes \mathscr{O}_W(2Y)$. Then Y is $\mathbb{Q}$-factorial.

The following result is proved in [63].

Theorem 15. *Let $\pi\colon Y \to \mathbb{P}^2$ be a blow-up of points $P_1, \dots, P_s$ such that*

$$s \leqslant \frac{d^2 + 9d + 10}{6},$$

and at most $k(d+3-k)-2$ points among the P_i lie in a curve of degree $k \leqslant (d+3)/2$ for some integer $d \geqslant 3$. Then the linear system

$$\left|\pi^*(\mathscr{O}_{\mathbb{P}^2}(d)) - \sum_{i=1}^{s} E_i\right|$$

where $E_i = \pi^{-1}(P_i)$, has no base points.

In the case $d = 3$ the assertion of Theorem 15 is the base-point freeness of the anticanonical system of a *weak del Pezzo surface* of degree $9 - s \geqslant 2$ (see [64]–[66]).

Corollary 16. *Let Σ be a finite subset of $\mathbb{P}^2$ and $d \geqslant 3$ an integer such that*

$$|\Sigma| \leqslant \frac{d^2 + 9d + 16}{6},$$

and at most $k(d+3-k)-2$ points in the set Σ lie on a (possibly reducible) curve $\mathbb{P}^2$ of degree $k \leqslant (d+3)/2$. Then for each point P in Σ there exists a curve $C \subset \mathbb{P}^2$ of degree d passing through all points in the set $\Sigma \setminus P$, but not passing through P.

Theorem 15 was improved in [67] in the following way.

Theorem 17. *Let $\pi\colon Y \to \mathbb{P}^2$ be a blow-up of points $P_1, \dots, P_s$ in $\mathbb{P}^2$ such that*

$$s \leqslant \max\left\{ \left\lfloor \frac{d+3}{2} \right\rfloor \left(d + 3 - \left\lfloor \frac{d+3}{2} \right\rfloor \right) - 1,\ \left\lfloor \frac{d+3}{2} \right\rfloor^2 \right\},$$

and at most $k(d+3-k)-2$ points among $\{P_1, \dots, P_s\}$ lie on a curve of degree $k \leqslant (d+3)/2$ for some integer $d \geqslant 3$. Then the linear system

$$\left|\pi^*(\mathscr{O}_{\mathbb{P}^2}(d)) - \sum_{i=1}^{s} E_i\right|$$

where $E_i = \pi^{-1}(P_i)$, has no base points.

§ 3. Connectedness principle

Let (X, B_X) be a log pair, that is, X is a variety and $B_X = \sum_{i=1}^{k} a_i B_i$, where a_i is a rational number and B_i is an effective irreducible reduced divisor. One usually assumes (see [68]) that for all indices i we either have $a_i \geqslant 0$ or $a_i \in [0, 1]$. We do not make this agreement, but assume for simplicity that X has $\mathbb{Q}$-factorial singularities.

In particular, the divisor K_X+B_X is $\mathbb{Q}$-Cartier. We observe that B_X is often called the boundary of the log pair (X, B_X).

Let $f\colon V\to X$ be a birational morphism such that V has $\mathbb{Q}$-factorial singularities. We set

$$B^V=f^{-1}(B_X)-\sum_{i=1}^{n}a(X,B_X,E_i)E_i,$$

where $a(X,B_X,E_i)\in\mathbb{Q}$, E_i is an f-exceptional divisor for each i, and we have the relation

$$K_V+B^V\sim_{\mathbb{Q}}f^*(K_X+B_X),$$

which is easily seen to define B^V uniquely. Then the log pair (V,B^V) is called the log pull-back of the log pair (X,B_X).

Definition 18. A proper irreducible subvariety $Y\subset X$ is called a *centre of log canonical singularities of the log pair* (X,B_X) if there exists a birational morphism $f\colon W\to X$ and a (not necessarily f-exceptional) divisor $E\subset W$ such that W has $\mathbb{Q}$-factorial singularities and E lies in the support of the effective part of the divisor $\lfloor B^Y\rfloor$.

We shall denote the set of all centres of log canonical singularities of the log pair (X,B_X) by $\mathbb{LCS}(X,B_X)$. In a similar way, the union of all centres of log canonical singularities of the log pair (X,B_X) regarded as a proper subset of X is usually called the locus of log canonical singularities and is denoted by $\mathrm{LCS}(X,B_X)$.

Example 19. Let O be a smooth point on X. Then it follows from the inequality $\mathrm{mult}_O(B_X)\geqslant\dim(X)$ that $O\in\mathbb{LCS}(X,B_X)$. Moreover, if $O\in\mathbb{LCS}(X,B_X)$ and the boundary B_X is effective, then $\mathrm{mult}_O(B_X)\geqslant 1$.

Remark 20. Let H be a general hyperplane section of the variety X and Z a subvariety of the variety X that is an element of $\mathbb{LCS}(X,B_X)$. Then each component of the intersection $Z\cap H$ belongs to $\mathbb{LCS}(H,B_X|_H)$.

Example 21. Let O be a smooth point of the variety X that is an element of $\mathbb{LCS}(X,B_X)$. Let $f\colon V\to X$ be the blow-up of the point O and E the f-exceptional divisor. Then either $E\in\mathbb{LCS}(V,B^V)$ or there exists a proper irreducible subvariety $Z\subset E$ that is a centre of log canonical singularities of the log pair (V,B^V). Moreover, the exceptional divisor E is a centre of log canonical singularities of the log pair (V,B^V) if and only if $\mathrm{mult}_O(B_X)\geqslant\dim(X)$.

Let $f\colon Y\to X$ be a birational morphism, where Y is a smooth variety and the union of all divisors $f^{-1}(B_i)$ and all f-exceptional divisors is a divisor with simple normal crossings. Then one usually calls f a log resolution of the log pair (X,B_X). For the log pull-back (Y,B^Y) of the log pair (X,B_X) we have the relation

$$K_Y+B^Y\sim_{\mathbb{Q}}f^*(K_X+B_X).$$

Definition 22. The subscheme associated with the ideal sheaf

$$\mathscr{I}(X,B_X)=f_*(\lceil -B^Y\rceil),$$

is called the *log canonical singularity subscheme of* (X,B_X); it is denoted by $\mathscr{L}(X,B_X)$.

