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Birationally rigid Fano varieties

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Birationally rigid Fano varieties

Ivan Chel'tsov

Abstract. The birational superrigidity and, in particular, the non-rationality of a smooth three-dimensional quartic was proved by V. Iskovskikh and Yu. Manin in 1971, and this led immediately to a counterexample to the three-dimensional Lüroth problem. Since then, birational rigidity and superrigidity have been proved for a broad class of higher-dimensional varieties, among which the Fano varieties occupy the central place. The present paper is a survey of the theory of birationally rigid Fano varieties.

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Introduction

§ 0.1. Non-rationality

The rationality problem for algebraic varieties[1] is one of the most deep and interesting problems in algebraic geometry. The global holomorphic differential forms give a natural birational invariant for smooth surfaces and completely solve the rationality problem for algebraic curves and surfaces (see [182], [160], [91]). However, even in the three-dimensional case there are non-rational varieties that are close to rational varieties in many respects, and the known discrete invariants are insufficient to establish whether or not these varieties are rational. In particular, the following well-known result, which was announced already in [61], was proved in [94].

Theorem 0.1.1. *Let V be a smooth hypersurface in $\mathbb{P}^4$ of degree 4. Then the group $\mathrm{Bir}(V)$ of birational automorphisms coincides with the group $\mathrm{Aut}(V)$ of biregular automorphisms.*

One can readily see that Theorem 0.1.1 implies the non-rationality of every smooth quartic threefold in $\mathbb{P}^4$. Indeed, in the notation of Theorem 0.1.1, the linear system $|\mathcal{O}_{\mathbb{P}^4}(1)|_V|$ is invariant under the action of the group $\mathrm{Aut}(V)$, because the divisor $-K_V$ is linearly equivalent to a hyperplane section of the quartic V. Therefore, the group of biregular automorphisms of the quartic hypersurface V consists of projective automorphisms, and hence is finite (see [127]). Thus, the group of birational automorphisms of the smooth quartic threefold V is finite, which implies that V is non-rational, because the group $\mathrm{Bir}(\mathbb{P}^3)$ is infinite. Later, the technique of [94] was usually called the *method of maximal singularities* (see 0.2).

The non-rationality of any smooth quartic threefold immediately implied the negative solution of the Lüroth problem in dimension 3. We recall that the Lüroth problem in dimension n is as follows: Is it true that all subfields of the field $\mathbb{C}(x_1,\dots,x_n)$ that contain the field $\mathbb{C}$ are of the form $\mathbb{C}(f_1,\dots,f_k)$, where $f_i = f_i(x_1,\dots,x_n)$ is a rational function? Thus, there are non-rational threefolds that are unirational.[2] For example, the quartic

$$x_0^4 + x_0x_4^3 + x_1^4 - 6x_1^2x_2^2 + x_2^4 + x_3^4 + x_3^3x_4 = 0 \subset \mathrm{Proj}(\mathbb{C}[x_0,x_1,x_2,x_3,x_4]) \cong \mathbb{P}^4$$

is unirational (see [87] and [123]), smooth, and hence non-rational by Theorem 0.1.1. We note that, by the rationality criterion in [182], every unirational curve or surface is rational. The existence of counterexamples to the Lüroth problem was conjectured long ago (see [61]). For example, in the book [16] it is claimed that the three-dimensional Lüroth problem has a negative solution and, as an argument,

[1] All varieties under consideration are assumed to be projective, normal, and defined over the field of complex numbers. A variety V is said to be *rational* if the field of rational functions on V is isomorphic to the field $\mathbb{C}(x_1,\dots,x_n)$, or, equivalently, if there is a birational map $\rho\colon \mathbb{P}^n \dashrightarrow V$. By a *divisor* we always mean a $\mathbb{Q}$-divisor, that is, a formal finite $\mathbb{Q}$-linear combination of subvarieties of codimension one.

[2] A variety V is said to be *unirational* if there is a dominant rational map $\rho\colon \mathbb{P}^n \dashrightarrow V$ or, equivalently, if the field of rational functions of V is a subfield of $\mathbb{C}(x_1,\dots,x_n)$. Some non-trivial constructions of higher-dimensional unirational varieties can be found in [87], [113], [123], [44].

reference is made to the papers [57] and [60], which do not meet modern criteria of mathematical rigour.

The approach to the proof of Theorem 0.1.1 goes back to the theorem on the generators of the two-dimensional Cremona group (see [138] and Theorem 1.6.15). Moreover, the technique in the proof of the latter theorem was used earlier to study the birational geometry of rational surfaces over algebraically non-closed fields, and thus a general approach to the proof of Theorem 0.1.1 was known long before the paper [94]. However, the realization of the approach in the three-dimensional case faced unexpected difficulties related to the exclusion of infinitely close maximal singularities.

There are no simple ways to prove non-rationality in a non-trivial situation, for instance, in the class of higher-dimensional rationally connected varieties[3] or in the class of unirational varieties (see [92]). We note that any smooth quartic threefold is rationally connected (see [114]), but the unirationality of the quartic has been shown only in some special cases (see [123]), and the unirationality of a general smooth quartic threefold is an open problem. Moreover, it is even unknown whether or not there is a rationally connected non-unirational variety. At present there are only four known ways to prove that a rationally connected variety is non-rational:

- the method of maximal singularities (see [87], [96], [152], [93]);
- the use of the Griffiths component of the intermediate Jacobian of a rationally connected threefold as a birational invariant (see [41], [175]);
- the use of the torsion subgroup of the third integral cohomology group as a birational invariant (see [3], [43], [140]);
- the reduction to a positive characteristic with the subsequent application of the degeneration method (see [109], [110], [112], [48], [38]) to prove the non-rationality of many rationally connected varieties, for instance, the non-rationality of a general hypersurface in $\mathbb{P}^n$ of degree $d \geqslant \frac{2}{3}(n+2) \geqslant 4$ (see [109]).

Each of the existing methods for proving the non-rationality of rationally connected varieties has some advantages and disadvantages:

- the method of the intermediate Jacobian can be applied only to threefolds and, except for a single known example (a double cover of $\mathbb{P}^3$ ramified over a quartic; see [179] and [125]), only to threefolds fibred into conics (see [175]);
- in many cases the method of the intermediate Jacobian together with the degeneration method is the only way to prove the non-rationality of a threefold (see [4], [28], [31]), for instance, important special cases of the rationality criterion for conic bundles were proved in [167] by the method of the intermediate Jacobian (see [90]);
- there are non-rational rationally connected threefolds with trivial intermediate Jacobian (see [163]);

[3] A variety V is said to be *rationally connected* if for any two sufficiently general points of V there is a rational curve passing through them (see [114], [115], [110]). Any unirational variety is rationally connected.

- the third integral cohomology group is torsion-free in many interesting cases, for instance, for smooth complete intersections, and hence the approach of [3] cannot be used to prove the non-rationality;
- the method of [109] works in all dimensions, but its direct application proves the non-rationality of only a very general variety in a given family (with a single exception; see [48]);
- the method of maximal singularities works in all dimensions (see [152]), but it can be applied effectively only to varieties which are very far in a sense from being rational, for instance, it is hard to suppose that one can use the approach of [94] to construct an example of a smooth deformation of a non-rational variety into a rational one (see [175]).

Thus, the method of maximal singularities is the only known way to prove that a given rationally connected variety of dimension at least four is non-rational.

§ 0.2. Maximal singularity

The technique used in [94] was applied later to prove the non-rationality of many higher-dimensional rationally connected varieties. Moreover, the finiteness of the group of birational automorphisms is in fact inessential in the proof of the non-rationality of a smooth quartic threefold! Namely, as was implicitly proved in [94], there are no non-biregular birational maps between a smooth quartic threefold and a very broad class of threefolds including, for instance, the projective space, any cubic in $\mathbb{P}^4$, any complete intersection of a cubic and a quadric in $\mathbb{P}^6$, any arbitrary conic bundle, and any threefold fibred into rational surfaces. It turned out that, in a sense, the birational properties of a smooth quartic threefold recall those of a variety of general type[4] with ample canonical divisor.[5]

It later became clear that not only the smooth quartic threefolds but also some other rationally connected threefolds have similar properties. In time these varieties came to be called *birationally rigid* varieties. The notion of birational rigidity can be formalized in a very simple way. However, before formalizing, we shall first describe the geometric nature of birational rigidity, the so-called *Noether–Fano–Iskovskikh inequality*, or the *existence of a maximal singularity*. Let us consider the following rather informal questions:

- When does a birational map realize an isomorphism?
- What birational invariant can be used to prove the birational inequivalence of two varieties in a general situation?

We do not consider curves and surfaces, due to their marginality. For the same reason, we ignore the group of birational automorphisms and neglect conditions like the condition that *the spaces of global holomorphic forms have different dimensions*,

---

[4]A variety X is said to be *of general type* if $\dim(\phi_{|nK_X|}(X)) = \dim(X)$ for all positive integers $n \gg 0$. The varieties of general type are not rationally connected (see [115]).

[5]A divisor D on a variety X is said to be *ample* if nD is very ample for some positive integer $n > 0$, that is, if nD is a hyperplane section of the variety X. The property of being ample for a divisor means geometrically that the divisor has positive intersections with all curves on X. It follows from the Kleiman ampleness criterion [108] that D is ample if and only if D defines a positive function on the closure of the cone of effective one-dimensional cycles on X, the so-called Mori cone $\overline{\mathbb{NE}}(X)$ (see [106]).

because there are no global holomorphic forms on rationally connected varieties. A possible answer to the first question is given by the following result.

Theorem 0.2.1. *Let $\rho\colon X \dashrightarrow Y$ be a birational map, where X and Y are smooth varieties such that the divisors K_X and K_Y are ample. Then ρ is an isomorphism.*

Proof. The groups $H^0(\mathcal{O}_X(nK_X))$ and $H^0(\mathcal{O}_Y(nK_Y))$ are canonically isomorphic, because there is a birational map ρ inducing an isomorphism of the corresponding graded algebras. However, since the divisors K_X and K_Y are ample, it follows that

$$X \cong \operatorname{Proj}\biggl(\bigoplus_{n\geqslant 0} H^0(\mathcal{O}_X(nK_X))\biggr) \cong \operatorname{Proj}\biggl(\bigoplus_{n\geqslant 0} H^0(\mathcal{O}_Y(nK_Y))\biggr) \cong Y.$$

We now present an alternative proof or, more precisely, a more geometric version of the above proof. To this end, we consider a *Hironaka hut* for the birational map ρ, that is, the commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "g"'] \arrow[dr, "f"] & \\ X \arrow[rr, dashed, "\rho"'] & & Y \end{tikzcd}$$

such that W is a smooth variety and g and f are birational morphisms. Then

$$K_W \sim g^*(K_X) + \Sigma_X \sim f^*(K_Y) + \Sigma_Y, \tag{0.2.2}$$

where Σ_X and Σ_Y are exceptional divisors of the morphisms g and f, respectively. Since the varieties X and Y are smooth, it follows from the elementary properties of blow-ups that both the divisors Σ_X and Σ_Y are effective. Moreover, to prove this fact, it suffices to use only the formula for the modification of the canonical divisor under a blow-up of a smooth variety along a smooth subvariety, because we can always assume that one of the morphisms f and g is a composition of blow-ups of smooth subvarieties (see [76]). Moreover, the coefficients of the irreducible exceptional components of the divisors Σ_X and Σ_Y depend only on the corresponding discrete valuations, which can always be realized by blow-ups along subvarieties that are smooth at a general point of the varieties, and this enables one to explicitly compute the coefficients of Σ_X and Σ_Y in terms of the corresponding graph of the blow-ups (see [152]).

We choose a sufficiently large positive integer n. Since the divisors Σ_X and Σ_Y are effective and exceptional, it follows that the divisors $n\Sigma_X$ and $n\Sigma_Y$ are fixed components of the complete linear system

$$|nK_W| = |n(g^*(K_X) + \Sigma_X)| = |n(f^*(K_Y) + \Sigma_Y)|$$

for any positive integer n. Thus, the rational maps given by the complete linear systems $|nK_W|$, $|g^*(nK_X)|$, and $|f^*(nK_Y)|$ coincide. However, for $n \gg 0$ the complete linear system $|g^*(nK_X)|$ determines the morphism g up to a twist by a biregular automorphism of the variety X, because the divisor K_X is ample. On the other hand, the complete linear system $|f^*(nK_Y)|$ determines the morphism f up to a twist by an automorphism of the variety Y, because the divisor K_Y is ample. Thus, $\rho = f \circ g^{-1}$ is an isomorphism.

We note that the assertion of Theorem 0.2.1 is the very uniqueness of a canonical model.[6]

The most well-known answer to the second question posed above is the so-called *Kodaira dimension*. Let us recall what this is.

Definition 0.2.3. By the Kodaira dimension $\kappa(X)$ of a smooth variety X one means the maximal dimension of the image $\phi_{|nK_X|}(X)$ for $n \gg 0$ if at least one of the complete linear systems $|nK_X|$ is non-empty. Otherwise, $\kappa(X) = -\infty$.

The Kodaira dimension is a birational invariant. To prove this well-known result, it suffices to only slightly modify the proof of Theorem 0.2.1. We note that, by definition, the Kodaira dimension of a variety which is a canonical model coincides with the dimension of the variety itself.

Both Theorem 0.2.1 and Definition 0.2.3 relate mainly to varieties whose canonical divisor is *positive* in a certain sense. However, we are interested in varieties whose canonical divisor is *negative*, for example, rationally connected varieties, or, to be even more precise, varieties whose anticanonical divisor is ample. A variety with ample anticanonical divisor is called a *Fano variety*. The smooth Fano threefolds were classified in [85], [86], [135], [136], where 105 deformation families of smooth Fano threefolds were found (see [95]). The main discrete invariant of a smooth Fano threefold is its genus, that is, the genus of a smooth curve obtained as the intersection of two general anticanonical divisors. If the Picard group is one-dimensional, then the known values of the genera of smooth Fano threefolds recall somewhat the classification of groups of rational points of elliptic curves defined over the field of rational numbers (see [128]). This phenomenon is possibly related to some unknown connections between birational geometry and number theory (see [69]). Let us now show for the example of a smooth quartic threefold how one can modify Theorem 0.2.1 and Definition 0.2.3 to make them useful in the modified form.

Let V be a smooth quartic threefold in $\mathbb{P}^4$. We want to prove, say, that V is non-rational. Suppose that V is rational; let $\rho\colon V \dashrightarrow \mathbb{P}^3$ be a birational map. We try to use the construction in the proof of Theorem 0.2.1 to show that ρ is an isomorphism. The last assertion is clearly absurd, and therefore the proof of this assertion means that the quartic V is non-rational. We use the notation of the proof of Theorem 0.2.1, keeping in mind that the role of the variety X is played in the new situation by the quartic threefold V and the role of the variety Y by the projective space $\mathbb{P}^3$.

In the proof of Theorem 0.2.1 we made essential use of the fact that the varieties X and Y are smooth and the divisors K_X and K_Y are ample. The smoothness of X and Y was really used, for instance, in the proof of the very existence of the relation (0.2.2). However, this is not very important, because instead of the assumption that the varieties X and Y are smooth, one can assume that K_X and K_Y are Cartier divisors or that some multiples of the canonical divisors K_X and K_Y

[6] A smooth variety X is called a *canonical model* if the divisor K_X is ample.

are Cartier divisors. The last condition is usually referred to as the $\mathbb{Q}$-Gorenstein property of the varieties X and Y, respectively. The rational equivalence in the relation (0.2.2) must also be replaced by $\mathbb{Q}$-rational equivalence[7] or even by numerical equivalence (see [106]). In fact, the smoothness of X and Y is used in an essential way in the proof of Theorem 0.2.1 to show that the exceptional divisors Σ_X and Σ_Y are effective. As we shall see below, this simple remark is very important.

The varieties V and $\mathbb{P}^3$ are indeed smooth, but the canonical divisors K_V and $K_{\mathbb{P}^3}$ are not ample. On the contrary, the anticanonical divisors

$$-K_V \sim \mathcal{O}_{\mathbb{P}^4}(1)|_V \quad \text{and} \quad -K_{\mathbb{P}^3} \sim \mathcal{O}_{\mathbb{P}^3}(4)$$

are very ample! The last problem can readily be resolved; however, this can lead to the loss in a sense of the existing smoothness of V and $\mathbb{P}^3$. To make a *negative* divisor *positive*, we must add a sufficiently *positive* divisor. For example, we can take an arbitrary hyperplane $H \subset \mathbb{P}^3$ and consider a *log pair* $(\mathbb{P}^3, 5H)$ instead of the projective plane $\mathbb{P}^3$ and the *log canonical* divisor $K_{\mathbb{P}^3} + 5H$, which is an ample divisor by construction, instead of the divisor $K_{\mathbb{P}^3}$. In these cases the additional divisor $5H$ is called a *boundary*. We now consider the commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "g"'] \arrow[dr, "f"] & \\ V \arrow[rr, dashed, "\rho"'] & & \mathbb{P}^3 \end{tikzcd},$$

where the variety W is smooth and $g\colon W \to V$ and $f\colon W \to \mathbb{P}^3$ are birational morphisms.

Let us try to literally repeat the scheme of the proof of Theorem 0.2.1. One must compare the log canonical divisor of the variety W with the pullback of the log canonical divisor $K_{\mathbb{P}^3} + 5H$ and with the pullback of the log canonical divisor of the quartic. However, the canonical divisor is said to be canonical for the very reason that it is canonically defined on all birationally equivalent models, and the log canonical divisor is no longer defined canonically!

We must choose how to define an appropriate log pair on the variety W and on the quartic V. The simplest way is to take as boundaries the proper transforms of the hyperplane H on W and V that are formally multiplied by 5, but the following problem can occur: in principle, the birational map ρ^{-1} can contract the hyperplane H to a point or to a curve. In this case the analogue of the relation (0.2.2) does not meet the necessary requirements. However, instead of the hyperplane H, we can take a hypersurface in $\mathbb{P}^3$ of a sufficiently large degree or take a hyperplane H in a sufficiently general way, which automatically ensures its incontractibility by the birational map ρ^{-1}. Nevertheless, it is better to proceed in another way.

Instead of the hyperplane $H \subset \mathbb{P}^3$, let us take the complete linear system $\Lambda = |H|$ as the boundary and consider a *movable log pair* $(\mathbb{P}^3, \mu\Lambda)$, where μ is an arbitrary

[7]Divisors D_1 and D_2 are said to be $\mathbb{Q}$-rationally equivalent if $nD_1 \sim nD_2$ for some integer $n \neq 0$.

positive rational number. In this case one can always take a proper transform of the linear system Λ on the variety W and on the quartic V. We denote these transforms by Λ_W and Λ_V, respectively.

The linear systems Λ, Λ_W, and Λ_V have the following properties:

- Λ, Λ_W, and Λ_V have no fixed components, which means that their base loci have codimension at least two;
- we can treat each of Λ, Λ_W, and Λ_V as a divisor, replacing the corresponding linear system by a sufficiently general divisor belonging to the system;
- the formal sums $K_{\mathbb{P}^3}+\mu\Lambda$, $K_W+\mu\Lambda_W$, and $K_V+\mu\Lambda_V$ are well defined as elements of the groups $\mathrm{Pic}(\mathbb{P}^3)\otimes\mathbb{Q}$, $\mathrm{Pic}(W)\otimes\mathbb{Q}$, and $\mathrm{Pic}(V)\otimes\mathbb{Q}$, respectively, for which the notion of ampleness is defined;
- by taking general divisors in each of Λ, Λ_W, and Λ_V, we can define in a natural way multiplicities of the linear systems Λ, Λ_W, and Λ_V at general points of subvarieties of the varieties $\mathbb{P}^3$, W, and V, respectively, and the multiplicity of Λ at every point or curve is zero, because the base locus of Λ is empty;
- by taking two general divisors in each of Λ, Λ_W, and Λ_V, we can define in a natural way one-dimensional effective cycles Λ^2, Λ_W^2, and Λ_V^2.

We can now repeat the arguments in the proof of Theorem 0.2.1 for the chosen movable log pairs, replacing the canonical divisors by the log canonical ones. We have

$$K_W+\mu\Lambda_W\sim_{\mathbb{Q}} g^*(K_V+\mu\Lambda_V)+\Sigma_V\sim_{\mathbb{Q}} f^*(K_{\mathbb{P}^3}+\mu\Lambda)+\Sigma_{\mathbb{P}^3}, \tag{0.2.4}$$

where Σ_V and $\Sigma_{\mathbb{P}^3}$ are exceptional divisors of the morphisms g and f, respectively. However, the divisors Σ_V and $\Sigma_{\mathbb{P}^3}$ can fail to be effective. To be precise, the divisor Σ_V can fail to be effective, whereas the divisor $\Sigma_{\mathbb{P}^3}$ is effective. Indeed, the linear system Λ is free, which readily implies that the exceptional divisor $\Sigma_{\mathbb{P}^3}$ in the equivalence (0.2.4) must be the same as that for $\Lambda=\varnothing$. On the other hand, by construction, the linear system Λ_V can have base points or base curves, which implies that the divisor Σ_V can also have negative coefficients.

The fact that the g-exceptional divisor Σ_V can have negative coefficients is directly related to the notion of singularities of the movable log pair $(V,\mu\Lambda_V)$. The singularities of log pairs generalize singularities of varieties (see [111]). In particular, the singularities of the log pair $(V,\mu\Lambda_V)$ are said to be canonical (see Definition 1.3.3) if the exceptional divisor Σ_V in the equivalence (0.2.4) is effective for any choice of the birational morphism g. If the boundary is empty, then this definition coincides with the classical definition of canonical singularities of a variety. Canonical singularities occur naturally on canonical models of smooth varieties of general type. For instance, let U be a smooth variety of dimension n having a big and nef canonical divisor K_U, that is, let $K_U\cdot C\geqslant 0$ for every curve $C\subset U$ and let $K_U^n>0$. Then the canonical model $\mathrm{Proj}(\oplus_{n\geqslant 0}H^0(\mathcal{O}_U(nK_U)))$ has canonical singularities. Canonical singularities are rational, whereas Gorenstein rational singularities are canonical (see [55]) and canonical singularities in dimension 2 are Du Val points (see [2]).

The singularities of the log pair $(\mathbb{P}^3,\mu\Lambda)$ are canonical, because the linear system Λ has no base points. On the other hand, one can readily make the divisors $K_{\mathbb{P}^3}+\mu\Lambda$

and $K_V + \mu\Lambda_V$ ample by choosing μ sufficiently large. Indeed, the group $\mathrm{Pic}(V)$ is generated by the anticanonical divisor $-K_V$ by the Lefschetz theorem, and hence the divisor $K_V + \mu\Lambda_V$ is ample for every rational $\mu > \frac{1}{n}$, where n is a positive integer such that $\Lambda_V \subset |-nK_V|$. That is, the surfaces in the linear system Λ_V are cut out by hypersurfaces of degree n on the quartic threefold $V \subset \mathbb{P}^4$. However, the divisor $K_{\mathbb{P}^3} + \mu\Lambda$ is ample for every $\mu > 4$. Therefore, if $\mu > 4$, then we have everything we need to use the arguments in the proof of Theorem 0.2.1, except for the effectiveness of the g-exceptional divisor Σ_V defined by the relation (0.2.4).

Since the divisor Σ_V can fail to be effective, we cannot efficiently apply the scheme of the proof of Theorem 0.2.1 to the quartic V and the projective space $\mathbb{P}^3$. This is not surprising, because otherwise we could apply the previous arguments to $\mathbb{P}^3$ instead of the quartic V and prove the coincidence of the groups $\mathrm{Bir}(\mathbb{P}^3)$ and $\mathrm{Aut}(\mathbb{P}^3)$, which is certainly absurd.

Thus, the scheme of the proof of Theorem 0.2.1 has logically led us to the constructions of the movable log pairs $(\mathbb{P}^3, \mu\Lambda)$ and $(V, \mu\Lambda_V)$, which can be called *birationally equivalent* pairs (see 1.3). Let us now recall the classical birational invariant, namely, the Kodaira dimension.

We can readily define the Kodaira dimension of the movable log pairs $(\mathbb{P}^3, \mu\Lambda)$ and (V, Λ_V) thus constructed by simply replacing the canonical divisor in Definition 0.2.3 by the corresponding log canonical divisor. The numbers (or the symbol $-\infty$) thus obtained can be denoted by $\kappa(\mathbb{P}^3, \mu\Lambda)$ and $\kappa(V, \Lambda_V)$ by analogy with the classical case. However, it is desirable that the Kodaira dimension of a movable log pair be a birational invariant, that is, be the same for birationally equivalent movable log pairs. For example, it would be desirable to have the equality $\kappa(\mathbb{P}^3, \mu\Lambda) = \kappa(V, \mu\Lambda_V)$. Although we face a small problem here, the problem can readily be resolved.

Above we defined the Kodaira dimension for smooth varieties (see Definition 0.2.3). However, due to possible non-effectiveness of the divisor Σ_V defined by the relation (0.2.4), one must define the Kodaira dimension of movable log pairs as a generalization of the Kodaira dimension of singular varieties. On the other hand, it is easy to see that the above classical definition of the Kodaira dimension is no longer a birational invariant if we omit the smoothness condition on X in Definition 0.2.3.

Example 0.2.5. Let S be a quartic surface in $\mathbb{P}^3$. Then it follows immediately from Definition 0.2.3 that $\kappa(S) = 0$. On the other hand, if S is a cone over a plane quartic curve C, then the quartic S is birationally equivalent to $\mathbb{P}^1 \times C$, but $\kappa(\mathbb{P}^1 \times C) = -\infty$ by Definition 0.2.3. Thus, the *correctly defined* Kodaira dimension $\kappa(S)$ vanishes if and only if the surface S has at most Du Val singularities (see Remarks 0.2.6 and 1.3.17).

Therefore, we must define the Kodaira dimension of a singular variety as the Kodaira dimension of its desingularization. Using the proof of Theorem 0.2.1, one can readily see that, under this definition, the Kodaira dimension is then well defined, that is, it does not depend on the choice of a desingularization and is a birational invariant.

Remark 0.2.6. The Kodaira dimension of a variety with canonical singularities can be defined at once in the same way as in Definition 0.2.3, that is, without passing to a desingularization (see Remark 1.3.17).

The definition of the Kodaira dimension of a singular variety easily generalizes to movable log pairs (see Definition 1.3.15). For example, one must set the number $\kappa(\mathbb{P}^3, \mu\Lambda)$ equal to the maximal dimension of the image $\dim(\phi_{|n(K_W+\mu\Lambda_W)|}(W))$ for a sufficiently divisible positive integer $n \gg 0$ and simply set $\kappa(\mathbb{P}^3, \mu\Lambda) = -\infty$ if $|n(K_W + M_W)| = \varnothing$. Then $\kappa(\mathbb{P}^3, \mu\Lambda) = \kappa(V, \mu\Lambda_V)$.

It follows from Remark 0.2.6 that we can define the Kodaira dimension of movable log pairs having canonical singularities without taking their desingularization. This implies the equality

$$\kappa(\mathbb{P}^3, \mu\Lambda) = \begin{cases} -\infty & \text{for } \mu < 4, \\ 0 & \text{for } \mu = 4, \\ 3 & \text{for } \mu > 4. \end{cases}$$

The elements of the linear system $\Lambda_V = \rho^{-1}(\Lambda)$ are cut out on $V \subset \mathbb{P}^4$ by hypersurfaces of degree $n \in \mathbb{N}$ in $\mathbb{P}^4$. We write $\mu = \frac{1}{n}$. Then

$$\kappa(V, \mu\Lambda_V) = \kappa(\mathbb{P}^3, \mu\Lambda) = -\infty,$$

because $\frac{1}{n} < 4$. On the other hand, the divisor $-K_V$ is rationally equivalent to a hyperplane section of the quartic V. Hence, the divisor $K_V + \mu\Lambda_V$ is numerically trivial by the very choice $\mu = \frac{1}{n}$, and the divisor $n(K_V + \mu\Lambda_V)$ is rationally equivalent to zero.

If we were to define the number $\kappa(V, \mu\Lambda_V)$ immediately by analogy with Definition 0.2.3, then we would obtain the equality $\kappa(V, \mu\Lambda_V) = 0$, which contradicts the above equality $\kappa(V, \mu\Lambda_V) = -\infty$. The contradiction is due to the fact that the singularities of the log pair $(V, \frac{1}{n}\Lambda_V)$ are not canonical (see Remark 0.2.6) and, to be more specific, the divisor Σ_V in the relation (0.2.4) is not effective.

Hence, the assumption that the quartic V is rational led us to the existence of a linear system Λ_V on V such that Λ_V has no fixed components and the singularities of the movable log pair $(V, \frac{1}{n}\Lambda_V)$ are not canonical, namely, there is a birational morphism $g\colon W \to V$ such that the following $\mathbb{Q}$-rational equivalence holds:

$$K_W + \frac{1}{n}\Lambda_W \sim_{\mathbb{Q}} g^*\left(K_V + \frac{1}{n}\Lambda_V\right) + \sum_{i=1}^{k} a_i E_i \sim_{\mathbb{Q}} \sum_{i=1}^{k} a_i E_i,$$

where $\Lambda_W = g^{-1}(\Lambda_V)$, E_i is an irreducible exceptional divisor of the morphism g, a_i is a rational number, and the inequality $a_r < 0$ holds for some r. The image $g(E_r)$ of the divisor E_r on the quartic threefold V is usually called a *maximal singularity* (see [87], [45], [152]).

The above arguments can be applied not only to a smooth quartic threefold but also to an arbitrary smooth Fano threefold whose Picard group is $\mathbb{Z}$ and, in particular, to the projective space $\mathbb{P}^3$. So what then is the specific feature of a smooth quartic threefold? The point is that a maximal singularity cannot exist on a smooth quartic threefold, because its anticanonical degree is relatively small.

We present an outline for proving that a smooth quartic threefold cannot have a maximal singularity (see 2.1). Let $C = g(E_r) \subset V$. Then:

- either the maximal singularity C is an irreducible curve;
- or the maximal singularity C is a point.

Suppose that C is a curve. In this case at a general point of C the morphism g is a composition of blow-ups of smooth curves dominating C and a blow-up of C itself. We can compute the coefficient a_r locally in a neighbourhood of a general point of C by induction on the number of blow-ups forming the morphism g at a general point of C. This implies the inequality

$$\mathrm{mult}_C(\Lambda_V) > n,$$

which means that the multiplicity of a general surface in the linear system Λ_V at a general point of C exceeds n, and the surfaces in Λ_V are cut out on V by hypersurfaces of degree n in $\mathbb{P}^4$. We use the original approach of [147] to prove that the inequality $\mathrm{mult}_C(\Lambda_V) > n$ is impossible (see Proposition 1.3.12).

Suppose that C is a line. Let Π be a general plane in $\mathbb{P}^4$ containing C. Then

$$\Pi \cap V = C \cup Z,$$

where Z is a planar cubic curve. The curves C and Z intersect at three points on the plane Π. It is not obvious that the points of intersection are distinct, but this follows from the fact that the plane is in general position and the quartic threefold V is smooth (see the proof of Proposition 1.3.12). It is also clear that the cubic curve Z is not contained in the base locus of the linear system Λ_V. Hence, by restricting a general surface S of the linear system Λ_V to the curve Z, we see immediately that

$$3n = \deg(Z)n = \deg(S|_Z) \geqslant \sum_{O \in Z \cap C} \mathrm{mult}_O(S)\,\mathrm{mult}_O(Z) > \sum_{O \in Z \cap C} n = 3n,$$

which is a contradiction. If C is not a line, then we can consider a general cone over the curve C instead of the plane Π and arrive at a contradiction in a similar way (see [147] and [48]).

Remark 0.2.7. In the situation treated above, the inequality $\mathrm{mult}_C(\Lambda_V) > n$ readily implies the inequality $\deg(C) < 4$. Thus, the above scheme of arriving at a contradiction can be replaced by considering all possible cases with respect to the degree of the curve $C \subset \mathbb{P}^4$, as in [94].

Thus, we must eliminate the case in which C is a point. This is the most complicated part of the proof of non-rationality of a quartic threefold. The fact that this case is impossible was proved in [94] by using global methods. However, a new local approach to the problem was found in [149]. Namely, it was proved that the non-canonical property of singularities of the movable log pair $(V, \frac{1}{n}\Lambda_V)$ at the point C implies the inequality

$$\mathrm{mult}_C(S_1 \cdot S_2) > 4n^2 \tag{0.2.8}$$

for general divisors S_1 and S_2 in the linear system Λ_V, where the intersection $S_1 \cdot S_2$ is understood in the scheme-theoretic sense (see Theorem 1.7.18). The inequality (0.2.8) leads immediately to a contradiction. Indeed, it suffices to intersect the one-dimensional cycle $S_1 \cdot S_2$ by a sufficiently general hyperplane section passing through C of the smooth quartic threefold V and to use the equality $-K_V^3 = 4$. The contradiction thus obtained implies the non-rationality of the smooth quartic threefold $V \subset \mathbb{P}^4$.

We note that the non-rationality of the quartic $V \subset \mathbb{P}^4$ holds due to the fact that the number $-K_V^3$, which is usually referred to as the degree of the Fano threefold V, is relatively small. For example, for the projective space $\mathbb{P}^3$ we have $-K_{\mathbb{P}^3}^3 = 64$, but all smooth Fano threefolds of degrees greater than 24 are rational (see [95]).

The inequality (0.2.8) is usually called the $4n^2$-inequality. We note that this inequality is local in nature and is not related to any smooth quartic threefold (see Theorem 1.7.18). Moreover, it turned out later that the inequality (0.2.8) has a very deep geometric meaning related to the classification of extremal birational contractions into a smooth threefold point (see [45] and [102]).

§ 0.3. Birational rigidity

To define a *birationally rigid* variety, we must recall what the *Minimal Model Programme* is (see 1.1), which is abbreviated to MMP everywhere below. The MMP is an algorithm birationally transforming any variety X into a variety Y for which one of the following possibilities holds:

- the canonical divisor K_Y is numerically effective;[8]
- there is a morphism $\xi\colon Y \to Z$ such that ξ has connected fibres and is not birational, the anticanonical divisor $-K_Y$ is ξ-ample, and $\operatorname{rk}\operatorname{Pic}(Y) = \operatorname{rk}\operatorname{Pic}(Z)+1$, where the ξ-ampleness of $-K_Y$ means that the divisor $-K_Y + \xi^*(D)$ is ample for some ample divisor D on the variety Z.

Generally, the output product of the MMP is not uniquely determined.

Example 0.3.1. The surfaces $\mathbb{P}^2$ and $\mathbb{F}_n$ are birationally equivalent, and each of them is an output product of the MMP. Moreover, when applying the MMP to any rational surface, we birationally transform this surface either to $\mathbb{P}^2$ or to $\mathbb{F}_n$.

On the other hand, the varieties satisfying the conditions of Theorem 0.2.1 are also output products of the MMP. However, these products are uniquely determined due to Theorem 0.2.1, even up to a biregular automorphism.

Example 0.3.2. Let V be a smooth quartic threefold in $\mathbb{P}^4$. It follows from the adjunction formula and the Lefschetz theorem that V is an output product of the MMP, and it follows from arguments in [94] that V is a unique output product of the MMP. In particular, all birational transformations of V used during the MMP lead back to the quartic threefold V, whereas the resulting birational transformation is certainly a biregular automorphism of V and is in turn projective.

In a sense the property of being a unique output product of the MMP was taken as a definition of a birationally rigid variety.

[8]A divisor D on a variety X is said to be *numerically effective* or, briefly, *nef* if the intersection of D with any curve on X is non-negative.

Definition 0.3.3. A Fano variety X with terminal and $\mathbb{Q}$-factorial singularities (see Definition 1.1.1; for example, a smooth variety) such that $\operatorname{rk}\operatorname{Pic}(X) = 1$ is said to be *birationally rigid* if the following conditions hold:

- X cannot be birationally transformed into a variety Y for which there is a non-birational surjective morphism $\xi\colon Y \to Z$ whose general fibre has Kodaira dimension $-\infty$;
- X cannot be birationally transformed into a variety Y such that Y is a Fano variety with terminal and $\mathbb{Q}$-factorial singularities, the equality $\operatorname{rk}\operatorname{Pic}(Y) = 1$ holds, and Y is not biregularly equivalent to X.

The birationally rigid Fano varieties are non-rational, and they cannot be birationally transformed into conic bundles or fibrations into rational surfaces. In particular, there are no birationally rigid del Pezzo surfaces defined over an algebraically closed field, though there are birationally rigid del Pezzo surfaces defined over a field which is not algebraically closed (see [120]–[122], [91], Theorem 1.5.1). We recall that del Pezzo surfaces are two-dimensional Fano varieties.