We point out that $\operatorname{Supp}(\mathscr{L}(X, B_X)) = \operatorname{LCS}(X, B_X) \subset X$. The following result is Shokurov's vanishing theorem (see [33]–[36]).

Theorem 23. *Assume that B_X is effective. Let H be an arbitrary nef and big divisor*[4] *on X such that $D = K_X + B_X + H$ is numerically equivalent to a Cartier divisor. Then $H^i(X, \mathscr{I}(X, B_X) \otimes D) = 0$ for all $i > 0$.*

Proof. It follows by the Kawamata–Vieweg vanishing theorem that

$$R^i f_*(f^*(K_X + B_X + H) + \lceil -B^W \rceil) = 0$$

for all $i > 0$ (see [68]–[70]). The degeneracy of the local-to-global spectral sequence and the equality

$$R^0 f_*(f^*(K_X + B_X + H) + \lceil -B^W \rceil) = \mathscr{I}(X, B_X) \otimes D$$

yield the equalities

$$H^i(X, \mathscr{I}(X, B_X) \otimes D) = H^i(W, f^*(K_X + B_X + H) + \lceil -B^W \rceil)$$

for $i \geqslant 0$. On the other hand,

$$H^i(W, f^*(K_X + B_X + H) + \lceil -B^W \rceil) = 0$$

for $i > 0$ by the Kawamata–Vieweg vanishing theorem.

For an arbitrary Cartier divisor D on the variety X consider the exact sequence of sheaves

$$0 \to \mathscr{I}(X, B_X) \otimes D \to \mathscr{O}_X(D) \to \mathscr{O}_{\mathscr{L}(X, B_X)}(D) \to 0$$

and the corresponding exact sequence of cohomology groups

$$H^0(\mathscr{O}_X(D)) \to H^0(\mathscr{O}_{\mathscr{L}(X, B_X)}(D)) \to H^1(\mathscr{I}(X, B_X) \otimes D).$$

Theorem 23 immediately yields the following result, usually called Shokurov's connectedness principle for the locus of log canonical singularities.

Theorem 24. *Let B_X be an effective boundary and let $-(K_X + B_X)$ be a nef and big divisor. Then the set $\operatorname{LCS}(X, B_X) \subset X$ is connected.*

We now consider the following application of Theorem 23 (see [32]).

Lemma 25. *Let Σ be a finite subset of $\mathbb{P}^n$ and $\mathscr{M}$ the linear system of all hypersurfaces of degree k passing through all points in Σ. Assume that the base locus of $\mathscr{M}$ is zero-dimensional. Then the points in Σ impose independent linear conditions on hypersurfaces in $\mathbb{P}^n$ of degree $n(k-1)$.*

[4] We point out that a $\mathbb{Q}$-Cartier divisor $H \in \operatorname{Div}(X) \otimes \mathbb{Q}$ is said to be *numerically effective* or *nef* if for each curve $C \subset X$ one has $H \cdot C \geqslant 0$. A numerically effective divisor H on a variety X is said to be *big* if $H^n > 0$, where $n = \dim(X)$.

Proof. Let Λ be the base locus of the linear system $\mathscr{M}$. Then $\Sigma \subseteq \Lambda$. Let $H_1, \dots, H_s$ be general divisors in $\mathscr{M}$, where $s \gg 0$. We set $X = \mathbb{P}^n$ and

$$B_X = \frac{n}{s} \sum_{i=1}^{s} H_i.$$

Then

$$\operatorname{Supp}(\mathscr{L}(X, B_X)) = \Lambda,$$

where $\mathscr{L}(X, B_X)$ is the subscheme of log canonical singularities of the log pair (X, B_X).

It is sufficient for the proof to construct for an arbitrary point $P \in \Sigma$ a hypersurface of degree $n(k-1)$ in $\mathbb{P}^n$ passing through all points in $\Sigma \setminus P$, but not through P.

Let $\Sigma \setminus P = \{P_1, \dots, P_k\}$, where the P_i are points in $X = \mathbb{P}^n$. Consider the blow-up $f\colon V \to X$ of all points in $\Sigma \setminus P$. Then

$$K_V + \left(B_V + \sum_{i=1}^{k} (\operatorname{mult}_{P_i}(B_X) - n) E_i\right) + f^*(H) = f^*(n(k-1)H) - \sum_{i=1}^{k} E_i,$$

where $E_i = f^{-1}(P_i)$, $B_V = f^{-1}(B_X)$, and H is a hypersurface in $\mathbb{P}^n$. By construction

$$\operatorname{mult}_{P_i}(B_X) = n \operatorname{mult}_{P_i}(\mathscr{M}) \geqslant n,$$

and the divisor

$$\widehat{B}_V = B_V + \sum_{i=1}^{k} (\operatorname{mult}_{P_i}(B_X) - n) E_i$$

is effective.

Let $\overline{P} = f^{-1}(P)$. Then

$$\overline{P} \in \mathbb{LCS}(V, \widehat{B}_V)$$

and $\overline{P}$ is an isolated centre of log canonical singularities of the log pair $(V, \widehat{B}_V)$ because the birational morphism $f\colon V \to X$ is an isomorphism in a neighbourhood of P.

On the other hand, the map

$$H^0\left(\mathscr{O}_V\left(f^*(n(k-1)H) - \sum_{i=1}^{k} E_i\right)\right) \to H^0\left(\mathscr{O}_{\mathscr{L}(V, \widehat{B}_V)} \otimes \mathscr{O}_V\left(f^*(n(k-1)H) - \sum_{i=1}^{k} E_i\right)\right)$$

is surjective by Theorem 23. However, in the neighbourhood of $\overline{P}$ the support of the scheme $\mathscr{L}(V, \widehat{B}_V)$ consists of the point $\overline{P}$ alone, therefore there exists a divisor

$$D \in \left| f^*(n(k-1)H) - \sum_{i=1}^{k} E_i \right|$$

not passing through $\overline{P}$. Hence $f(D)$ is a hypersurface in $\mathbb{P}^n$ of degree $n(k-1)$ passing through all the points in $\Sigma \setminus P$, but avoiding $P \in \Sigma$. The proof is complete.

§ 4. Complete intersections in $\mathbb{P}^5$

Let X be a complete intersection of hypersurfaces F and G in $\mathbb{P}^5$ such that the singularities of X are nodal. We set $n = \deg(F)$ and $k = \deg(G)$ and assume that $n \geqslant k$.

Example 26. Let F and G be general hypersurfaces in $\mathbb{P}^5$ containing a plane $\Pi \subset \mathbb{P}^5$. Then X is nodal and not $\mathbb{Q}$-factorial, both F and G are smooth, and $|\mathrm{Sing}(X)| = (n + k - 2)^2$.

The following result is proved in [71].

Theorem 27. *Suppose that G is smooth and $|\mathrm{Sing}(X)| \leqslant 3n/8$. Then X is $\mathbb{Q}$-factorial.*

In this section we prove the following result.

Theorem 28. *Suppose that G is smooth and $|\mathrm{Sing}(X)| \leqslant (n + k - 2)(n - 1)/5$. Then X is $\mathbb{Q}$-factorial.*

Theorem 28 fails in the case when the hypersurface G is singular.

Example 29. Let Q be a smooth quadric surface in $\mathbb{P}^5$, G a cone over Q the vertex of which is a general line $L \subset \mathbb{P}^5$, F a general hypersurface of degree n, and X the complete intersection of the hypersurfaces G and F. Then X is a nodal threefold of degree $2n$ and $|\mathrm{Sing}(X)| = n$. Let Ω be a linear subspace of $\mathbb{P}^5$ spanned by L and a line lying in Q. Then $\Omega \subset G$, the surface $\Omega \cap F$ has degree n and is not a $\mathbb{Q}$-Cartier divisor on X.

For $k = 1$ the assertion of Theorem 28 follows from [32].

Conjecture 30. *Suppose that G is smooth and $|\mathrm{Sing}(X)| \leqslant (n + k - 2)^2$. Then the threefold X is $\mathbb{Q}$-factorial.*

The following result is a consequence of Corollary 14.

Proposition 31. *Suppose that G is smooth. Then X is $\mathbb{Q}$-factorial if its singular points impose independent linear conditions on sections in $H^0(\mathscr{O}_{\mathbb{P}^5}(2n + k - 6)|_G)$.*

Corollary 32. *Suppose that G is smooth and that $|\mathrm{Sing}(X)| \leqslant 2n+k-5$. Then X is $\mathbb{Q}$-factorial.*

The variety X is $\mathbb{Q}$-factorial if and only if the group $\mathrm{Cl}(X)$ is generated by the class of a hyperplane section of X (see Remark 4). In particular, if X is $\mathbb{Q}$-factorial, then each surface in X is a complete intersection in $\mathbb{P}^5$.

We now prove Theorem 28.

Proof of Theorem 28. Suppose that $|\mathrm{Sing}(X)| \leqslant (n + k - 2)(n - 1)/5$, and let G be a smooth hypersurface. We observe that $n = \deg(F) \geqslant k = \deg(G)$. We claim that the singular points of the complete intersection $X \subset \mathbb{P}^5$ impose independent linear conditions on a hypersurface in $\mathbb{P}^5$ of degree $2n + k - 6$, which yields the result of Theorem 28. We shall assume that $k \geqslant 2$ and $n \geqslant 5$, since for $k = 1$ the result of Theorem 28 is a consequence of [32], and for $4 \geqslant n \geqslant k \geqslant 2$ it is an easy consequence of Corollary 32.

Lemma 33. *There exists a hypersurface $\widehat{F} \subset \mathbb{P}^5$ of degree n such that the threefold X is a complete intersection of $\widehat{F}$ and G, but* $\mathrm{Sing}(\widehat{F}) \subseteq \mathrm{Sing}(X)$.

Proof. Assume that X is given by a system of equations

$$\begin{cases} f(x_0, x_1, x_2, x_3, x_4, x_5) = 0, \\ g(x_0, x_1, x_2, x_3, x_4, x_5) = 0 \end{cases} \subset \mathbb{P}^5 \cong \mathrm{Proj}\,(\mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5]),$$

where f and g are homogeneous polynomials of degrees n and k defining the hypersurfaces F and G, respectively. Consider the linear system

$$\mathscr{L} = \left|\lambda f + h(x_0, x_1, x_2, x_3, x_4, x_5)g\right| \subset |\mathscr{O}_{\mathbb{P}^5}(n)|,$$

where $\lambda \in \mathbb{C}$ and h is a homogeneous polynomial of degree $n-k$. Then the base locus of the linear system $\mathscr{L}$ is the variety X. By Bertini's theorem there exists a hypersurface $\widehat{F} \subset \mathscr{L}$ with the required properties.

We shall assume that $\mathrm{Sing}(F) \subseteq \mathrm{Sing}(X)$.

Definition 34. The points in a subset Γ of $\mathbb{P}^r$ have *property* $(\star)$ if a curve of degree $t \in \mathbb{N}$ in $\mathbb{P}^r$ contains at most $t(n+k-2)$ points in Γ.

Let $\Sigma = \mathrm{Sing}(X) \subset \mathbb{P}^5$.

Proposition 35. *The points in $\Sigma \subset \mathbb{P}^5$ have property $(\star)$.*

Proof. The hypersurface $F \subset \mathbb{P}^5$ can be defined by an equation

$$f(x_0, x_1, x_2, x_3, x_4, x_5) = 0 \subset \mathbb{P}^5 \cong \mathrm{Proj}\,(\mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5]),$$

where f is a homogeneous polynomial of degree n and $G \subset \mathbb{P}^5$ can be defined by an equation

$$g(x_0, x_1, x_2, x_3, x_4, x_5) = 0 \subset \mathbb{P}^5 \cong \mathrm{Proj}\,(\mathbb{C}[x_0, x_1, x_2, x_3, x_4, x_5]),$$

where g is a homogeneous polynomial of degree k. Then the set Σ is defined by the vanishing of the polynomials f and g and of all the minors of order 1 of the matrix

$$\begin{pmatrix} \dfrac{\partial f}{\partial x_0} & \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} & \dfrac{\partial f}{\partial x_3} & \dfrac{\partial f}{\partial x_4} & \dfrac{\partial f}{\partial x_5} \\ \dfrac{\partial g}{\partial x_0} & \dfrac{\partial g}{\partial x_1} & \dfrac{\partial g}{\partial x_2} & \dfrac{\partial g}{\partial x_3} & \dfrac{\partial g}{\partial x_4} & \dfrac{\partial g}{\partial x_5} \end{pmatrix},$$

so that Σ is a set-theoretical intersection of hypersurfaces of degree $n+k-2$ in $\mathbb{P}^5$, which completes the proof.