Definition 0.3.4. A birationally rigid Fano variety X having terminal and $\mathbb{Q}$-factorial singularities (for example, a smooth variety) such that $\operatorname{rk}\operatorname{Pic}(X) = 1$ is said to be *birationally superrigid* if the groups $\operatorname{Bir}(X)$ (of birational automorphisms) and $\operatorname{Aut}(X)$ (of biregular automorphisms) coincide.

There are birationally rigid Fano varieties that are not birationally superrigid.

Example 0.3.5. Let X be a nodal three-dimensional quartic in $\mathbb{P}^4$, that is, the singularities of the variety X are isolated ordinary double points. Suppose that $\operatorname{rk}\operatorname{Cl}(X) = 1$. Then X is a birationally rigid Fano variety with terminal and $\mathbb{Q}$-factorial singularities (see [94], [146], [46], [129]), and the quartic X is birationally superrigid if and only if X is smooth. The equality $\operatorname{rk}\operatorname{Cl}(X) = 1$ holds for $|\operatorname{Sing}(X)| \leqslant 8$ (see Corollary 2.1.10), but nodal rational quartic threefolds exist, for example, determinantal quartics (see [139]).

The proof of the non-rationality of a smooth quartic threefold V sketched in 0.2 can readily be modified to show that the threefold V is birationally superrigid (see Theorem 1.4.1). Moreover, as was already mentioned in 0.2, the non-rationality of V is due mainly to the inequality $-K_V^3 \leqslant 4$, because the proof of the non-rationality of V made essential use of the inequality (0.2.8).

Remark 0.3.6. Among the smooth three-dimensional Fano varieties, only the following four varieties are birationally rigid:

- a smooth quartic in $\mathbb{P}^4$ (see 2.1),
- a double cover of $\mathbb{P}^3$ ramified along a smooth sextic surface (see 2.2; this variety can also be realized as a smooth hypersurface of degree 6 in $\mathbb{P}(1^4, 3)$),
- a double cover of a smooth quadric in $\mathbb{P}^4$ ramified along a smooth surface of degree 8 (see 2.3; this variety can also be realized as a smooth complete intersection of a quadric cone and a quartic in $\mathbb{P}(1^5, 2)$),
- a smooth complete intersection of a quadric and a cubic in $\mathbb{P}^5$ (see 2.4),

where only the smooth three-dimensional quartic and the double cover of $\mathbb{P}^3$ ramified along a smooth sextic surface are birationally superrigid.

Let us now consider an arbitrary n-dimensional smooth Fano variety X such that $\operatorname{rk}\operatorname{Pic}(X)=1$. It is natural to expect that X is birationally superrigid for relatively small values of $(-K_X)^n$, because the inequality (0.2.8) holds in any dimension greater than 2 (see Theorem 1.7.18). Moreover, many examples confirm this assumption (see [145], [154], [63], [29]). However, the proof of birational superrigidity of a smooth quartic threefold makes essential use of the projective geometry of the threefold. Hence, the birational superrigidity of X for small values of $(-K_X)^n$ is not obvious in general. For example, the following very natural conjecture is still open for any n, though it seems to be obvious.

Conjecture 0.3.7. *If $(-K_X)^n=1$, then X is birationally superrigid.*

The equality $(-K_X)^n=1$ is impossible if $n=3$. If n is even, then the equality $(-K_X)^n=1$ holds, for example, for smooth hypersurfaces of degree $2(n+1)$ in the weighted projective space $\mathbb{P}(1^4,2,n+1)$. For $n=2$ the condition $\operatorname{rk}\operatorname{Pic}(X)=1$ can hold if the field of definition of the variety X is not algebraically closed (see [120], [121]). There are some much more general conjectures similar to Conjecture 0.3.7, and one of them is as follows (see [156]).

Conjecture 0.3.8. *Suppose that $n\geqslant 5$ and the group $\operatorname{Pic}(X)$ is generated by the anticanonical divisor of the variety X. Then X is birationally superrigid.*

The number $(-K_X)^n$ is called the *degree* of the Fano variety X. The inequality (0.2.8) readily implies the birational rigidity of X if $(-K_X)^n\leqslant 4$ whenever the linear system $|-K_X|$ has no base points (see [87], [145], [29]). However, there are few Fano varieties satisfying the last two conditions (for example, double spaces and double quadrics). The first example of a birationally superrigid Fano variety of degree greater than 4 was found in [144], where it was proved that a smooth quintic hypersurface in $\mathbb{P}^5$, which is a Fano variety of degree 5, is birationally superrigid. Informally speaking, this overcame the *degree barrier* for the first time, but a qualitative step in overcoming the degree barrier was made in [149], where the following result was proved (see 3.1).

Theorem 0.3.9. *Let $X\subset\mathbb{P}^n$ be a general hypersurface of degree $n\geqslant 5$. Then X is a birationally superrigid Fano variety of degree n.*

The methods of [149] were later used in [151], [153], [156], and an $8n^2$-inequality proved in [21] logically generalized the $4n^2$-inequality to the case of smooth varieties of dimension 4 and higher. This result led to a new proof of birational superrigidity for a smooth quintic fourfold in $\mathbb{P}^5$ and to a proof of birational superrigidity for smooth hypersurfaces of degree n in $\mathbb{P}^n$ when $n\leqslant 8$. The methods of [21] were used later in [27] and [34]. The birational superrigidity of any smooth hypersurface of degree n in $\mathbb{P}^n$ for all $n\geqslant 6$ was proved in [154]. As was indicated in [63], the proof of the main result in [154] contained a small gap, but the gap was removed in [63] for $n\leqslant 12$.

PART 1. PRELIMINARIES

§ 1.1. Minimal Model Programme

The paper [106] and the book [126] give a very good introduction to the Minimal Model Programme, which is abbreviated to MMP in what follows. Informally, we can describe the MMP as a *black box* into which you can put a variety X and then take out a birationally equivalent variety Y for which one of the following two possibilities holds:

- the divisor K_Y is numerically effective;
- there is a morphism $\xi\colon Y \to Z$ such that ξ has connected fibres, the inequality $\dim(Z) < \dim(Y)$ holds, the anticanonical divisor $-K_Y$ is ξ-ample, the equality $\operatorname{rk}\operatorname{Pic}(Y/Z) = 1$ holds, and, in particular, the variety Y is covered by curves having negative intersection with the divisor K_Y.

If the canonical divisor K_Y is numerically effective, then it is customary to refer to the variety Y as a *minimal model*. If the morphism ξ exists, then it is customary to call ξ a *Mori fibration* and to call the variety Y a *Mori fibred space*. The output product of the MMP can fail to be uniquely determined (see Example 0.3.1), but there are varieties for which the output product of the MMP is uniquely determined, even up to biregular automorphism: for example, two-dimensional minimal models or canonical models (see Theorem 0.2.1). At the moment, the MMP has been proved for varieties of dimension not exceeding 4 (see [106], [134], [171]).

Beginning with dimension 3, the variety Y can be singular in each possible case, and singular varieties must be admissible. The following types of singularities can occur during the MMP.

Definition 1.1.1. A variety X is said to *have terminal singularities* if the canonical divisor K_X is a $\mathbb{Q}$-Cartier divisor (that is, some multiple of K_X is a Cartier divisor) and for any birational morphism $f\colon W \to X$ we have

$$K_W \sim_{\mathbb{Q}} f^*(K_X) + \sum_{i=1}^{k} a(X, E_i)E_i,$$

where E_i is an irreducible f-exceptional divisor and $a(X, E_i)$ is a positive rational number.

It can readily be seen that a smooth variety has terminal singularities and that surfaces with terminal singularities are smooth. Terminal singularities have been extensively studied in dimension 3 (see [133] and [159]). Isolated ordinary double points are terminal singularities starting from dimension 3.

Remark 1.1.2. It is usually assumed when using the MMP that the singularities of varieties are $\mathbb{Q}$-factorial, because $\mathbb{Q}$-factoriality is preserved by the MMP (we recall that a variety is said to have $\mathbb{Q}$-factorial singularities if every Weil divisor on the variety has a multiple which is a Cartier divisor).

We describe an iterative application of the MMP to a three-dimensional variety X, assuming that the singularities of X are terminal and $\mathbb{Q}$-factorial. Let $\overline{\mathbb{NE}}(X)$ be the closure of the cone of one-dimensional effective cycles on the variety X. Then the divisor K_X can be regarded as a linear function on the cone $\overline{\mathbb{NE}}(X)$.

If $K_X \cdot Z \geqslant 0$ for every one-dimensional cycle $Z \in \overline{\mathbb{NE}}(X)$, then the divisor K_X is numerically effective and the variety X is an output product of the MMP. In this case it is natural to conjecture that the linear system $|nK_X|$ is free for $n \gg 0$, which is indeed the case even in a more general context (see [113] and [107]). Moreover, the dimension of the variety $\phi_{|nK_X|}(X)$ is the Kodaira dimension $\kappa(X)$ of the variety X. If $\kappa(X) = \dim(X)$, then the variety $\phi_{|nK_X|}(X)$ has canonical singularities[9] and its canonical divisor is ample, and $\phi_{|nK_X|}(X)$ is called a canonical model.

If the canonical divisor K_X is not numerically effective, then there is an extremal ray R of the cone $\overline{\mathbb{NE}}(X)$ such that $K_X \cdot R < 0$, and there is a surjective morphism $\psi\colon X \to Y$ with connected fibres such that Y is normal and any curve $C \subset X$ is contracted by ψ to a point if and only if $C \in R$. The morphism ψ is usually called a contraction of the extremal ray R. We have the following possibilities:

- ψ is birational and contracts a curve $C \subset X$;
- ψ is birational and contracts an irreducible divisor $E \subset X$;
- the variety Y is a smooth curve, the morphism ψ is a fibration into del Pezzo surfaces, and the equality $\operatorname{rk} \operatorname{Pic}(X) = 2$ holds;
- Y is a surface having cyclic quotient-singularities, ψ is a conic bundle, and $\operatorname{rk} \operatorname{Pic}(X) = \operatorname{rk} \operatorname{Pic}(Y) + 1$, and moreover, it is conjectured that the singularities of Y are Du Val singularities (see Conjecture III in [90], [141], [142]);
- Y is a point, X is a Fano variety with terminal $\mathbb{Q}$-factorial singularities, and $\operatorname{rk} \operatorname{Pic}(X) = 1$.

If $\dim(Y) < \dim(X)$, then the variety X is an output product of the MMP, and the Kodaira dimension of X is assumed to be equal to $-\infty$. If ψ contracts an irreducible divisor $E \subset X$, then the singularities of the variety Y are also terminal and $\mathbb{Q}$-factorial, and $\operatorname{rk} \operatorname{Pic}(Y) = \operatorname{rk} \operatorname{Pic}(X) - 1$. In this case the birational morphism ψ is an iterative step of the MMP, because we can apply the above considerations to the variety Y. If the morphism ψ contracts a (possibly reducible) curve $C \subset X$, then K_Y is no longer a $\mathbb{Q}$-Cartier divisor.

Definition 1.1.3. Let V be a threefold, let the divisor K_V be a $\mathbb{Q}$-Cartier divisor, let $\gamma\colon V \to U$ be a birational contraction of an extremal ray of the cone $\overline{\mathbb{NE}}(V)$, and let γ contract a curve $\Delta \subset V$. Then a map $\rho\colon V \dashrightarrow \widehat{V}$ is called a flip (antiflip, flop, respectively) if there is a curve $\widehat{\Delta} \subset \widehat{V}$ such that ρ induces an isomorphism $V \setminus \Delta \cong \widehat{V} \setminus \widehat{\Delta}$ and the diagram

$$\begin{tikzcd} V \arrow[rr, dashed, "\rho"] \arrow[dr, "\gamma"'] & & \widehat{V} \arrow[dl, "\widehat{\gamma}"] \\ & U & \end{tikzcd}$$

is commutative, where $\widehat{\gamma}$ is a contraction of an extremal ray of the cone $\overline{\mathbb{NE}}(\widehat{V})$, and the inequalities $K_V \cdot \Delta < 0$ and $K_{\widehat{V}} \cdot \widehat{\Delta} > 0$ hold (the inequalities $K_V \cdot \Delta > 0$ and $K_{\widehat{V}} \cdot \widehat{\Delta} < 0$ hold, the equalities $K_V \cdot \Delta = K_{\widehat{V}} \cdot \widehat{\Delta} = 0$ hold, respectively).

If the morphism ψ constructed is birational and contracts a curve $C \subset X$, then there is a flip $\eta\colon X \to \widehat{X}$ in the curve C (see [134], [168]), and η can be regarded as

[9] A variety V is said to have canonical singularities if K_V is a $\mathbb{Q}$-Cartier divisor and if the relation $K_W \sim_{\mathbb{Q}} f^*(K_X) + \sum_{i=1}^{k} a(X, E_i)E_i$ holds for every birational morphism $f\colon W \to X$, where E_i is an exceptional divisor of the birational morphism f and $a(X, E_i)$ is a non-negative rational number.

an iterative step of the MMP, because the variety $\widehat{X}$ has terminal and $\mathbb{Q}$-factorial singularities. Moreover, one can show that there is no infinite sequence of flips of three-dimensional varieties with terminal singularities. Thus, after finitely many iterations we obtain a birational transformation of the three-dimensional variety X either into a minimal model or into a Mori fibred space.

The MMP algorithm seems to be rather abstract. However, this is not the case, as can be seen from the following example of application of the MMP to a specific problem (see [8], [83], [17], [18]).

Proposition 1.1.4. *Let X be a normal three-dimensional variety containing an ample effective Cartier divisor S which has Du Val singularities and is a minimal surface of Kodaira dimension zero. Then either X has canonical singularities or X is a contraction of a section on the variety $\mathbb{P}(\mathcal{O}_S \oplus \mathcal{O}_S(S|_S))$.*

Proof. It is easy to see that the divisor $-K_X$ is a $\mathbb{Q}$-Cartier divisor and $-K_X \sim_{\mathbb{Q}} S$ (see [18]). Suppose that the singularities of the threefold X are not canonical. Then the MMP implies the existence of a birational morphism $\pi\colon V \to X$ such that the variety V has canonical singularities and the divisor K_V is π-ample (in particular, the divisor K_V has positive intersection with every curve contracted by π). This implies the relation $-K_V \sim_{\mathbb{Q}} \pi^*(S) - B$, where B is an effective and non-zero π-exceptional divisor (see [113], Proposition 2.18). Then there is an extremal ray $R \in \overline{\mathbb{NE}}(V)$ such that $-B \cdot R < 0$. Let $\psi\colon V \to Z$ be a contraction of the ray R. Then the following cases are possible:

- the morphism ψ is birational;
- the variety Z is a curve;
- Z is a surface.

It follows from the π-ampleness of K_V, the ψ-ampleness of B, and the formula (2.3.2) in [134] that ψ is a conic bundle. We set $\widehat{S} = \pi^{-1}(S)$. Then $\widehat{S}$ is a section of the morphism ψ. The equalities $R^1\psi_*(\mathcal{O}_V) = \operatorname{Ext}^1(\mathcal{O}_S(S|_S), \mathcal{O}_S) = 0$ readily imply the isomorphisms $V \cong \mathbb{P}(\psi_*(\mathcal{O}_V(\widehat{S})))$ and $\psi_*(\mathcal{O}_V(\widehat{S})) \cong \mathcal{O}_S \oplus \mathcal{O}_S(S|_S)$.

We note that the assertion of Proposition 1.1.4 generalizes the following two-dimensional results: the singularities of a cubic surface in $\mathbb{P}^3$ with isolated singularities are not Du Val singularities if and only if the cubic surface is a cone; the singularities of a double cover of $\mathbb{P}^2$ ramified along a reduced quartic curve are not Du Val singularities if and only if the quartic curve is a union of four lines passing through a single point.

Corollary 1.1.5. *Let X be a normal subvariety in $\mathbb{P}^n$ such that some hyperplane section S of X is a minimal smooth surface of Kodaira dimension zero. It follows from the classification of surfaces that the section S can be one of the following surfaces*: *an Abelian surface*, *a bi-elliptic surface*, *a $K3$ surface*, *and an Enriques surface. In this case either the variety X has canonical singularities* (*and the surface S is either a $K3$ surface or an Enriques surface*) *or X is a cone.*

In particular, it follows from Corollary 1.1.5 that an Abelian surface cannot be a hyperplane section of any threefold but a cone (see [65]).

§ 1.2. Log Minimal Model Programme

For several important reasons, a study of the birational properties of a pair consisting of a variety and some divisor on the variety is often needed. One can consider the birational transforms thus arising by analogy with the birational transforms arising during the MMP. For example, let X be a three-dimensional variety and let B_X be an effective divisor on X such that $B_X = \sum_{i=1}^{r} a_i B_i$, where B_i is an irreducible reduced subvariety of codimension 1 on X and a_i is a non-negative rational number less than 1. A pair of this kind (consisting of a variety X and a divisor B_X) is usually called a *log pair* and denoted by (X, B_X). The divisor B_X is usually called the *boundary* of the log pair (X, B_X) and the divisor $K_X + B_X$ is called a *log canonical divisor* of the log pair (X, B_X).

Definition 1.2.1. A log pair (X, B_X) is said to have *log terminal* (*log canonical*) singularities if the divisor $K_X + B_X$ is a $\mathbb{Q}$-Cartier divisor and for any birational morphism $f\colon W \to X$ there is an equivalence

$$K_W + B_W \sim_{\mathbb{Q}} f^*(K_X + B_X) + \sum_{i=1}^{k} a(X, B_X, E_i)E_i,$$

where $B_W = \sum_{i=1}^{r} a_i f^{-1}(B_i)$, E_i is an f-exceptional divisor, and $a(X, B_X, E_i)$ is a rational number such that $a(X, B_X, E_i) > -1$ ($a(X, B_X, E_i) \geqslant -1$, respectively).

Suppose that a log pair (X, B_X) has log terminal singularities, but the log canonical divisor $K_X + B_X$ is not numerically effective. Then there is an extremal ray R of the cone $\overline{\mathbb{NE}}(X)$ such that $(K_X + B_X)\cdot R < 0$, and there is a surjective morphism with connected fibres $\psi\colon X \to Y$ such that the variety Y is normal and any curve $C \subset X$ is contracted by ψ to a point if and only if C is contained in the extremal ray R. Moreover, the following cases are possible:

- the morphism ψ is birational and contracts a curve $C \subset X$;
- ψ is birational and contracts an irreducible divisor $E \subset X$;
- ψ is a Mori fibration.

If ψ contracts an irreducible divisor $E \subset X$, then one can consider a new log pair (Y, B_Y), where $B_Y = \sum_{i=1}^{r} a_i \psi(B_i)$. Then the singularities of the log pair (Y, B_Y) are also log terminal, and we have $\mathrm{rk}\ \mathrm{Pic}(Y) = \mathrm{rk}\ \mathrm{Pic}(X) - 1$.

If ψ contracts a (possibly reducible) curve $C \subset X$, then there is a birational map $\rho\colon X \dashrightarrow \widehat{X}$ and a (possibly reducible) curve $\widehat{C} \subset \widehat{X}$ such that ρ induces an isomorphism $X \setminus C \cong \widehat{X} \setminus \widehat{C}$ and the diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\rho"] \arrow[dr, "\psi"'] & & \widehat{X} \arrow[dl, "\widehat{\psi}"] \\ & Y & \end{tikzcd}$$

is commutative, where $\widehat{\psi}$ is a birational morphism contracting the curve $\widehat{C}$ to a point and is a contraction of an extremal ray of the cone $\overline{\mathbb{NE}}(\widehat{X})$; furthermore, the strict inequality $(K_{\widehat{X}} + B_{\widehat{X}}) \cdot \widehat{C} > 0$ holds for $B_{\widehat{X}} = \sum_{i=1}^{r} a_i \rho(B_i)$. Moreover, the log pair $(\widehat{X}, B_{\widehat{X}})$ also has log terminal singularities. The birational map ρ is usually called a *log flip in the curve* C for the log pair (X, B_X). The existence of ρ was proved in [171].

Thus, if the morphism ψ is birational, then we can apply the above arguments either to the log pair (Y, B_Y) or to the log pair $(\widehat{X}, B_{\widehat{X}})$. Moreover, one can show that there is no infinite sequence of three-dimensional log flips (see [106]). Hence, after finitely many iterations we obtain a birational map $\sigma\colon X \dashrightarrow V$ and a log pair (V, B_V), where $B_V = \sum_{i=1}^{r} a_i\sigma(B_i)$, such that the singularities of the log pair (V, B_V) are log terminal and one of the following two possibilities holds:

- the log canonical divisor $K_V + B_V$ is numerically effective;
- there is a Mori fibration $\xi\colon V \to Z$ such that the divisor $-(K_V + B_V)$ is ample with respect to the morphism ξ.

The above way to construct the birational map σ is a general scheme of the Log Minimal Model Programme (see [106], [168], [169]), which is briefly denoted below by log MMP. There is a relative version of the log MMP for varieties admitting a morphism to a given variety (see [106]).

Remark 1.2.2. One can readily see that log terminality and $\mathbb{Q}$-factoriality are preserved by the log MMP. Moreover, terminality of singularities is also preserved by the log MMP if no component of a boundary is contracted during the log MMP.

There are specific applications of the log MMP (see [19], [20], [35]). For example, the log MMP implies the following result (see [35] and [62]).

Proposition 1.2.3. *Let S be a hypersurface in $\mathbb{P}^3$ of degree 4 having isolated singularities. Then the singularities of the log pair $(\mathbb{P}^3, \frac{3}{4}S)$ are not log terminal if and only if the surface S is a cone.*

Proof. Suppose that the singularities of the log pair $(\mathbb{P}^3, \frac{3}{4}S)$ are not log terminal at a point O. One can show that such a point is unique and the log pair $(\mathbb{P}^3, \frac{3}{4}S)$ has log canonical singularities (see Theorem 1.7.10). Let $h\colon Y \to \mathbb{P}^3$ be a log terminal modification (see [113], Proposition 2.18) of the log pair $(\mathbb{P}^3, \frac{3}{k}S)$. Consider a terminal modification $t\colon V \to Y$ of the variety Y. Then the composition $f = h\circ t$ is biregular outside the point O, the variety V has terminal $\mathbb{Q}$-factorial singularities, and

$$K_V + \frac{3}{k}\widetilde{S} \sim_{\mathbb{Q}} f^*\biggl(K_{\mathbb{P}^3} + \frac{3}{k}S\biggr) - E,$$

where $\widetilde{S} = f^{-1}(S)$, E is an effective f-exceptional divisor whose support coincides with $f^{-1}(O)$, and $\lfloor E\rfloor \neq 0$. There is an extremal contraction $g\colon V \to W$ such that the divisor $-(K_V + \frac{3}{k}\widetilde{S} + E)$ is g-ample. One can readily see that the morphism g is either a contraction of the surface $\widetilde{S}$ to a curve or a conic bundle contracting $\widetilde{S}$ to a curve. Let C be a general fibre of the morphism g. Then elementary computations show that $f(C)$ is a line, which implies that S is a cone.

It follows from Proposition 1.2.3 and Theorem 1.7.10 that the cones have the highest log canonical threshold (see [111]) among the two-dimensional quartics in $\mathbb{P}^3$ having isolated singularities. Using results obtained in [177], [74], [75], [52], [84], [143], one can find all log canonical thresholds of quartic surfaces in $\mathbb{P}^3$ with isolated singularities. This is important for a deeper understanding of the birational geometry of Fano threefolds, because quartic surfaces with isolated singularities arise as hyperplane sections of smooth quartic threefolds.

§ 1.3. Movable log pairs

The solution of many important problems of birational geometry of higher-dimensional algebraic varieties has showed that it is natural to consider global and local properties of pairs consisting of a variety and a divisor on the variety (see 1.2). On the other hand, it is often convenient to consider pairs of this kind formed by a variety and some (not necessarily complete) linear system on it without fixed components.

Definition 1.3.1. A *movable log pair* (X, M_X) consists of a variety X and a movable boundary M_X on X, where $M_X = \sum_{i=1}^{n} a_i \mathcal{M}_i$ is a formal finite linear combination, $\mathcal{M}_i$ is a linear system on X without fixed components, and a_i is a non-negative rational number for any i.

A movable log pair can always be regarded as an ordinary log pair with effective boundary by replacing every linear system by a weighted sum of its general elements. In particular, for a movable log pair (X, M_X) the formal sum $K_X + M_X$ can be regarded as a divisor on X usually called the log canonical divisor of the log pair (X, M_X). In the rest of this section we assume that the log canonical divisors of all movable log pairs are $\mathbb{Q}$-Cartier divisors.

Remark 1.3.2. For a movable log pair (X, M_X) one can treat the self-intersection M_X^2 as an effective cycle on X of codimension 2 if X has $\mathbb{Q}$-factorial singularities. Namely, let $M_X = \sum_{i=1}^{n} a_i \mathcal{M}_i$, where $\mathcal{M}_i$ is a linear system on X without fixed components. For every index i we choose two sufficiently general divisors S_i and $\widehat{S}_i$ in the linear system $\mathcal{M}_i$ and set $M_X^2 = \sum_{i,\, j=1}^{n} a_i a_j\, S_i \cdot \widehat{S}_j$.

The image of a movable boundary under a birational map is well defined. Movable log pairs (X, M_X) and (Y, M_Y) are said to be *birationally equivalent* if there is a birational map $\rho\colon X \dashrightarrow Y$ such that $M_Y = \rho(M_X)$. For a movable log pair we can define some rational numbers, so-called *discrepancies*, and several classes of singularities, as done in the case of ordinary log pairs (see [106], [113], [111]); however, the most natural classes of singularities in this case are terminal and canonical singularities.

Definition 1.3.3. A log pair (X, M_X) is said to have *canonical* (*terminal*) singularities if every rational number $a(X, M_X, E_i)$ determined by the equivalence

$$K_W + f^{-1}(M_X) \sim_{\mathbb{Q}} f^*(K_X + M_X) + \sum_{i=1}^{k} a(X, M_X, E_i) E_i$$

is non-negative (positive) for every birational morphism $f\colon W \to X$, where E_i is an exceptional divisor of the birational morphism f. The number $a(X, M_X, E_i)$ is called the *discrepancy* of the movable log pair (X, M_X) at the divisor E_i.

Remark 1.3.4. If the singularities of a log pair (X, M_X) are terminal, then the singularities of the log pair $(X, \varepsilon M_X)$ are also terminal for a sufficiently small rational number $\varepsilon > 1$.

The singularities of a given movable log pair coincide with the singularities of the variety outside the union of the base loci of the components of the movable boundary. Thus, it follows from the existence of a resolution of singularities of an algebraic variety (see [76]) that every movable log pair is birationally equivalent to a movable log pair with terminal singularities.

Example 1.3.5. Let us consider a movable log pair $(\mathbb{P}^2, \gamma\mathcal{H})$, where $\mathcal{H}$ is a complete linear system $|\mathcal{O}_{\mathbb{P}^2}(1)|$ and γ is a rational number. Let $\tau\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a rational map such that

$$\tau(x:y:z) = (yz:xz:xy),$$

where $(x:y:z)$ are homogeneous coordinates on $\mathbb{P}^2$. In this case the map τ is undefined at the points $(1:0:0)$, $(0:1:0)$, and $(0:0:1)$ and is an involution, which is usually called a *Cremona involution*. By construction, $\tau(\mathcal{H})$ is a linear system formed by conics passing through the points $(1:0:0)$, $(0:1:0)$, and $(0:0:1)$. Moreover, the singularities of the movable log pair $(\mathbb{P}^2, \gamma\mathcal{H})$ are terminal for any choice of γ, and the singularities of the movable log pair $(\mathbb{P}^2, \gamma\tau(\mathcal{H}))$ are terminal for $\gamma < 1$ and canonical for $\gamma \leqslant 1$.

Example 1.3.6. Let $\mathcal{M}$ be a linear system without fixed components on a threefold X which has $\mathbb{Q}$-factorial and terminal singularities. Then, according to [168] and [124], the singularities of the log pair $(X, \mathcal{M})$ are terminal if and only if all base points of the linear system $\mathcal{M}$ are isolated, smooth on a general divisor in the linear system $\mathcal{M}$, and smooth on the variety X.

Example 1.3.7. Let X be a normal variety such that the dualizing sheaf ω_X is locally free, let $\mathcal{M}$ be a linear system on X having no fixed components, and let S be a general divisor in the linear system $\mathcal{M}$. Then the variety S has canonical singularities if and only if the log pair $(X, \mathcal{M})$ has canonical singularities (see [111], Theorems 4.5.1 and 7.9).

Any application of the log MMP to canonical and terminal movable log pairs preserves their canonicity and terminality, respectively (see [106]).

Definition 1.3.8. A proper irreducible subvariety Y of a variety X is called a *centre of canonical singularities of a movable log pair* (X, M_X) if there exist a birational morphism $f\colon W \to X$ and an f-exceptional divisor $E_1 \subset W$ such that

$$K_W + f^{-1}(M_X) \sim_{\mathbb{Q}} f^*(K_X + M_X) + \sum_{i=1}^{k} a(X, M_X, E_i)E_i,$$

where $a(X, M_X, E_i)$ is a rational number, E_i is an exceptional divisor of the birational morphism $f\colon W \to X$, $f(E_1) = Y$, and $a(X, M_X, E_1) \leqslant 0$. The set of all centres of canonical singularities of the log pair (X, M_X) is denoted by $\mathbb{CS}(X, M_X)$.

Remark 1.3.9. Let us consider a movable log pair (X, M_X) and a proper irreducible subvariety $Z \subset X$ such that X is smooth at a general point of Z. Then

$$Z \in \mathbb{CS}(X, M_X) \Rightarrow \operatorname{mult}_Z(M_X) \geqslant 1,$$

and we also have $\operatorname{mult}_Z(M_X) \geqslant 1 \Rightarrow Z \in \mathbb{CS}(X, M_X)$ if $\operatorname{codim}(Z \subset X) = 2$.

Remark 1.3.10. Let (X, M_X) be a movable log pair, let H be a sufficiently general hyperplane section of the variety X, and let Z be a proper irreducible subvariety such that $Z \in \mathbb{CS}(X, M_X)$. Then every component of the intersection $Z \cap H$ is contained in the set $\mathbb{CS}(H, M_X|_H)$.

Example 1.3.11. Let S be a smooth cubic surface in $\mathbb{P}^3$, let P be a point on S, and let $\psi\colon S \dashrightarrow \mathbb{P}^2$ be the projection from the point P. Then ψ is of degree 2 at a general point of S and induces an involution $\tau \in \mathrm{Bir}(S)$. Let us consider the linear system $\mathcal{H}$ consisting of all hyperplane sections of S. Let $\mathcal{M} = \tau(\mathcal{H})$. Suppose that P is not an Eckardt point, namely, P is not an intersection of three lines contained in S. Then one can readily show that a general curve in the linear system $\mathcal{M}$ has multiplicity three at the point P and is cut out on the surface S by a quadric in $\mathbb{P}^3$. Moreover, it can be shown that the relation $K_S + \frac{1}{2}\mathcal{M} \sim_{\mathbb{Q}} 0$ holds, the set $\mathbb{CS}(S, \frac{1}{2}\mathcal{M})$ consists of the point P, and the singularities of the log pair $(S, \frac{1}{2}\mathcal{M})$ are not canonical (see 1.5).

Thus, the singularities of movable log pairs depend essentially on the multiplicities of the corresponding movable boundaries. An effective way to bound these multiplicities is to use the following result (see [147] and [27]).

Proposition 1.3.12. *Let X be a smooth complete intersection $\cap_{i=1}^{k} G_i \subset \mathbb{P}^m$ and let D be an effective divisor on the variety X such that $D \sim \mathcal{O}_{\mathbb{P}^m}(n)|_X$, where G_i is a hypersurface. Then* $\mathrm{mult}_S(D) \leqslant n$ *for every irreducible subvariety $S \subset X$ such that* $\dim(S) \geqslant k$.

Proof. One can assume that $\dim(S) = k \leqslant (m-1)/2$. We consider a cone C over S with vertex at a general point $P \in \mathbb{P}^m$. Then $C \cap X = S \cup R$, where R is a curve on the complete intersection X.

Let $\pi\colon X \to \mathbb{P}^{m-1}$ be the projection from the point P and let $D_\pi \subset X$ be the corresponding subvariety along which the projection π is ramified. We show that $R \cap S = D_\pi \cap S$ in the set-theoretic sense. Let $C \cap G_i = S \cup R^i$. Then $R^i \cap S = D_\pi^i \cap S$ by Lemma 3 in [154], where $D_\pi^i \subset G_i$ is the ramification divisor of the projection $\pi^i\colon G_i \to \mathbb{P}^{m-1}$ from the point P. Thus, the set-theoretic equalities $R = \cap_{i=1}^{k} R^i$ and $D_\pi = \cap_{i=1}^{k} D_\pi^i$ imply that $R \cap S = D_\pi \cap S$.

We consider homogeneous coordinates $(z_0 : \ldots : z_m)$ on $\mathbb{P}^m$ such that the point P has some coordinates $(p_0 : \ldots : p_m)$ and the hypersurface G_j is given by the equation $F_j(z_0, \ldots, z_m) = 0$. In this case the subvariety D_π is given by the k equations

$$\sum_{i=0}^{m} \frac{\partial F_j}{\partial z_i} p_i = 0,$$

and the linear systems of divisors of the form $\sum_{i=0}^{m} \lambda_i \frac{\partial F_j}{\partial z_i} = 0$, where $\lambda_i \in \mathbb{C}$, do not have base points on X because X is smooth. Thus, the set-theoretic intersection $D_\pi \cap S$ consists of $d \prod_{i=1}^{k} (d_i - 1)$ distinct points, where d_i is the degree of the hypersurface G_i and d is the degree of the variety $S \subset \mathbb{P}^m$. Therefore, we have the inequality

$$nd\prod_{i=1}^{k}(d_i-1)=\deg(D|_R)\geqslant\sum_{O\in R\cap S}\mathrm{mult}_O(D)\geqslant\sum_{O\in R\cap S}\mathrm{mult}_S(D)$$
$$=\mathrm{mult}_S(D)d\prod_{i=1}^{k}(d_i-1),$$

which shows that $\mathrm{mult}_S(D)\leqslant n$.

Corollary 1.3.13. *Under the assumptions and with the notation of* Proposition 1.3.12, *suppose that μ is a positive rational number such that $\mu<\frac{1}{n}$. Then the set $\mathbb{CS}(X,\mu D)$ contains no subvarieties of X of dimension greater than or equal to k.*

The claim in Proposition 1.3.12 cannot be sharpened in general.

Example 1.3.14. Let us consider a smooth complete intersection $X\in\mathbb{P}^5$ of a quadric hypersurface Q and a cubic hypersurface V. Let C be an irreducible conic on the variety X such that the plane $\Pi\subset\mathbb{P}^5$ containing C is contained in Q. In this case $\Pi\cap V$ consists of the conic C and a line L. On the other hand, the general fibre F of the projection from Π is a smooth plane cubic curve which intersects L at a single point. There is a natural involution of the elliptic curve F: reflection in the point $L\cap F$, which induces a birational involution τ. We take the complete linear system $\mathcal{H}$ consisting of hyperplane sections of the variety X, let $\mathcal{M}=\tau(\mathcal{H})$, and consider a general surface D in the linear system $\mathcal{M}$. Then one can show that $D\sim\mathcal{O}_{\mathbb{P}^5}(13)|_X$ (see [87], Proposition 4.5); however, $\mathrm{mult}_C(D)=14$.

Definition 1.3.15. For a movable log pair (X,M_X) we consider a birationally equivalent movable log pair (W,M_W) with canonical singularities and a positive integer m such that the divisor $m(K_W+M_W)$ is a Cartier divisor. We define the Kodaira dimension $\kappa(X,M_X)$ of the movable log pair (X,M_X) as the maximal dimension of the variety $\phi_{|nm(K_W+M_W)|}(W)$ for $n\gg 0$ if at least one complete linear system $|nm(K_W+M_W)|$ is non-empty and we set $\kappa(X,M_X)=-\infty$ otherwise.

The Kodaira dimension of a movable log pair is a birational invariant and a non-decreasing function of the boundary coefficients.

Lemma 1.3.16. *The Kodaira dimension of a movable log pair is well defined, namely, it does not depend on the choice of a birationally equivalent movable log pair with canonical singularities in Definition* 1.3.15.