Consider an arbitrary point $P \in \Sigma$. We must show the existence of a hypersurface in $\mathbb{P}^5$ of degree $2n+k-6$ that contains the set $\Sigma \setminus P$ and does not contain P, which will prove Theorem 28 since P can be arbitrary.

Lemma 36. *Let $\Pi \subset \mathbb{P}^5$ be a plane such that $\Sigma \subset \Pi \subset \mathbb{P}^5$. Then there exists a hypersurface of degree $2n+k-6$ in $\mathbb{P}^5$ containing the set $\Sigma \setminus P$ and not containing $P \in \Sigma$.*

Proof. We wish to apply Corollary 16 to $\Sigma \subset \Pi$ and $d = 2n + k - 6 \geqslant 6$. We shall verify that all the assumptions of Corollary 16 are satisfied.

We must show that $|\Sigma| \leqslant (d^2 + 9d + 16)/6$. Assume that $|\Sigma| > (d^2 + 9d + 16)/6$. Then

$$\frac{(n+k-2)(n-1)}{5} > \frac{(2n+k-6)^2 + 9(2n+k-6) + 16}{6}$$

for $n \geqslant 5$ and $k \geqslant 2$. We set $A = n + k \geqslant 7$; then

$$\begin{aligned} 0 &> (A+n-6)^2 + 9(A+n-6) + 16 - 6An \\ &= 5A^2 - 3A - 10 + 5n^2 - 3n + 4An \geqslant 464, \end{aligned}$$

which is a contradiction.

We must now show that at most $t(2n + k - 3 - t) - 2$ points in the set Σ lie on a curve of degree $t \leqslant (2n + k - 3)/2$. However, at most $t(n + k - 2)$ points of Σ lie on a curve of degree t by Proposition 35. In particular, for $t = 1$ we have

$$t(2n + k - 3 - t) - 2 = 2n + k - 6 \geqslant n + k - 2 = t(n-1)$$

because $n \geqslant 5$. In the case when $t > 1$ it is sufficient to show that

$$t(2n + k - 3 - t) - 2 \geqslant t(n + k - 2)$$

for each $t \leqslant (2n + k - 3)/2$ such that $t(2n + k - 3 - t) - 2 < |\Sigma|$. We have

$$t(2n + k - 3 - t) - 2 \geqslant t(n + k - 2) \quad \Longleftrightarrow \quad n - 1 > t$$

for $t > 1$. We can therefore assume that $t \geqslant n - 1$, in which case

$$t(2n + k - 3 - t) - 2 \geqslant (n-1)(n + k - 2) > |\Sigma|.$$

It therefore follows by Corollary 16 that there exists a curve $C \subset \Pi$ of degree $2n + k - 6$ containing the set $\Sigma \setminus P$, but not containing P. Let Y be a general four-dimensional cone in $\mathbb{P}^5$ over the curve C. Then Y is the required hypersurface.

Let Π and Γ be sufficiently general hypersurfaces in $\mathbb{P}^5$ and

$$\psi \colon \mathbb{P}^5 \dashrightarrow \Pi$$

the projection from Γ. We set $\Sigma' = \psi(\Sigma) \subset \Pi \cong \mathbb{P}^2$ and $\widehat{P} = \psi(P) \in \Sigma'$.

Lemma 37. *Assume that the points of* $\Sigma' \subset \Pi$ *have property* $(\star)$. *Then there exists a hypersurface of degree* $2n + k - 6$ *in* $\mathbb{P}^5$ *containing the set* $\Sigma \setminus P$ *and not containing* $P \in \Sigma$.

Proof. The proof of Lemma 36 yields the existence of a curve $C \subset \Pi$ of degree $2n + k - 6$ containing $\Sigma' \setminus \widehat{P}$ but not passing through the point $\widehat{P}$. Let $Y \subset \mathbb{P}^5$ be the four-dimensional cone over C whose vertex is Γ. Then Y is the required hypersurface.

We can thus assume that the points of the set

$$\Sigma' \subset \Pi \cong \mathbb{P}^2$$

do not have property $(\star)$. Hence there exists $\Lambda_r^1 \subset \Sigma$ such that $|\Lambda_r^1| > r(n+k-2)$, while the set

$$\widetilde{\Lambda}_r^1 = \psi(\Lambda_r^1) \subset \Sigma' \subset \Pi \cong \mathbb{P}^2$$

lies in a curve $C \subset \Pi$ of degree r. Moreover, we can assume that r is the smallest positive integer with this property, so that the curve C is irreducible and reduced.

We iterate the construction of $\Lambda_r^1 \subset \Sigma$ and obtain a disjoint union of subsets $\Lambda_j^i \subset \Sigma$, $j = r, \dots, l \geqslant r$, such that $|\Lambda_j^i| > j(n+k-2)$, the points of the set

$$\widetilde{\Lambda}_j^i = \psi(\Lambda_j^i) \subset \Sigma'$$

lie on an irreducible curve in $\Pi \cong \mathbb{P}^2$ of degree j, and the points of the subset

$$\overline{\Sigma} = \Sigma' \setminus \bigcup_{j=r}^{l} \bigcup_{i=1}^{c_j} \widetilde{\Lambda}_j^i \subsetneq \Sigma' \subset \Pi \cong \mathbb{P}^2$$

satisfy property $(\star)$, where $c_j \geqslant 0$ is the number of the subsets $\widetilde{\Lambda}_j^i$.

By construction $c_r > 0$ and

$$|\overline{\Sigma}| < \frac{(n+k-2)(n-1)}{5} - \sum_{i=r}^{l} c_i(n-1)i = \frac{n+k-2}{5}\biggl(n-1-\sum_{i=r}^{l} 5ic_i\biggr). \tag{1}$$

Corollary 38. *The inequality $\sum_{i=r}^{l} ic_i < (n-1)/5$ holds.*

In particular, if $\Lambda_j^i \neq \varnothing$, then we have $j < (n-1)/5$.