Proof. Let (X,M_X) and (Y,M_Y) be movable log pairs having canonical singularities such that $M_X=\rho(M_Y)$ for some birational map $\rho\colon Y\dashrightarrow X$ and let m be a positive integer such that $m(K_X+M_X)$ and $m(K_Y+M_Y)$ are Cartier divisors. To prove the desired assertion, it suffices to show that either the divisors $|nm(K_X+M_X)|$ and $|nm(K_Y+M_Y)|$ are empty for any $n\in\mathbb{N}$ or

$$\phi_{|nm(K_X+M_X)|}(X)\cong\phi_{|nm(K_Y+M_Y)|}(Y)$$

for any $n \gg 0$. Let us consider the commutative diagram

$$\begin{tikzcd} & W \arrow[dl, "g"'] \arrow[dr, "f"] & \\ X \arrow[rr, dashed, "\rho"'] & & Y, \end{tikzcd}$$

where W is smooth and $g\colon W \to X$ and $f\colon W \to Y$ are birational morphisms. Then

$$K_W + M_W \sim_{\mathbb{Q}} g^*(K_X + M_X) + \sum_{i=1}^{k} a_i G_i \sim_{\mathbb{Q}} f^*(K_Y + M_Y) + \sum_{i=1}^{l} b_i F_i,$$

where $M_W = g^{-1}(M_X)$, a_i and b_j are rational numbers, and G_i and F_i are irreducible exceptional divisors of g and f, respectively. Moreover, since the movable log pairs (X, M_X) and (Y, M_Y) are canonical, it follows that $a_i \geqslant 0$ and $b_j \geqslant 0$. It follows from Lemma 2.19 in [113] that the dimensions of $|nm(K_W + M_W)|$, $|g^*(nm(K_X + M_X))|$, and $|f^*(nm(K_Y + M_Y))|$ are equal and that

$$\phi_{|nm(K_W+M_W)|}(W) \cong \phi_{|g^*(nm(K_X+M_X))|}(W) \cong \phi_{|f^*(nm(K_Y+M_Y))|}(W)$$

provided that these varieties are non-empty, which implies the desired assertion.

If the boundary is empty, then Definition 1.3.15 coincides with the classical definition of the Kodaira dimension (see Definition 0.2.3). One can define the Kodaira dimension for arbitrary log pairs exactly as for movable log pairs (see [173] and [106]), and the assertion of Lemma 1.3.16 remains valid. The Kodaira dimension can be used for the birational classification of planar curves (see [80] and [81]). For example, it follows from the results of [173] that $\kappa(\mathbb{P}^2, C) = -\infty$ for an irreducible curve $C \subset \mathbb{P}^2$ if and only if there is a $\sigma \in \mathrm{Bir}(\mathbb{P}^2)$ such that $\sigma(C)$ is a line. As was proved in [79], for two irreducible planar curves C_1 and C_2 there is a $\rho \in \mathrm{Bir}(\mathbb{P}^2)$ such that $\rho(C_1) \cup \rho(C_2)$ is a union of two lines if and only if $\kappa(\mathbb{P}^2,\, C_1 + C_2) = -\infty$.

Remark 1.3.17. Let (X, M_X) be a movable log pair for which the equivalence $K_X + M_X \sim_{\mathbb{Q}} 0$ holds. Then $\kappa(X, M_X) \leqslant 0$, and the equality $\kappa(X, M_X) = 0$ holds if and only if the singularities of the movable log pair (X, M_X) are canonical.

Example 1.3.18. We consider a smooth three-dimensional quartic $X \subset \mathbb{P}^4$ and a line L on X. Let $\psi\colon W \to X$ be a blow-up of the line L. We note that a smooth quartic X contains a one-dimensional family of lines (see [42] and [166]). In this case the linear system $|-K_W|$ is free and induces an elliptic fibration $\phi\colon W \to \mathbb{P}^2$. We set $M_X = \mu\psi(|-K_W|)$ for $\mu \in \mathbb{Q}$. Then

$$\kappa(X, M_X) = \begin{cases} -\infty & \text{for } \mu < 1, \\ 0 & \text{for } \mu = 1, \\ 2 & \text{for } \mu > 1. \end{cases}$$

Definition 1.3.19. A movable log pair (V, M_V) is called a *canonical model of a movable log pair* (X, M_X) if there is a birational map $\psi\colon X \dashrightarrow V$ such that $M_V = \psi(M_X)$, the divisor $K_V + M_V$ is ample, and the singularities of the log pair (V, M_V) are canonical.

This definition of canonical model coincides with the classical definition of canonical model if the boundary is empty, and the following assertion results from the proof of Lemma 1.3.16.[10]

Theorem 1.3.20. *A canonical model is unique whenever it exists.*

In most applications one uses movable log pairs whose boundary consists of a single linear system without fixed components which is multiplied by a positive rational number (see [46]). On the other hand, all assertions which can be proved for movable log pairs can be derived by linearity from analogous assertions for movable log pairs whose boundaries have a single component. However, sometimes one must use log pairs of more complicated structure.

§ 1.4. Noether–Fano–Iskovskikh inequality

Let us consider Fano varieties X having terminal $\mathbb{Q}$-factorial singularities such that $\operatorname{rk}\operatorname{Pic}(X) = 1$. The following theorem was proved in [45], although many special cases of the assertion can be found in [120], [121], and [94].

Theorem 1.4.1. *Suppose that every movable log pair (X, M_X) such that $K_X + M_X \sim_{\mathbb{Q}} 0$ has canonical singularities. Then the Fano variety X is birationally superrigid.*

Proof. Let $\rho\colon X \dashrightarrow Y$ be a birational map such that either the variety Y admits a structure of a fibration $\tau\colon Y \to Z$ into varieties whose Kodaira dimension is equal to $-\infty$, or Y is a Fano variety of Picard rank 1 with terminal $\mathbb{Q}$-factorial singularities. We claim that the first case is impossible, while the rational map ρ is biregular in the second case.

Suppose that we have a fibration $\tau\colon Y \to Z$ into varieties of Kodaira dimension $-\infty$. We arbitrarily choose a sufficiently general very ample divisor H on the variety Z and consider the movable boundary $M_Y = \mu|\tau^*(H)|$ for an arbitrary positive rational number μ. By construction, the Kodaira dimension $\kappa(Y, M_Y)$ of the movable log pair (Y, M_Y) is equal to $-\infty$. Let $M_Y = \mu\rho^{-1}(|\tau^*(H)|)$. Then

$$\kappa(X, M_X) = \kappa(Y, M_Y) = -\infty$$

by the definition of the Kodaira dimension (see Lemma 1.3.16). Let us now choose a rational number μ in such a way that $K_X + M_X \sim_{\mathbb{Q}} 0$. Then $\kappa(X, M_X) = 0$, because the log pair (X, M_X) has canonical singularities by the assumption of the theorem. This is a contradiction.

[10] In the case of an empty movable boundary the assertion of Theorem 1.3.20 about the uniqueness of a canonical model of an algebraic variety is well known. In particular, it follows from Theorem 1.3.20 that all birational automorphisms of canonical models are biregular, but it is this property that is the classical attribute of a birationally superrigid variety (see Definition 0.3.4). Moreover, Theorem 1.3.20 explains the geometric nature of this phenomenon in both cases (see 1.4).

Suppose that Y is a Fano variety of Picard rank 1 with $\mathbb{Q}$-factorial terminal singularities. We set $M_Y = \frac{\mu}{n}|-nK_Y|$ and $M_X = \rho^{-1}(M_Y)$ for $n \gg 0$ and $\mu \in \mathbb{Q}$ and choose a μ in such a way that the relation $K_X + M_X \sim_{\mathbb{Q}} 0$ holds. Then the singularities of the movable log pair (X, M_X) are canonical by the assumption of the theorem. In particular, $\kappa(X, M_X) = \kappa(Y, M_Y) = 0$, and hence $\mu = 1$.

Let us consider the commutative diagram (see [76])

$$\begin{tikzcd} & W \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, dashed, "\rho"'] & & Y \end{tikzcd}$$

such that W is smooth and $g\colon W \to X$ and $f\colon W \to Y$ are birational morphisms. Then

$$\sum_{j=1}^{k} a(X, M_X, F_j)F_j \sim_{\mathbb{Q}} \sum_{i=1}^{l} a(Y, M_Y, G_i)G_i,$$

where G_i is an irreducible g-exceptional divisor and F_j is an f-exceptional divisor.

The singularities of the log pairs (X, M_X) and (Y, M_Y) are canonical. Moreover, the singularities of the movable log pair (Y, M_Y) are terminal. In particular, all the numbers $a(X, M_X, F_j)$ are non-negative and all the numbers $a(Y, M_Y, G_i)$ are strictly positive. It follows from Lemma 2.19 in [113] that $a(X, M_X, E) = a(Y, M_Y, E)$ for any divisor E on W, which implies that the singularities of the movable log pair (X, M_X) are terminal.

We take a rational number $\zeta > 1$ such that the singularities of (X, M_X) and (Y, M_Y) are terminal (see Remark 1.3.4). In this case the divisors $K_X + \zeta M_X$ and $K_Y + \zeta M_Y$ are ample and each of the movable log pairs (X, M_X) and (Y, M_Y) is a canonical model. Thus, the rational map ρ is biregular by Theorem 1.3.20.

The assertion of Theorem 1.4.1 holds over any field of characteristic zero.

Example 1.4.2. Let us consider a smooth del Pezzo surface S defined over $\mathbb{Q}$ and such that $K_S^2 = 1$ and $\mathrm{rk}\,\mathrm{Pic}(S) = 1$. Then it follows from Theorem 1.4.1, the equality $K_S^2 = 1$, and Remark 1.3.9 that S is birationally superrigid (see Theorem 1.5.1 and [120], [121], [91], [23], [48]).

One can show that the assertion of Theorem 1.4.1 is a criterion for the birational superrigidity of the Fano variety X if the log MMP exists.

Corollary 1.4.3. *Let* $\dim(X) = 3$. *Then X is birationally superrigid if and only if the singularities of every movable log pair (X, M_X) with $K_X + M_X \sim_{\mathbb{Q}} 0$ are canonical.*

The following generalization of Theorem 1.4.1 was obtained in [23].

Theorem 1.4.4. *Suppose that $\zeta\colon X \dashrightarrow Z$ is a map such that the normalization of a general fibre of ζ is an elliptic curve. Let us consider the commutative diagram*

$$\begin{tikzcd} & V \arrow[dl, "\rho"'] \arrow[dr, "\tau"] & \\ X \arrow[rr, dashed, "\zeta"'] & & Z \end{tikzcd},$$

where V is a smooth variety, τ is an elliptic fibration, and the morphism ρ is birational. Take a very ample divisor D on Z, consider the linear system $\mathcal{D} = |\tau^*(D)|$, and let $\mathcal{M} = \rho(\mathcal{D})$. Then $\mathbb{CS}(X, \gamma\mathcal{M}) \neq \varnothing$, where γ is a positive rational number such that $K_X + \gamma\mathcal{M} \sim_{\mathbb{Q}} 0$.

Proof. Let $\mathbb{CS}(X, \gamma\mathcal{M}) = \varnothing$. Then the singularities of the log pair $(X, \gamma\mathcal{M})$ are terminal and the log pair $(X, \varepsilon\mathcal{M})$ is a canonical model for some rational number $\varepsilon > \gamma$ (see Remark 1.3.4). In particular, we have the equalities

$$\kappa(V, \varepsilon\mathcal{D}) = \kappa(X, \varepsilon\mathcal{M}) = \dim(X);$$

however, $\kappa(V, \varepsilon\mathcal{D}) \leqslant \dim(Z) = \dim(X) - 1$, a contradiction.

One can readily see that the assertion of Theorem 1.4.4 can be generalized to the case of a birational transformation into a fibration whose general fibre has Kodaira dimension zero, and to any birational transformation into a Fano variety with arbitrary canonical singularities (see [23]).

§ 1.5. Cubic surfaces

Let S be a smooth cubic surface in $\mathbb{P}^3$ defined over a field $\mathbb{F}$ of characteristic zero. Then the adjunction formula implies the rational equivalence $-K_S \sim \mathcal{O}_{\mathbb{P}^3}(1)|_S$. In particular, the surface S is a del Pezzo surface. As is known, the surface S is rational if the field $\mathbb{F}$ is algebraically closed. However, S need not be rational if $\mathbb{F}$ is not algebraically closed. The following result holds (see [120]–[122], [45], [91]).

Theorem 1.5.1. *Suppose that* rk Pic$(S) = 1$. *Then S is a birationally rigid del Pezzo surface.*

In particular, a surface S is non-rational if rk Pic$(S) = 1$.

Corollary 1.5.2. *Let Y be a smooth cubic surface in $\mathbb{P}^3$ such that* rk Pic$(S) =$ rk Pic$(Y) = 1$. *In this case the cubic surfaces S and Y are birationally equivalent if and only if they are projectively equivalent.*

Smooth cubic surfaces whose Picard group is $\mathbb{Z}$ do indeed exist.

Example 1.5.3. Let $\mathbb{F} = \mathbb{Q}$ and let the equation

$$2x^3 + 3y^3 + 5z^3 + 7w^3 = 0 \subset \operatorname{Proj}(\mathbb{Q}[x, y, z, w]) \cong \mathbb{P}^3$$

define a surface $S \subset \mathbb{P}^3$. Then rk Pic$(S) = 1$ (see [164], [122], [48]).

We show how to prove Theorem 1.5.1 using Theorem 1.4.1, for example, by proving Corollary 1.5.2. Suppose that rk Pic$(S) = 1$. In particular, the field $\mathbb{F}$ is not algebraically closed. Let $\rho\colon S \dashrightarrow Y$ be a birational map such that Y is a smooth cubic surface with rk Pic$(Y) = 1$. We must show that S and Y are projectively equivalent; we note that the surfaces Y and S are projectively equivalent if they are isomorphic.

We set $\mathcal{M} = \rho^{-1}(|-K_Y|)$. Then there is a positive integer n such that the equivalence $\mathcal{M} \sim -nK_S$ holds. Let us consider a movable log pair $(S, \frac{1}{n}\mathcal{M})$. The singularities of the log pair $(S, \frac{1}{n}\mathcal{M})$ are canonical if and only if ρ is an isomorphism (see the proof of Theorem 1.4.1). Suppose that ρ is not an isomorphism. Then there is an irreducible (but perhaps geometrically reducible) zero-dimensional subvariety $Z \subset S$ defined over $\mathbb{F}$ and such that $\mathrm{mult}_Z(\mathcal{M}) > n$ (see Remark 1.3.9).

Let $\overline{\mathbb{F}}$ be the algebraic closure of the field $\mathbb{F}$. Then the subvariety Z defined over $\overline{\mathbb{F}}$ consists of finitely many $\overline{\mathbb{F}}$-points $O_1, \dots, O_r$ such that $\mathrm{mult}_{O_i}(\mathcal{M}) > n$. In particular, for two general curves C_1 and C_2 in the linear system $\mathcal{M}$ we have

$$3n^2 = C_1 \cdot C_2 \geqslant \sum_{P \in C_1 \cap C_2} \mathrm{mult}_P(C_1)\,\mathrm{mult}_P(C_2) \geqslant \sum_{i=1}^{r} \mathrm{mult}_{O_i}(C_1)\,\mathrm{mult}_{O_i}(C_2) > n^2 r,$$

which implies that $r \leqslant 2$.

Let $r = 2$, let $\alpha\colon W \to S$ be a blow-up of the surface S at Z, and let E be an exceptional divisor of α which splits into a disjoint union of smooth rational curves over the field $\overline{\mathbb{F}}$. Then it follows immediately from the conditions

$$\mathrm{mult}_{O_1}(\mathcal{M}) = \mathrm{mult}_{O_2}(\mathcal{M}) > n$$

that W is a smooth del Pezzo surface defined over $\mathbb{F}$ and such that $K_W^2 = 1$. As is well known, the surface W is a hypersurface of degree 6 in $\mathbb{P}(1,1,2,3)$, and the natural projection $W \to \mathbb{P}(1,1,2)$ is a double cover and coincides with the map $\phi_{|-2K_W|}$. Let τ be the biregular involution of W interchanging the fibres of the double cover $\phi_{|-2K_W|}$ and let $\sigma_1 = \alpha \circ \tau \circ \alpha^{-1}$. In this case,

$$\begin{cases} \tau^*(\alpha^*(-K_S)) = 5\alpha^*(-K_S) - 6E, \\ \tau^*(E) = 4\alpha^*(-K_S) - 5E. \end{cases} \tag{1.5.4}$$

Let $r = 1$, that is, let Z be an $\mathbb{F}$-point. We consider a blow-up $\beta\colon U \to S$ of the cubic surface S at the point Z and denote the exceptional divisor of the birational morphism β by F. Then it follows immediately from the inequality $\mathrm{mult}_Z(\mathcal{M}) > n$ that U is a smooth del Pezzo surface such that U is defined over the field $\mathbb{F}$, $K_U^2 = 2$, the surface U is a hypersurface of degree 4 in $\mathbb{P}(1,1,1,2)$, and the natural projection of U on $\mathbb{P}^2$ is a double cover which coincides with the anticanonical map $\phi_{|-K_U|}$. Let η be the involution of U interchanging the fibres of the double cover $\phi_{|-K_U|}$ and let $\sigma_1 = \beta \circ \eta \circ \beta^{-1}$. Then

$$\begin{cases} \eta^*(\beta^*(-K_S)) = 2\beta^*(-K_S) - 3F, \\ \eta^*(F) = 4\beta^*(-K_S) - 2F. \end{cases} \tag{1.5.5}$$

The involution σ_1 is birational and non-biregular. If $r = 2$, then σ_1 is called a Bertini involution, and if $r = 1$, then the birational involution σ_1 is called a Geiser involution. Let $\mathcal{B} = \sigma_1(\mathcal{M})$ and let k be a positive integer such that the rational equivalence $\mathcal{B} \sim -kK_S$ holds. Then the relations (1.5.4) and (1.5.5) immediately imply the inequality $k < n$. Thus, the proof of Theorem 1.4.1 implies that either the birational map $\rho \circ \sigma_1\colon S \dashrightarrow Y$ is an isomorphism or the singularities of the movable log pair $(S, \frac{1}{k}\mathcal{B})$ are not canonical. In the latter case we can apply to the movable log pair $(S, \frac{1}{k}\mathcal{B})$ all the arguments used above for $(S, \frac{1}{n}\mathcal{M})$. Thus, we have proved that there are $s \leqslant n$ birational (Bertini or Geiser) involutions $\sigma_1, \dots, \sigma_s$ such that the map $\rho \circ \sigma_1 \circ \dots \circ \sigma_s$ is an isomorphism. In particular, the surfaces S and Y are isomorphic.

It is easy to see that the arguments in the above proof of Corollary 1.5.2 prove both Theorem 1.5.1 and also the following well-known result.

Theorem 1.5.6. *Let* $\operatorname{rk}\operatorname{Pic}(S) = 1$. *Then the group* $\operatorname{Bir}(S)$ *is generated by the projective automorphisms of the surface* S, *the Bertini involutions*, *and the Geiser involutions.*

The possible relations between Bertini and Geiser involutions of the surface S are described in Theorem 7.8 of Chapter V in the book [122]. One can readily see from the proof of Corollary 1.5.2 that the results analogous to Theorems 1.5.1 and 1.5.6 hold for del Pezzo surfaces of degrees 1 and 2 (see [120] and [121]).

§ 1.6. Sarkisov Programme

Let us consider a morphism $\pi \colon X \to Z$ such that the variety X has $\mathbb{Q}$-factorial and terminal singularities, the divisor $-K_X$ is π-ample, the inequality $\dim(Z) < \dim(X)$ and the equality $\operatorname{rk}\operatorname{Pic}(X) = \operatorname{rk}\operatorname{Pic}(Z) + 1$ hold, and the fibres of π are connected. In this case the variety X is usually called a *Mori fibred space* and the morphism π is called a *Mori fibration*. If Z is a point, then X is a Fano variety with $\mathbb{Q}$-factorial terminal singularities such that $\operatorname{rk}\operatorname{Pic}(X) = 1$.

If Z is a point, then Theorem 1.4.1 gives a necessary condition for the birational superrigidity of the variety X, and this condition is even a criterion in the three-dimensional case (see Corollary 1.4.3). It is natural to consider the following two problems:

- how to generalize the assertions of Theorem 1.4.1 and Corollary 1.4.3 to the case of $\dim(Z) > 0$;
- how to generalize the assertion of Theorem 1.4.1 to the case in which Z is a point and the variety X is not birationally rigid.

In a sense the answer to these questions is the so-called Sarkisov Programme (see [45]) which has so far been proved only for surfaces and threefolds, and its proof is non-trivial even in the three-dimensional case. However, one can readily understand and even feel the geometric meaning of the Sarkisov Programme without going into the details of proving that every sequence of elementary links terminates. In the rest of this section we briefly show how to logically approach the three-dimensional Sarkisov Programme starting from the two problems posed above.

Let $\dim(X) = 3$. Then the following result (whose proof readily follows from the proof of Theorem 1.4.1, which in turn is Theorem 4.2 in [45]) is a natural generalization of Theorem 1.4.1.

Theorem 1.6.1. *Let* $\overline{\pi} \colon \overline{X} \to \overline{Z}$ *be a Mori fibration and let* $\rho \colon X \dashrightarrow \overline{X}$ *be a birational map. Take a very ample divisor* D *on* $\overline{Z}$ *and choose an* $n \in \mathbb{N}$ *such that the linear system* $|-nK_{\overline{X}} + \overline{\pi}^*(nD)|$ *has no base points. Let* $\mathcal{M}$ *be the proper transform of the linear system* $|-nK_{\overline{X}} + \overline{\pi}^*(nD)|$ *on the variety* X *and* *consider a positive rational number μ such that $K_X + \mu\mathcal{M} \sim_{\mathbb{Q}} \pi^*(H)$ for some divisor H on the variety Z. Suppose that the movable log pair $(X, \mu\mathcal{M})$ has canonical singularities and that the divisor H is numerically effective. In this case ρ is an isomorphism and there is an isomorphism $\alpha\colon Z \to \overline{Z}$ such that the diagram*

$$\begin{tikzcd} X \arrow[r, "\rho"] \arrow[d, "\pi"'] & \overline{X} \arrow[d, "\overline{\pi}"] \\ Z \arrow[r, "\alpha"'] & \overline{Z} \end{tikzcd} \tag{1.6.2}$$

is commutative.

If $\dim(Z) \neq 0$, then it is impossible to obtain a criterion (similar to Corollary 1.4.3) for the birational superrigidity of a Mori fibration $\pi\colon X \to Z$ in terms of numerical properties of movable log pairs on the variety X. The main reason for this fact is that the birational superrigidity (respectively, rigidity) of the Mori fibration π is defined as the uniqueness of a Mori fibred space in a given class of birational equivalence up to the commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "\rho"] \arrow[d, "\pi"'] & \overline{X} \arrow[d, "\overline{\pi}"] \\ Z \arrow[r, dashed, "\alpha"] & \overline{Z}\,, \end{tikzcd} \tag{1.6.3}$$

where ρ and α are birational but not biregular isomorphisms and the rational map ρ induces a biregular isomorphism of the general fibres of the fibrations π and $\overline{\pi}$ (respectively, a birational isomorphism of the general fibres of the fibrations π and $\overline{\pi}$, which must be isomorphic in this case). In particular, if π is birationally superrigid (respectively, rigid), then the general fibre of the fibration π is a birationally superrigid (respectively, rigid) Fano variety, where the general fibre of π is regarded as a variety defined over the field of rational functions of the variety Z. However, it can be seen from Theorem 1.6.1 that imposing specific numerical conditions on movable log pairs on the Mori fibred space X can rather imply the existence of the commutative diagram (1.6.2).

It is easy to prove the following criterion.

Proposition 1.6.4. *A Mori fibration $\pi\colon X \to Z$ is birationally superrigid if and only if $\kappa(X, M_X) \geqslant 0$ for every movable log pair (X, M_X) satisfying the relation $K_X + M_X \sim_{\mathbb{Q}} \pi^*(H)$, where H is a divisor on the variety Z.*

For $\dim(Z) = 1$ the assertion of Proposition 1.6.4 implicitly contains the non-existence of a so-called *super-maximal singularity* (see [150]), and for $\dim(Z) = 2$ the geometric meaning of Proposition 1.6.4 is revealed in [162]. For $\dim(Z) = 0$ the assertion of Proposition 1.6.4 coincides with Corollary 1.4.3.

Suppose now that the Mori fibred space X is not birationally superrigid, that is, there is a Mori fibration $\overline{\pi}\colon \overline{X} \to \overline{Z}$ but there is no commutative diagram (1.6.3). Let D be a very ample divisor on $\overline{Z}$ and let

$$\mathcal{M} = \rho^{-1}(|-nK_{\overline{X}} + \overline{\pi}^*(nD)|)$$

for $n \gg 0$. We consider a positive rational number μ such that $K_X + \mu\mathcal{M} \sim_{\mathbb{Q}} \pi^*(H)$ for a divisor H on the variety Z. Then it follows from Theorem 1.6.1 that either the singularities of the movable log pair $(X, \mu\mathcal{M})$ are not canonical or the divisor H is not numerically effective.

Remark 1.6.5. One can readily see that $6/\mu \in \mathbb{N}$ if $\dim(Z) > 0$. If the variety Z is a point, then it follows from [103] that the variety X belongs to finitely many deformation families, which implies the existence of an absolute constant λ such that $\lambda/\mu \in \mathbb{N}$. A similar result holds under much more general assumptions, for example, for all Fano threefolds with canonical singularities (see [12], [116], [13]). The boundedness of smooth Fano varieties of any fixed dimension was proved in [137] and [114].

We first consider the case in which the singularities of the movable log pair $(X, \mu\mathcal{M})$ are canonical and the divisor H is not numerically effective. In particular, the variety Z is not a point, and the following two cases are possible:

- $\dim(Z) = 1$, which means that π is a del Pezzo fibration;
- $\dim(Z) = 2$, which means that π is a conic bundle.

Suppose that $\dim(Z) = 1$. In this case $\operatorname{rk}\operatorname{Pic}(X) = 2$. Let $\overline{\mathbb{NE}}(X) \subset \mathbb{R}^2$ be the closure of the cone of one-dimensional effective cycles on the variety X. Then the cone $\overline{\mathbb{NE}}(X)$ is two-dimensional and, in particular, has exactly two extremal rays R_1 and R_2 of which one, say R_1, is generated by a curve contained in the fibres of the fibration π. On the other hand, it follows from the numerical non-effectiveness of the divisor H that $(K_X + \mu\mathcal{M}) \cdot R_2 < 0$. The log MMP now implies that there is a contraction $\psi \colon X \to Y$ such that Y is normal, the morphism ψ is surjective and has connected fibres, and any curve $C \subset X$ is contracted by ψ to a point if and only if $C \in R_2$. The possible cases are

- ψ is birational and contracts a curve $C \subset X$,
- ψ is birational and contracts an irreducible divisor $E \subset X$,
- ψ is a fibration into del Pezzo surfaces,
- ψ is a conic bundle.

Moreover, if ψ is a small contraction, then there is a log flip $\gamma \colon X \dashrightarrow \widehat{X}$ with respect to the movable log pair $(X, \mu\mathcal{M})$, and we can apply the above arguments to the movable log pair $(\widehat{X}, \mu\gamma(\mathcal{M}))$. Hence, there is a diagram

$$\begin{tikzcd} & X \arrow[r, dashed, "\xi"] \arrow[dl, "\pi"'] & W \arrow[dr, "\upsilon"] & \\ Z & & & U \end{tikzcd} \tag{1.6.6}$$

where ξ is a composition of flips, flops, and antiflips, and υ is a divisorial contraction, a del Pezzo fibration, or a conic bundle. If υ is not birational, then we set $X' = W$, $Z' = U$, and $\pi' = \upsilon$; otherwise we set $X' = U$ and take Z' to be a point and $\pi' \colon X' \to Z'$ to be a surjection. In this case, π' is a Mori fibration.

Remark 1.6.7. In the terminology of [45] and [46] or that of Theorem 1.6.14, the birational diagram (1.6.6) is a Sarkisov link of type IV if υ is a Mori fibration and a Sarkisov link of type III if υ is birational.

Suppose now that $\dim(Z) = 2$. As above, we assume that the divisor H is not numerically effective. This case is similar to the previous one, but the number of possible birational combinations is greater, because the variety Z is a surface (rather than a curve) and has its own birational geometry. There is an effective divisor Δ on Z for which the rational equivalence $H \sim_{\mathbb{Q}} K_Z + \Delta$ holds and the singularities of the log pair (Z, Δ) are log terminal (see [107] and [105]). Hence, we can apply the log MMP to the log pair (Z, Δ) constructed, and the relation

$$K_X + \mu\mathcal{M} \sim_{\mathbb{Q}} \pi^*(K_Z + \Delta)$$

immediately ensures the *synchronous* applicability of the log MMP both to the log pair (Z, Δ) and to the movable log pair $(X, \mu\mathcal{M})$. In particular, there is a surjective map with connected fibres $\tau\colon Z \to S$ for which the following cases are possible:

- $\dim(S) = 2$, that is, the morphism τ contracts an irreducible rational curve;
- $\dim(S) = 1$, that is, the morphism τ is a $\mathbb{P}^1$-fibration over the curve S;
- $\dim(S) = 0$, that is, the surface Z is a del Pezzo surface with log terminal singularities and such that $\operatorname{rk}\operatorname{Pic}(Z) = 1$.

Here the case $\dim(S) = 0$ is quite similar to the case $\dim(Z) = 1$, because in this case we have $\operatorname{rk}\operatorname{Pic}(X) = 2$. Applying the log MMP, we obtain the commutative diagram

$$\begin{tikzcd} & X \arrow[r, dashed, "\xi"] \arrow[dl, "\pi"'] & W \arrow[dr, "\upsilon"] & \\ Z \arrow[dr, "\tau"'] & & & U \arrow[dll, "\zeta"] \\ & S & & \end{tikzcd} \qquad (1.6.8)$$

where the map ξ is a birational isomorphism in codimension one, that is, ξ is a composition of finitely many flops, flips, and antiflips. Moreover, if $\dim(S) = 2$, then the morphism υ is a divisorial contraction and ζ is a conic bundle. If $\dim(S) = 1$, then υ is either a conic bundle or a del Pezzo fibration, and in the latter case the map ζ is simply an isomorphism. If $\dim(S) = 0$, then υ is either a birational map or a Mori fibration. If υ is a Mori fibration, then we set $X' = W$, $Z' = U$, and $\pi' = \upsilon$. If υ is birational, then we set $X' = U$, $Z' = S$, and $\pi' = \zeta$. In this case, π' is a Mori fibration.

Remark 1.6.9. In the terminology of [45] and [46] or that of Theorem 1.6.14, the birational diagram (1.6.8) is a Sarkisov link of type III if υ is birational and a Sarkisov link of type IV otherwise.

Thus, the assumption that the singularities of the log pair $(X, \mu\mathcal{M})$ are canonical and the divisor H is not numerically effective leads to the construction of a birational transformation of the Mori fibred space X into the Mori fibred space X'. Let $\sigma\colon X \dashrightarrow X'$ be the birational map thus constructed, let $\mathcal{M}' = \sigma(\mathcal{M})$, and let μ' be a rational number for which the relation $K_{X'} + \mu'\mathcal{M}' \sim_{\mathbb{Q}} \pi'^*(H')$ holds for some divisor H' on the variety Z'. Then one of the following three cases holds: there is a commutative diagram

$$
\begin{tikzcd}
X' \arrow[r, "\rho\circ\sigma^{-1}"] \arrow[d, "\pi'"'] & \overline{X} \arrow[d, "\overline{\pi}"] \\
Z' \arrow[r, "\alpha"'] & \overline{Z}\,,
\end{tikzcd}
\tag{1.6.10}
$$

where $\rho\circ\sigma^{-1}$ and α are isomorphisms, the singularities of the movable log pair $(X',\mu'\mathcal{M}')$ are not canonical, or the divisor H' is not numerically effective.

Remark 1.6.11. We have the inequality $\mu'>\mu$ by construction, where μ' belongs to a set satisfying the ascending chain condition (see Remark 1.6.5).

Let us now assume that the singularities of the movable log pair $(X,\mu\mathcal{M})$ are not canonical. In this case the movable log pair $(X,\nu\mathcal{M})$ is canonical (but need not be terminal) for some rational number $\nu<\mu$, and the singularities of $(X,\mu\mathcal{M})$ are either canonical over a general point of the variety Z or not canonical. In the former case the log MMP implies the existence of the diagram

$$
\begin{tikzcd}
X \arrow[r, dashed, "\sigma"] \arrow[d, "\pi"'] & X' \arrow[d, "\pi'"] \\
Z \arrow[r, dashed, "\alpha"] & Z'\,,
\end{tikzcd}
$$

where π' is a Mori fibration, σ and α are birational maps, and σ induces a biregular isomorphism of the general fibres of the fibrations π and π'. On the other hand, if the variety Z is a point, then it follows immediately from the log MMP and from the existence of extremal blow-up for log pairs having canonical singularities that the following commutative diagram exists:

$$
\begin{tikzcd}
 & V \arrow[r, dashed, "\xi"] \arrow[dl, "\chi"'] & W \arrow[dr, "\upsilon"] & \\
X & & & U\,,
\end{tikzcd}
\tag{1.6.12}
$$

where χ is a divisorial contraction such that

$$K_V+\nu\chi^{-1}(\mathcal{M})\sim_{\mathbb{Q}}\chi^*(K_X+\nu\mathcal{M}),$$

the map ξ is an isomorphism in codimension one, and υ is a divisorial contraction, a del Pezzo fibration, or a conic bundle.

Remark 1.6.13. In the terminology of [45] and [46] or that in Theorem 1.6.14, the birational diagram (1.6.12) is a Sarkisov link of type III.

One can readily generalize the arguments used in the above cases to the remaining cases and obtain some more examples of elementary non-biregular birational transformations of a Mori fibred space X into some other Mori fibred space.

Here Theorem 1.6.1 either ensures a decomposition of the birational map ρ into a composition of finitely many elementary birational transformations of this kind or gives necessary conditions for the existence of another elementary birational transformation. The only non-trivial question is as follows: Is it possible to obtain an infinite sequence of elementary birational transformations of this kind? The answer to this question is certainly negative (see [45]). Thus, the following theorem holds, whose assertion is the three-dimensional Sarkisov Programme.

Theorem 1.6.14. *Let* $\pi\colon X \to Z$ *and* $\overline{\pi}\colon \overline{X} \to \overline{Z}$ *be Mori fibrations, let* $\dim(X) = \dim(\overline{X}) = 3$, *and let there be a birational map* $\rho\colon X \dashrightarrow \overline{X}$. *Then the map* ρ *is a composition of the form* $\eta \circ \rho_n \circ \cdots \rho_1$, *where* $\eta\colon \overline{X} \to \overline{X}$ *is a biregular automorphism for which there is an isomorphism* $\omega : \overline{Z} \to \overline{Z}$ *such that the diagram*

$$\begin{tikzcd}
\overline{X} \arrow[r, "\eta"] \arrow[d, "\overline{\pi}"'] & \overline{X} \arrow[d, "\overline{\pi}"] \\
\overline{Z} \arrow[r, "\omega"'] & \overline{Z}
\end{tikzcd}$$

is commutative, *and* $\rho_{i+1}\colon X_i \dashrightarrow X_{i+1}$ *is a birational map*, *a so-called elementary link* (*or Sarkisov link*), *defined in one of the following ways*:

- *a link of type* I, $\rho_{i+1} = \xi_{i+1} \circ \chi_i^{-1}$, *and one has the commutative diagram*

$$\begin{tikzcd}
& V_i \arrow[dl, "\chi_i"'] \arrow[r, dashed, "\xi_{i+1}"] & X_{i+1} \arrow[dd, "\pi_{i+1}"] \\
X_i \arrow[d, "\pi_i"'] & & \\
Z_i & & Z_{i+1} \arrow[ll, "\tau_{i+1}"']
\end{tikzcd},$$

- *a link of type* II, $\rho_{i+1} = \check{\chi}_{i+1} \circ \xi_{i+1} \circ \widehat{\chi}_i^{-1}$, *and one has the commutative diagram*

$$\begin{tikzcd}
& \widehat{V}_i \arrow[dl, "\widehat{\chi}_i"'] \arrow[r, dashed, "\xi_{i+1}"] & \check{V}_{i+1} \arrow[dr, "\check{\chi}_{i+1}"] & \\
X_i \arrow[dr, "\pi_i"'] & & & X_{i+1} \arrow[dl, "\pi_{i+1}"] \\
& Z_i \arrow[r, Rightarrow] & Z_{i+1} &
\end{tikzcd},$$

- *a link of type* III, $\rho_{i+1} = \chi_{i+1} \circ \xi_{i+1}$, *and one has the commutative diagram*

$$
\begin{tikzcd}
X_i \arrow[r, dashed, "\xi_{i+1}"] \arrow[dd, "\pi_i"'] & V_i \arrow[dr, "\chi_{i+1}"] & \\
& & X_{i+1} \arrow[d, "\pi_{i+1}"] \\
Z_i \arrow[rr, "\tau_{i+1}"] & & Z_{i+1}\ ,
\end{tikzcd}
$$

- *a link of type* IV, $\rho_{i+1} = \xi_{i+1}$, *and one has the commutative diagram*

$$
\begin{tikzcd}
X_i \arrow[rr, dashed, "\xi_{i+1}"] \arrow[d, "\pi_i"'] & & X_{i+1} \arrow[d, "\pi_{i+1}"] \\
Z_i \arrow[dr, "\alpha_i"'] & & Z_{i+1}\ , \arrow[dl, "\beta_i"] \\
& S &
\end{tikzcd}
$$

where $X_0 = X$, $X_n = \overline{X}$, $\pi_0 = \pi$, $\pi_n = \overline{\pi}$, $Z_0 = Z$, $Z_n = \overline{Z}$, *the varieties* V_i *and* X_i *have terminal* $\mathbb{Q}$*-factorial singularities*, π_i *is a Mori fibration*, ξ_{i+1} *is a composition of finitely many flips*, *flops*, *and antiflips*, *the maps* χ_i, $\widehat{\chi}_i$, *and* $\check{\chi}_i$ *are divisorial contractions of an irreducible divisor*, *and* τ_{i+1}, α_i, *and* β_i *are surjective morphisms with connected fibres.*

Of course, Theorem 1.6.14 holds for algebraic surfaces as well. In the two-dimensional case, the assertion of Theorem 1.6.14 is much simpler and more specific, because there are only two types of Mori fibred surfaces, namely, $\mathbb{P}^2$ and $\mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a vector bundle on some smooth curve, the only extremal birational contractions of surfaces are ordinary contractions of smooth rational curves into smooth points, and the isomorphisms in codimension two are isomorphisms. Instead of rewriting the assertion of Theorem 1.6.14 for algebraic surfaces, we shall now prove the following classical result by using the Sarkisov Programme and following the arguments in [1].

Theorem 1.6.15. *The group* $\mathrm{Bir}(\mathbb{P}^2)$ *is generated by the elements of the group* $\mathrm{Aut}(\mathbb{P}^2) \cong \mathrm{PGL}(3, \mathbb{C})$ *together with an involution* τ *such that* $\tau(x : y : z) = (yz : xz : xy)$, *where* $(x : y : z)$ *are homogeneous coordinates on* $\mathbb{P}^2$.

Proof. Let σ be a birational automorphism of $\mathbb{P}^2$. We recall that a Cremona map of $\mathbb{P}^2$ is a birational map which blows up three points on the plane $\mathbb{P}^2$ not belonging to a single line and contracts the proper transforms of three lines passing through these points into three points on $\mathbb{P}^2$. We show that σ is a composition of Cremona maps.

Let τ be a Cremona map. Then there is a commutative diagram

$$\begin{tikzcd}
 & & \widehat{S} \arrow[dl, "\widehat{\alpha}"'] \arrow[dr, "\widehat{\beta}"] & & & \check{S} \arrow[dl, "\check{\beta}"'] \arrow[dr, "\check{\alpha}"] & & \\
 & \mathbb{F}_1 \arrow[dr, "\pi"] \arrow[ddl, "\widehat{\gamma}"'] & & \mathbb{F}_0 \arrow[dl, "\widehat{\pi}"] \arrow[r, Rightarrow] & \mathbb{F}_0 \arrow[dr, "\check{\pi}"'] & & \mathbb{F}_1 \arrow[dl, "\pi"'] \arrow[ddr, "\check{\gamma}"] & \\
 & & \mathbb{P}^1 & & & \mathbb{P}^1 & & \\
\mathbb{P}^2 \arrow[rrrrrrr, dashed, "\tau"] & & & & & & & \mathbb{P}^2
\end{tikzcd} \tag{1.6.16}$$

where $\widehat{\gamma}$ and $\check{\gamma}$ are blow-ups of points, π is the natural projection, $\widehat{\pi}$ and $\check{\pi}$ are distinct projections of the surface $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$, $\widehat{\alpha}$ and $\check{\alpha}$ are blow-ups of points not belonging to the exceptional section of the surface $\mathbb{F}_1$, and $\widehat{\beta}$ and $\check{\beta}$ are contractions of proper transforms of the fibres in which the points were blown up by the morphisms $\widehat{\alpha}$ and $\check{\alpha}$, respectively. The commutative diagram (1.6.16) is a factorization of the map τ into elementary links. The birational maps $\widehat{\beta} \circ \widehat{\alpha}^{-1}$ and $\check{\alpha} \circ \check{\beta}^{-1}$ are elementary transformations of the surface $\mathbb{F}_1$ into the surface $\mathbb{F}_0$ and of the ruled surface $\mathbb{F}_0$ into the ruled surface $\mathbb{F}_1$, respectively, and the construction of these transformations can readily be generalized to the general situation to define an elementary transformation of a ruled surface $\mathbb{F}_k$ into a ruled surface $\mathbb{F}_{k\pm 1}$.

The map σ admits the following factorization into elementary links:

$$\begin{tikzcd}
\mathbb{F}_{k_0} \arrow[r, dashed, "\psi_1"] \arrow[d, "\widehat{\upsilon}"'] & \mathbb{F}_{k_1} \arrow[r, dashed, "\psi_2"] & \cdots \arrow[r, dashed, "\psi_{n-1}"] & \mathbb{F}_{k_{n-1}} \arrow[r, dashed, "\psi_n"] & \mathbb{F}_{k_n} \arrow[d, "\check{\upsilon}"] \\
\mathbb{P}^2 \arrow[rrrr, dashed, "\sigma"] & & & & \mathbb{P}^2
\end{tikzcd} \tag{1.6.17}$$

where $k_i \geqslant 0$ is an integer such that $k_{i+1} = k_i \pm 1$ and $k_0 = k_n = 1$, ψ_{k_i} is an elementary transformation of the ruled surface $\mathbb{F}_{k_{i-1}}$ into the surface $\mathbb{F}_{k_i}$, and $\widehat{\upsilon}$ and $\check{\upsilon}$ are blow-ups of points.

We call the number $\max(k_1, \ldots, k_n) \geqslant 1$ the *height* of this factorization and say that this factorization is *minimal* if $k_i \neq 1$ for all i. It follows from the commutative diagram (1.6.16) that the map σ is a composition of Cremona maps if the height of the factorization (1.6.17) is 1. We proceed by induction on the height of the factorization (1.6.17). Suppose that height of (1.6.17) is not less than two. One can assume that (1.6.17) is minimal, because otherwise one can represent the map σ as a composition of birational automorphisms for which there is a minimal factorization whose height does not exceed the height of (1.6.17), and the desired inductive assertion can be proved for each of the birational automorphisms separately.

It follows from the above assumptions that $k_1 = 2$. Let $\xi \colon \mathbb{F}_1 \dashrightarrow \mathbb{F}_0$ be an elementary transformation undefined at a sufficiently general point of the surface $\mathbb{F}_1$.

In this case the factorization (1.6.17) induces the commutative diagram

$$
\begin{tikzcd}
& \mathbb{F}_1 \arrow[r, dashed, "\xi"] \arrow[dl, "\overline{\upsilon}"'] & \mathbb{F}_0 \arrow[r, dashed, "\eta"] & \mathbb{F}_1 \arrow[dr, "\widetilde{\upsilon}"] & & \mathbb{F}_1 \arrow[r, dashed, "\xi^{-1}"] \arrow[dl, "\breve{\upsilon}"'] & \mathbb{F}_0 \arrow[r, dashed, "\phi_1"] & \cdots \arrow[r, dashed, "\phi_n"] & \mathbb{F}_1 \arrow[dr, "\acute{\upsilon}"] & \\
\mathbb{P}^2 \arrow[rrrr, dashed, "\nu"] & & & & \mathbb{P}^2 \arrow[rrrrr, dashed, "\chi"] & & & & & \mathbb{P}^2,
\end{tikzcd}
\tag{1.6.18}
$$

where $\overline{\upsilon}$, $\widetilde{\upsilon}$, $\breve{\upsilon}$, and $\acute{\upsilon}$ are blow-ups of points, ϕ_i and η are elementary transformations of ruled surfaces, $\sigma = \chi \circ \nu$, and ν is a Cremona map.

We can assume that the commutative diagram (1.6.18) contains a factorization of the birational map χ into elementary links; however, this factorization may no longer be minimal. Nevertheless, by construction, the height of the factorization (1.6.18) is strictly less than that of the factorization (1.6.17). Thus, the birational automorphism χ is a composition of finitely many Cremona maps by the induction hypothesis, and $\sigma = \chi \circ \nu$, where ν is a Cremona map.

The defining relations for the generators of the group $\mathrm{Bir}(\mathbb{P}^2)$ were found in [67], and a simpler solution of this problem was obtained in [88]. The generators of the group $\mathrm{Bir}(\mathbb{P}^2)$ and the relations among them were described in [89] and [97], respectively, in the case when the field of definition is not algebraically closed. Nevertheless, there are many open questions related to the two-dimensional Cremona group. For example, the following conjecture was expressed in [68].

Conjecture 1.6.19. *The group* $\mathrm{Bir}(\mathbb{P}^2)$ *is simple.*

We note that the assertion of Theorem 1.6.14 remains valid for surfaces defined over an arbitrary (not necessarily algebraically closed) perfect field (see [91]).

§ 1.7. Log adjunction and connectedness

For some natural reasons, movable log pairs (see Definition 1.3.1) reflect well the birational geometry of a given variety (see Theorem 1.3.20), the canonical and terminal singularities form the most natural class of singularities for these log pairs, and many important problems in birational geometry can be formulated in a very simple way in terms of movable log pairs (see Theorem 1.4.1).

On the other hand, one can consider log pairs which have both movable and fixed components, by analogy with the existence of linear systems having both movable and fixed parts. Moreover, in principle one can also consider log pairs with negative coefficients of components of the boundary, and these assumptions are not only admissible but also necessary for the following reasons:

- even under elementary blow-ups, the log pullback of a movable log pair (see Definition 1.7.1), which can have both fixed components and negative coefficients of components of the boundaries, reflects the properties of the pair on the blown-up variety more exactly than the naturally defined proper transform;
- the geometric properties of the centres of canonical singularities (see Definition 1.3.8) are not good when regarded outside the birational context, in contrast to centres of log canonical singularities (see Definition 1.7.2), whose role in geometry is great (see [106], [113], [111], [168], [154], [63]).

In the present section we impose no restrictions on the coefficients of the boundaries, although this is not very convenient and sometimes even excessive. In particular, boundaries can fail to be effective, unless otherwise stated explicitly; however, all log canonical divisors are assumed to be $\mathbb{Q}$-Cartier divisors.

Definition 1.7.1. A log pair (V, B^V) is called a *log pullback* of a log pair (X, B_X) with respect to a birational morphism $f\colon V \to X$ if

$$B^V = f^{-1}(B_X) - \sum_{i=1}^{n} a(X, B_X, E_i)E_i \quad \text{and} \quad K_V + B^V \sim_{\mathbb{Q}} f^*(K_X + B_X),$$

where the coefficients $a(X, B_X, E_i)$ are in $\mathbb{Q}$ and the divisor E_i is f-exceptional.

Definition 1.7.2. A proper irreducible subvariety $Y \subset X$ is called a *centre of log canonical singularities* of a log pair (X, B_X) if there exist a birational morphism $f\colon W \to X$ and a (not necessarily f-exceptional) divisor $E \subset W$ such that E is contained in the support of the effective part of the divisor $\lfloor B^Y \rfloor$. The set of all centres of log canonical singularities of the log pair (X, B_X) is denoted by $\mathbb{LCS}(X, B_X)$, and the set-theoretic union of the centres of log canonical singularities of (X, B_X) (regarded as a proper subset of the variety X) is called the *locus of log canonical singularities* of (X, B_X) and is denoted by the symbol $LCS(X, B_X)$.

Remark 1.7.3. Let (X, B_X) be a log pair, let H be a sufficiently general hyperplane section of X, and let $Z \in \mathbb{LCS}(X, B_X)$. Then $Z \cap H \in \mathbb{LCS}(H, B_X|_H)$.

We consider a log pair (X, B_X), where $B_X = \sum_{i=1}^{n} a_i B_i$, $a_i \in \mathbb{Q}$, and B_i is either an effective, irreducible, and reduced divisor on the variety X or a linear system on X not having fixed components. We say that the boundary B_X is *effective* if $a_i \geqslant 0$ for all i. We say that the boundary B_X is *movable* if B_i is a linear system on X having no fixed components for every subscript i.

Example 1.7.4. Let us consider a smooth point O on a variety X. Suppose that the point O belongs to the set $\mathbb{LCS}(X, B_X)$. Let $f\colon V \to X$ be an ordinary blow-up of the point O and let E be an exceptional divisor of the birational morphism f. Then either $E \in \mathbb{LCS}(V, B^V)$ or there is a subvariety $Z \subsetneq E$ such that $Z \in \mathbb{LCS}(V, B^V)$, and $E \in \mathbb{LCS}(V, B^V)$ if and only if $\operatorname{mult}_O(B_X) \geqslant \dim(X)$.

Let $f\colon Y \to X$ be a birational morphism, let the variety Y be smooth, and let the union of all the divisors $f^{-1}(B_i)$ and all the f-exceptional divisors form a divisor with simple normal crossings. In this case the birational morphism f is called a *log resolution* of the log pair (X, B_X), and the relation $K_Y + B^Y \sim_{\mathbb{Q}} f^*(K_X + B_X)$ holds for the log pullback (Y, B^Y) of the log pair (X, B_X).

Definition 1.7.5. The subscheme $\mathcal{L}(X, B_X)$ associated with the sheaf of ideals

$$\mathcal{I}(X, B_X) = f_*(\mathcal{O}_Y(\lceil -B^V \rceil))$$

is called the *subscheme of log canonical singularities*.

We note that $\operatorname{Supp}(\mathcal{L}(X, B_X)) = LCS(X, B_X) \subset X$. The following result is the Shokurov vanishing theorem (see [168]).

Theorem 1.7.6. *Suppose that the boundary B_X is effective and H is a nef and big divisor on a variety X such that the divisor $D = K_X + B_X + H$ is a Cartier divisor. Then $H^i(X, \mathcal{I}(X, B_X) \otimes D) = 0$ for any $i > 0$.*

Proof. It follows from the Kawamata–Viehweg vanishing theorem that

$$R^i f_*\bigl(f^*(K_X + B_X + H) + \lceil -B^W \rceil\bigr) = 0$$

for any $i > 0$ (see Theorem 1.2.3 in [106]). It follows from the degeneration of the corresponding spectral sequence and from the coincidence of the sheaves

$$R^0 f_*\bigl(f^*(K_X + B_X + H) + \lceil -B^W \rceil\bigr) = \mathcal{I}(X, B_X) \otimes D$$

that the cohomology groups coincide, that is,

$$H^i\bigl(X, \mathcal{I}(X, B_X) \otimes D\bigr) = H^i\bigl(W, f^*(K_X + B_X + H) + \lceil -B^W \rceil\bigr)$$

for any $i \geqslant 0$. However. the cohomology groups

$$H^i\bigl(W, f^*(K_X + B_X + H) + \lceil -B^W \rceil\bigr)$$

are trivial for $i > 0$ by the Kawamata–Viehweg vanishing theorem.

We consider the following simple application of Theorem 1.7.6 (see [33]).

Lemma 1.7.7. *Let Σ be a finite subset of $\mathbb{P}^n$ and let $\mathcal{M}$ be a linear system formed by the hypersurfaces of degree k in $\mathbb{P}^n$ that contain the set Σ. Suppose that the base locus of the linear system $\mathcal{M}$ is zero-dimensional. Then the points of the set Σ impose independent linear conditions on the hypersurfaces in $\mathbb{P}^n$ of degree $n(k-1)$.*

Proof. Let Λ be the finite subset of $\mathbb{P}^n$ which is the base locus of the linear system $\mathcal{M}$. Then we have $\Sigma \subseteq \Lambda$. Let us consider general divisors $H_1, \dots, H_s$ in the linear system $\mathcal{M}$ for $s \gg 0$ and write $X = \mathbb{P}^n$ and $B_X = \frac{n}{s}\sum_{i=1}^{s} H_i$. In this case,

$$\operatorname{Supp}(\mathcal{L}(X, B_X)) = \Lambda,$$

where $\mathcal{L}(X, B_X)$ stands for the subscheme of log canonical singularities of the log pair (X, B_X).

To prove the desired assertion, it suffices to construct for an arbitrary point $P \in \Sigma$ a hypersurface in $\mathbb{P}^n$ of degree $n(k-1)$ that passes through all points in $\Sigma \setminus P$ but does not pass through the point P.

Let $\Sigma \setminus P = \{P_1, \dots, P_k\}$, where P_i are points of the variety $X = \mathbb{P}^n$, and let $f\colon V \to X$ be a blow-up of all points of the set $\Sigma \setminus P$. Then

$$K_V + \Bigl(B_V + \sum_{i=1}^{k}\bigl(\operatorname{mult}_{P_i}(B_X) - n\bigr)E_i\Bigr) + f^*(H) = f^*(n(k-1)H) - \sum_{i=1}^{k} E_i,$$

where $E_i = f^{-1}(P_i)$, $B_V = f^{-1}(B_X)$, and H is a hyperplane in $\mathbb{P}^n$. By construction, we have

$$\operatorname{mult}_{P_i}(B_X) = n \operatorname{mult}_{P_i}(\mathcal{M}) \geqslant n,$$

and the divisor $\widehat{B}_V = B_V + \sum_{i=1}^{k}(\operatorname{mult}_{P_i}(B_X) - n)E_i$ is effective.

Let $\overline{P} = f^{-1}(P)$. Then $\overline{P} \in \mathbb{LCS}(W, \overline{B}_W)$, and the point $\overline{P}$ is an isolated centre of log canonical singularities of the log pair $(W, \overline{B}_W)$, because the birational morphism $f\colon V \to X$ is an isomorphism in a neighbourhood of the point P. On the other hand, the map

$$H^0\left(\mathcal{O}_V\left(f^*(n(k-1)H) - \sum_{i=1}^{k} E_i\right)\right) \to H^0\left(\mathcal{O}_{\mathcal{L}(V,\widehat{B}_V)} \otimes \mathcal{O}_V\left(f^*(n(k-1)H) - \sum_{i=1}^{k} E_i\right)\right)$$

is surjective by Theorem 1.7.6. However, in a neighbourhood of the point $\overline{P}$ the support of the scheme $\mathcal{L}(V, \widehat{B}_V)$ consists solely of $\overline{P}$, which implies that there is a divisor $D \in |f^*(n(k-1)H) - \sum_{i=1}^{k} E_i|$ that does not contain $\overline{P}$. The divisor $f(D)$ is a hypersurface in $\mathbb{P}^n$ of degree $n(k-1)$ and passes through all points of the set $\Sigma \setminus P$ but not through the point $P \in \Sigma$.

Corollary 1.7.8. *Let V be a hypersurface in $\mathbb{P}^4$ of degree n such that the singularities of V are isolated ordinary double points. As is well known* [51], *V is $\mathbb{Q}$-factorial if and only if its singular points impose independent linear conditions on the hypersurfaces in $\mathbb{P}^4$ of degree $2n-5$. Suppose that the set* $\operatorname{Sing}(V)$ *is a set-theoretic intersection of hypersurfaces in $\mathbb{P}^4$ whose degrees are less than $n/2$. Then V is $\mathbb{Q}$-factorial.*

We consider another elementary application of Theorem 1.7.6 (see [25]).

Lemma 1.7.9. *Let $V = \mathbb{P}^1 \times \mathbb{P}^1$ and let B_V be an effective boundary on V of bi-degree (a, b), where $a, b \in \mathbb{Q} \cap [0, 1)$. Then $\mathbb{LCS}(V, B_V) = \varnothing$.*

Proof. Let $B_V = \sum_{i=1}^{k} a_i B_i$ for positive rational numbers a_i and for irreducible reduced curves $B_i \subset V$. Then $a_i \leqslant \max(a, b) < 1$ for any i, because the intersections of the boundary B_V with the fibres of the two projections on $\mathbb{P}^1$ are equal to a and b, respectively, which implies that the set $\mathbb{LCS}(V, B_V)$ contains no curves.

Suppose that the set $\mathbb{LCS}(V, B_V)$ contains a point $O \in V$. Let us consider a divisor H on V of bi-degree $(1-a, 1-b)$. Then $H^0(\mathcal{O}_V(K_V + B_V + H)) = 0$. However, Theorem 1.7.6 implies that the map

$$H^0(\mathcal{O}_V(K_V + B_V + H)) \to H^0(\mathcal{O}_{\mathcal{L}(V,\, B_V)}(K_V + B_V + H))$$

is surjective, which is impossible because

$$H^0(\mathcal{O}_{\mathcal{L}(V,B_V)}(K_V + B_V + H)) = H^0(\mathcal{O}_{\mathcal{L}(V,B_V)}) \neq 0,$$

a contradiction.

The assertion of Lemma 1.7.9 is a special case of the following result (see [25]).

Theorem 1.7.10. *Let X be a smooth hypersurface in $\mathbb{P}^n$ of degree k with $n \geqslant 3$, let the boundary B_X be effective, and let $B_X \sim_{\mathbb{Q}} rH$ for a rational number $r > 0$, where H is a hyperplane section of X. Then* $\operatorname{lct}(X, B_X) \geqslant \min(\frac{n-1}{rk}, \frac{1}{r})$.

We recall that the *log-canonical threshold* $\operatorname{lct}(X, B_X)$ of the log pair (X, B_X) is the greatest real number λ such that the singularities of the log pair $(X, \lambda B_X)$

are log canonical (see [111]). In the notation and assumptions of Theorem 1.7.10, it follows from [25], [35], [63] that the equality $\mathrm{lct}(X,B_X)=\min(\frac{n-1}{rk},\frac{1}{r})$ holds if and only if one of the following possible cases is valid: $k\geqslant n$ and $B_X=rS$, where S is a hyperplane section of X which is a cone (see Proposition 1.2.3); $n>k$ and $B_X=rS$, where S is an arbitrary hyperplane section of X; $n=3>k$ and $B_X=rS+\Sigma$, where S is a curve and Σ is an effective boundary with $\lfloor\Sigma\rfloor=\varnothing$.

The ideas for the proofs of Lemmas 1.7.7 and 1.7.9 can be used to obtain a more general result. Namely, we take an arbitrary Cartier divisor D on the variety X, consider the exact sequence of cohomology groups

$$H^0(\mathcal{O}_X(D))\to H^0(\mathcal{O}_{\mathcal{L}(X,\,B_X)}(D))\to H^1(\mathcal{J}(X,B_X)\otimes D),$$

apply Theorem 1.7.6, and obtain the following two theorems on connectedness (see [168]).

Theorem 1.7.11. *Let the boundary B_X be effective and let the divisor $-(K_X+B_X)$ be nef and big. Then the set $LCS(X,B_X)\subset X$ is connected.*

Theorem 1.7.12. *Let the boundary B_X be effective and let the divisor $-(K_X+B_X)$ be nef and big with respect to some morphism $g\colon X\to Z$ with connected fibres. Then the set $LCS(X,B_X)$ is connected in a neighbourhood of each fibre of the morphism g.*

Similar arguments imply the following result (see [113], Theorem 17.4).

Theorem 1.7.13. *Let $g\colon X\to Z$ be a surjective morphism with connected fibres, let the divisor $-(K_X+B_X)$ be nef and big with respect to g, and let* $\mathrm{codim}(g(B_i)\subset Z)\geqslant 2$ *if $b_i<0$. In this case the set $LCS(Y,B^Y)$ is connected in a neighbourhood of each fibre of the morphism $g\circ f\colon Y\to Z$.*

We have defined centres of canonical singularities and the set of centres of canonical singularities for movable log pairs (see Definition 1.3.8). However, the fact that the boundary is movable was in fact not used in the definition. The main application of Theorem 1.7.13 gives the following inductive result.

Theorem 1.7.14. *Suppose that the boundary B_X is effective. Let Z be an element of $\mathbb{CS}(X,B_X)$ contained in the support of an effective and reduced Cartier divisor $H\subset X$ which is not a component of the boundary B_X and is smooth at a general point of the subvariety Z. Then $\mathbb{LCS}(H,B_X|_H)\neq\varnothing$.*

Proof. Let us consider the log pair (X,B_X+H). We have

$$\{Z,H\}\subset\mathbb{LCS}(X,B_X+H).$$

Let $f\colon W\to X$ be a log resolution of the log pair (X,B_X+H). Then

$$K_W+\widehat{H}\sim_{\mathbb{Q}}f^*(K_X+B_X+H)+\sum_{E\neq\widehat{H}}a(X,B_X+H,E)E,$$

where $\widehat{H}=f^{-1}(H)$. Applying Theorem 1.7.13 to the log pullback of the log pair (X,B_X+H) on the variety W, we see that $\widehat{H}\cap E\neq\varnothing$ for some f-exceptional divisor E on the variety W such that $f(E) = Z$ and $a(X, B_X, E) \leqslant -1$. The equivalence

$$K_{\widehat{H}} \sim (K_W + \widehat{H})|_{\widehat{H}} \sim_{\mathbb{Q}} f|_{\widehat{H}}^*(K_H + B_X|_H) + \sum_{E \neq \widehat{H}} a(X, B_X + H, E)E|_{\widehat{H}}$$

now gives the desired assertion.

The proof of Theorem 1.7.14 implies the following assertions.

Corollary 1.7.15. *Let the boundary B_X be effective and let the log pair (X, B_X) have log terminal singularities in a punctured neighbourhood of a smooth point $O \in X$ belonging to the set $\mathbb{CS}(X, B_X)$. Then $O \in \mathbb{LCS}(H, B_H)$, where H is a reduced effective divisor on X containing the point O, and $B_H = B_X|_H$.*

Corollary 1.7.16. *Let the boundary B_X be effective, let the log pair (X, B_X) have log terminal singularities in a punctured neighbourhood of an isolated hypersurface singular point O of the variety X, and let $O \in \mathbb{LCS}(S, B_S)$. Then O belongs to the set $\mathbb{LCS}(S, B_S)$, where $S = \cap_{i=1}^k H_i$, $B_S = B_X|_S$, and H_i is a general hyperplane section of X that passes through O.*

The following result is Theorem 3.1 in [46].

Theorem 1.7.17. *Let H be a surface, let O be a smooth point of H, let M_H be an effective movable boundary on H, and let Δ_1 and Δ_2 be irreducible and reduced curves on H intersecting normally at the point O. Suppose that*

$$O \in \mathbb{LCS}(H, (1 - a_1)\Delta_1 + (1 - a_2)\Delta_2 + M_H)$$

for some positive rational numbers a_1 and a_2. Then

$$\mathrm{mult}_O(M_H^2) \geqslant \begin{cases} 4a_1a_2 & \text{if } a_1 \leqslant 1 \text{ or } a_2 \leqslant 1, \\ 4(a_1 + a_2 - 1) & \text{if } a_1 > 1 \text{ and } a_2 > 1. \end{cases}$$

Proof. Let $D = (1 - a_1)\Delta_1 + (1 - a_2)\Delta_2 + M_H$. We consider a birational morphism $f \colon S \to H$ such that the surface S is smooth and

$$K_S + f^{-1}(D) \sim_{\mathbb{Q}} f^*(K_H + D) + \sum_{i=1}^{k} a(H, D, E_i)E_i,$$

where E_i is an f-exceptional curve, $a(H, D, E_i)$ is a rational number, the inequality $a(H, D, E_1) \leqslant -1$ holds, and the birational morphism f is a composition of k blow-ups of smooth points. If $k = 1$, then the assertion is obvious. Suppose that the assertion we need has already been proved for all cases with $a_1 \leqslant 1$ or $a_2 \leqslant 1$. Let $a_1 > 1$ and $a_2 > 1$. Then

$$O \in \mathbb{LCS}(H, (2 - a_1 - a_2)\Delta_2 + M_H),$$

which implies that $\mathrm{mult}_O(M_H^2) \geqslant 4(a_1 + a_2 - 1)$, because the assertion of the theorem holds for the log pair $(H, (2 - a_1 - a_2)\Delta_2 + M_H)$ by assumption. Thus, one can assume that $a_1 \leqslant 1$. The desired assertion now readily follows by induction on k.

Theorem 1.7.17 and Corollary 1.5.15 now imply the following result (see [149], [46], [152], [102]).

Theorem 1.7.18. *Suppose that* $\dim(X) \geqslant 3$, *the boundary* B_X *is effective and movable, and* O *is a smooth point on the variety* X *such that* $O \in \mathbb{CS}(X, B_X)$. *Then* $\mathrm{mult}_O(B_X^2) \geqslant 4$, *and if* $\mathrm{mult}_O(B_X^2) = 4$, *then* $\mathrm{mult}_O(B_X) = 2$ *and* $\dim(X) = 3$.

A natural four-dimensional generalization of Theorem 1.7.18 is the following assertion, which we informally call the $8n^2$-*inequality*.

Theorem 1.7.19. *Let* Y *be a variety of dimension* $r \geqslant 4$, *let* $\mathcal{M}$ *be a linear system on* Y *without fixed components, let* S_1 *and* S_2 *be general divisors in the linear system* $\mathcal{M}$, *let* P *be a smooth point on* Y *belonging to the set* $\mathbb{CS}(Y, \frac{1}{n}\mathcal{M})$, *where* $n \in \mathbb{N}$, *and let the singularities of the log pair* $(Y, \frac{1}{n}\mathcal{M})$ *be canonical in a punctured neighbourhood of the point* P. *Let* $\pi \colon \widehat{Y} \to Y$ *be a blow-up of the point* P *and let* Π *be an exceptional divisor of the morphism* π. *Then there is a linear subspace* $\Lambda \subset \Pi \cong \mathbb{P}^{r-1}$ *of codimension* 2 *such that the inequality*

$$\mathrm{mult}_P(S_1 \cdot S_2 \cdot \Delta) \geqslant 8n^2$$

holds for every effective divisor Δ *on* Y *such that the following conditions hold*:

- *the divisor* Δ *contains* P *and is smooth at this point*;
- *the linear subspace* $\Lambda \subset \Pi \cong \mathbb{P}^{r-1}$ *is contained in the divisor* $\pi^{-1}(\Delta)$;
- Δ *contains no subvarieties of* Y *of codimension* 2 *that are contained in the base locus of the linear system* $\mathcal{M}$;

moreover, the equality is strict for $r \geqslant 5$.

Proof. See [28] and [34].

Theorem 1.7.14 implies the following result (see [46], Theorem 3.10).

Theorem 1.7.20. *Let* B_X *be an effective boundary, let* O *be an isolated ordinary double point on* X, *let* $O \in \mathbb{CS}(X, B_X)$, *and let* $\dim(X) \geqslant 3$. *Then* $\mathrm{mult}_O(B_X) \geqslant 1$, *and the inequality is strict for* $\dim(X) \geqslant 4$.

Proof. Let $f \colon W \to X$ be a blow-up of the point O. Then

$$B_W \sim_{\mathbb{Q}} f^*(B_X) - \mathrm{mult}_O(B_X)E,$$

where $B_W = f^{-1}(B_X)$ and E is an exceptional divisor of the birational morphism f. Suppose that the inequality $\mathrm{mult}_O(B_X) < 1$ is satisfied. By Corollary 1.7.16, we can assume that $\dim(X) = 3$. Then the relation

$$K_W + B_W \sim_{\mathbb{Q}} f^*(K_X + B_X) + (1 - \mathrm{mult}_O(B_X))E$$

implies the existence of a proper subvariety $Z \subset E$ which is a centre of canonical singularities of the log pair (W, B_W). Hence, the set $\mathbb{LCS}(E, B_W|_E)$ is not empty by Theorem 1.7.14, which contradicts Lemma 1.7.9, because $E \cong \mathbb{P}^1 \times \mathbb{P}^1$.

The assertion of Theorem 1.7.20 can be generalized in several ways.

Proposition 1.7.21. *Let the boundary B_X be effective, let $\dim(X)=3$, and let the set $\mathbb{CS}(X,B_X)$ contain a singular point O of a variety X which is locally isomorphic to the hypersurface*

$$y^3=\sum_{i=1}^{3}x_i^2\subset\mathbb{C}^4\cong\operatorname{Spec}(\mathbb{C}[x_1,x_2,x_3,y])$$

in a neighbourhood of O. Then $\operatorname{mult}_O(B_X)\geqslant\frac{1}{2}$.

Proof. Let $f\colon W\to X$ be a blow-up of the variety X at the point O. Then

$$B_W\sim_{\mathbb{Q}}f^*(B_X)-\operatorname{mult}_O(B_X)E,$$

where $B_W=f^{-1}(B_X)$ and E is an exceptional divisor of the birational morphism f. Then the three-dimensional variety W is smooth, and the exceptional divisor E is a cone in $\mathbb{P}^3$ over a smooth conic. Moreover, the restriction $-E|_E$ is rationally equivalent to a hyperplane section of the cone $E\subset\mathbb{P}^3$, and the following relation holds:

$$K_W+B_W\sim_{\mathbb{Q}}f^*(K_X+B_X)+(1-\operatorname{mult}_O(B_X))E.$$

Suppose that $\operatorname{mult}_O(B_X)<\frac{1}{2}$. Then

$$\mathbb{CS}(W,B_W)\subset\mathbb{CS}(W,B_W+(\operatorname{mult}_O(B_X)-1)E),$$

because $\operatorname{mult}_O(B_X)-1<0$. On the other hand, the log pair

$$(W,B_W+(\operatorname{mult}_O(B_X)-1)E)$$

is the log pullback of the log pair (X,B_X) and $O\in\mathbb{CS}(X,B_X)$, Therefore, there is a proper irreducible subvariety $Z\subset E$ such that $Z\in\mathbb{CS}(W,B_W)$, which immediately implies that $\mathbb{LCS}(E,B_W|_E)\neq\varnothing$ by Theorem 1.7.14.

Let $B_E=B_W|_E$. Then the set $\mathbb{LCS}(E,B_E)$ contains no curves on the quadric cone E, because otherwise the intersection of the boundary B_E with a ruling of the cone E is greater than $1/2$, which is impossible because $\operatorname{mult}_O(B_X)<1/2$. Thus, we have the equality $\dim(\operatorname{Supp}(\mathcal{L}(E,B_E)))=0$.

Let H be a hyperplane section of the cone $E\subset\mathbb{P}^3$. Then

$$K_E+B_E+(1-\operatorname{mult}_O(B_X))H\sim_{\mathbb{Q}}-H$$

and $H^0(\mathcal{O}_E(-H))=0$. However, the sequence of cohomology groups

$$H^0(\mathcal{O}_E(-H))\to H^0(\mathcal{O}_{\mathcal{L}(E,B_E)})\to H^1(E,\mathcal{I}(E,B_E)\otimes\mathcal{O}_E(-H))$$

is exact, and $H^1(E,\mathcal{I}(E,B_E)\otimes\mathcal{O}_E(-H))=0$ by Theorem 1.7.6. Therefore, $H^0(\mathcal{O}_{\mathcal{L}(E,\,B_E)})=0$, which is impossible because $\mathbb{LCS}(E,B_E)\neq\varnothing$.

Corollary 1.7.6 and Theorem 1.7.20 imply the following result.

Proposition 1.7.22. *Let the boundary B_X be effective, let $\dim(X) \geqslant 4$, and let O be a singular point of the variety X which is locally isomorphic to the hypersurface*

$$y^3 = \sum_{i=1}^{\dim(X)} x_i^2 \subset \mathbb{C}^{\dim(X)+1} \cong \operatorname{Spec}(\mathbb{C}[x_1, \dots, x_{\dim(X)}, y])$$

in a neighbourhood of the point O, where $O \in \mathbb{CS}(X, B_X)$. Then $\operatorname{mult}_O(B_X) > 1$.

One can also prove the following generalization of Theorem 1.7.20.