Lemma 39. *Suppose that $\Lambda_j^i \neq \varnothing$. Let $\mathscr{M}$ be a linear system of hypersurfaces in $\mathbb{P}^5$ of degree j containing Λ_j^i. Then the base locus of $\mathscr{M}$ is zero-dimensional.*

Proof. It follows by the construction of the set Λ_j^i that all points in the subset

$$\widetilde{\Lambda}_j^i = \psi(\Lambda_j^i) \subset \Sigma' \subset \Pi \cong \mathbb{P}^2$$

lie in an irreducible curve $C \subset \Pi$ of degree j. Let Y be a cone over C in $\mathbb{P}^5$ whose vertex is some plane $\Upsilon \subset \mathbb{P}^5$. Then Y is a hypersurface in $\mathbb{P}^5$ of degree j containing all the points of the set Λ_j^i, therefore $Y \in \mathscr{M}$.

Assume that the base locus of the linear system $\mathscr{M}$ contains an irreducible curve $Z \subset \mathbb{P}^5$. Then $Z \subset Y$. However, it follows from the generality of ψ and the irreducibility of Z and C that $\psi(Z) = C$ and

$$\Lambda_j^i \subset Z,$$

and the restriction $\psi|_Z \colon Z \to C$ is a birational morphism; in particular, we have the equality $\deg(Z) = j$; but Z contains at least $|\Lambda_j^i| > j(n+k-2)$ points of $\Sigma \subset \mathbb{P}^4$, which is impossible by Proposition 35.

Corollary 40. *The inequality $r \geqslant 2$ holds.*

Let $\Xi_j^i \subset \mathbb{P}^5$ be the base locus of the linear system of hypersurfaces of degree j in $\mathbb{P}^5$ containing the set Λ_j^i. For $\Lambda_j^i = \varnothing$ we set $\Xi_j^i = \varnothing$. Then Ξ_j^i is a finite subset of $\mathbb{P}^5$ by Lemma 39, and we have $\Lambda_j^i \subseteq \Xi_j^i$.

Lemma 41. *Suppose that $\Xi_j^i \neq \varnothing$. Then the points in the set Ξ_j^i impose independent linear conditions on hypersurfaces of degree $5(j-1)$ in $\mathbb{P}^5$.*

This follows from Lemma 25.

In particular, the points in Λ_j^i impose independent linear conditions on the hypersurfaces in $\mathbb{P}^5$ of degree $5(j-1)$, provided that $\Lambda_j^i \neq \varnothing$.

Lemma 42. *Suppose that $\overline{\Sigma} = \varnothing$. Then there exists a hypersurface in $\mathbb{P}^5$ of degree $2n+k-6$ containing all points in the set $\Sigma \setminus P$ and not containing $P \in \Sigma$.*

Proof. We have a disjoint union of subsets

$$\Sigma = \bigcup_{j=r}^{l} \bigcup_{i=1}^{c_j} \Lambda_j^i,$$

and therefore there exists a unique set Λ_a^b containing the point P. In particular, P also lies in Ξ_a^b, although it is possible in principle that P lies in several sets Ξ_j^i.

It follows from Lemma 41 that for each non-empty set Ξ_j^i containing P there exists a hypersurface of degree $5(j-1)$ passing through all points in the set $\Xi_j^i \setminus P$, but not containing the point P. On the other hand, we see from the construction of the sets Ξ_j^i that for each non-empty set Ξ_j^i not containing P there exists a hypersurface of degree j passing through all the points in Ξ_j^i and not containing P.

We have $j < 5(j-1)$ because $j \geqslant r \geqslant 2$ (see Corollary 40).

Thus, for each Ξ_j^i containing P there exists a hypersurface $F_j^i \subset \mathbb{P}^5$ of degree $5(j-1)$ that contains the set $\Xi_j^i \setminus P$, but does not contain the point P. Consider the hypersurface

$$F = \bigcup_{j=r}^{l} \bigcup_{i=1}^{c_j} F_j^i \subset \mathbb{P}^5$$

of degree $\sum_{i=r}^{l} 5(i-1)c_i$. Then F contains $\Sigma \setminus P$ and does not contain P; moreover,

$$\deg(F) = \sum_{i=r}^{l} 5(i-1)c_i < \sum_{i=r}^{l} 5ic_i \leqslant n-1 \leqslant 2n+k-6$$

by Corollary 38 because $n \geqslant 5$.

Let $\widehat{\Sigma} = \bigcup_{j=r}^{l} \bigcup_{i=1}^{c_j} \Lambda_j^i$ and $\check{\Sigma} = \Sigma \setminus \widehat{\Sigma}$. Then $\Sigma = \widehat{\Sigma} \cup \check{\Sigma}$ and $\psi(\check{\Sigma}) = \overline{\Sigma} \subset \Pi$.

Remark 43. It follows from the proof of Lemma 42 that there exists a hypersurface $F \subset \mathbb{P}^5$ of degree $\sum_{i=r}^{l} 5(i-1)c_i$ such that F passes through all the points of the subset $\widehat{\Sigma} \setminus P \subsetneq \Sigma$ and does not contain the point $P \in \Sigma$.

We set $d = 2n+k-6-\sum_{i=r}^{l} 5(i-1)c_i$. We shall verify that the subset $\overline{\Sigma} \subset \Pi \cong \mathbb{P}^2$ and the positive integer d satisfy all conditions of Theorem 15. We can assume that $\widehat{\Sigma} \neq \varnothing$ and $\check{\Sigma} \neq \varnothing$.

Lemma 44. *The inequality $d \geqslant 5$ holds.*

This follows from Corollary 38 because $c_r \geqslant 1$.

Lemma 45. *The inequality $|\overline{\Sigma}| \leqslant (d^2+9d+10)/6$ holds.*

Proof. We shall show that $6(n+k-2)\big(n-1-\sum_{i=r}^{l} 5ic_i\big)$ does not exceed

$$5\left(2n+k-6-\sum_{i=r}^{l} 5(i-1)c_i\right)^2 + 45\left(2n+k-6-\sum_{i=r}^{l} 5(i-1)c_i\right) + 50,$$

which will yield the required result because

$$|\overline{\Sigma}| < \frac{(n+k-2)}{5}\left(n-1-5\sum_{i=r}^{l} ic_i\right)$$

by inequality (1).