Theorem 1.7.23. *Let $r = \dim(X) \geqslant 4$, let O be an isolated ordinary double point of the variety X, let $O \in \mathbb{CS}(X, \frac{1}{n}\mathcal{M})$, and let the log pair $(X, \frac{1}{n}\mathcal{M})$ be canonical in a punctured neighbourhood of O, where $\mathcal{M}$ is a linear system without fixed components and n is a positive integer. Let $\pi\colon V \to X$ be a blow-up of the point O and let E be a π-exceptional divisor, which can be identified with a smooth quadric in $\mathbb{P}^r$. Then there is a linear subspace $\Lambda \subset E$ of codimension 3 in $\mathbb{P}^r$ for which*[11]

$$\operatorname{mult}_O(S_1 \cdot S_2 \cdot \Delta) \geqslant 6n^2$$

for every effective divisor Δ on X such that the following conditions are satisfied:

- *Δ contains the point O;*
- *O is an ordinary double point on Δ;*
- *the divisor $\pi^{-1}(\Delta)$ contains Λ;*
- *Δ contains no subvarieties of X of codimension 2 that are contained in the base locus of the linear system $\mathcal{M}$;*

moreover, the inequality is strict for $\dim(X) \geqslant 5$.

In fact, the Lefschetz theorem and the proof of Theorem 1.7.23 imply the following corollary.

Corollary 1.7.24. *Under the assumptions and in the notation of* Theorem 1.7.23, *suppose that $\dim(X) \geqslant 5$. Then $\operatorname{mult}_O(S_1 \cdot S_2) > 6n^2$.*

The idea of the proofs of Theorems 1.7.19 and 1.7.23 is similar to that of the proof of the following result, which is Corollary 3.5 in [46].

Proposition 1.7.25. *Let B_X be movable and effective, let $\dim(X) = 3$, let O be a smooth point on X belonging to $\mathbb{CS}(X, B_X)$, let $\operatorname{mult}_O(B_X) < 2$, and let $f\colon V \to X$ be a blow-up of the variety X at the point O. Then there is a line $L \subset E \cong \mathbb{P}^2$ in the set $\mathbb{LCS}(V, B_V + (\operatorname{mult}_O(B_X) - 1)E)$, where $B_V = f^{-1}(B_X)$ and $E = f^{-1}(O)$.*

Proof. The assertion is local with respect to X. Hence, we can assume that $X \cong \mathbb{C}^3$ and that O is the origin in $\mathbb{C}^3$. Consider a general hyperplane section H of X that passes through the point O and set $T = f^{-1}(H)$. Then we have the relation

$$K_V + B_V + (\operatorname{mult}_O(B_X) - 1)E + T \sim_{\mathbb{Q}} f^*(K_X + B_X + H)$$

[11] The multiplicities $\operatorname{mult}_O(S_1 \cdot S_2 \cdot \Delta)$ and $\operatorname{mult}_O(S_1|_\Delta \cdot S_2|_\Delta)$ can be formally defined by means of the numerical relationship between the full and proper transforms of the boundary on the blow-up of the corresponding singular point.

and $\mathrm{mult}_O(B_X) \geqslant 1$ (see Remark 1.3.9). Moreover, since the section H is general, it follows immediately from Theorem 1.7.14 that $O \in \mathbb{LCS}(H, B_X|_H)$, and

$$\mathbb{LCS}(T, B_V|_T + (\mathrm{mult}_O(B_X) - 1)E|_T) \neq \varnothing,$$

because $\mathrm{mult}_O(B_X) < 2$ by assumption (see Example 1.7.4). Applying Theorem 1.7.12 to the morphism f, we now see immediately that the set

$$\mathbb{LCS}(T, B_V|_T + (\mathrm{mult}_O(B_X) - 1)E|_T)$$

consists of a single point $P \in E \cap T$. On the other hand, since the section H was chosen to be general, it follows that the point P is the intersection of the surface T with some element of the set $\mathbb{LCS}(V, B_V + (\mathrm{mult}_O(B_X) - 1)E)$. Thus, the set

$$\mathbb{LCS}(T, B_V + (\mathrm{mult}_O(B_X) - 1)E)$$

contains some curve $L \subset E$ whose intersection with the surface T is exactly the point P, and hence the curve L is a line in $E \cong \mathbb{P}^2$.

Corollary 1.7.26. *Suppose that the boundary B_X is movable and effective, the variety X is three-dimensional, and there is a smooth point $O \in X$ such that $\mathrm{mult}_O(B_X) \leqslant 2$ but the log pair (X, B_X) is not canonical at O, that is, $O \in \mathbb{CS}(X, \mu B_X)$ for some rational $\mu < 1$. Let $f\colon V \to X$ be a blow-up of the point O. Then there is a line $L \subset E \cong \mathbb{P}^2$ such that the log pair $(V, B_V + (\mathrm{mult}_O(B_X) - 1)E)$ is not log canonical at a general point of the line L, where $B_V = f^{-1}(B_X)$ and $E = f^{-1}(O)$.*

Using Theorem 1.7.10, one can readily generalize Theorem 1.7.20 to the case of an ordinary hypersurface singular point of arbitrary multiplicity.

PART 2. THREEFOLDS

§ 2.1. Quartic threefold

Let $X \in \mathbb{P}^4$ be a hypersurface of degree 4 having at most isolated singularities. Then X is a three-dimensional Fano variety of degree 4 such that $-K_X \sim \mathcal{O}_{\mathbb{P}^4}(1)|_X$. The following result was proved in [94].

Theorem 2.1.1. *Suppose that X is smooth. Then X is birationally superrigid.*

Proof. Suppose that X is not birationally superrigid (see Definitions 0.3.3 and 0.3.4). Let us show that this assumption leads to a contradiction.

It follows from the Lefschetz theorem that the group $\mathrm{Pic}(X)$ is generated by the anticanonical divisor $-K_X$. Thus, by Theorem 1.4.1, there is a linear system $\mathcal{M}$ on X without fixed components such that the singularities of the movable log pair $(X, \frac{1}{n}\mathcal{M})$ are not canonical, where n is a positive integer such that $\mathcal{M} \sim -nK_X$. In particular, the set $\mathbb{CS}(X, \mu\mathcal{M})$ is not empty for some positive rational number $\mu < \frac{1}{n}$.

Let $P \subset X$ be an irreducible subvariety contained in $\mathbb{CS}(X, \mu\mathcal{M})$, that is, the movable log pair $(X, \frac{1}{n}\mathcal{M})$ is not canonical at P. Then it follows from Remark 1.3.9 that $\mathrm{mult}_P(\mathcal{M}) > n$, and it follows from Proposition 1.3.12 that P is a point.

Let S_1 and S_2 be sufficiently general surfaces in the linear system $\mathcal{M}$. In this case, by Theorem 1.7.18, we have the inequality $\operatorname{mult}_P(S_1 \cdot S_2) > 4n^2$. Let H be a general hyperplane section of X that passes through the point P. Then

$$4n^2 = S_1 \cdot S_2 \cdot H \geqslant \operatorname{mult}_P(S_1 \cdot S_2) \operatorname{mult}_P(H) > 4n^2,$$

which is a contradiction.

The following result, which was proved in [146] and [129], is a generalization of Theorem 2.1.1.

Theorem 2.1.2. *Suppose that X is nodal and $\mathbb{Q}$-factorial. Then X is birationally rigid.*

Let us recall that the $\mathbb{Q}$-*factoriality* of a nodal[12] quartic X means that all Weil divisors on the quartic X are Cartier divisors, which is equivalent to the topological condition that $\operatorname{rk} H_4(X, \mathbb{Z}) = 1$ and to the condition that the group $\operatorname{Cl}(X)$ is generated by the class of a hyperplane section of X. The assertion of Theorem 2.1.2 can simply fail without the $\mathbb{Q}$-factoriality condition. For example, the Burkhardt quartic

$$w^4 - w(x^3 + y^3 + z^3 + t^3) + 3xyzt = 0 \subset \mathbb{P}^4 \cong \operatorname{Proj}(\mathbb{C}[x, y, z, t, w])$$

is nodal and determinantal, which implies that it is rational. A nodal three-dimensional quartic cannot have more than 45 singular points (see [178]), and there are nodal quartics with arbitrarily many singular points up to 45. Moreover, it follows from [100] that the Burkhardt quartic is the only nodal quartic threefold with exactly 45 singular points (see [174] and [139]). In fact, the Burkhardt quartic is a unique invariant of degree 4 of the simple group $\operatorname{PSp}(4, \mathbb{Z}_3)$ of order 25920 (see [66] and [77]), and the singular points of the Burkhardt quartic correspond canonically to 45 tritangents of a smooth cubic surface, which is related to the fact that the Weyl group of the simple root system $\mathbb{E}_6$ is an extension of the group $\operatorname{PSp}(4, \mathbb{Z}_3)$ by the group $\mathbb{Z}_2$.

Remark 2.1.3. A nodal hypersurface in $\mathbb{P}^4$ of degree 5 or higher is non-rational, the nodal quadrics in $\mathbb{P}^4$ are rational, and a nodal cubic hypersurface in $\mathbb{P}^4$ is non-rational if and only if it is smooth (see [41]). Thus, the rationality problem for nodal quartic threefolds is the only problem on rationality of three-dimensional nodal hypersurfaces which remains unsolved.

There are non-rational singular quartic threefolds which are not birationally rigid. For example, the following result was proved in [47].

Theorem 2.1.4. *Let X be a sufficiently general quartic given by an equation of the form*

$$x^2 + yz + xf_3(y, z, t, w) + g_4(y, z, t, w) = 0 \subset \mathbb{P}^4 \cong \operatorname{Proj}(\mathbb{C}[x, y, z, t, w]),$$

where f_3 and g_4 are homogeneous polynomials of degree 3 *and* 4, *respectively. In this case the singularities of the quartic threefold X are terminal and $\mathbb{Q}$-factorial,*

[12]Nodality of a hypersurface X means that all the singular points of X are isolated ordinary double points.

and rk $\mathrm{Pic}(X) = 1$. *The threefold X can be birationally transformed into a complete intersection V of a quartic and a cubic in $\mathbb{P}(1^4, 2, 2)$ such that* rk $\mathrm{Pic}(V) = 1$ *and V has $\mathbb{Q}$-factorial terminal singularities. Moreover, the varieties X and V are the only Mori fibred spaces in the birational equivalence class of X. In particular, X is not birationally rigid; however, it is non-rational and is birationally isomorphic neither to a conic bundle nor to a fibration into rational surfaces.*

The following result was proved in [31].

Theorem 2.1.5. *Let X be a very general quartic threefold containing a two-dimensional linear subspace in $\mathbb{P}^4$. Then X is nodal and non-rational.*

Any quartic X satisfying the assumptions of Theorem 2.1.5 can be birationally transformed by means of *antiprojections* (see Example 2.1.7) into a fibration into cubic surfaces and into a complete intersection of two cubics in $\mathbb{P}(1^5, 2)$ that are Mori fibred spaces.

Conjecture 2.1.6. *Let X be a nodal three-dimensional quartic containing a two-dimensional linear subspace $\Pi \subset \mathbb{P}^4$ and such that* rk $\mathrm{Cl}(X) = 2$, *let $\eta\colon X \dashrightarrow V$ be an antiprojection, where V stands for a complete intersection of two cubic hypersurfaces in $\mathbb{P}(1^5, 2)$, and let $\pi\colon X \dashrightarrow \mathbb{P}^1$ be the projection from the plane Π. Suppose that there is a Mori fibration $\tau\colon Y \to Z$ and a birational map $\rho\colon X \dashrightarrow Y$. Then there is a birational automorphism σ of the quartic X such that either $\rho = \eta\circ\sigma$ or $\rho \circ \tau = \pi \circ \sigma$.*

Of course, there are nodal quartic threefolds that do not contain planes and are not $\mathbb{Q}$-factorial.

Example 2.1.7. Let X be a general three-dimensional quartic that passes through a given smooth quadric surface $Q \subset \mathbb{P}^4$. Then X can be given by an equation of the form

$$\begin{aligned} &a_2(x, y, z, t, w)h_2(x, y, z, t, w) \\ &\qquad = b_3(x, y, z, t, w)g_1(x, y, z, t, w) \subset \mathbb{P}^4 \cong \mathrm{Proj}(\mathbb{C}[x, y, z, t, w]), \end{aligned}$$

where a_2, h_2, b_3, and g_1 are homogeneous polynomials of degrees 2, 2, 3, and 1, respectively, and Q is given by the equations $h_2 = g_1 = 0$. Moreover, X is nodal and has 12 singular points given by the equations $h_2 = g_1 = a_2 = b_3 = 0$. Obviously, X is not $\mathbb{Q}$-factorial.

After introducing a new variable $\alpha = a_2/g_1$, the quartic X can be antiprojected onto a complete intersection $V \subset \mathbb{P}^5$ given by the equations

$$\alpha g_1(x, y, z, t, w) - a_2(x, y, z, t, w) = \alpha h_2(x, y, z, t, w) - b_3(x, y, z, t, w) = 0 \subset \mathbb{P}^5,$$

such that the commutative diagram

$$\begin{tikzcd} & Y \arrow[dl, "\psi"'] \arrow[drr, "\phi"] & & \\ X \arrow[rrr, dashed, "\rho"'] & & & V \end{tikzcd}$$

holds, where ρ is an antiprojection, the variety V is smooth outside an isolated ordinary double point P given by the equations $x = y = z = t = w = 0$, the morphism ϕ contracts a surface isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and the morphism ψ contracts the proper transforms of 12 lines on the complete intersection V that pass through the point P. Similarly, we can introduce a new variable $\beta = h_2/g_1$, and then the quartic X can be antiprojected to a complete intersection $V' \subset \mathbb{P}^5$ given by the equations

$$\beta g_1(x,y,z,t,w) - h_2(x,y,z,t,w) = \beta a_2(x,y,z,t,w) - b_3(x,y,z,t,w) = 0 \subset \mathbb{P}^5.$$

It is unknown whether or not any quartic threefold $X \subset \mathbb{P}^4$ satisfying all the conditions in Example 2.1.7 is rational (see [87], [96], [46]).

Conjecture 2.1.8. *In the notation and under the assumptions of Example* 2.1.7, *the complete intersections V and V' are the only Mori fibred spaces birationally equivalent to the quartic threefold $X \subset \mathbb{P}^4$.*

Let us prove the following result to make the application of Theorem 2.1.2 effective (see [31]).

Proposition 2.1.9. *Let a quartic X be nodal and let $|\operatorname{Sing}(X)| \leqslant 9$. Then X is $\mathbb{Q}$-factorial if and only if it contains no two-dimensional linear subspaces of $\mathbb{P}^4$.*

Proof. It follows from [40] and [51] that X is $\mathbb{Q}$-factorial if and only if its singular points impose independent linear conditions on the cubic hypersurfaces in $\mathbb{P}^4$. Suppose that X contains no planes. We show that the singular points of X do impose independent linear conditions on the cubic hypersurfaces in $\mathbb{P}^4$.

Let the quartic X be given by an equation of the form

$$f(x_0,x_1,x_2,x_3,x_4) = 0 \subset \mathbb{P}^4 \cong \operatorname{Proj}(\mathbb{C}[x_0,x_1,x_2,x_3,x_4]),$$

where f is a homogeneous polynomial of degree 4. Consider the linear system

$$\mathcal{L} = \left| \sum_{i=0}^{4} \lambda_i \frac{\partial f}{\partial x_i} = 0 \right| \subset |\mathcal{O}_{\mathbb{P}^4}(3)|,$$

where $\lambda_i \in \mathbb{C}$. Then the base locus of the linear system $\mathcal{L}$ is exactly formed by the singular points of X. On the other hand, any curve in $\mathbb{P}^4$ of degree d intersects a general surface in $\mathcal{L}$ at not more than $3d$ points. Thus,

- at most three points of the set $\operatorname{Sing}(X)$ belong to a line;
- at most six points of the set $\operatorname{Sing}(X)$ belong to a conic.

Let Σ be a plane in $\mathbb{P}^4$ and let $T = \Sigma \cap X$. Then T is a planar curve of degree 4, and T can be reducible and non-reduced. Nevertheless, one can readily see that

$$\operatorname{Sing}(V) \cap \Sigma \subset \operatorname{Sing}(T),$$

which implies that $|\operatorname{Sing}(X) \cap \Sigma| \leqslant 6$ if T is non-reduced, because at most three points of the set $\operatorname{Sing}(X)$ can belong to a line and at most six points of $\operatorname{Sing}(X)$ can belong to a conic. On the other hand, if T is reduced, then $|\mathrm{Sing}(T)| \leqslant 6$. Thus, at most eight singular points of X can belong to a given plane in $\mathbb{P}^4$.

Let us consider an arbitrary subset $\Sigma \subsetneq \mathrm{Sing}(X)$ and a point $P \in \mathrm{Sing}(X) \setminus \Sigma$. We must show that there is a cubic hypersurface in $\mathbb{P}^4$ that passes through all points of the set Σ and does not contain the point P. Without loss of generality, we can assume that $|\mathrm{Sing}(X)| = |\Sigma| + 1 = 9$.

Suppose that there is a hyperplane $\Gamma \subset \mathbb{P}^4$ containing $\mathrm{Sing}(X)$. We claim that there is a cubic hypersurface in $\mathbb{P}^4$ passing through all the points of Σ but not containing P. To this end, it suffices to prove that there is a cubic surface in Γ passing through all the points of Σ but not through P.

Let $\varepsilon \geqslant 2$ be the maximal number of points of Σ that are contained in a two-dimensional linear subspace $\Pi \subset \Gamma$ containing the point P. Then $\varepsilon \leqslant 7$. We set

$$\Sigma = \{P_1, \dots, P_8\},$$

where the points $P_1, \dots, P_\varepsilon$ are contained in Π together with P. Then the points P and $P_1, \dots, P_\varepsilon$ do not belong to a single line. We consider only two cases, namely, $\varepsilon = 2$ and $\varepsilon = 7$.

Suppose that $\varepsilon = 2$. In this case we single out in Σ three subsets, possibly not disjoint, such that each of them contains three points of Σ and their union is the entire set Σ. Then the hyperplanes in Γ generated by each of these subsets cannot contain P, because $\varepsilon = 2$. Therefore, the union of these three hyperplanes is the desired cubic surface.

Let $\varepsilon = 7$. In this case both the results of [9] and the elementary properties of *weak del Pezzo surfaces* (see [73] and [119]) imply the existence of a cubic curve C on Π passing through all points of the set $\Sigma \cap \Pi$ but not through P. Hence, the cone in Γ over C with vertex at the point P_8 is a cubic surface in Γ that contains the set Σ and not the point P.

Thus, one can assume that there is no hyperplane in $\mathbb{P}^4$ containing all the singular points of the quartic X. Let $\delta \geqslant 3$ be the maximal number of points in Σ contained in some hyperplane $\Xi \subset \mathbb{P}^4$ such that $P \in \Xi$. Then $\delta \leqslant 7$. We set $\Sigma = \{P_1, \dots, P_8\}$, where the points $P_1, \dots, P_\delta$ are contained in Ξ together with the point P. We consider only the case $\delta = 4$.

Suppose that $\delta = 4$. Then there are lines L_1 and L_2 in the hyperplane Ξ such that each contains a pair of points in $\Sigma \cap \Xi$ and does not pass through P. We may assume that L_1 contains the points P_1 and P_2 and L_2 contains the points P_3 and P_4. Moreover, at most two points of the set $\{P_5, P_6, P_7, P_8\}$ can belong to a line that passes through P. Therefore, there are two points in the set $\Sigma \setminus \Xi$, say P_5 and P_6, such that the line passing through them does not contain the point P. Hence, the desired cubic hypersurface passing through all points of the set Σ and not passing through P can be obtained as the union of a general hyperplane passing through the line L_1 and the point P_7, a general hyperplane passing through the line L_2 and the point P_8, and a general hyperplane passing through the points P_5 and P_6.

Corollary 2.1.10. *Suppose that a quartic threefold X is nodal and $|\mathrm{Sing}(X)| \leqslant 8$. Then the quartic X is $\mathbb{Q}$-factorial.*

Using Lemma 1.7.7, one can generalize Proposition 2.1.9 in the following non-trivial way (see [35]).

Theorem 2.1.11. *Let V be a nodal hypersurface in $\mathbb{P}^4$ of degree n such that $|\operatorname{Sing}(V)| \leqslant (n-1)^2/4$. Then V is $\mathbb{Q}$-factorial.*

It follows from [51] that, in the notation and under the assumptions of Theorem 2.1.11, a nodal hypersurface in $\mathbb{P}^4$ is $\mathbb{Q}$-factorial if and only if its singular points impose independent linear conditions on hypersurfaces in $\mathbb{P}^4$ of degree $2n-5$. It can be seen from the assertion of Proposition 2.1.9 that the inequality in Theorem 2.1.11 is not sharp; nevertheless, it is sufficiently sharp from the asymptotic point of view, as can be seen from the following example.

Example 2.1.12. Let V be a nodal hypersurface given by the equation

$$xg_{n-1}(x,y,z,t,w)+yf_{n-1}(x,y,z,t,w)=0\subset\mathbb{P}^4\cong\operatorname{Proj}(\mathbb{C}[x,y,z,t,w]),$$

where g_{n-1} and f_{n-1} are general homogeneous polynomials of degree $n-1$. Then $|\operatorname{Sing}(V)|=(n-1)^2$ and the hypersurface V is not $\mathbb{Q}$-factorial.

As was shown in [39], every smooth surface contained in a nodal hypersurface $V\subset\mathbb{P}^4$ of degree n is a Cartier divisor on the hypersurface V if the inequality $|\operatorname{Sing}(V)|<(n-1)^2$ is satisfied. It is natural to express the following conjecture, which has been proved only for cubics and quartics (see Proposition 2.1.9 and [64]).

Conjecture 2.1.13. *Let V be a nodal hypersurface in $\mathbb{P}^4$ of degree n such that the inequality $|\operatorname{Sing}(V)|<(n-1)^2$ is satisfied. Then V is $\mathbb{Q}$-factorial.*

In the conclusion of this section we present an analogue of Theorem 2.2.10 for smooth three-dimensional quartics (see [23], [26]).

Theorem 2.1.14. *Suppose that X is smooth. Let $\rho\colon X\dashrightarrow\mathbb{P}^2$ be a map for which the normalization of a general fibre is an elliptic curve. Then $\rho=\sigma\circ\gamma$, where $\gamma\colon X\dashrightarrow\mathbb{P}^2$ is the projection from a line contained in the quartic X and σ is a birational automorphism of $\mathbb{P}^2$.*

We note that a smooth quartic threefold contains a one-dimensional family of lines (see [42]).

At present, the classification of birational transformations into elliptic fibrations has been considered for many higher-dimensional rationally connected varieties (see [53], [21], [22], [24], [161], [27], [29], [30], [37]).

§ 2.2. Sextic double solid

Let $\pi\colon X\to\mathbb{P}^3$ be a double cover ramified along a surface $S\subset\mathbb{P}^3$ of degree 6 such that S has at most isolated ordinary double points. As is known, S cannot have more than 65 ordinary double points (see [98], [180]), and for every positive integer $m\leqslant 65$ there is a surface S having m singular points (see Example 2.2.2, [5], and [15]), the variety X is a terminal Fano variety of degree 2, $-K_X\sim\pi^*(\mathcal{O}_{\mathbb{P}^3}(1))$, and the threefold X is $\mathbb{Q}$-factorial if and only if $\operatorname{rk}\operatorname{Cl}(X)=1$. The following result holds (see [87], [148], [36]).

Theorem 2.2.1. *Suppose that* $\operatorname{rk}\operatorname{Cl}(X) = 1$. *Then X is birationally superrigid.*

Proof. Suppose not. Let us show that this assumption leads to a contradiction. There is a movable log pair (X, M_X) on X with an effective boundary M_X such that the set $\mathbb{CS}(X, M_X)$ is non-empty and the divisor $-(K_X + M_X)$ is ample (see Theorem 1.4.1). In particular, we have $M_X \sim_{\mathbb{Q}} -rK_X$ for some positive rational number $r < 1$. Let $C \subset X$ be an element of $\mathbb{CS}(X, M_X)$.

Suppose that C is a smooth point of the variety X. Then $\operatorname{mult}_C(M_X^2) \geqslant 4$ by Theorem 1.7.18, which implies the inequalities $2 > M_X^2 \cdot H \geqslant 4$ for any sufficiently general divisor H in the complete linear system $|\pi^*(\mathcal{O}_{\mathbb{P}^3}(1))|$ provided that H contains the point C. This is a contradiction.

Suppose that C is an ordinary double point on X. Then Theorem 1.7.20 implies the inequality $\operatorname{mult}_C(M_X) \geqslant 1$. We consider two sufficiently general divisors H_1 and H_2 in the linear system $|\pi^*(\mathcal{O}_{\mathbb{P}^3}(1))|$ that pass through the point C. In this case,

$$2 > M_X \cdot H_1 \cdot H_2 \geqslant 2 \operatorname{mult}_C(M_X) \operatorname{mult}_C(H_2) \operatorname{mult}_C(H_2) \geqslant 2,$$

which is a contradiction.

Thus, we have proved that C is an irreducible curve. Let H be a sufficiently general divisor in the complete linear system $|\pi^*(\mathcal{O}_{\mathbb{P}^3}(1))|$ and let H contain Z. Then

$$2 = H^3 > 2r^2 = M_X^2 \cdot H \geqslant \operatorname{mult}_C(M_X^2) H \cdot C \geqslant -K_X \cdot C,$$

because $\operatorname{mult}_C(M_X^2) \geqslant \operatorname{mult}_C^2(M_X) \geqslant 1$. Hence, the equality $-K_X \cdot C = 1$ holds. Therefore, the curve $\pi(C)$ is a line, the curve C is smooth and rational, and $\pi|_C$ is an isomorphism.

Suppose that C is entirely contained in the set of non-singular points of the variety X. Let $f\colon W \to X$ be a blow-up of C, let $E = f^{-1}(C)$, and let $M_W = f^{-1}(M_X)$. Then

$$M_W \sim_{\mathbb{Q}} f^*(M_X) - \operatorname{mult}_C(M_X)E,$$

where $\operatorname{mult}_C(M_X) \geqslant 1$. However, the base locus of the pencil $|-K_W|$ consists of a curve $\widetilde{C}$ which is smooth and rational and satisfies the condition $\pi \circ f(\widetilde{C}) = \pi(C)$. Moreover, we have $\widetilde{C} \subset E$ if and only if $\pi(C) \subset S$. Let $H = f^*(-K_X)$. Then the intersection of the divisor $3H - E$ with any curve on the variety W, except for the curve $\widetilde{C}$, is certainly non-negative. Let us show that the divisor $3H - E$ is numerically effective. Obviously, $(3H - E) \cdot \widetilde{C} = 0$ if the base curve $\widetilde{C}$ is not contained in the exceptional divisor E. Therefore, we can assume that $\widetilde{C}$ is contained in E. Consider the normal sheaf $\mathcal{N}_{C/X}$ of the curve C on the variety X. Since the curve C is rational, it follows that

$$\mathcal{N}_{C/X} \cong \mathcal{O}_C(a) \oplus \mathcal{O}_C(b)$$

for some integers a and b, $a \geqslant b$. It follows from the exact sequence

$$0 \to \mathcal{T}_C \to \mathcal{T}_X|_C \to \mathcal{N}_{C/X} \to 0$$

that $\deg(\mathcal{N}_{C/X}) = a + b = -K_X \cdot C + 2g(C) - 2 = -1$. However, C is entirely contained in the smooth locus of the surface $\widehat{S} = \pi^{-1}(S) \cong S$. It follows from the exact sequence

$$0 \to \mathcal{N}_{C/\widehat{S}} \to \mathcal{N}_{C/X} \to \mathcal{N}_{\widehat{S}/X}|_C \to 0$$

and from the equivalence $\mathcal{N}_{C/\widehat{S}} \cong \mathcal{O}_C(-4)$ that $b \geqslant -4$. In particular, $a-b \leqslant 7$. Let s_∞ be the exceptional section of the rationally ruled surface $f|_E \colon E \to C$. Then the equalities

$$E^3 = -\deg(\mathcal{N}_{C/X}) = 1$$

and $-K_X \cdot C = 1$ yield

$$(3H - E) \cdot s_\infty = \frac{7+b-a}{2} \geqslant 0,$$

which implies that $3H-E$ is numerically effective. In particular, we have

$$0 \leqslant (3H-E) \cdot M_W^2 = 6r^2 - 4\,\mathrm{mult}_C^2(M_X) - 2r\,\mathrm{mult}_C(M_X) < 0,$$

because $r \in \mathbb{Q} \cap (0,1)$ and $M_X \sim_{\mathbb{Q}} -rK_X$. This is a contradiction. Therefore, the irreducible rational curve C contains a singular point of the variety X.

Suppose that $\pi(C) \not\subset S$. Let $\mathcal{H}$ be a linear system consisting of surfaces in the linear system $|-K_X|$ that contain the curve C. Then $\mathcal{H}$ is a pencil, and its base locus consists of the curve C and a curve $\widetilde{C}$ such that $\pi(C) = \pi(\widetilde{C})$. Let D be a general surface in the pencil $\mathcal{H}$. Then the restriction $M_X|_D$ need not be a movable boundary, but we have

$$M_X|_D = \mathrm{mult}_C(M_X)C + \mathrm{mult}_{\widetilde{C}}(M_X)\widetilde{C} + R_D,$$

where R_D is a movable boundary on the surface D. The surface D is smooth outside the singular points P_i of the variety X that belong to the curve C. Since the divisor D is general, each of the points P_i is an ordinary double point on D. We have the equalities

$$C^2 = \widetilde{C}^2 = -2 + \frac{k}{2},$$

where k is the number of singular points P_i. However, $k \leqslant 3$, because $\pi(C)$ is a line which is not contained in the surface S, and the degree of the surface $S \subset \mathbb{P}^3$ is equal to six. Thus, $C^2 = \widetilde{C}^2 < 0$, which implies that

$$(1 - \mathrm{mult}_{\widetilde{C}}(M_X))\widetilde{C}^2 \geqslant (\mathrm{mult}_C(M_X) - 1)C \cdot \widetilde{C} + R_D \cdot \widetilde{C} \geqslant 0$$

and, in particular, $\mathrm{mult}_{\widetilde{C}}(M_X) \geqslant 1$. Let H be a general divisor in $|-K_X|$. Then

$$2 = H \cdot K_X^2 > H \cdot M_X^2 \geqslant \mathrm{mult}_C^2(M_X)H \cdot C + \mathrm{mult}_{\widetilde{C}}^2(M_X)H \cdot \widetilde{C} \geqslant 2,$$

which is impossible. Thus, the curve $\pi(C)$ is contained in the surface S.

Let P be a sufficiently general point on the curve C, let $L \subset \mathbb{P}^3$ be a general line tangent to the surface S at the point $\pi(P)$, and let $\widetilde{L} = \pi^{-1}(L)$. We note that the curve $\widetilde{L}$ is irreducible and singular at the point P. Moreover, $\widetilde{L}$ is not contained in the base locus of any component of the movable boundary M_X, because the point P and the line L were chosen to be sufficiently general. Hence, we have the inequalities

$$2 > \widetilde{L} \cdot M_X \geqslant \mathrm{mult}_P(\widetilde{L})\,\mathrm{mult}_P(M_X) \geqslant 2\,\mathrm{mult}_C(M_X) \geqslant 2,$$

which is a contradiction.

The variety X can even be rational if X is not $\mathbb{Q}$-factorial.

Example 2.2.2. Let S be a Barth sextic (see [5])

$$4(\tau^2x^2-y^2)(\tau^2y^2-z^2)(\tau^2z^2-x^2)=t^2(1+2\tau)(x^2+y^2+z^2-t^2)^2\subset\mathbb{P}^3 \cong \operatorname{Proj}(\mathbb{C}[x,y,z,t]),$$

where $\tau=\frac{1+\sqrt{5}}{2}$. Then X is nodal and has 65 singular points, and one can show that no nodal sextic in $\mathbb{P}^3$ can have more than 65 ordinary double points (see [98], [180]). Moreover, there is a determinantal nodal quartic $Y\subset\mathbb{P}^3$ having 42 singular points and such that the diagram

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, dashed, "\rho"'] & \mathbb{P}^4 \arrow[d, dashed, "\gamma"] \\ X \arrow[r, "\pi"'] & \mathbb{P}^3 \end{tikzcd}$$

is commutative (see [56], [139]), where γ is the projection from a singular point of the quartic Y and ρ is a birational map. In particular, the variety X is rational, because the determinantal quartics are rational.

It is not hard to construct an example of a variety X which is not $\mathbb{Q}$-factorial and has 15 singular points (see Example 2.2.6). On the other hand, by analogy with Proposition 2.1.9, one can prove the following result (see [36]).

Proposition 2.2.3. *The variety X is $\mathbb{Q}$-factorial if $|\operatorname{Sing}(S)|\leqslant 14$.*

Corollary 2.2.4. *Suppose that $|\operatorname{Sing}(X)|\leqslant 14$. Then X is birationally superrigid, non-rational, and not birationally isomorphic to a conic bundle.*

The assertion of Proposition 2.2.3 can be generalized as follows (see [33]).

Theorem 2.2.5. *Let $\gamma\colon V\to\mathbb{P}^3$ be a double cover ramified along a nodal surface $F\subset\mathbb{P}^3$ of degree $2r$ and let $|\operatorname{Sing}(V)|\leqslant(2r-1)r/3$. Then V is $\mathbb{Q}$-factorial.*

It follows from [40] that, in the notation and under the assumptions of Theorem 2.2.5, the nodal variety V is $\mathbb{Q}$-factorial if and only if the ordinary double points of the surface F impose independent linear conditions on the hypersurfaces in $\mathbb{P}^3$ of degree $3r-4$. The inequality in Theorem 2.2.5 is not sharp (see Proposition 2.2.3); nevertheless, it correctly reflects the general picture asymptotically.

Example 2.2.6. Let us consider a hypersurface $V\subset\mathbb{P}(1^4,r)$ given by the equation

$$u^2=g_r^2(x,y,z,t)+h_1(x,y,z,t)f_{2r-1}(x,y,z,t)\subset\mathbb{P}(1^4,r)\cong\operatorname{Proj}(\mathbb{C}[x,y,z,t,u]),$$

where g_i, h_i, and f_i are sufficiently general polynomials of degree i. Let $\gamma\colon V\to\mathbb{P}^3$ be the natural projection. Then γ is a double cover ramified over a nodal surface of degree $2r$ are such that $|\operatorname{Sing}(V)|=(2r-1)r$, but the variety V is not $\mathbb{Q}$-factorial.

One can express the following conjecture.

Conjecture 2.2.7. *Let $\gamma\colon V\to\mathbb{P}^3$ be a double cover ramified along a nodal surface of degree $2r$ and let $|\operatorname{Sing}(V)|<(2r-1)r$. Then the variety V is $\mathbb{Q}$-factorial.*

Along with birational superrigidity, the variety X has other interesting properties.

Example 2.2.8. Let O be a singular point of the variety X and let $\gamma\colon \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ be the projection from the point $\pi(O)$. Then the general fibre of the map $\gamma\circ\pi$ is a smooth elliptic curve.

Example 2.2.9. Let L be a line on S such that $|\operatorname{Sing}(X)\cap L| = 4$. Then for a sufficiently general point P of the variety X there is a unique hyperplane $H\subset\mathbb{P}^3$ containing the point $\pi(P)$ and the line L. Let C be a curve of degree 5 in the plane H such that $S\cap H = L\cup C$. Then the intersection $L\cap(C\setminus\operatorname{Sing}(X))$ consists of a single point Q, and hence there is a unique line $Z\subset\mathbb{P}^3$ that passes through the points Q and $\pi(P)$. Thus, we have constructed a rational map $\Xi_L\colon X\dashrightarrow \operatorname{Gr}(2,4)$ taking any point P of X to the line $\Xi_L(P) = Z$, and the normalization of the general fibre of the rational map Ξ_L is an elliptic curve.

The following result holds (see [23], [24], [36]).

Theorem 2.2.10. *Suppose that the singularities of a variety X are $\mathbb{Q}$-factorial, there is a birational map $\rho\colon X\dashrightarrow Y$ such that the variety Y is smooth, and there is a morphism $\tau\colon Y\to Z$ whose general fibre is an elliptic curve. Then one of the following assertions holds*:

- *for some singular point O on X there exists a commutative diagram*

$$\begin{tikzcd} & X \arrow[dl, "\pi"'] \arrow[rr, dashed, "\rho"] & & Y \arrow[d, "\tau"] \\ \mathbb{P}^3 \arrow[dr, dashed, "\gamma"'] & & & \\ & \mathbb{P}^2 \arrow[rr, dashed, "\beta"] & & Z\,, \end{tikzcd} \tag{2.2.11}$$

  where γ is the projection from the point $\pi(O)$ and β is a birational map;
- *the surface S contains a line $L\subset\mathbb{P}^3$ that passes through exactly* 4 *singular points of S, and there exists a commutative diagram*

$$\begin{tikzcd} & X \arrow[dl, dashed, "\rho"'] \arrow[r, dashed, "\Xi_L"] & \mathbb{P}^2 \arrow[dr, dashed, "\beta"] & \\ Y \arrow[rrr, "\tau"] & & & Z\,, \end{tikzcd} \tag{2.2.12}$$

  where β is a birational map and Ξ_L is the rational map constructed in Example 2.2.9.

Proof. We consider a very ample divisor $\check{H}$ on Z. Let $\mathcal{M}$ be a proper transform of the linear system $|\tau^*(\check{H})|$ on the variety X. In this case we have $\mathcal{M}\subset|-nK_X|$ for some $n\in\mathbb{N}$, and $\mathcal{M}$ is not composed of a pencil. We set $M_X = \frac{1}{n}\mathcal{M}$. Then $\mathbb{CS}(X, M_X)\neq\varnothing$ by Theorem 1.4.4, and it follows from the proof of Theorem 2.2.1 that the set $\mathbb{CS}(X, M_X)$ cannot contain smooth points of X.

We assume that the set $\mathbb{CS}(X, M_X)$ contains a singular point $O \in X$. Suppose that $f\colon W \to X$ is a blow-up of O, C is a general fibre of the elliptic fibration $\phi_{|-K_W|}\colon W \to \mathbb{P}^2$, and D is a general surface in the linear system $f^{-1}(\mathcal{M})$. Then

$$2(n - \mathrm{mult}_O(\mathcal{M})) = C \cdot D \geqslant 0.$$

However, we have $\mathrm{mult}_O(\mathcal{M}) \geqslant n$ by Theorem 1.7.20. Hence, the linear system $f^{-1}(\mathcal{M})$ belongs to the fibres of the elliptic fibration $\phi_{|-K_W|}$, which implies the existence of the commutative diagram (2.2.11).

Therefore, we can assume that the set $\mathbb{CS}(X, M_X)$ contains no points of the variety X. Hence, this set contains an irreducible curve $C \subset X$ such that $\mathrm{mult}_C(\mathcal{M}) \geqslant n$. Let H be a general divisor in $|-K_X|$. Then

$$2 = H \cdot K_X^2 \geqslant H \cdot M_X^2 \geqslant \mathrm{mult}_C(M_X^2) H \cdot C \geqslant -K_X \cdot C,$$

which implies the inequality $-K_X \cdot C \leqslant 2$.

Let $-K_X \cdot C = 2$. Then $\mathrm{Supp}(M_X^2) = C$ and $\mathrm{mult}_C(M_X^2) = \mathrm{mult}_C^2(M_X) = 1$. This implies that the equalities

$$\mathrm{mult}_C(M_1 \cdot M_2) = \mathrm{mult}_C(M_1) = \mathrm{mult}_C(M_2) = n$$

and the set-theoretic identity $\mathrm{Supp}(M_1 \cap M_2) = C$ hold for any two distinct surfaces M_1 and M_2 in the linear system $\mathcal{M}$. Let P be a sufficiently general point in the complement $X \setminus C$ and let $\mathcal{D}$ be the linear subsystem of $\mathcal{M}$ consisting of the surfaces passing through P. Then the linear system $\mathcal{D}$ has no fixed components, because the linear system $\mathcal{M}$ is not composed of a pencil. Let D_1 and D_2 be two general surfaces in $\mathcal{D}$. Then

$$P \in D_1 \cap D_1 = M_1 \cap M_2 = C$$

in the set-theoretic sense, because D_1 and D_2 belong to $\mathcal{M}$. This is a contradiction.

Thus, we have proved the equality $-K_X \cdot C = 1$. This means that $\pi(C)$ is a line in $\mathbb{P}^3$ and the restriction $\pi|_C\colon C \to \pi(C)$ is an isomorphism. Moreover, it follows from the above arguments that the set $\mathbb{CS}(X, M_X)$ contains no subvarieties of X except for the curve C, and the inequality $\mathrm{mult}_C(\mathcal{M}^2) < 2n^2$ holds.

Suppose that $\pi(C) \not\subset S$. Let $\mathcal{H}$ be the pencil in the linear system $|-K_X|$ formed by the surfaces containing the curve C. Then the base locus of the pencil $\mathcal{H}$ consists of the curve C and a curve $\widetilde{C}$ such that $\pi(C) = \pi(\widetilde{C})$. Let D be a general divisor in $\mathcal{H}$. Then $M_X|_D$ is no longer a movable boundary; however,

$$M_X|_D = \mathrm{mult}_C(M_X)C + \mathrm{mult}_{\widetilde{C}}(M_X)\widetilde{C} + R_D,$$

where R_D is a movable boundary on D. By construction,

$$\mathrm{Sing}(D) \cap C = \{P_1, \dots, P_k\},$$

where the points P_i are the singular points of X belonging to the curve C. Moreover, every point P_i is an ordinary double point on D because of the general choice of D in the linear system $\mathcal{H}$. On the surface D we have the equalities $C^2 = \widetilde{C}^2 = -2+k/2$; however, $k \leqslant 3$, and hence

$$(1 - \mathrm{mult}_{\widetilde{C}}(M_X))\widetilde{C}^2 = (\mathrm{mult}_C(M_X) - 1)C \cdot \widetilde{C} + R_D \cdot \widetilde{C} \geqslant 0,$$

which implies that $\widetilde{C} \in \mathbb{CS}(X, M_X)$, a contradiction. Hence, the line $\pi(C)$ is contained in the surface $S \subset \mathbb{P}^3$.

Let H be a general hyperplane in $\mathbb{P}^3$ containing the line $\pi(C)$. Then the curve

$$D = H \cap S = \pi(C) \cup Q$$

is reduced and $\pi(C) \not\subset \mathrm{Supp}(Q)$, where Q is a planar curve of degree 5. Moreover, the curve D is singular at every singular point P_i of S belonging to the line $\pi(C)$, where $i = 1, \dots, k$. However, the set $\pi(C) \cap Q$ contains at most 5 points, and

$$\mathrm{Sing}(D) \cap \pi(C) = Q \cap \pi(C),$$

which implies that $k = |\,\mathrm{Sing}(X) \cap C| \leqslant 5$.

Suppose that $k \leqslant 3$. Then it follows from the Bertini theorem that the intersection $\pi(C) \cap Q$ contains at least two distinct points, say O_1 and O_2, which differ from the points P_i. In this case the hyperplane H is tangent to the surface S at the points O_1 and O_2. We write $\widetilde{O}_j = \pi^{-1}(O_j)$. Let L_j be a general line in H passing through the point O_j and let $\widetilde{L}_j$ be the proper transform of L_j on X. Then the line L_j is tangent to S at the point O_j, the curve $\widetilde{L}_j$ is irreducible and singular at the point $\widetilde{O}_j$, and on the other hand, $-K_X \cdot \widetilde{L}_j = 2$. Let $\widetilde{H} = \pi^{-1}(H)$ and let M be a general surface in $\mathcal{M}$. Then

$$M|_{\widetilde{H}} = \mathrm{mult}_C(\mathcal{M})C + R,$$

where R is an effective divisor on $\widetilde{H}$ such that $C \not\subset \mathrm{Supp}(R)$, and

$$2n = M \cdot \widetilde{L}_j \geqslant \mathrm{mult}_{\widetilde{O}_j}(\widetilde{L}_j)\,\mathrm{mult}_C(M) + \sum_{P \in (M \setminus C) \cap \widetilde{L}_j} \mathrm{mult}_P(M) \cdot \mathrm{mult}_P(\widetilde{L}_j) \geqslant 2n,$$

which implies that $M \cap \widetilde{L}_j \subset C$. As the lines L_1 and L_2 vary in the plane H, the curves $\widetilde{L}_1$ and $\widetilde{L}_2$ generate two distinct pencils on $\widetilde{H}$ whose base loci are the points $\widetilde{O}_1$ and $\widetilde{O}_1$, respectively. The latter implies that $R = \varnothing$. Thus, the equality $M \cap \widetilde{H} = C$ holds in the set-theoretic sense for a general divisor $\widetilde{H}$ in $|-K_X|$ passing through C and for any divisor M in the linear system $\mathcal{M}$ such that $\widetilde{H} \not\subset \mathrm{Supp}(M)$. Let $\widetilde{P}$ be a general point in $\widetilde{H} \setminus C$ and let $\widehat{\mathcal{M}}$ be the linear subsystem of $\mathcal{M}$ consisting of the surfaces containing the point $\widetilde{P}$. Then $\widehat{\mathcal{M}}$ has no fixed components, because the linear system $\mathcal{M}$ is not composed of a pencil. Let $\widehat{M}$ be a general surface in $\widehat{\mathcal{M}}$. Then $\widetilde{P} \in \widehat{M} \cap \widetilde{H} = C$, because $\widetilde{H} \not\subset \mathrm{Supp}(\widehat{M})$. This is a contradiction.

Thus, we have proved that either $k = 4$ or $k = 5$. Let $g\colon V \to X$ be a blow-up of the sheaf of ideals of the curve C and let F be an exceptional divisor of g. Then

$$-K_V \sim_{\mathbb{Q}} M_V \sim_{\mathbb{Q}} g^*(-K_X) - F,$$

where $M_V = g^{-1}(M_X)$, because $\mathrm{mult}_C(M_X) = 1$ (this relation holds since the variety X is birationally superrigid by Theorem 2.2.1). One can show (see [36] and [176]) that the following assertions hold:

- the variety V has k singular points of type $\frac{1}{2}(1,1,1)$, and each of these points dominates a singular point of X that belongs to the curve C;
- the morphism $g|_F\colon F \to C$ has four reducible fibres each consisting of two smooth rational curves transversally intersecting at a singular point of V which is an ordinary double point on F;
- the linear system $|-K_V|$ is a pencil and its base locus consists of a smooth curve $\widetilde{C}$ contained in F which is a section of the morphism $g|_F$;
- the relation $\widetilde{C} \equiv K_V^2$ holds, and the divisor $-K_V$ is numerically effective if and only if $-K_V^3 = -2 + \frac{k}{2} \geqslant 0$.

Therefore, the divisor $-K_V$ is numerically effective. On the other hand, the image of every element of the set $\mathbb{CS}(V, M_V)$ on the variety X must be contained in the set $\mathbb{CS}(X, M_X)$, because

$$K_V + M_V \sim_{\mathbb{Q}} g^*(K_X + M_X),$$

which implies that every element of the set $\mathbb{CS}(V, M_V)$ is a curve in F dominating the curve C. Therefore, the existence of such an element implies the inequality $\mathrm{mult}_C(\mathcal{M}) \geqslant 2n^2$, which is impossible, as was proved above. Thus, the singularities of the log pair $(V, \lambda M_V)$ are terminal for some rational number $\lambda > 1$. This means that the linear system $|-rK_V|$ is free for $r \gg 0$ (see [106], Theorem 3.1.1).

Suppose that $k = 4$. Then the morphism $\phi_{|-rK_V|}$ is a fibration into elliptic curves, and it follows from the equality $K_V^2 \cdot M_V = 0$ that the linear system $g^{-1}(\mathcal{M})$ is contained in the fibres of the fibration $\phi_{|-rK_V|}$, which readily implies the existence of the commutative diagram (2.2.12).

Suppose that $k = 5$. Then there is a birational morphism $\phi_{|-rK_V|}\colon V \to U$ such that U is a Fano variety with canonical singularities, $-K_U^2 = 1/2$, and the relation

$$K_V + \lambda M_V \sim_{\mathbb{Q}} \phi^*_{|-rK_V|}(K_U + \lambda M_U)$$

holds, where $M_U = \phi_{|-rK_V|}(M_V)$. Thus, the log pair $(U, \lambda M_U)$ is a canonical model. In particular, the equality $\kappa(U, \lambda M_U) = 3$ holds; however,

$$\kappa(U, \lambda M_U) = \kappa\biggl(Y, \frac{\lambda}{n}|\tau^*(\check{H})|\biggr) \leqslant \dim(Z) = 2,$$

which is a contradiction.

The assertion of Theorem 2.2.10 remains valid over any field of definition $\mathbb{F}$ of characteristic zero. One can show (see [36]) that the assertion of Theorem 2.2.10 holds over any perfect field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \notin \{2,3,5\}$ and fails for $\mathrm{char}(\mathbb{F}) = 5$.

Corollary 2.2.13. *Suppose that X is smooth. Then X cannot be birationally transformed into a fibration into elliptic curves.*

Birational transformations of smooth Fano threefolds into elliptic fibrations and the properties of torsion points of an elliptic curve that is defined over a number field (see [130]) were used in [10], [11], [72] to prove the potential density[13] of rational points on Fano threefolds, and the following assertion was proved.

Theorem 2.2.14. *The rational points are potentially dense on all smooth Fano threefolds, except possibly for a double cover of $\mathbb{P}^3$ ramified along a sextic curve.*

The possible exception in Theorem 2.2.14 is related to the fact that a double cover of $\mathbb{P}^3$ ramified along a smooth sextic curve is the only smooth Fano threefold for which there is no birational transform to an elliptic fibration. It follows from the classification of smooth Fano threefolds that a smooth double cover of $\mathbb{P}^3$ ramified along a sextic curve is the only smooth Fano threefold which cannot be transformed birationally to an elliptic fibration (see Corollary 2.2.13 and [95]). The rational points are always potentially dense on a geometrically uniruled variety. There is the well-known *weak Lang conjecture* stating that the rational points are not potentially dense on varieties of general type, and this conjecture is proved for subvarieties of Abelian varieties (see [58] and [59]). On the other hand, the geometry of birationally superrigid varieties is very similar to that of varieties of general type, which gives hope that rational points are not potentially dense on some birationally superrigid varieties, and this would imply the existence of non-unirational rationally connected varieties.

§ 2.3. Double cover of a quadric

Let $\psi\colon X \to Q$ be a double cover of a three-dimensional quadric $Q \subset \mathbb{P}^4$ ramified along a surface $S \subset Q$ cut out on Q by a quartic hypersurface in $\mathbb{P}^4$ and such that the threefold X has at most isolated ordinary double singular points. In this case X is a Fano variety of degree 4, the relation $-K_X \sim \psi^*(\mathcal{O}_{\mathbb{P}^4}(1)|_Q)$ holds, the variety X is unirational, and the following result is valid (see [87]).

Theorem 2.3.1. *Suppose that X is smooth. Then X is birationally rigid.*

Proof. Suppose that X is not birationally rigid. For example, suppose for clarity that there is a birational map $\rho\colon X \dashrightarrow \mathbb{P}^3$. We claim that this assumption leads to a contradiction, and we shall then omit the proofs of the remaining possible cases.

Let σ be a birational automorphism of the variety X. Consider the linear system $\mathcal{M} = (\rho\circ\sigma)^{-1}(|\mathcal{O}_{\mathbb{P}^3}(1)|)$. It follows from the Lefschetz theorem that the group $\operatorname{Pic}(X)$ is generated by the anticanonical divisor $-K_X$. Thus, there is a positive integer n such that $\mathcal{M} \sim -nK_X$. The number n depends on the birational automorphism σ as well. Hence, one can assume that σ is chosen in such a way that n takes the least possible value.

The singularities of the log pair $(X, \frac{1}{n}\mathcal{M})$ are not canonical (see the proof of Theorem 1.4.1). Moreover, it follows immediately from the proofs of Theorems 2.1.1 and 2.2.1 that the singularities of the movable log pair $(X, \frac{1}{n}\mathcal{M})$ are not canonical at a general point of some irreducible curve $C \subset X$ such that $-K_X\cdot C \leqslant 3$. In particular, $\operatorname{mult}_C(\mathcal{M}) > n$.

Suppose that $-K_X\cdot C = 2$ and that the curve $\psi(C)$ is a line. Let η be the composition of the double cover ψ with the projection from the line $\psi(C)$,

[13] The rational points of a variety V defined over a number field $\mathbb{F}$ are said to be *potentially dense* if there is a finite extension $\mathbb{F} \subset \mathbb{K}$ of fields such that the set of $\mathbb{K}$-points of the variety V is Zariski dense on the variety V.

and let L be a sufficiently general curve on X which is contracted to a point by η. In this case L is a smooth elliptic curve such that $-K_X \cdot L = 2$, and the intersection $L \cap C$ consists of two distinct points. Let S be a general surface in the linear system $\mathcal{M}$. Then $L \not\subset S$, because the linear system $\mathcal{M}$ has no fixed components. On the other hand, we have

$$2n = S \cdot L \geqslant \sum_{O \in L \cap C} \mathrm{mult}_O(S)\,\mathrm{mult}_O(L) \geqslant \sum_{O \in L \cap C} \mathrm{mult}_C(S) > 2n,$$

which is a contradiction.

Suppose that $-K_X \cdot C = 1$ and $\psi(C) \subset S$. Then $\psi(C)$ is a line. Let L be a general curve on X such that $-K_X \cdot L = 2$ and the image $\psi(L)$ is a line on the quadric Q that is tangent to the surface S at some point of the line $\psi(C)$. Then it follows from the condition $\psi(C) \subset S$ that the curve L is not contained in the base locus of the linear system $\mathcal{M}$. Let O be the point of intersection of the curves C and L. Then $\mathrm{mult}_O(L) \geqslant 2$, which immediately contradicts the inequality $\mathrm{mult}_C(\mathcal{M}) > n$. The same holds if $-K_X \cdot C = 2$ and $\psi(C)$ is a line.

Suppose that $-K_X \cdot C = 1$ and $\psi(C) \not\subset S$. Then $\psi(C)$ is a line, and there is a smooth curve $Z \subset X$ which differs from the curve C and satisfies the condition $\psi(Z) = \psi(C)$. Let η be the composition of the double cover ψ with the projection from the line $\psi(C)$, let $\pi\colon V \to X$ be the composition of a blow-up of the smooth curve C with a subsequent blow-up of the proper transform of the smooth curve Z, let E be an exceptional divisor of π that dominates C, and let G be an exceptional divisor of π that dominates Z. Then the map

$$\eta \circ \pi\colon V \dashrightarrow \mathbb{P}^2$$

is a morphism over a general point of $\mathbb{P}^2$, its general fibre is an elliptic curve, and the surfaces E and G are sections of the rational map $\eta \circ \pi$. Let γ be the reflection of a general fibre of the map $\eta \circ \pi$ in the section G, which can be regarded as a point on the corresponding elliptic curve. Then the involution γ induces a birational involution $\tau = \pi \circ \gamma \circ \pi^{-1}$ of the variety X. One can readily see that the involution τ is not biregular. Hence, we have a rational equivalence

$$\tau(\mathcal{M}) \sim -n' K_X,$$

where n' is a positive integer. Moreover, arguing as in the proof of Theorem 1.5.1, we can see that $n' < n$, which contradicts the minimality of the number n.

Therefore, we have proved that either $-K_X \cdot C = 2$ or $-K_X \cdot C = 3$ and the restriction $\psi|_C$ is an isomorphism. In particular, the curve C is smooth and the curve $\psi(C)$ is either an irreducible smooth conic or a smooth rational curve of degree 3.

Let $\xi\colon W \to X$ be a blow-up of the curve C, let F be an exceptional divisor of the birational morphism ξ, let $\mathcal{B} = \xi^{-1}(\mathcal{M})$, and let $d = -K_X \cdot C$. In this case, as in the proof of Theorem 2.2.1, one can readily show that the divisor $\xi^*(-dK_X) - F$ is numerically effective. Thus, for any two general surfaces S_1 and S_2 in the linear system $\mathcal{B}$ we have

$$0 > (\xi^*(-dK_X) - F)(\xi^*(-nK_X) - \mathrm{mult}_C(\mathcal{M})F)^2 = S_1 \cdot S_2 \cdot (\xi^*(-dK_X) - F) \geqslant 0,$$

because $\mathrm{mult}_C(\mathcal{M}) > 0$. This is a contradiction.

For every curve $L \subset X$ such that $-K_X \cdot L = 1$ and $\psi(L) \not\subset S$ there is a birational involution τ of X which is not biregular. In the smooth case we have a one-dimensional family of curves of this kind, and the corresponding birational involutions do not admit any relations (see [87]). Let Γ be the subgroup of $\mathrm{Bir}(X)$ generated by these birational involutions. Then the proof of Theorem 2.3.1 implies the following result.

Theorem 2.3.2. *Suppose that X is smooth. Then there is an exact sequence of groups of the form*

$$1 \to \Gamma \to \mathrm{Bir}(X) \to \mathrm{Aut}(X) \to 1.$$

Conjecture 2.3.3. *Suppose that X is $\mathbb{Q}$-factorial. Then X is birationally rigid.*

Conjecture 2.3.3 was proved in [70] under the additional assumption that the variety X has a single ordinary double point and satisfies some generality assumptions. Moreover, the following result was proved in [71].

Theorem 2.3.4. *Suppose that Q is a cone, the surface S is smooth and does not pass through the vertex of Q, and $\alpha\colon X \dashrightarrow \mathbb{P}^1$ and $\beta\colon X \dashrightarrow \mathbb{P}^1$ are rational maps induced by distinct projections of the base of Q, which is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Then the general fibres of the rational maps α and β are del Pezzo surfaces of degree 2, and X is not a $\mathbb{Q}$-factorial variety. Suppose that there is a Mori fibration $\tau\colon Y \to Z$ and a birational map $\rho\colon X \dashrightarrow Y$ and that the surface S is sufficiently general. Then there is a birational automorphism σ of X such that $\rho \circ \tau = \alpha \circ \sigma$ or $\rho \circ \tau = \beta \circ \sigma$.*

Theorem 2.3.1 was generalized in [148] as follows.

Theorem 2.3.5. *Let $\zeta\colon V \to Y$ be a double cover of a smooth quadric $Y \subset \mathbb{P}^n$ ramified over a smooth divisor $D \subset Y$ such that $n \geqslant 5$ and $D \sim \mathcal{O}_{\mathbb{P}^n}(2n-4)|_Y$. Then V is a birationally superrigid Fano variety of degree 4.*

One can show that the assertion of Theorem 1.7.23 implies the assertion of Theorem 2.3.5 in the nodal case. A result similar to Theorem 2.2.10 was proved in [26] for smooth varieties X.

§ 2.4. Intersection of a quadric and a cubic

Let $X = Q \cap V \subset \mathbb{P}^5$ be a smooth complete intersection, where Q is a quadric hypersurface and V is a cubic hypersurface. Then $-K_X \sim \psi^*(\mathcal{O}_{\mathbb{P}^5}(1)|_X)$. Therefore, X is a smooth Fano variety of degree 6, and it follows from the Lefschetz theorem that the group $\mathrm{Pic}(X)$ is generated by the divisor $-K_X$. As is known, X is unirational (see [87]) and has non-biregular birational automorphisms (see Example 1.3.14).

Example 2.4.1. Let L be a line on the variety X and let $\psi\colon X \dashrightarrow \mathbb{P}^3$ be the projection from L. Then the rational map ψ is a double cover over a general point of $\mathbb{P}^3$, and this map induces a birational involution τ of X which is clearly not biregular.

The following result holds (see [87], [96]).

Theorem 2.4.2. *Suppose that the complete intersection X is sufficiently general. Then X is birationally rigid, and there is an exact sequence of groups*

$$1 \to \Gamma \to \mathrm{Bir}(X) \to \mathrm{Aut}(X) \to 1,$$

where Γ is the free product of the groups generated by the involutions constructed in Examples 2.4.1 *and* 1.3.14.

In the remaining part of this section we sketch the proof of the birational rigidity of the variety X. For example, we assume that there is a birational map $\rho\colon X \dashrightarrow Y$ for which there is a morphism $\pi\colon Y \to \mathbb{P}^2$ defining a conic bundle. Let us show that this assumption leads to a contradiction.

We consider the linear system

$$\mathcal{M} = (\rho \circ \sigma)^{-1}(|\pi^*(\mathcal{O}_{\mathbb{P}^2}(1))|),$$

where σ is a birational automorphism of X, and we choose a positive integer n such that $\mathcal{M} \sim -nK_X$; let σ be such that n takes the least possible value. Then the singularities of the movable log pair $(X, \frac{1}{n}\mathcal{M})$ are not canonical (see the proof of Theorem 1.4.1). In particular, the set $\mathbb{CS}(X, \mu\mathcal{M})$ is non-empty for some positive rational number $\mu < \frac{1}{n}$.

Suppose that $\mathbb{CS}(X, \mu\mathcal{M})$ contains a curve Z on the variety X. Then $\mathrm{mult}_Z(\mathcal{M}) > n$, which implies that $\deg(Z) \leqslant 5$. If the curve Z is a line, then we assume that τ is the corresponding involution in Example 2.4.1. If Z is a conic such that the plane containing Z is contained in the quadric Q, then we assume that τ is the involution constructed in Example 1.3.14. In this case, $\tau(\mathcal{M}) \sim -\widehat{n}K_X$ for a positive integer $\widehat{n} < n$, which contradicts the minimality of n. In all remaining cases we arrive at a contradiction by using the inequality $\mathrm{mult}_Z(\mathcal{M}) > n$ as in the proof of Theorem 2.3.1 (see [87]).

Therefore, the set $\mathbb{CS}(X, \mu\mathcal{M})$ contains a point $P \in X$. We use the arguments of [46]. Let T be the intersection of two hyperplanes in $\mathbb{P}^5$ that are tangent to the quadric Q and the cubic V, respectively, at the point P. Suppose now that the variety X is sufficiently general in the following sense: the curve $T \cap X$ is a curve of degree 6 such that $\mathrm{mult}_P(T \cap X) = 4$, the singularity of $T \cap X$ at P is analytically equivalent to a cone over four general points in $\mathbb{P}^2$, and one of the following possible cases holds:

- $T \cap X$ is an irreducible curve Δ of degree 6 such that $\mathrm{mult}_P(\Delta) = 4$;
- $T \cap X$ consists of the union of an irreducible curve Δ of degree 5 and a line Γ_1 that passes through the point P, and $\mathrm{mult}_P(\Delta) = 3$;
- $T \cap X$ consists of the union of an irreducible curve Δ of degree 4 and lines Γ_1 and Γ_2 passing through P, and $\mathrm{mult}_P(\Delta) = 2$;
- $T \cap X$ consists of the union of an irreducible smooth rational curve Δ of degree 3 and lines Γ_1, Γ_2, and Γ_3 passing through P.

We consider only the case in which the intersection $T \cap X$ consists of the union of an irreducible smooth rational curve Δ of degree 3 and lines Γ_1, Γ_2, Γ_3 passing through the point P. Let $\xi\colon U \to X$ be a blow-up of P, let F be an exceptional divisor of the birational morphism ξ, and let S be a general hyperplane section of the complete intersection X that contains $T\cap X$. We write $\overline{S}=\xi^{-1}(S)$, $\Phi=F|_{\overline{S}}$, $\overline{\Delta}=\xi^{-1}(\Delta)$, and $\overline{\Gamma}=\xi^{-1}(\Gamma)$. Then the surface $\overline{S}$ is smooth and the following equalities hold on $\overline{S}$:

$$\begin{cases} \Phi^2=\overline{\Delta}^2=\overline{\Gamma}_i^2=-2, \\ \overline{\Delta}\cdot\overline{\Gamma}_i=\Phi\cdot\overline{\Gamma}_i=\overline{\Delta}\cdot\Phi=1, \\ \overline{\Gamma}_i\cdot\overline{\Gamma}_j=0 \iff i\neq j. \end{cases}$$

Let $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety U. Then

$$\mathcal{B}|_{\overline{S}}-m\overline{\Delta}-\sum_{i=1}^{3}m_i\Gamma_i=\mathcal{D}\sim\varepsilon\Phi+\delta\overline{\Delta}+\sum_{i=1}^{3}\gamma_i\Gamma_i,$$

where $\mathcal{D}$ is a linear system without fixed components, m and m_i are non-negative integers such that $m<n$ and $m_i<n$, $\varepsilon=2n-\operatorname{mult}_P(\mathcal{M})$, $\delta=n-m$, and $\gamma_i=n-m_i$. Therefore, for any two sufficiently general curves D_1 and D_2 in the linear system $\mathcal{D}$ we have

$$D_1\cdot D_2=2n^2\biggl(\varepsilon\delta-\varepsilon^2-\delta^2-\sum_{i=1}^{3}\gamma_i^2+(\varepsilon+\delta)\sum_{i=1}^{3}\gamma_i\biggr)\geqslant 0. \tag{2.4.3}$$

Suppose that $\operatorname{mult}_P(\mathcal{M})>2n$. Then

$$D_1\cdot D_2=n^2\bigl(6\varepsilon\delta-\gamma_3^2-(\gamma_1-\gamma_2)^2-(\varepsilon+\delta-\gamma_1-\gamma_2)^2-(\varepsilon+\delta-\gamma_3)^2\bigr)<0,$$

because $\varepsilon\leqslant 0$. Hence, the inequality $\operatorname{mult}_P(\mathcal{M})\leqslant 2n$ holds, and it follows from Corollary 1.7.26 that there is a line $\Lambda\subset F\cong\mathbb{P}^2$ such that the log pair

$$\Bigl(U,\frac{1}{n}\mathcal{B}+(1-\operatorname{mult}_P(\mathcal{M})/n)F\Bigr)$$

is not log canonical in Λ and the surface $\overline{S}$ still intersects Λ at two distinct points. Hence, the log pair

$$\biggl(\overline{S},\frac{1}{n}\mathcal{D}+(1-\delta/n)\overline{\Delta}+\sum_{i=1}^{3}(1-\gamma_i/n)\Gamma_i+(1-\operatorname{mult}_P(\mathcal{M})/n)\Phi\biggr)$$

is not log canonical at two distinct points O_1 and O_2 belonging to the curve Φ, because the choice of the surface S was general. It follows now from Theorem 1.7.17 that

- $D_1\cdot D_2>8n^2\varepsilon\delta$ if $O_1\notin\Gamma_1\cup\Gamma_2\cup\Gamma_3$ and $O_2\notin\Gamma_1\cup\Gamma_2\cup\Gamma_3$,
- $D_1\cdot D_2>4n^2\varepsilon(\gamma_i+\gamma_j)$ if $O_1\in\Gamma_i$ and $O_2\in\Gamma_j$, where $i\neq j$,
- $D_1\cdot D_2>4n^2\varepsilon(\delta+\gamma_i)$ if $O_1\notin\Gamma_1\cup\Gamma_2\cup\Gamma_3$ and $O_2\in\Gamma_i$,
- $D_1\cdot D_2>4n^2\varepsilon(\delta+\gamma_i)$ if $O_1\in\Gamma_i$ and $O_2\notin\Gamma_1\cup\Gamma_2\cup\Gamma_3$,

and, as one can readily see, this contradicts the inequality (2.4.3).

§ 2.5. Weighted hypersurfaces

Let X be a quasi-smooth hypersurface in $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ of degree d such that

$$-K_X \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}(1,\, a_1,\, a_2,\, a_3,\, a_4)}(1),$$

the inequalities $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$ hold, and the hypersurface X has terminal singularities. Then the quintuple (d, a_1, a_2, a_3, a_4) admits the 95 possibilities which were found in [78] by computer processing. The completeness of the list obtained in [78] was proved in [99].

Let S be a sufficiently general surface in the linear system $|-K_X|$. Then S is a quasi-smooth hypersurface in $\mathbb{P}(a_1, a_2, a_3, a_4)$ of degree d, S has Du Val singularities, and $K_S \sim 0$. Therefore, S is a $K3$ surface. Surprisingly, the converse is also true, that is, every quasi-smooth hypersurface in a weighted projective space and having type $K3$ with Du Val singularities can be obtained by the above construction. These surfaces were classified by Reid, but he did not publish this result, and an independent classification was obtained in [181]. As was proved in [7], the equality $\operatorname{rk}\operatorname{Pic}(S) = 1$ holds if the threefold X is very general. This is a generalization of the classical result that the Picard group of a very general hypersurface in $\mathbb{P}^3$ of degree more than 3 is generated by a hyperplane section (see [132]). One can conjecture that a similar assertion holds for any very general quasi-smooth hypersurface in $\mathbb{P}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ of degree δ if $\delta \geqslant \sum_{i=1}^{4} \alpha_i$, but the last conjecture has been proved only in some special cases (see [172], [50], [101]).

Let n be the ordinal number of X in the notation of the paper [78].

Remark 2.5.1. The hypersurface X is a smooth quartic threefold if $n = 1$, X is a double cover of $\mathbb{P}^3$ ramified over a smooth sextic if $n = 3$, and X is singular in the other cases.

Suppose that X is sufficiently general. As is well known, X is rationally connected (see [170]). On the other hand, the following result was obtained in [49].

Theorem 2.5.2. *The variety X is birationally rigid.*

The assertion of Theorem 2.5.2 can be regarded as a generalization of Theorems 2.1.1 and 2.2.1. Moreover, the lines of the proof of Theorem 2.5.2 are similar to those of the proofs of Theorems 2.1.1, 2.2.1, 2.3.1, and 2.4.2, namely, the exclusion of maximal singularities and the *untwisting of birational automorphisms* are involved. However, the proof of Theorem 2.5.2 contains many substantial technical modifications. We do not describe the scheme for proving Theorem 2.5.2, but to illustrate the methods of [49], we do prove the following natural generalization (obtained in [37]) of Corollary 2.2.13.

Theorem 2.5.3. *A weighted hypersurface X can be birationally transformed into an elliptic fibration if and only if $n \notin \{3, 60, 75, 84, 87, 93\}$.*

If $n = 1$, then the general fibre of the projection of the smooth quartic $X \subset \mathbb{P}^4$ from a line in X is an elliptic curve. If $n = 2$, then the threefold X is birationally equivalent to a double cover of $\mathbb{P}^3$ ramified over a singular sextic surface with 15 ordinary double points (see [49], [36]), which implies that X can be birationally transformed into a fibration into elliptic curves (see Example 2.2.8).

Lemma 2.5.4. *Suppose that* $n \notin \{1, 2, 3, 7, 11, 19, 60, 75, 84, 87, 93\}$. *In this case the normalization of the general fibre of the natural projection* $X \dashrightarrow \mathbb{P}(1, a_1, a_2)$ *is a smooth elliptic curve.*

Proof. Let C be the general fibre of the projection $X \dashrightarrow \mathbb{P}(1, a_1, a_2)$. Then C is not a rational curve by Theorem 2.5.2. On the other hand, C is a hypersurface of degree d in $\mathbb{P}(1, a_3, a_4) \cong \operatorname{Proj}(\mathbb{C}[x, t, w])$, where $\operatorname{wt}(x) = 1$, $\operatorname{wt}(t) = a_3$, and $\operatorname{wt}(w) = a_4$ and either $\lceil \frac{d}{a_3} \rceil \leqslant 3$ or $\lceil \frac{d}{a_3} \rceil \leqslant 4$ and $2a_5 \leqslant d \leqslant 2a_5 + a_4$. Let V be the subset of $\mathbb{P}(1, a_3, a_4)$ given by the inequality $x \neq 0$. Then $V \cong \mathbb{C}^2$, and the affine curve $V \cap C$ is either a cubic curve if $\lceil \frac{d}{a_3} \rceil \leqslant 3$ or a double cover of $\mathbb{C}$ ramified at no more than four points if $\lceil \frac{d}{a_3} \rceil \leqslant 4$ and $2a_5 \leqslant d \leqslant 2a_5 + a_4$. Thus, the curve C is elliptic.

Therefore, if $n \notin \{1, 2, 3, 7, 11, 19, 60, 75, 84, 87, 93\}$, then the variety X is birationally equivalent to an elliptic fibration having a section.

Lemma 2.5.5. *Suppose that* $n \in \{7, 11, 19\}$. *Then* X *can be birationally transformed into a fibration into elliptic curves.*

Proof. We consider only the case $n = 19$, because the other cases are similar. Thus, let $n = 19$. In this case $d = 12$ and the variety X is a hypersurface in $\mathbb{P}(1, 2, 3, 3, 4)$ which can be given by an equation of the form

$$\begin{aligned} &wf_8(x, y, z, t, w) + zf_3(z, t) \\ &\qquad + yf_{10}(x, y, z, t, w) + xf_{11}(x, y, z, t, w) = 0 \subset \operatorname{Proj}(\mathbb{C}[x, y, z, t, w]), \end{aligned}$$

where $\operatorname{wt}(x) = 1$, $\operatorname{wt}(y) = 2$, $\operatorname{wt}(z) = \operatorname{wt}(t) = 3$, $\operatorname{wt}(w) = 4$, and f_i is a general quasi-homogeneous polynomial of degree i. Let $\mathcal{H}$ and $\mathcal{B}$ be the pencils of surfaces that are cut out on the variety X by the equations

$$\lambda x^2 + \mu y = 0 \quad \text{and} \quad \delta x^3 + \gamma z = 0,$$

respectively, where $(\lambda : \mu) \in \mathbb{P}^1$ and $(\delta : \gamma) \in \mathbb{P}^1$. In this case the pencils $\mathcal{H}$ and $\mathcal{B}$ determine a map $\rho \colon X \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Let C be the general fibre of the map ρ. Then C is a hypersurface of degree 12 in $\mathbb{P}(1, 3, 4)$ containing the point $(0 : 1 : 0)$, which implies that the affine part of C is a cubic curve in $\mathbb{C}^2$. However, the curve C is non-rational by Theorem 2.5.2 and is thus elliptic.