Assume that the inequality in question fails. We set $A = n-1-\sum_{i=r}^{l} 5ic_i$ and $B = \sum_{i=r}^{l} 5c_i$. Then

$$6A(n+k-2) > 5(A+n+k-5+B)^2 + 45(A+n+k-5+B) + 50,$$

which is impossible since $A > 0$ by Corollary 38 and $n \geqslant 5$.

Lemma 46. *At most $t(d+3-t)-2$ points in the set $\overline{\Sigma}$ lie on a curve of degree t in $\mathbb{P}^2$ for each $t \leqslant (d+3)/2$.*

Proof. First, let $t = 1$. Then

$$\begin{aligned} t(d+3-t)-2 = d = 2n+k-6-\sum_{i=r}^{l} 5(i-1)c_i &\geqslant n+k-5+\sum_{i=r}^{l} 5c_i \\ &\geqslant n+k-5+5c_r > n+k-2 \end{aligned}$$

by Corollary 38. This means that at most d points in $\overline{\Sigma}$ lie on a line in $\mathbb{P}^2$ by Proposition 35.

Assume now that $t > 1$. The points in $\overline{\Sigma} \subset \mathbb{P}^2$ have property $(\star)$, therefore at most $(n+k-2)t$ points in $\overline{\Sigma}$ lie on a curve of degree t in $\mathbb{P}^2$. It is therefore sufficient to show that

$$t(d+3-t)-2 \geqslant (n+k-2)t$$

for all $t > 1$ such that $t \leqslant (d+3)/2$ and $t(d+3-t)-2 < |\overline{\Sigma}|$.

It is easy to see that

$$t(d+3-t)-2 \geqslant t(n+k-2) \quad \Longleftrightarrow \quad n-1-\sum_{i=r}^{l} 5(i-1)c_i > t$$

because $t > 1$. Assume that

$$n - 1 - \sum_{i=r}^{l} 5(i-1)c_i \leqslant t \leqslant \frac{d+3}{2}$$

and $t(d+3-t) - 2 < |\overline{\Sigma}|$. We shall show that this leads to a contradiction.

Let $g(x) = x(d+3-x) - 2$. Then $g(x)$ increases for $x \leqslant (d+3)/2$. Hence

$$g(t) \geqslant g\biggl(n - 1 - \sum_{i=r}^{l} 5(i-1)c_i\biggr),$$

therefore

$$\frac{n+k-2}{5}\biggl(n - 1 - \sum_{i=r}^{l} 5ic_i\biggr) > |\overline{\Sigma}| > g(t) \geqslant g\biggl(n - 1 - \sum_{i=r}^{l} 5(i-1)c_i\biggr).$$

Let $A = n - 1 - \sum_{i=r}^{l} 5ic_i$ and $B = \sum_{i=r}^{l} 5c_i$. Then

$$A\,\frac{n+k-2}{5} > g(A+B),$$

where $A > 0$ by Corollary 38. Hence

$$0 > 4(n+k-2)(A+B) + 5(A+B) - 2 \geqslant 118,$$

which is a contradiction.

It follows by Lemmas 44–46 that we can apply Theorem 15 to $\overline{\Sigma} \setminus \widehat{P} \subset \Pi \cong \mathbb{P}^2$ and the positive integer d. Hence there exists a curve $C \subset \Pi$ of degree

$$2n + k - 6 - \sum_{i=r}^{l} 5(i-1)c_i$$

containing $\overline{\Sigma} \setminus \widehat{P}$, but not containing $\widehat{P} = \psi(P)$. Let G be the four-dimensional cone in $\mathbb{P}^5$ over C with vertex Γ. Then G is a hypersurface of degree

$$2n + k - 6 - \sum_{i=r}^{l} 5(i-1)c_i$$

containing $\check{\Sigma} \setminus P$ and avoiding P. On the other hand, it follows from Remark 43 that there exists a hypersurface $F \subset \mathbb{P}^5$ of degree

$$\sum_{i=r}^{l} 5(i-1)c_i$$

containing $\widehat{\Sigma} \setminus P$ and not containing P. Then $F \cup G$ is a hypersurface of degree $2n + k - 6$ in $\mathbb{P}^5$ containing $\Sigma \setminus P$ and not containing $P \in \Sigma$, which completes the proof of Theorem 28.

§ 5. Double hypersurfaces in $\mathbb{P}^4$

Let $\eta\colon X \to F$ be a double cover such that F is a smooth hypersurface of degree $n \geqslant 2$ and η is branched in a nodal surface $S \subset F$ cut on the hypersurface F by a hypersurface $G \subset \mathbb{P}^4$ of degree $2r \geqslant n$. In this section we prove the following result.

Theorem 47. *Suppose that* $|\mathrm{Sing}(X)| \leqslant (2r+n-2)r/4$. *Then* X *is* $\mathbb{Q}$*-factorial.*

The following result is a consequence of Corollary 14.

Proposition 48. *The three-dimensional variety* X *is* $\mathbb{Q}$*-factorial if and only if the singular points of the surface* S *impose independent linear conditions on the sections in* $H^0(\mathscr{O}_{\mathbb{P}^4}(3r+n-5)|_F)$.

Corollary 49. *Suppose that* $|\mathrm{Sing}(X)| \leqslant 3r+n-4$. *Then* X *is* $\mathbb{Q}$*-factorial.*

We now prove Theorem 47. Assume that

$$|\mathrm{Sing}(X)| \leqslant \frac{(2r+n-2)r}{4}.$$

We shall show that the singular points of $S \subset \mathbb{P}^4$ impose independent linear conditions on hypersurfaces of degree $3r-n-5$. We can assume that $r \geqslant 3$ and $n \geqslant 2$ because otherwise the assertion of Theorem 47 follows from Corollary 49 and [32].

Lemma 50. *There exists a hypersurface* $\widehat{G} \subset \mathbb{P}^4$ *of degree* $2r$ *such that the surface* S *is a complete intersection of* $\widehat{G}$ *and* F, *but* $\mathrm{Sing}(\widehat{G}) \subseteq \mathrm{Sing}(S)$.

Proof. See the proof of Lemma 33.

We can thus assume that $\mathrm{Sing}(G) \subseteq \mathrm{Sing}(S)$.