Corollary 2.5.6. *The threefold* X *can be birationally transformed into an elliptic fibration if* $n \notin \{3, 60, 75, 84, 87, 93\}$.

Let us now prove Theorem 2.5.3. Suppose that $n \in \{3, 75, 84, 87, 93\}$ and that there exist a birational map $\rho \colon X \dashrightarrow V$ and a morphism $\nu \colon V \to \mathbb{P}^2$ for which V is smooth and the general fibre of ν is an elliptic curve. We claim that these assumptions lead to a contradiction. Let $\mathcal{D} = |\nu^*(\mathcal{O}_{\mathbb{P}^2}(1))|$ and $\mathcal{M} = \rho^{-1}(\mathcal{D})$. Then $\mathcal{M} \sim -kK_X$ for some positive integer k, and the singularities of the log pair $(X, \frac{1}{k}\mathcal{M})$ are not terminal by Theorem 1.4.4. In particular, the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ is non-empty. On the other hand, the singularities of the log pair $(X, \frac{1}{k}\mathcal{M})$ are canonical by Corollary 1.4.3, because X is birationally superrigid (see [49]).

Lemma 2.5.7. *The set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains no smooth points of the variety X.*

Proof. Suppose that $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains a smooth point $P \in X$. Consider quasi-homogeneous coordinates (x, y, z, t, w) on $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ with $\mathrm{wt}(x) = 1$, $\mathrm{wt}(y) = a_1$, $\mathrm{wt}(z) = a_1$, $\mathrm{wt}(t) = a_3$, and $\mathrm{wt}(w) = a_4$ such that $P = (\xi_0, \xi_1, \xi_2, \xi_3, \xi_4)$. We set $\varepsilon = a_3$ if $\xi_0 \neq 0$, $\varepsilon = a_1 a_3$ if $\xi_0 = 0$ and $\xi_1 \neq 0$, and

$$\varepsilon = \min\Big\{\big\{\mathrm{LCM}(a_i, a_j)\big\} \cup \big\{a_i + a_j \mid a_k \text{ divides } a_i + a_j\big\}\Big\}$$

if $\xi_0 = \xi_1 = 0$, where the minimum is taken over all triples $\{i, j, k\} = \{2, 3, 4\}$. In this case $-\varepsilon K_X^3 < 4$.

Let λ be a positive integer and let $\mathcal{H}$ be the linear subsystem of $|-\varepsilon\lambda K_X|$ that consists of the surfaces whose multiplicity at the point P is not less than λ. It follows from Theorem 5.6.2 in [49] that there is a λ such that the point P is an isolated base point of the linear system $\mathcal{H}$. Thus, for a general surface S in the linear system $\mathcal{H}$ and for general surfaces D_1 and D_2 in the linear system $\mathcal{M}$ we have

$$\mathrm{mult}_P(D_1 \cdot D_2) \leqslant \frac{D_1 \cdot D_2 \cdot S}{\mathrm{mult}_P(S)} = \frac{-\lambda\varepsilon K_X^3 k^2}{\mathrm{mult}_P(S)} \leqslant -\varepsilon K_X^3 k^2 < 4k^2,$$

which contradicts Theorem 1.7.18.

We can assume that $n \neq 3$ by Corollary 2.2.13.

Lemma 2.5.8. *Let C be a curve on X such that $C \cap \mathrm{Sing}(X) = \varnothing$. Then $C \notin \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$.*

Proof. Let H be a very ample divisor on X. Then $H \sim -\lambda K_X$ for some $\lambda \in \mathbb{N}$ and

$$\frac{\lambda k^2}{60} \geqslant -\lambda k^2 K_X^3 = H \cdot S_1 \cdot S_2 \geqslant \mathrm{mult}_C^2(\mathcal{M}) H \cdot C \geqslant \lambda\, \mathrm{mult}_C^2(\mathcal{M}),$$

where S_1 and S_2 are general surfaces in $\mathcal{M}$. Thus, $\mathrm{mult}_C(\mathcal{M}) < k$.

The following result was obtained in [104].

Theorem 2.5.9. *Let $\pi \colon U \to V$ be a birational morphism such that V and U are threefolds with terminal $\mathbb{Q}$-factorial singularities, π contracts exactly one exceptional divisor E, and $\pi(E)$ contains a singular point O of V which is locally isomorphic to a terminal quotient singularity of type $\frac{1}{r}(1, a, r-a)$, where a and r are coprime positive integers with $r \geqslant 2$. Then π is a weighted blow-up of O with weights $(1, a, r-a)$.*

Corollary 2.5.10. *Let C be a curve on X such that $C \in \mathbb{CS}(X, \frac{1}{k}\mathcal{M})$. Then every point in the intersection $C \cap \mathrm{Sing}(X)$ is contained in the set $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$.*

Thus, $\mathbb{CS}(X, \frac{1}{k}\mathcal{M})$ contains a point P of X which is locally isomorphic to a terminal quotient singularity of type $\frac{1}{r}(1, a, r-a)$, where a and r are coprime positive integers such that $r \geqslant 2$. Let $\pi \colon Y \to X$ be a weighted blow-up of P with weights $(1, a, r-a)$, let E be an exceptional divisor of the blow-up π, and let $\mathcal{B}$ be the proper transform of the linear system $\mathcal{M}$ on the variety Y. Then $\mathcal{B} \sim_{\mathbb{Q}} -kK_Y$ by Theorem 2.5.9.

Remark 2.5.11. $-K_Y^3 < 0$ (see Proposition 3.4.6 in [49]).

Let $\overline{\mathbb{NE}}(Y) \subset \mathbb{R}^2$ be the closure of the cone of one-dimensional cycles on the variety Y. Then the class of the cycle $-E \cdot E$ generates one of the extremal rays of the cone $\overline{\mathbb{NE}}(Y)$, and the other extremal ray is defined by the following lemma, which is Corollary 5.4.6 in [49].

Lemma 2.5.12. *There are integers $b > 0$ and $c \geqslant 0$ such that the cycle $-K_Y \cdot (-bK_Y + cE)$ is numerically equivalent to an effective, irreducible, and reduced curve Γ on Y which generates an extremal ray of $\overline{\mathbb{NE}}(Y)$ different from that generated by the cycle $-E \cdot E$.*

Let S_1 and S_2 be distinct surfaces in $\mathcal{B}$. Then $S_1 \cdot S_2 \in \overline{\mathbb{NE}}(Y)$, but we have the numerical equivalence $S_1 \cdot S_2 \equiv k^2 K_Y^2$, which implies that the cycle $S_1 \cdot S_2$ generates an extremal ray of $\overline{\mathbb{NE}}(Y)$ containing the curve Γ. However, for each effective cycle $C \in \mathbb{R}^+\Gamma$ we have

$$\operatorname{Supp}(C) = \operatorname{Supp}(S_1 \cdot S_2),$$

because $S_1 \cdot \Gamma < 0$ and $S_2 \cdot \Gamma < 0$.

Let Q be a sufficiently general point in $Y \setminus \operatorname{Supp}(S_1 \cdot S_2)$ and let $\mathcal{P}$ be the linear subsystem of $\mathcal{B}$ formed by the divisors passing through the point Q. Then the linear system $\mathcal{P}$ has no fixed components, because by construction the linear system $\mathcal{B}$ does not consist of a pencil. Thus, we can apply the above arguments to $\mathcal{P}$ instead of $\mathcal{B}$ and show that the cycle $D_1 \cdot D_2$ generates an extremal ray of $\overline{\mathbb{NE}}(Y)$ containing the curve Γ for any two general surfaces D_1 and D_2 in $\mathcal{P}$. In particular, we have

$$\operatorname{Supp}(D_1 \cdot D_2) \subseteq \operatorname{Supp}(S_1 \cdot S_2),$$

but $Q \in \operatorname{Supp}(D_1 \cdot D_2)$ and $Q \in Y \setminus \operatorname{Supp}(S_1 \cdot S_2)$, a contradiction.

Hence, we have proved Theorem 2.5.3 for $n \neq 60$. For $n = 60$ the proof is similar but more technical (for details, see [37]). We note that it is impossible to arrive at a contradiction in the proof of Theorem 2.5.3 without using the condition that the linear system $\mathcal{M}$ is not formed by a pencil, because any hypersurface X under consideration can be birationally transformed into a fibration into $K3$ surfaces (see [37]).

Conjecture 2.5.13. *Suppose that $n \notin \{1, 2, 3, 7, 9, 11, 17, 19, 20, 26, 30, 36, 44, 49, 51, 60, 64, 75, 84, 87, 93\}$. Let $\rho\colon X \dashrightarrow \mathbb{P}^2$ be a rational map such that the normalization of the general fibre of ρ is an irreducible elliptic curve. Then $\rho = \phi \circ \psi$, where $\psi\colon X \dashrightarrow \mathbb{P}(1, a_1, a_2)$ is the natural projection and $\phi\colon \mathbb{P}(1, a_1, a_2) \dashrightarrow \mathbb{P}^2$ is a birational map.*

Conjecture 2.5.13 was proved in [161] for $n = 5$ and in [37] for $n \in \{14, 22, 28, 34, 37, 39, 52, 53, 57, 59, 66, 70, 72, 73, 78, 81, 86, 88, 89, 90, 92, 94, 95\}$.

Birational involutions $\tau_1, \dots, \tau_k$ of the variety X such that there is an exact sequence of groups of the form $1 \to \Gamma \to \operatorname{Bir}(X) \to \operatorname{Aut}(X) \to 1$, where Γ is the group generated by $\tau_1, \dots, \tau_k$, were explicitly constructed in [49] for any possible value of n. The following result holds (see [37]).

Theorem 2.5.14. *There are no relations among the birational involutions* $\tau_1, \dots,$ τ_k *when* $k \neq 3$ *or* $n = 20$, *and if* $k = 3$ *and* $n \neq 20$, *then the only relation among the involutions* τ_1, τ_2, *and* τ_3 *is* $\tau_1 \circ \tau_2 \circ \tau_3 = \tau_3 \circ \tau_2 \circ \tau_1$.

The possible values of the number k are as follows:

- $k = 5$ for $n = 7$;
- $k = 3$ for $n \in \{4, 9, 17, 20, 27\}$;
- $k = 2$ for $n \in \{5, 6, 12, 13, 15, 23, 25, 30, 31, 33, 36, 38, 40, 41, 42, 44, 58, 61, 68, 76\}$;
- $k = 1$ for $n \in \{2, 8, 16, 18, 24, 26, 32, 43, 45, 46, 47, 48, 54, 56, 60, 65, 69, 74, 79\}$;
- $k = 0$ in the other birationally superrigid cases.

PART 3. HIGHER-DIMENSIONAL VARIETIES

§ 3.1. Hypersurfaces

Let X be a smooth hypersurface in $\mathbb{P}^n$ of degree $n \geqslant 5$. Then $-K_X \sim \mathcal{O}_{\mathbb{P}^n}(1)|_X$, the group $\operatorname{Pic}(X)$ is generated by a hyperplane section of the hypersurface X (see [14]), and X is a Fano variety of degree n.

Proof of Theorem 0.3.9. The variety X is said to be *k-regular* if the sequence of homogeneous polynomials $q_1(x_1, \dots, x_n), \dots, q_k(x_1, \dots, x_n)$ is regular for every point P on X, where

$$\begin{aligned} &x_0^{n-1} q_1(x_1, \dots, x_n) + x_0^{n-2} q_2(x_1, \dots, x_n) \\ &\qquad + \dots + x_0 q_{n-1}(x_1, \dots, x_n) + q_n(x_1, \dots, x_n) = 0 \end{aligned} \tag{$*$}$$

is the equation of the hypersurface X in $\mathbb{P}^n \cong \operatorname{Proj}(\mathbb{C}[x_0, \dots, x_n])$ and the point P is given by the equations $x_1 = \dots = x_n = 0$. The hypersurface X is never n-regular, because it contains a line (see [6] or [110], Chap. V, Theorem 4.3). As was proved in [149], a sufficiently general hypersurface X is $(n-1)$-regular.

Suppose now that X is $(n-1)$-regular and not birationally superrigid. In this case Theorem 1.4.1 implies the existence of a linear system $\mathcal{M}$ without fixed components on X such that the singularities of the movable log pair $(X, \frac{1}{r}\mathcal{M})$ are not canonical, where r is a positive integer such that $\mathcal{M} \sim -rK_X$. Moreover, it follows from Proposition 1.3.12 that the log pair $(X, \frac{1}{r}\mathcal{M})$ is not canonical at some point P of X.

The hypersurface X can be given in $\mathbb{P}^n \cong \operatorname{Proj}(\mathbb{C}[x_0, \dots, x_n])$ by the above equation $(*)$ in such a way that P is given by the equations $x_1 = \dots = x_n = 0$. It follows now from the $(n-1)$-regularity of X that the sequence of homogeneous polynomials $q_1, \dots, q_{n-1}$ is a regular sequence. Let $\mathcal{H}_k$ be the linear system on X consisting of the divisors cut out by all possible equations of the form

$$f_0(x_1, \dots, x_n) \sum_{i=1}^{k} q_i + f_1(x_1, \dots, x_n) \sum_{i=1}^{k-1} q_i + \dots + f_{k-1}(x_1, \dots, x_n) q_1 = 0,$$

for $k = 1, \dots, n-1$, where f_i is a homogeneous polynomial of degree i. Then the regularity of the sequence $q_1, \dots, q_{n-1}$ implies the inequality $\operatorname{codim}(\operatorname{Bs}(\mathcal{H}_k) \subset X) \geqslant k-1$. Let S_k be a general divisor in the linear system $\mathcal{H}_k$, let D_1 and D_2 be general divisors in $\mathcal{M}$, and let H be a general hyperplane section of X containing the point P. We set $Z = S_4 \cdot \ldots \cdot S_{n-1} \cdot D_1 \cdot D_2$. Then the rational equivalence $S_k \sim -kK_X$ and the inequality $\mathrm{mult}_P(S_k) \geqslant k+1$ hold and the one-dimensional cycle Z is effective. Therefore, we have

$$\frac{n!}{3!}r^2 = H \cdot Z \geqslant \mathrm{mult}_P(Z) \geqslant \mathrm{mult}_P(D_1 \cdot D_2)\,\mathrm{mult}_P(S_4)\cdots\mathrm{mult}_P(S_{n-1}) \geqslant \mathrm{mult}_P(D_1 \cdot D_2)\frac{n!}{4!},$$

which gives $\mathrm{mult}_P(D_1 \cdot D_2) \leqslant 4r^2$, and this is impossible by Theorem 1.7.18.

Let us now prove the following result (see [144]).

Theorem 3.1.1. *If* $n = 5$, *then* X *is birationally superrigid.*

Proof. Suppose that a smooth hypersurface X is not birationally superrigid. Then it follows from Theorem 1.4.1 and Proposition 1.3.12 that there is a linear system $\mathcal{M}$ on X without fixed components and such that the log pair $(X, \frac{1}{r}\mathcal{M})$ is not canonical at some point $P \in X$, where r is a positive integer such that $\mathcal{M} \sim -rK_X$. In particular, we have the inequality $\mathrm{mult}_P(\mathcal{M}) > r$.

Let $\pi\colon V \to X$ be a blow-up of the point P and let E be an exceptional divisor of π. Then

$$K_V + \frac{1}{r}\widehat{\mathcal{M}} \sim_{\mathbb{Q}} \pi^*\biggl(K_X + \frac{1}{r}\mathcal{M}\biggr) + \biggl(3 - \frac{1}{r}\,\mathrm{mult}_P(\mathcal{M})\biggr)E,$$

where $\widehat{\mathcal{M}}$ is the proper transform of the linear system $\mathcal{M}$ on V.

Let S_1 and S_2 be general divisors in $\mathcal{M}$ and let H_1 and H_2 be general hyperplane sections of the hypersurface X which pass through P. Then

$$5r^2 = S_1 \cdot S_2 \cdot H_1 \cdot H_2 \geqslant \mathrm{mult}_P(S_1 \cdot S_2) \geqslant \mathrm{mult}^2_P(\mathcal{M}),$$

which implies that $\mathrm{mult}_P(\mathcal{M}) \leqslant \sqrt{5}r < 3r$.

Theorem 1.7.19 implies the existence of a line $L \subset E \cong \mathbb{P}^3$ such that

$$\mathrm{mult}_P(S_1 \cdot S_2 \cdot H) > 8r^2,$$

where H is an arbitrary hyperplane section of the hypersurface X satisfying the following conditions:

- the divisor H contains the point P and is smooth at P;
- the line L is contained in the divisor $\pi^{-1}(H)$;
- the divisor H does not contain surfaces in $\mathrm{Bs}(\mathcal{M})$.

We consider a linear system $\mathcal{D}$ consisting of the hyperplane sections H of the hypersurface X such that $L \subset \pi^{-1}(H)$. The base locus of the linear system $\mathcal{D}$ is the intersection of the quintic X with the plane $\Pi \subset \mathbb{P}^5$ that corresponds to the line $L \subset E$. In particular, the above conditions are consistent if the plane Π is not contained in X, and we have

$$5r^2 = S_1 \cdot S_2 \cdot H \cdot H_1 \geqslant \mathrm{mult}_P(S_1 \cdot S_2 \cdot H) > 8r^2,$$

which is impossible. Hence, the quintic X contains the plane Π. Similar arguments show that Π is contained in the base locus of the linear system $\mathcal{M}$. We note that a general quintic in $\mathbb{P}^5$ contains no planes (see [118]). One can show [21] that the hypersurface X is birationally equivalent to a fibration into $K3$ surfaces if and only if X contains planes (see Theorem 3.2.4), and if this is the case, then the corresponding fibrations into $K3$ surfaces are induced by the projections from planes contained in X.

Let Y be a sufficiently general hyperplane section of X which passes through the point P and is such that $L \subset \pi^{-1}(Y)$. Then Y is a hypersurface in $\mathbb{P}^4$ of degree 5, and by construction the variety Y is smooth at the point P. Moreover, it follows from Proposition 1.3.12 that Y has isolated singularities (this also follows from the finiteness of the Gauss map for any smooth hypersurface; see [82]).

Let $\overline{\pi}\colon \widehat{Y} \to Y$ be a blow-up of the point P and let $\overline{E} = \overline{\pi}^{-1}(P)$. Then the diagram

$$\begin{tikzcd} \widehat{Y} \arrow[r, hook] \arrow[d, "\overline{\pi}"'] & V \arrow[d, "\pi"] \\ Y \arrow[r, hook] & X \end{tikzcd}$$

is commutative, and one can identify $\widehat{Y}$ with the divisor $\pi^{-1}(Y) \subset V$ and $\overline{E}$ with $E \cap \widehat{Y}$.

We write $\overline{\mathcal{M}} = \mathcal{M}|_Y$ and note that the linear system $\overline{\mathcal{M}}$ has fixed components, because the plane Π is contained in the base locus of $\mathcal{M}$. Indeed, we have $\Pi \subset Y$ by construction, the plane Π is a fixed component of the linear system $\overline{\mathcal{M}}$, and this component is unique. Therefore, $\overline{\mathcal{M}} = \mathcal{B} + \alpha\Pi$, where $\mathcal{B}$ is a linear system on the threefold Y, $\mathcal{B}$ has no fixed components, and α is the multiplicity of a general divisor of the linear system $\mathcal{M}$ at a general point of the plane Π.

Let $\widehat{\mathcal{B}} = \overline{\pi}^{-1}(\mathcal{B})$ and $\widehat{\Pi} = \overline{\pi}^{-1}(\Pi)$. Then $L \subset \widehat{\Pi}$ and

$$K_{\widehat{Y}} + \frac{1}{r}\widehat{\mathcal{B}} + \frac{\alpha}{r}\widehat{\Pi} \sim_{\mathbb{Q}} \overline{\pi}^*\biggl(K_Y + \frac{1}{r}\mathcal{B} + \frac{\alpha}{r}\Pi\biggr) + \biggl(2 - \frac{1}{r}\operatorname{mult}_P(\mathcal{B}) - \frac{\alpha}{r}\biggr)\overline{E}.$$

It follows from the proof of Theorem 1.7.19 that the set

$$\mathbb{LCS}\biggl(\widehat{Y}, \frac{1}{r}\widehat{\mathcal{B}} + \frac{\alpha}{r}\widehat{\Pi} + \biggl(\frac{1}{r}\operatorname{mult}_P(\mathcal{B}) - \frac{\alpha}{r} - 2\biggr)\overline{E}\biggr)$$

contains a line $L \subset \overline{E} \subset E$. Therefore, we can apply Theorem 1.7.17 to the log pair $\bigl(\widehat{Y}, \frac{1}{r}\widehat{\mathcal{B}} + \frac{\alpha}{r}\widehat{\Pi} + (\frac{1}{r}\operatorname{mult}_P(\mathcal{B}) - \frac{\alpha}{r} - 2)\overline{E}\bigr)$ at a general point of the curve L, which immediately implies the strict inequality

$$\operatorname{mult}_L(\mathcal{B}^2) > 4(3r - \operatorname{mult}_P(\mathcal{B}) - \alpha)(r - \alpha),$$

which, in turn, immediately implies the inequality

$$\operatorname{mult}_P(\mathcal{B}^2) \geqslant \operatorname{mult}_P^2(\mathcal{B}) + \operatorname{mult}_L(\mathcal{B}^2) > \operatorname{mult}_P^2(\mathcal{B}) + 4(3r - \operatorname{mult}_P(\mathcal{B}) - \alpha)(r - \alpha)$$

and the inequality $\operatorname{mult}_P(\mathcal{B}^2) > (\operatorname{mult}_P(\mathcal{B}) - r + \alpha)^2 + 8(r - \alpha)$.

We now find an upper bound for $\mathrm{mult}_P(\mathcal{B}^2)$. Let Z be a sufficiently general hyperplane section of the quintic Y and let Z contain the point P. Then Z is a smooth quintic in $\mathbb{P}^3$ and, in particular, the inequality $(\mathcal{B}|_Z)^2 \geqslant \mathrm{mult}_P(\mathcal{B}^2)$ holds. However, $(\mathcal{B}|_Z)^2 = 5r^2 - 2r\alpha - 3\alpha^2$. Hence,

$$5r^2 - 2r\alpha - 3\alpha^2 > (\mathrm{mult}_P(\mathcal{B}) - r + \alpha)^2 + 8(r - \alpha),$$

and it is easy to see that this is a contradiction.

Omitting the last part of the proof of Theorem 3.1.1, we obtain the following result.

Theorem 3.1.2. *Let $n \in \{6, 7, 8\}$. Then X is birationally superrigid.*

The technique of *hypertangent linear systems* and the $8n^2$-inequality have been successfully used to prove the birational superrigidity of a broad class of higher-dimensional Fano varieties (see [151], [153], [155]–[157], [27], [34], [158]). Nevertheless, neither of these approaches can be used to prove the birational superrigidity of an arbitrary smooth hypersurface in $\mathbb{P}^n$ for $n \geqslant 9$. A new approach, found in [154], to the solution of the last problem is based on the following result (see [154], [63]).

Proposition 3.1.3. *Let V be a smooth variety, let $\mathcal{M}$ be linear system on V without fixed components, let D_1 and D_2 be general divisors in $\mathcal{M}$, let $Z = D_1 \cdot D_2$, and let P be a point of V that belongs to the set $\mathbb{LCS}(V, \frac{1}{r}\mathcal{M})$ for some positive integer r. Suppose that there is a morphism $\xi\colon V \to U$ such that $\dim(U) = \dim(V) - 1$, ξ is smooth in a neighbourhood of P, and ξ is finite on every irreducible component of the cycle Z. Then $\xi(P) \in \mathbb{LCS}(U, \frac{1}{4r^2}\xi(Z))$.*

Proposition 3.1.3 implies the following result (see [154], [63]).

Theorem 3.1.4. *Suppose that $n \leqslant 12$. Then X is birationally superrigid.*

We note that results in [109] imply the non-rationality of a hypersurface X under the assumption that it is very general (in the sense of the complement of countably many Zariski closed sets)† and, in particular, non-singular.

§ 3.2. Complete intersections

Let $X = \cap_{i=1}^{k} F_i \subset \mathbb{P}^r$ be a smooth complete intersection such that F_i is a hypersurface of degree d_i, where $d_k \geqslant \dots \geqslant d_1 \geqslant 2$ and $\dim(X) = r - k \geqslant 4$. In this case the group $\mathrm{Pic}(X)$ is generated by a hyperplane section of the variety X in view of the Lefschetz theorem, and the following equivalence holds:

$$-K_X \sim \mathcal{O}_{\mathbb{P}^r}\biggl(\sum_{i=1}^{k} d_i - r - 1\biggr)\bigg|_X,$$

which implies that

- X is not rationally connected if $\sum_{i=1}^{k} d_i > r$;
- X is a Fano variety if $\sum_{i=1}^{k} d_i \leqslant r$;
- X is not birationally rigid if $\sum_{i=1}^{k} d_i < r$.

† *Russian editor's note*: In more detail, a hypersurface is said to be very general if it corresponds to a point in the complement of countably many closed subvarieties in the space of all hypersurfaces.

Conjecture 3.2.1. *Suppose that $\sum_{i=1}^{k} d_i = r$ and $r-k > 4$. Then X is birationally superrigid.*

As was proved in [38], if $\sum_{i=1}^{k} d_i = n$, $d_k \geqslant 4$, there is an index j such that $d_j \neq 5$, and X is very general (in the sense of a complement of countably many Zariski closed sets), then the variety X is non-rational. The following result was obtained in [153] by using the technique of hypertangent linear systems (see the proof of Theorem 0.3.9).

Theorem 3.2.2. *Suppose that X is sufficiently general. Then X is birationally superrigid if $\sum_{i=1}^{k} d_i = r > 3k$.*

On the other hand, Theorem 1.7.19 implies the following result (see [27]).

Theorem 3.2.3. *Suppose that $r = 6$, $k = 2$, $d_1 = 2$, $d_2 = 4$, and the variety X contains no two-dimensional linear subspace of $\mathbb{P}^6$. Then X is birationally superrigid.*

Proof. Suppose that X is not birationally superrigid and contains no two-dimensional linear subspace of $\mathbb{P}^6$. Then by Theorem 1.4.1, there is a linear system $\mathcal{M}$ on X having no fixed components and such that the singularities of the movable log pair $(X, \frac{1}{n}\mathcal{M})$ are not canonical for a positive integer n for which the rational equivalence $\mathcal{M} \sim -nK_X$ holds. In particular, the set $\mathbb{CS}(X, \mu\mathcal{M})$ is not empty for some positive rational number $\mu < \frac{1}{n}$.

Let C be an irreducible subvariety of X that is a centre of canonical singularities of the log pair $(X, \mu\mathcal{M})$. Then $\mathrm{mult}_C(\mathcal{M}) > n$ (see Remark 1.3.9). In particular, if follows from Proposition 1.3.12 that $\dim(C) \leqslant 1$. On the other hand, it follows from Theorem 1.7.19 that C is a curve. Moreover, Theorem 1.7.18 gives us that $\mathrm{mult}_C(S_1 \cdot S_2) > 4n^2$ for sufficiently general divisors S_1 and S_2 in the linear system $\mathcal{M}$.

Suppose that C is not a line. Let L be a line in $\mathbb{P}^6$ passing through two sufficiently general points of the curve C. Then $L \not\subset X$, because otherwise the inequality $\mathrm{mult}_C(\mathcal{M}) > n$ would imply that $L \subset \mathrm{Bs}(\mathcal{M})$, which is impossible because the line L sweeps out either a plane in $\mathbb{P}^6$ (if the curve C is planar) or a variety of dimension 3; however, the linear system $\mathcal{M}$ has no fixed components, and X contains no two-dimensional linear subspaces of $\mathbb{P}^6$. Thus, we have

$$8n^2 = H_1 \cdot H_2 \cdot S_1 \cdot S_2 \geqslant 2\,\mathrm{mult}_C(\mathcal{M}^2) > 8n^2,$$

where H_1 and H_2 are sufficiently general hyperplane sections of X passing through the intersection $L \cap X$, and S_1 and S_2 are general divisors in $\mathcal{M}$. This is a contradiction. Thus, the curve C is a line in $\mathbb{P}^6$.

Let S be an intersection of two general hyperplane sections of $X \subset \mathbb{P}^6$ that pass through the line C. Then S is a smooth surface, the divisor K_S is rationally equivalent to a hyperplane section of S, and $K_S^2 = 8$. We have the relation

$$\mathcal{M}|_S = \mathcal{B} + \mathrm{mult}_C(\mathcal{M})C,$$

where $\mathcal{B}$ is a linear system on S without fixed components. Then

$$(nK_S - \mathrm{mult}_C(\mathcal{M})C)^2 = 8n^2 - 2n\,\mathrm{mult}_C(\mathcal{M}) - 3\,\mathrm{mult}_C^2(\mathcal{M}) = \mathcal{B}^2 \geqslant 0,$$

because $K_S \cdot C = 1$. However, $C^2 = -3$, and hence $\operatorname{mult}_C(\mathcal{M}) \leqslant \frac{4}{3}n$. Let $f\colon V \to X$ be a blow-up of the line C, let $E = f^{-1}(C)$, let $\widehat{\mathcal{M}} = f^{-1}(\mathcal{M})$, let H be a sufficiently general hyperplane section of X that passes through the line C, let $B_X = \frac{1}{n}\mathcal{M} + H$, and let $\widehat{H} = f^{-1}(H)$. Then

$$K_V + \frac{1}{n}\widehat{\mathcal{M}} + \widehat{H} \sim_{\mathbb{Q}} f^*\biggl(K_X + \frac{1}{n}\mathcal{M} + H\biggr) + \biggl(1 - \frac{1}{n}\operatorname{mult}_C(\mathcal{M})\biggr)E$$

and $C \in \mathbb{LCS}(X, B_X)$. Let D be a general hyperplane section of X passing through some point $P \in C$. Then $P \in \mathbb{LCS}(D, B_X|_D)$ and $P \in \mathbb{CS}(D, \frac{1}{n}\mathcal{M}|_D)$. By applying Proposition 1.7.25 directly, we see that the set

$$\mathbb{LCS}\biggl(V, \frac{1}{n}\widehat{\mathcal{M}} + \widehat{H} + \biggl(\frac{1}{n}\operatorname{mult}_C(\mathcal{M}) - 1\biggr)E\biggr)$$

contains a surface $S \subset E$ such that the fibre of the morphism $f|_S\colon S \to C$ over every point of the line C is a line in the fibre of f over the point, and this fibre is isomorphic to $\mathbb{P}^2$, because $\operatorname{mult}_C(\mathcal{M}) \leqslant \frac{4}{3}n$. We can now apply Theorem 1.7.17 to the log pair $(V, \frac{1}{n}\widehat{\mathcal{M}} + \widehat{H} + (\frac{1}{n}\operatorname{mult}_C(\mathcal{M}) - 1)E)$ at a general point of the surface S, and we get the inequality

$$\operatorname{mult}_S(\widehat{\mathcal{M}}^2) > 4(2n^2 - n\operatorname{mult}_C(\mathcal{M})) \geqslant \frac{8}{3}n^2,$$

because $\operatorname{mult}_C(\mathcal{M}) \leqslant \frac{4}{3}n$. On the other hand, the equality $\widehat{H}^4 = 3$ holds, and $\widehat{H}$ is a general divisor in the free linear system $|\widehat{H}|$. This system induces a morphism $\phi_{|\widehat{H}|}\colon V \to \mathbb{P}^4$ of degree 3 at a general point of the variety V, and the morphism $\phi_{|\widehat{H}|}$ does not contract any surface to a point, because X contains no planes. Let $\overline{H}$ be another general divisor in the linear system $|\widehat{H}|$. Then simple manipulations show that $\overline{H}$ intersects the curve $\widehat{H} \cap S$ at at least two distinct points, which implies that

$$\frac{16}{3}n^2 < \overline{H} \cdot \widehat{H} \cdot \widehat{\mathcal{M}}^2 = (f^*(H) - E)^2 \cdot (f^*(nH) - \operatorname{mult}_C(\mathcal{M})E)^2 < 3n^2,$$

because $\operatorname{mult}_C(\mathcal{M}) > n$. This is a contradiction.

A general complete intersection of a quadric and a quartic in $\mathbb{P}^6$ contains no planes (see [118]). On the other hand, if the complete intersection of a quadric and a quartic in $\mathbb{P}^6$ contains a two-dimensional linear subspace, then the birational geometry of this intersection is more complicated than that in the general case, because the general fibre of the projection from this linear subspace is a smooth elliptic curve, and the following result was proved in [27].

Theorem 3.2.4. *Suppose that $r = 6$, $k = 2$, $d_1 = 2$, and $d_2 = 4$. Then X is birationally equivalent to an elliptic fibration if and only if X contains a plane.*

It would be of interest to construct an example of a smooth deformation of a birationally superrigid smooth Fano variety into a Fano variety which is not birationally superrigid. Most probably, this is impossible for hypersurfaces; however, despite the above Conjecture 3.2.1, it can happen that there is a smooth complete intersection X such that $\sum_{i=1}^{k} d_i = r$, $r-k>4$, and X is still not birationally superrigid. On the other hand, it follows from Theorem 3.2.2 that a sufficiently general complete intersection X is most probably birationally superrigid if $\sum_{i=1}^{k} d_i = r$. As a pilot step, one can try to show the birational superrigidity of a sufficiently general complete intersection of four quadrics in $\mathbb{P}^8$ or a sufficiently general complete intersection of two quadrics and a cubic in $\mathbb{P}^7$, because there are smooth complete intersections in the corresponding families which are not birationally superrigid.

§ 3.3. Double spaces

Let $\pi\colon X\to\mathbb{P}^n$ be a double cover ramified along an irreducible reduced hypersurface $F\subset\mathbb{P}^n$ of degree $2n$ with at most isolated ordinary singular points[14] with multiplicities at most $2(n-2)$. Let $n\geqslant 4$. Then

$$-K_X\sim\pi^*(\mathcal{O}_{\mathbb{P}^n})(1),$$

and the group $\mathrm{Cl}(X)$ is generated by the divisor $-K_X$. In particular, X is a Fano variety with terminal $\mathbb{Q}$-factorial singularities. It follows from [178] that the number of singular points of X does not exceed the number of points $(a_1,\dots,a_n)\subset\mathbb{Z}^n$ such that $(n-1)^2<\sum_{i=1}^{n} a_i\leqslant n^2$, where $a_i\in(0,2n)$. In particular, this implies that the number of singular points of X does not exceed 1190 and 27237 when n is 4 and 5, respectively. In this section we prove the following result (see [148] and [32]).

Theorem 3.3.1. *The variety X is birationally superrigid.*

Example 3.3.2. Let $n=2k$ for $k\in\mathbb{N}$ and let F be a general hypersurface of degree $4k$ containing a linear subspace of dimension k. Then F is nodal and has $(4k-1)^k$ singular points. Hence, the variety X is birationally superrigid for $k\geqslant 2$ by Theorem 3.3.1.

Example 3.3.3. Let $n=2k+1$ for $k\in\mathbb{N}$ and let F be a sufficiently general hypersurface of degree $4k+2$ given by an equation of the form

$$\begin{aligned} g^2(x_0,\dots,x_{2k+2}) &= \sum_{i=1}^{k} a_i(x_0,\dots,x_{2k+2})b_i(x_0,\dots,x_{2k+2})\subset\mathbb{P}^n \\ &\cong \mathrm{Proj}(\mathbb{C}[x_0,\dots,x_{2k+2}]), \end{aligned}$$

where g, a_i, and b_i are homogeneous polynomials of degree $2k+1$. Then the hypersurface F is nodal and has $(2k+1)^{2k+1}$ singular points and the hypersurface X is birationally superrigid for $k\geqslant 2$ by Theorem 3.3.1.