Let $\Sigma = \mathrm{Sing}(S) \subset \mathbb{P}^4$, and let P be an arbitrary point in Σ. We must prove the existence of a hypersurface of degree $3r+n-5$ in $\mathbb{P}^4$ that contains $\Sigma \setminus P$ and does not contain P. It follows from the proof of Proposition 35 that at most $t(2r+n-2)$ points in the set Σ can lie on a curve of degree $t \in \mathbb{N}$ in $\mathbb{P}^4$.

Lemma 51. *Let* $\Pi \cong \mathbb{P}^2$ *be a plane such that* $\Sigma \subset \Pi \subset \mathbb{P}^4$. *Then there exists a hypersurface of degree* $3r+n-5$ *in* $\mathbb{P}^4$ *containing* $\Sigma \setminus P$ *and not containing* $P \in \Sigma$.

Proof. We shall verify all the conditions of Corollary 16 for the set $\Sigma \subset \Pi$ and the integer $d = 3r+n-5 \geqslant 6$.

The inequality

$$|\Sigma| \leqslant \frac{d^2+9d+16}{6}$$

is obvious because $r \geqslant 3$, $2r \geqslant n$, and $|\Sigma| \leqslant (2r+n-2)r/4$. We must therefore show that at most $t(3r+n-2-t)-2$ points in Σ lie on a curve of degree $t \leqslant (3r+n-2)/2$ in $\mathbb{P}^2$. It is sufficient to show that

$$t(3r+n-2-t)-2 \geqslant t(2r+n-2)$$

for all t such that $t \leqslant (3r+n-2)/2$ and $t(3r+n-2-t)-2 < |\Sigma|$.

We can assume that $t \geqslant 2$ because $3r+n-5 \geqslant 2r+n-2$. Then

$$t(3r+n-2-t)-2 \geqslant t(2r+n-2) \quad \Longleftrightarrow \quad r > t.$$

Assume that $r \leqslant t$ for some positive integer t such that

$$t \leqslant \frac{3r+n-2}{2}$$

and $t(3r+n-2-t)-2 < |\Sigma|$. Let $g(x) = x(3r+n-2-x)-2$. Then $g(x)$ increases for $x < (3r+n-2)/2$, and therefore $g(t) \geqslant g(r)$. Thus,

$$\frac{(2r+n-1)r}{4} \geqslant |\Sigma| > g(t) \geqslant g(r) = r(2r+n-2)-2,$$

which is impossible for $r \geqslant 3$.

It now follows by Corollary 16 that there exists a curve $C \subset \Pi$ of degree $3r+n-5$ passing through all the points in $\Sigma \setminus P$ and not passing through P. Let Y be a sufficiently general three-dimensional cone in $\mathbb{P}^4$ over C. Then Y is the required hypersurface.

Let Π and Γ be a general plane and a line in $\mathbb{P}^4$, respectively. Let $\psi\colon \mathbb{P}^4 \dashrightarrow \Pi$ be the projection from the line Γ. We set $\Sigma' = \psi(\Sigma) \subset \Pi \cong \mathbb{P}^2$ and $\widehat{P} = \psi(P) \in \Sigma'$.

Lemma 52. *Suppose that at most $t(2r+n-2)$ points of the set Σ' can lie on a (possibly reducible) curve $\Pi \cong \mathbb{P}^2$ of degree $t \in \mathbb{N}$. Then there exists a hypersurface in $\mathbb{P}^4$ of degree $3r+n-5$ that contains the set $\Sigma \setminus P$ and does not contain $P \in \Sigma$.*

Proof. It follows by the proof of Lemma 51 that there exists a curve $C \subset \Pi$ of degree $3r+n-5$ containing the set $\Sigma' \setminus \widehat{P}$ and not containing the point $\widehat{P}$. Let Y be a three-dimensional cone over C the vertex of which is the line Γ. Then Y is the required hypersurface in $\mathbb{P}^4$.

We can thus assume that the points in the set

$$\Sigma' \subset \Pi \cong \mathbb{P}^2$$

fail the conditions of Lemma 52. Hence there exists a subset $\Lambda^1_k \subset \Sigma$ such that $|\Lambda^1_k| > k(2r+n-2)$, but all points of the set

$$\widetilde{\Lambda}^1_k = \psi(\Lambda^1_k) \subset \Sigma' \subset \Pi \cong \mathbb{P}^2$$

lie in a curve $C \subset \Pi$ of degree k. Moreover, we can assume that k is the minimum positive integer with this property, so that C is irreducible and reduced.

We can iterate the construction of the set $\Lambda^1_k \subset \Sigma$ to obtain a disjoint union of subsets Λ^i_j of Σ, $j = k, \ldots, l \geqslant k$, such that $|\Lambda^i_j| > j(2r+n-2)$, the points of the set

$$\widetilde{\Lambda}^i_j = \psi(\Lambda^i_j) \subset \Sigma'$$

lie on an irreducible curve of degree j in Π and at most $t(2r+n-2)$ points of the set

$$\overline{\Sigma} = \Sigma' \setminus \bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \widetilde{\Lambda}^i_j \subsetneq \Sigma' \subset \Pi \cong \mathbb{P}^2$$

lie in a curve of degree t in $\mathbb{P}^2$, where $c_j \geqslant 0$ is the number of subsets $\widetilde{\Lambda}^i_j$.

By construction we have $c_k > 0$ and

$$|\overline{\Sigma}| < \frac{(2r+n-2)r}{4} - \sum_{i=k}^{l} c_i(2r+n-2)i = \frac{2r+n-2}{4}\left(r - \sum_{i=k}^{l} 4ic_i\right). \qquad (2)$$

Corollary 53. *The inequality $\sum_{i=k}^{l} ic_i < r/4$ holds.*

Lemma 54. *Assume that $\Lambda_j^i \neq \varnothing$. Let $\mathscr{M}$ be the linear system of hypersurfaces of degree j in $\mathbb{P}^4$ containing all points in the set Λ_j^i. Then the base locus of the linear system $\mathscr{M}$ is zero-dimensional.*

Proof. See the proof of Lemma 39.

Corollary 55. *The inequality $k \geqslant 2$ holds.*

Let $\Xi_j^i \subset \mathbb{P}^4$ be the base locus of the linear system of hypersurfaces of degree j in $\mathbb{P}^4$ that contain the set Λ_j^i. For $\Lambda_j^i = \varnothing$ we set $\Xi_j^i = \varnothing$. Then Ξ_j^i is a finite subset of $\mathbb{P}^4$ by Lemma 54, and $\Lambda_j^i \subseteq \Xi_j^i$.

Lemma 56. *Suppose that $\Xi_j^i \neq \varnothing$. Then the points of Ξ_j^i impose independent linear conditions on hypersurfaces of degree $4(j-1)$ in $\mathbb{P}^4$.*

Proof. This follows by Lemma 25.

In particular, the points in Λ_j^i impose independent linear conditions on hypersurfaces of degree $4(j-1)$ in $\mathbb{P}^4$ if $\Lambda_j^i \neq \varnothing$.

Lemma 57. *Let $\overline{\Sigma} = \varnothing$. Then there exists a hypersurface of degree $3r+n-5$ in $\mathbb{P}^4$ containing the set $\Sigma \setminus P$, but not containing the point $P \in \Sigma$.*

Proof. See the proof of Lemma 42.

Let $\widehat{\Sigma} = \bigcup_{j=k}^{l} \bigcup_{i=1}^{c_j} \Lambda_j^i$ and $\check{\Sigma} = \Sigma \setminus \widehat{\Sigma}$. Then $\Sigma = \widehat{\Sigma} \cup \check{\Sigma}$ and $\psi(\check{\Sigma}) = \overline{\Sigma} \subset \Pi$. Moreover, it follows from the proof of Lemma 57 that there exists a hypersurface $\Upsilon \subset \mathbb{P}^4$ of degree $\sum_{i=k}^{l} 4(i-1)c_i$ containing all points in the subset $\widehat{\Sigma} \setminus P \subsetneq \Sigma$ and not containing $P \in \Sigma$.

Let $d = 3r+n-5-\sum_{i=k}^{l} 4(i-1)c_i$. We shall verify that we can apply Theorem 15 to the subset $\overline{\Sigma} \subset \Pi \cong \mathbb{P}^2$ and the positive integer d; we can assume here that $\widehat{\Sigma} \neq \varnothing$ and $\check{\Sigma} \neq \varnothing$.

Lemma 58. *The inequality $d \geqslant 3$ holds.*

This follows from Corollary 53 because $r \geqslant 3$ and $c_k \geqslant 1$.

Lemma 59. *The inequality $|\overline{\Sigma}| \leqslant (d^2+9d+10)/6$ holds.*

Proof. We shall show that

$$6(2r+n-2)\left(r - \sum_{i=k}^{l} 4ic_i\right) \leqslant 4(d^2+9d+10),$$

which yields the required result. Assume that

$$6(2r+n-2)\left(r-\sum_{i=k}^{l}4ic_i\right)>4(d^2+9d+10),$$

and let $A=r-\sum_{i=k}^{l}4ic_i$ and $B=\sum_{i=k}^{l}c_i$. Then we see that

$$6A(2r+n-2)>4(2r+n-5+A+4B)^2+36(2r+n-5+A+4B)+40,$$

where $A>0$ by Corollary 53 and $r\geqslant 3$, which is a contradiction.

Lemma 60. *At most $t(d+3-t)-2$ points in the set $\overline{\Sigma}$ lie on a (possibly reducible) curve of degree t in $\mathbb{P}^2$ for each $t\leqslant (d+3)/2$.*

Proof. We start with the case $t=1$. Then it follows from Corollary 53 that

$$t(d+3-t)-2=d=3r+n-5-\sum_{i=k}^{l}4(i-1)c_i\geqslant 2r+n-5+4c_k\geqslant 2r+n-2.$$

Assume now that $t>1$. Then at most $(2r+n-2)t$ points in the set $\overline{\Sigma}$ lie on a curve of degree t in $\mathbb{P}^2$. It is therefore sufficient to show that

$$t(d+3-t)-2\geqslant (2r+n-2)t$$

for all $t>1$ such that $t\leqslant (d+3)/2$ and $t(d+3-t)-2<|\overline{\Sigma}|$. However,

$$t(d+3-t)-2\geqslant t(2r+n-2)\quad\Longleftrightarrow\quad r-\sum_{i=k}^{l}4(i-1)c_i>t$$

because $t>1$. We can thus assume that

$$r-\sum_{i=k}^{l}4(i-1)c_i\leqslant t\leqslant \frac{d+3}{2}$$

and $t(d+3-t)-2<|\overline{\Sigma}|$. We shall now derive a contradiction.

Let $g(x)=x(d+3-x)-2$. Then $g(x)$ increases for $x\leqslant (d+3)/2$. Hence

$$g(t)\geqslant g\left(r-\sum_{i=k}^{l}4(i-1)c_i\right).$$

Let $A=r-\sum_{i=k}^{l}4ic_i$ and $B=\sum_{i=k}^{l}c_i$. Then

$$A\frac{2r+n-2}{4}>g(A+4B)=(A+4B)(2r+n-5)-2,$$

which is impossible because $A>0$ by Corollary 53.

We have thus proved that we can apply Theorem 15 to the subset $\overline{\Sigma}\setminus\widehat{P}\subset\Pi\cong\mathbb{P}^2$ and the positive integer d. Hence there exists a curve $C\subset\Pi$ of degree

$$3r+n-5-\sum_{i=k}^{l}4(i-1)c_i$$

containing the set $\overline{\Sigma}\setminus\widehat{P}$ and not containing $\widehat{P}=\psi(P)$. Let Φ be the three-dimensional cone in $\mathbb{P}^4$ over C with vertex Γ. Then Φ is a hypersurface of degree

$$3r+n-5-\sum_{i=k}^{l}4(i-1)c_i$$

containing $\check{\Sigma}\setminus P$ and not containing P. On the other hand, we already have a hypersurface $\Upsilon\subset\mathbb{P}^4$ of degree

$$\sum_{i=k}^{l}4(i-1)c_i$$

that contains $\widehat{\Sigma}\setminus P$ and does not contain P. Hence $\Phi\cup\Upsilon$ is a hypersurface in $\mathbb{P}^4$ of degree $3r+n-5$ that contains $\Sigma\setminus P$ and does not contain the point $P\in\Sigma$.

The proof of Theorem 47 and therefore also of Theorem 8 is now complete.

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