Let us now prove Theorem 3.3.1. Suppose that the variety X is not birationally superrigid. We show that this assumption leads to a contradiction. There is a movable log pair (X,M_X) on X such that the boundary M_X is effective, the set $\mathbb{CS}(X,M_X)$ is not empty, and $M_X\sim_{\mathbb{Q}}-rK_X$ for some rational number $r<1$ (see Theorem 1.4.1). Let Z be an element of $\mathbb{CS}(X,M_X)$.

[14] An isolated singular point O of a variety V is said to be ordinary if O is a hypersurface singular point and the projectivization of the tangent cone to V at O is smooth.

Lemma 3.3.4. *The subvariety Z is not a smooth point of X.*

Proof. Suppose that Z is a smooth point of X. Then

$$\operatorname{mult}_Z(M_X^2) > 4$$

by Theorem 1.7.18. Let $H_1, \dots, H_{n-2}$ be sufficiently general divisors in the complete linear system $|\pi^*(\mathcal{O}_{\mathbb{P}^n}(1))|$ that pass through the point Z. Then

$$2 > M_X^2 \cdot H_1 \cdots H_{n-2} \geqslant \operatorname{mult}_Z(M_X^2)\operatorname{mult}_Z(H_1)\cdots\operatorname{mult}_Z(H_{n-2}) > 4,$$

a contradiction.

The next logical step is to prove that Z cannot be a singular point on the variety X. To this end, we need two very special results on global log canonical thresholds. A natural place for these results was in 1.7, because they are applications of Theorem 1.7.6. However, since they have a very narrow domain of application, we present them here without proofs (see [32]).

Proposition 3.3.5. *Let $\tau\colon V \to \mathbb{P}^k$ be a double cover ramified over a smooth hypersurface $S \subset \mathbb{P}^k$ of degree $2d$ such that $2 \leqslant d \leqslant k-1$ and let B_V be an effective boundary on V such that $B_V \sim_{\mathbb{Q}} \tau^*(\lambda H)$ for some positive rational number $\lambda < 1$, where H is a hyperplane. Then $\mathbb{LCS}(V, B_V) = \varnothing$.*

Proposition 3.3.6. *Let $B_{\mathbb{P}^n}$ be an effective boundary on $\mathbb{P}^n$ that is $\mathbb{Q}$-rationally equivalent to λH for some rational number $\lambda < 1$ and some hyperplane $H \subset \mathbb{P}^n$, and let S be a smooth hypersurface in $\mathbb{P}^n$ of degree d such that $2(n-1) \geqslant d \geqslant 2$. Then $\mathbb{LCS}(\mathbb{P}^n, B_{\mathbb{P}^n} + \frac{1}{2}S) = \varnothing$.*

Everything is now ready to prove the following result.

Lemma 3.3.7. *The subvariety Z is not a singular point of the variety X.*

Proof. Suppose that $Z \in \operatorname{Sing} X$. Then $O = \pi(Z)$ is a singular point of the hypersurface $F \subset \mathbb{P}^n$. Two cases are possible, namely, the multiplicity $\operatorname{mult}_O(F)$ can be even or odd. We treat these cases separately. The proof uses Proposition 3.3.5 in the first case and Proposition 3.3.6 in the second case.

We note that the variety X is a hypersurface of the form

$$y^2 = f_{2n}(x_0, \dots, x_n) \subset \mathbb{P}(1^{n+1}, n) \cong \operatorname{Proj}(\mathbb{C}[x_0, \dots, x_n, y])$$

where f_{2n} is a homogeneous polynomial of degree $2n$.

Suppose that $\operatorname{mult}_O(F) = 2m \geqslant 2$ for some $m \in \mathbb{N}$. Then $m \leqslant n-2$, and there is a weighted blow-up

$$\beta\colon U \to \mathbb{P}(1^{n+1}, n)$$

of the point Z with weights $(m, 1^n)$ and such that the proper transform $V \subset U$ of the variety X is non-singular in a neighbourhood of the β-exceptional divisor E. The birational morphism β induces a birational morphism $\alpha\colon V \to X$ with an exceptional divisor $G \subset V$.

We note that $E|_V = G$ and G is a smooth hypersurface in $E \cong \mathbb{P}(1^n, m)$ which can be given by the equation

$$z^2 = g_{2m}(t_1, \dots, t_n) \subset \mathbb{P}(1^n, m) \cong \mathrm{Proj}(\mathbb{C}[t_1, \dots, t_n, z]),$$

where g_{2m} is a homogeneous polynomial of degree $2m$.

Let $\mathrm{mult}_Z(M_X)$ be a positive rational number such that

$$M_V \sim_{\mathbb{Q}} \alpha^*(M_X) - \mathrm{mult}_Z(M_X)G,$$

where $M_V = \alpha^{-1}(M_X)$. Then

$$K_V + M_V \sim_{\mathbb{Q}} \alpha^*(K_X + M_X) + (n - 1 - m - \mathrm{mult}_Z(M_X))G.$$

However, the linear system $|\alpha^*(-K_X) - G|$ is free and determines a morphism ψ such that the diagram

$$\begin{tikzcd} V \arrow[rr, "\psi"] \arrow[d, "\alpha"'] & & \mathbb{P}^{n-1} \\ X \arrow[r, "\pi"'] & \mathbb{P}^n \arrow[ur, dashed, "\chi"'] & \end{tikzcd}$$

is commutative, where χ is the projection from the point O. Let C be a general fibre of ψ. Then

$$0 \leqslant M_V \cdot C = 2(1 - \mathrm{mult}_Z(M_X)) + \alpha^*(K_X + M_X) \cdot C < 2(1 - \mathrm{mult}_Z(M_X)),$$

because the divisor $-(K_X + M_X)$ is ample, and hence $\mathrm{mult}_Z(M_X) < 1$.

If $m = 1$, then the inequality $\mathrm{mult}_Z(M_X) < 1$ contradicts Theorem 1.7.20. Thus, $m > 1$. But it follows from the inequality $n - 1 - m > \mathrm{mult}_Z(M_X)$ that there is a proper subvariety $\Delta \subset G$ that is a centre of canonical singularities of the log pair (V, M_V). Therefore, the set $\mathbb{LCS}(G, M_V|_G)$ is not empty by Theorem 1.7.14, which contradicts Proposition 3.3.5.

Thus, $\mathrm{mult}_O(F) = 2k + 1 \geqslant 3$ for $k \in \mathbb{N}$ such that $k \leqslant n - 3$.

Let $\alpha\colon W \to \mathbb{P}^n$ be a blow-up of the point O, let Λ be an exceptional divisor of the birational morphism α, and let $\widetilde{F} \subset W$ be the proper transform of the hypersurface F. Then $\widetilde{F}$ is smooth in a neighbourhood of Λ. Let $S = \widetilde{F} \cap \Lambda$. Then the variety S is a smooth hypersurface in $\Lambda \cong \mathbb{P}^{n-1}$ of degree $2k + 1$.

Let $\widetilde{\pi}\colon \widetilde{X} \to W$ be a double cover ramified at the effective divisor

$$\widetilde{F} \cup \Lambda \sim 2(\alpha^*(\mathcal{O}_{\mathbb{P}^n}(n)) - k\Lambda),$$

which is smooth outside S. Let $\widetilde{S} = \widetilde{\pi}^{-1}(S)$. Then the variety W is smooth outside the subvariety $\widetilde{S} \subset W$, and the singularities of W along $\widetilde{S}$ are locally isomorphic to $\mathbb{A}_1 \times \mathbb{C}^{n-2}$, that is, the variety W has a two-dimensional ordinary double point along the subvariety $\widetilde{S}$.

Let $\Xi = \widetilde{\pi}^{-1}(\Lambda)$. Then $\Xi \cong \mathbb{P}^{n-1}$, and there is a birational morphism $\xi\colon \widetilde{X} \to X$ contracting the divisor Ξ to the point Z in such a way that the diagram

$$\begin{tikzcd}
V \arrow[r, "\widetilde{\pi}"] \arrow[d, "\xi"'] & W \arrow[dr, dashed, "\alpha"] & \\
X \arrow[rr, "\pi"'] & & \mathbb{P}^n
\end{tikzcd}$$

is commutative. One can readily see that ξ is the restriction to X of the weighted blow-up, with weights $(2k+1, 2^n)$, of the projective space $\mathbb{P}(1^{n+1}, n)$ at the smooth point Z.

Let $\operatorname{mult}_Z(M_X)$ be a positive rational number such that

$$M_{\widetilde{X}} \sim_{\mathbb{Q}} \xi^*(M_X) - \operatorname{mult}_Z(M_X)\Xi,$$

where $M_{\widetilde{X}} = \xi^{-1}(M_X)$. Then

$$K_{\widetilde{X}} + M_{\widetilde{X}} \sim_{\mathbb{Q}} \xi^*(K_X + M_X) + (2(n-1-k) - \operatorname{mult}_Z(M_X))\Xi,$$

but the linear system $|\xi^*(-K_X) - 2\Xi|$ is free and determines a morphism ω such that the diagram

$$\begin{tikzcd}
V \arrow[rr, "\omega"] \arrow[d, "\xi"'] & & \mathbb{P}^{n-1} \\
X \arrow[r, "\pi"'] & \mathbb{P}^n \arrow[ur, dashed, "\chi"'] &
\end{tikzcd}$$

is commutative, where χ is the projection from the point O.

Intersecting the boundary $M_{\widetilde{X}}$ with a sufficiently general fibre of ω, we immediately obtain the inequality $\operatorname{mult}_Z(M_X) < 2$, which implies that $2(n-1-k) > \operatorname{mult}_Z(M_X)$, and hence there is a centre of canonical singularities

$$\nabla \in \mathbb{CS}(\widetilde{X}, M_{\widetilde{X}} - (2(n-1-k) - \operatorname{mult}_Z(M_X))\Xi)$$

such that $\nabla \subset G$. Thus,

$$\nabla \in \mathbb{LCS}(\widetilde{X}, M_{\widetilde{X}} - (2(n-1-k) - \operatorname{mult}_Z(M_X))\Xi + 2\Xi),$$

because 2Ξ is a Cartier divisor. However,

$$\mathbb{LCS}(\widetilde{X}, M_{\widetilde{X}} - (2(n-2-k) - \operatorname{mult}_Z(M_X))\Xi) \subset \mathbb{LCS}(\widetilde{X}, M_{\widetilde{X}} + \Xi),$$

because $2k+1 \leqslant 2(n-2)$. This implies that

$$\mathbb{LCS}(\Xi, \operatorname{Diff}_{\Xi}(M_{\widetilde{X}})) = \mathbb{LCS}(\Xi, M_{\widetilde{X}}|_{\Xi} + \operatorname{Diff}_{\Xi}(0)) \neq \varnothing$$

by Theorem 7.5 in [111]. In this case we have $\mathrm{Diff}_{\Xi}(0)=\frac{1}{2}\widetilde{S}$ (see [113]) and

$$M_{\widetilde{X}}|_{\Xi}\sim_{\mathbb{Q}}-\mathrm{mult}_{Z}(M_X)\Xi|_{\Xi}\sim_{\mathbb{Q}}\frac{\mathrm{mult}_Z(M_X)}{2}H,$$

where H is a hyperplane section of the hypersurface $\Xi\cong\mathbb{P}^{n-1}$. Therefore, we have proved that the set $\mathbb{LCS}(\Xi, M_{\widetilde{X}}|_{\Xi}+\frac{1}{2}\widetilde{S})$ is empty, which contradicts Proposition 3.3.6.

Thus, we have proved that $\dim(Z)>0$. Moreover, it follows immediately from the proof of Lemma 3.3.4 that $\mathrm{codim}(Z\subset X)=2$. Let $H_1,\dots,H_{n-2}$ be general divisors in the linear system $|\pi^*(\mathcal{O}_{\mathbb{P}^n}(1))|$. Then

$$2>M_X^2\cdot H_1\cdot\ldots\cdot H_{n-2}\geqslant \mathrm{mult}_Z^2(M_X)Z\cdot H_1\cdot\ldots\cdot H_{n-2}\geqslant Z\cdot H_1\cdot\ldots\cdot H_{n-2},$$

which implies that $\pi(Z)$ is a linear subspace of $\mathbb{P}^n$ of dimension $n-2$, and the induced map $\pi|_Z\colon Z\to\pi(Z)$ is an isomorphism.

Let $V=\cap_{i=1}^{n-3}H_i$, $C=Z\cap V$, $M_V=M_X|_V$, $\tau=\pi|_V$, and $S=F\cap\pi(V)$. In this case,

- the threefold V is smooth;
- the curve $C\subset V$ is irreducible;
- the boundary M_V is effective and movable;
- the surface $S\subset\mathbb{P}^3$ is smooth and of degree $2n$;
- the morphism $\tau\colon V\to\mathbb{P}^3$ is a double cover;
- the morphism τ is ramified over the surface S;
- the curve $\tau(C)$ is a line in $\mathbb{P}^3$;
- the morphism $\tau|_C$ is am isomorphism;
- the divisor $\tau^*(\mathcal{O}_{\mathbb{P}^3}(1))-M_V$ is ample;
- the equality $\mathrm{mult}_C(M_V)=\mathrm{mult}_Z(M_X)$ holds.

Suppose that $\tau(C)\not\subset S$. Then there is an irreducible curve $\widetilde{C}\subset V$ which differs from the curve C and is such that $\tau(C)=\tau(\widetilde{C})$. Let us take a general divisor $D\in|\tau^*(\mathcal{O}_{\mathbb{P}^3}(1))|$ passing through the curve C. Then D is a smooth surface and C and $\widetilde{C}$ are smooth rational curves on D. Let $M_D=M_V|_D$. Then

$$M_D=\mathrm{mult}_C(M_V)C+\mathrm{mult}_{\widetilde{C}}(M_V)\widetilde{C}+\Delta,$$

where Δ is a movable boundary on D. On the other hand, we have the equivalence $M_V\sim_{\mathbb{Q}}rD$ for some rational number $r<1$. Hence, we have the equivalence

$$(r-\mathrm{mult}_{\widetilde{C}}(M_V))\widetilde{C}\sim_{\mathbb{Q}}(\mathrm{mult}_C(M_V)-r)C+\Delta,$$

and therefore $\mathrm{mult}_{\widetilde{C}}(M_V)\geqslant r$, because $\widetilde{C}^2<0$. Suppose now that H is a sufficiently general divisor in the linear system $|\tau^*(\mathcal{O}_{\mathbb{P}^3}(1))|$. Then

$$2r^2=M_V^2\cdot H\geqslant \mathrm{mult}_C^2(M_V)+\mathrm{mult}_{\widetilde{C}}^2(M_V)\geqslant 1+r^2,$$

which is impossible by virtue of the inequality $r<1$. We have thus proved that $\tau(C)\subset S$.

Let O be a sufficiently general point on the line $\tau(C)$ and let T be a hyperplane in $\mathbb{P}^3$ tangent to S at the point O. Consider a sufficiently general line $L \subset T$ passing through O. Let $\widehat{L} = \tau^{-1}(L)$. By construction, the curve $\widehat{L}$ is irreducible and is singular at the point $\widehat{O} = \tau^{-1}(O)$. This implies that $\widehat{L} \subset \operatorname{Supp}(M_V)$, because otherwise we would have the incompatible inequalities

$$2 > \widehat{L} \cdot M_V \geqslant \operatorname{mult}_{\widehat{O}}(\widehat{L}) \operatorname{mult}_C(M_V) \geqslant 2.$$

On the other hand, $\widehat{L}$ sweeps out a divisor on V as the line L varies in the hyperplane T, which contradicts the condition $\widehat{L} \subset \operatorname{Supp}(M_V)$ and the fact that the boundary M_V is movable. This contradiction proves Theorem 3.3.1.

§ 3.4. Triple spaces

Let $\pi\colon X \to \mathbb{P}^{2n}$ be a cyclic triple cover (see [131]) ramified over an irreducible and reduced hypersurface $S \subset \mathbb{P}^{2n}$ of degree $3n$ such that S has isolated ordinary double points and $n \geqslant 2$. In this case X is a Fano variety with terminal $\mathbb{Q}$-factorial singularities, and the groups $\operatorname{Pic}(X)$ and $\operatorname{Cl}(X)$ are generated by the anticanonical divisor $-K_X \sim \pi^*(\mathcal{O}_{\mathbb{P}^{2n}}(1))$. The following assertion holds (see [29]).

Theorem 3.4.1. *The variety X is birationally superrigid.*

Let $\xi\colon V \to \mathbb{P}^k$ be a cyclic triple cover ramified along a hypersurface of degree $3n$ all of whose possible singularities are isolated ordinary double or triple points, and let $k \geqslant 3$. In this case the variety V is not birationally rigid for $k < 2n$, because V contains pencils of varieties of Kodaira dimension $-\infty$. On the other hand, V has non-negative Kodaira dimension if $k > 2n$. In particular, Theorem 3.4.1 describes all birationally superrigid smooth cyclic triple spaces.

Example 3.4.2. Let X be a hypersurface in $\mathbb{P}(1^{2n+1}, n)$ of degree $3n$,

$$y^3 = \sum_{i=0}^{2n} x_i^{3n} \subset \mathbb{P}(1^{2n+1}, n) \cong \operatorname{Proj}(\mathbb{C}[x_0, \dots, x_{2n}, y]),$$

and let $n \geqslant 2$. Then the natural projection $\pi\colon X \to \mathbb{P}^{2n} \cong \operatorname{Proj}(\mathbb{C}[x_0, \dots, x_{2n}])$ is a cyclic triple cover ramified along a smooth hypersurface $\sum_{i=0}^{2n} x_i^{3n} = 0$. Moreover, the variety X is birationally superrigid by Theorem 3.4.1, and

$$\operatorname{Bir}(X) = \operatorname{Aut}(X) \cong \mathbb{Z}_3 \oplus \operatorname{Aut}\Bigl(\sum_{i=0}^{2n} x_i^{3n} = 0\Bigr) \cong \mathbb{Z}_3 \oplus (\mathbb{Z}_{3n}^{2n} \rtimes \mathrm{S}_{2n+1}),$$

where S_{2n+1} stands for the corresponding symmetric group (see [165], [117]).

Let us prove Theorem 3.4.1. Suppose that X is not birationally superrigid. We claim that this assumption leads to a contradiction. There is a movable log pair (X, M_X) such that the boundary M_X is effective, the set $\mathbb{CS}(X, M_X)$ is non-empty, and the divisor $-(K_X + M_X)$ is ample (see Theorem 1.4.1). In particular, the equivalence $M_X \sim_{\mathbb{Q}} -rK_X$ holds for some rational number $1 > r \gg 0$. Let Z be an element of the set $\mathbb{CS}(X, M_X)$ of greatest dimension.

Lemma 3.4.3. *The variety Z is not a smooth point on X.*

Proof. Suppose that Z is a smooth point of X. Then $\mathrm{mult}_Z(M_X^2) > 4$ by Theorem 1.7.18. Consider general divisors $H_1, \dots, H_{2n-2}$ in the linear system $|\pi^*(\mathcal{O}_{\mathbb{P}^{2n}}(1))|$ that pass through the smooth point Z. Then $3 > M_X^2 \cdot H_1 \cdots H_{2n-2} > 4$, a contradiction.

Lemma 3.4.4. *The variety Z is not a singular point on X.*

Proof. Suppose that $Z \subset X$ is a singular point on X. Then $\pi(Z)$ is an ordinary double point of the hypersurface $S \subset \mathbb{P}^{2n}$. Let $\alpha\colon V \to X$ be a blow-up of the point Z and let $G \subset V$ be an α-exceptional divisor. Then V is smooth and G is a quadric of dimension $2n-1$ having one singular point $O \in G$, that is, the variety $G \subset V$ is a quadric cone with the vertex $O \in V$.

Let $M_V = \alpha^{-1}(M_X)$. Then we have the relation

$$M_V \sim_{\mathbb{Q}} \alpha^*(M_X) - \mathrm{mult}_Z(M_X)G,$$

where $\mathrm{mult}_Z(M_X) > 1$ is a rational number (see Proposition 1.7.22).

Let $H = \alpha^*(-K_X)$ and consider the linear system $|H - G|$. By construction, the map $\phi_{|H-G|}$ coincides with the map $\gamma \circ \pi \circ \alpha$, where $\gamma\colon \mathbb{P}^{2n} \dashrightarrow \mathbb{P}^{2n-1}$ is the projection from the point $\pi(Z)$. The linear system $|H - G|$ is not free: namely, its base locus is exactly the vertex O of the quadric cone G, the blow-up of the point O resolves the indeterminacies of the cone, and the proper transform of G is contracted to a smooth quadric of dimension $2n-2$.

Let C be a general curve contained in the fibres of the map $\phi_{|H-G|}$. Then C is irreducible and reduced, and the curve $\pi \circ \alpha(C)$ is a line passing through the point $\pi(Z)$. Moreover, we have

$$C \cdot G = 2, \qquad C \cdot (H - G) = 1,$$

and the point O belongs to the curve C. Intersecting the boundary M_V with the curve C, we obtain the inequalities

$$1 > 3 - 2\,\mathrm{mult}_Z(M_X) > M_V \cdot C \geqslant \mathrm{mult}_O(M_V),$$

which implies that $\mathrm{mult}_Z(M_X) \leqslant \frac{3}{2}$ and $\mathrm{mult}_O(M_V) < 1$. The equivalence

$$K_V + M_V \sim_{\mathbb{Q}} \alpha^*(K_X + M_X) + (2n - 2 - \mathrm{mult}_Z(M_X))G$$

and the inequality $\mathrm{mult}_Z(M_X) \leqslant \frac{3}{2}$ imply the existence of a subvariety $Y \subset G$ such that

$$Y \in \mathbb{CS}(V, M_V - (2n - 2 - \mathrm{mult}_Z(M_X))G)$$

and, in particular, $\dim(Y) \leqslant 2n-2$, $\mathrm{mult}_Y(M_V) > 1$, and $Y \in \mathbb{CS}(V, M_V)$.

Let $\dim(Y) = 2n-2$. If $O \in Y$, then

$$1 > \mathrm{mult}_O(M_V) \geqslant \mathrm{mult}_Y(M_V) > 1,$$

which shows that $O \notin Y$. Let L be a general ruling of the cone G. Then

$$\frac{3}{2} \geqslant \mathrm{mult}_Z(M_X) = M_V \cdot L \geqslant \mathrm{mult}_Y(M_V) L \cdot Y,$$

where $L \cdot Y$ is treated as the intersection on G. Hence, $L \cdot Y = 1$, and the variety Y is a hyperplane section of the quadric $G \subset \mathbb{P}^{2n}$. We note that

$$Y \in \mathbb{LCS}(V, M_V - (2n-3-\mathrm{mult}_Z(M_X))G),$$

and hence at a general point of the subvariety $Y \subset V$ we can apply Theorem 1.7.17 to the log pair $(V, M_V - (2n-3-\mathrm{mult}_Z(M_X))G)$. This gives the inequality

$$\mathrm{mult}_Y(M_V^2) \geqslant 4(2n-2-\mathrm{mult}_Z(M_X)) \geqslant 2,$$

because $\mathrm{mult}_Z(M_X) \leqslant \frac{3}{2}$ and $n \geqslant 2$. Let $H_1, \dots, H_{2n-2}$ be general divisors in the linear system $|H - G|$. Then

$$1 > 3 - 2\,\mathrm{mult}_Z^2(M_X) > H_1 \cdot H_2 \cdots H_{2n-2} \cdot M_V^2 \geqslant \mathrm{mult}_Y(M_V^2)(H-G)^{2n-2} \cdot Y \geqslant 2,$$

a contradiction. Thus, we have proved that $\dim(Y) < 2n-2$.

The inequality $\mathrm{mult}_O(M_V) < 1$ implies that $O \notin Y$. It follows from Theorem 1.7.18 that $\mathrm{mult}_P(M_V^2) > 4$ for any general point $P \in Z$. Let us now consider the linear subsystem $\mathcal{D} \subset |H - G|$ formed by the divisors containing the point P. In this case the base locus of the linear system $\mathcal{D}$ consists of the following curves:

- a ruling L_P of the quadric cone G passing through P,
- a (possibly reducible) curve C_P such that its image $\pi \circ \alpha(C_P)$ is a line passing through the point $\pi(Z)$.

Consider $2n-2$ general divisors $D_1, \dots, D_{2n-2}$ in the linear system $\mathcal{D}$ and an effective one-dimensional cycle $T = H_1 \cdots H_{2n-3} \cdot M_V^2$. Then $\mathrm{mult}_P(T) > 4$. However, the divisor H_{2n-2} can contain components of the cycle T if the curve L_P or one of the components of C_P is contained in $\mathrm{Supp}(T)$.

Suppose that the curve C_P is irreducible. Let

$$T = \mu L_P + \lambda C_P + \Gamma,$$

where μ and λ are non-negative rational numbers and Γ is an effective one-dimensional cycle whose support does not contain the curves L_P and C_P. Then

$$\mathrm{mult}_P(\Gamma) > 4 - \mathrm{mult}_P(L_P)\mu - \mathrm{mult}_P(C_P)\lambda = 4 - \mu - \mathrm{mult}_P(C_P)\lambda \geqslant 4 - \mu - 3\lambda$$

because $\mathrm{mult}_P(C_P) \leqslant 3$, and we have the inequalities

$$3 - 2\,\mathrm{mult}_Z^2(M_X) - \mu > \Gamma \cdot H_{2n-2} \geqslant \mathrm{mult}_P(\Gamma) > 4 - \mu - 3\lambda$$

because $C_P \cdot H_{2n-2} = 0$, and hence $\lambda > 1$. Intersecting the cycle T with a general divisor in the linear system $|\alpha^*(-K_X)|$, we immediately arrive at a contradiction.

Suppose now that the curve C_P is reducible. Then

$$C_P = C_1 + C_2 + C_3,$$

where C_i is a non-singular rational curve such that $\pi \circ \alpha(C_P)$ is a line, the restriction $\pi \circ \alpha|_{C_i}$ is an isomorphism, $-K_X \cdot \alpha(C_i) = 1$, and $C_i \neq C_j$ for $i \neq j$. We write

$$T = \mu L_P + \sum_{i=1}^{3} \lambda_i C_i + \Gamma,$$

where μ and λ_i are non-negative rational numbers and Γ is an effective one-dimensional cycle whose support does not contain the curves L_P and C_i. Intersecting the cycle Γ with the divisor H_{2n-2}, we immediately obtain the inequality $\sum_{i=1}^{3} \lambda_i > 1$. Intersecting the cycle T with a sufficiently general divisor in the linear system $|\alpha^*(-K_X)|$, we arrive at a contradiction.

Thus, we have proved that $\dim(Z) > 0$. Moreover, it follows immediately from the proof of Lemma 3.4.3 that the inequality $\operatorname{codim}(Z \subset X) > 2$ is impossible. We thus have $\operatorname{codim}(Z \subset X) = 2$. On the other hand, it follows from the equality $K_X^{2n} = 3$ and from the inequality $\operatorname{mult}_Z(M_X) \geqslant 1$ that $K_X^{2n-2} \cdot Z \leqslant 2$.

Lemma 3.4.5. *The equality* $n = 2$ *holds*, *that is*, $\dim(X) = 4$.

Proof. Suppose that $n > 2$. We take a sufficiently general divisor V in the free linear system $|-K_X|$. In this case V is a smooth hypersurface of degree $3n$ in the weighted projective space $\mathbb{P}(1^{2n}, n)$ of dimension $2n - 1 \geqslant 5$. Hence, the homology group $H_{4n-6}(V, \mathbb{C})$ is one-dimensional (see [78], Theorem 7.2, and [54], § 4).

We show that the subvariety $Y = Z \cap V \subset V$ cannot generate the homology group $H_{4n-6}(V, \mathbb{C})$. Let $Y \equiv \lambda D^2$ in $H_{4n-6}(V, \mathbb{C})$ for some $\lambda \in \mathbb{C}$, where $D = -K_X|_V$. We note that $\pi(Z) \not\subset S$ by the Lefschetz theorem. Thus, the image $\pi(Z)$ is either a linear subspace of dimension $2n - 2$ or a quadric of dimension $2n - 2$. Moreover, the subvariety $\pi^{-1}(\pi(Z))$ splits into three irreducible subvarieties conjugate by the action of the group $\mathbb{Z}_3$ on the variety X (this action transposes the fibres of π), and therefore $\lambda = \frac{\alpha}{3}$, where $\alpha = K_X^{2n-2} \cdot Z$. We have the equality

$$\alpha = Y \cdot D^{2n-3} = \lambda^{2-n} D \cdot Y^{n-2},$$

which yields $D \cdot Y^{n-2} = \frac{\alpha^{n-1}}{3} \notin \mathbb{Z}$, a contradiction.

We can now arrive at the desired contradiction in just the same way as in the proof of Theorem 3.3.1. Therefore, Theorem 3.4.1 is proved. It is natural to try to generalize the assertion of Theorem 3.4.1 as follows: remove the assumption that the triple cover is cyclic, assume that the hypersurface S has ordinary isolated singularities of multiplicity at most $3n - 3$ (see [29]), and admit degrees of the cover π that do not exceed 8.

§ 3.5. Cyclic covers

Let $\psi\colon X \to V \subset \mathbb{P}^n$ be a cyclic cover of degree $d \geqslant 2$ ramified along a smooth divisor $R \subset V$ such that the variety V is a hypersurface of degree m. Let $n \geqslant 5$. In this case the group $\mathrm{Pic}(V)$ is generated by the divisor $\psi^*(\mathcal{O}_{\mathbb{P}^n}(1)|_V)$, and

$$-K_X \sim \psi^*(\mathcal{O}_{\mathbb{P}^n}(m-n-1+(d-1)k)|_V),$$

where k is a positive integer such that $R \sim \mathcal{O}_{\mathbb{P}^n}(dk)|_V$.

Remark 3.5.1. We are interested in finding conditions for X to be a birationally superrigid Fano variety. Hence, we can assume that $m+(d-1)k = n$.

The birational superrigidity of X follows from Theorem 3.3.1 if $m = 1$ and $d = 2$. The birational superrigidity of X was proved in Theorem 2.3.5 for $m = 2$ and $d = 2$. Moreover, the following result was proved in [151] and [158] by using the technique of hypertangent linear systems.

Theorem 3.5.2. *Suppose that X is sufficiently general and either $d = 2$ or $n \geqslant 6$. Then X is birationally superrigid.*

Let us show that Theorem 1.7.19 implies the following result (see [34]).

Theorem 3.5.3. *Suppose that $d = 2$, $n \geqslant 8$, and either $m = 3$ or $m = 4$. Then X is birationally superrigid.*

Proof. Suppose that the variety X is not birationally superrigid. Then it follows from Theorem 1.4.1 that there is a linear system $\mathcal{M}$ on X that has no fixed components and is such that the singularities of the log pair $(X, \frac{1}{r}\mathcal{M})$ are not canonical, where r is a positive integer such that $\mathcal{M} \sim -rK_X$. The set $\mathbb{CS}(X, \frac{\mu}{r}\mathcal{M})$ contains a subvariety $Z \subset X$ for some rational number $\mu < 1$. We can assume that Z is a subvariety of the largest dimension having such a property. In particular, $\mathrm{mult}_Z(S) > r$ for every divisor S in $\mathcal{M}$, and we have $\dim(Z) \leqslant \dim(X) - 2 = n-3$.

Suppose that Z is a point. Let S_1 and S_2 be sufficiently general divisors in $\mathcal{M}$, let $f\colon U \to X$ be a blow-up of the variety X at the point Z, and let E be an exceptional divisor of the morphism f. Then it follows from Theorem 1.7.19 that there is a linear subspace $\Pi \subset E$ of codimension 2 such that $\mathrm{mult}_Z(S_1 \cdot S_2 \cdot D) > 8r^2$ for every divisor $D \in |-K_X|$ with the following properties: D contains the point Z and is smooth at Z, the proper transform $f^{-1}(D)$ of D contains the subvariety $\Pi \subset U$, and D contains no subvarieties of X of codimension 2 that are contained in the base locus of the linear system $\mathcal{M}$.

Let us consider a linear subsystem $\mathcal{H} \subset |\mathcal{O}_{\mathbb{P}^n}(1)|_V|$ such that

$$H \in \mathcal{H} \iff \Pi \subset (\psi \circ f)^{-1}(H).$$

In this case there is a linear subspace $\Sigma \subset \mathbb{P}^n$ of dimension $n-3$ such that all the divisors in the linear system $\mathcal{H}$ are cut out on the hypersurface V by hyperplanes in $\mathbb{P}^n$ that contain Σ. In particular, the base locus of the linear system $\mathcal{H}$ consists of the intersection $\Sigma \cap V$; however, the Lefschetz theorem implies that $\Sigma \not\subset V$, and hence $\dim(\Sigma \cap V) = n-4$.

Let H be a general divisor in $\mathcal{H}$ and let $D = \psi^{-1}(H) \in |-K_X|$. In this case D contains the point Z and is smooth at Z and the divisor $f^{-1}(D)$ contains the subvariety $\Pi \subset U$. Suppose that D contains a subvariety $\Gamma \subset X$ of codimension 2 which is contained in the base locus of $\mathcal{M}$. Then

$$\dim(\psi(\Gamma)) = n - 3.$$

However, $\psi(\Gamma) \subset \Sigma \cap V$ and $\dim(\Sigma \cap V) = n - 4$, a contradiction. Thus, we have proved that the divisor D contains no subvarieties of X of codimension 2 that are contained in the base locus of the linear system $\mathcal{M}$. Let $H_1, H_2, \dots, H_t$ be general divisors in $|-K_X|$ passing through the point Z, where $t = \dim(X) - 3$. Then

$$8r^2 \geqslant 2mr^2 = H_1 \cdot \ldots \cdot H_t \cdot S_1 \cdot S_2 \cdot D \geqslant \operatorname{mult}_Z(S_1 \cdot S_2 \cdot D) > 8r^2,$$

a contradiction. Hence, $\dim(Z) \neq 0$.

Suppose that $\dim(Z) \leqslant \dim(X) - 5$ and that $H_1, H_2, \dots, H_t$ are general hyperplane sections of the hypersurface V, where $t = \dim(Z) > 0$. We write

$$\overline{V} = \bigcap_{i=1}^{t} H_i, \qquad \overline{X} = \psi^{-1}(\overline{V}), \qquad \overline{\psi} = \psi|_{\overline{X}} \colon \overline{X} \to \overline{V}$$

and $\overline{\mathcal{M}} = \mathcal{M}|_{\overline{X}}$. Then $\overline{V} \subset \mathbb{P}^{n-t}$ is a smooth hypersurface, the morphism $\overline{\psi}$ is a double cover ramified along a smooth divisor $R \cap \overline{V}$, the linear system $\mathcal{M}$ has no fixed components, and $\overline{V}$ contains no linear subspaces of $\mathbb{P}^{n-t}$ of dimension $n-t-3$ by the Lefschetz theorem, because $n - t \geqslant 6$. Let P be a point in $Z \cap \overline{X}$. Then P is a centre of canonical singularities of the log pair $\mathbb{CS}(\overline{X}, \frac{1}{r}\overline{\mathcal{M}})$. Moreover, the above arguments lead to a contradiction of Theorem 1.7.19. Hence, $\dim(Z) \geqslant \dim(X) - 4$.

Suppose that $\dim(Z) = \dim(X) - 2$. Let S_1 and S_2 be general divisors in $\mathcal{M}$ and let $H_1, H_2, \dots, H_{n-3}$ be general divisors in the linear system $|-K_X|$. Then

$$2mr^2 = H_1 \cdot \ldots \cdot H_{n-3} \cdot S_1 \cdot S_2 > r^2(-K_X)^{n-3} \cdot Z,$$

because $\operatorname{mult}_Z(\mathcal{M}) > r$. Therefore, $(-K_X)^{n-3} \cdot Z < 2m$, where

$$(-K_X)^{n-3} \cdot Z = \begin{cases} \deg(\psi(Z) \subset \mathbb{P}^n) & \text{if } \psi|_Z \text{ is birational,} \\ 2\deg(\psi(Z) \subset \mathbb{P}^n) & \text{otherwise,} \end{cases}$$

and, by the Lefschetz theorem, the number $\deg(\psi(Z))$ must be divisible by a power of m. Hence, the morphism $\psi|_Z$ is birational and $\deg(\psi(Z)) = m$. This immediately implies that either the scheme-theoretic intersection $\psi(Z) \cap R$ is singular at every point or $\psi(Z) \subset R$. Therefore, applying the Lefschetz theorem to the smooth complete intersection $R \subset \mathbb{P}^n$, we immediately arrive at a contradiction. This proves the inequality $\dim(Z) \geqslant \dim(X) - 4 \geqslant 3$.

Let S be a general divisor in $\mathcal{M}$, let $\widehat{S} = \psi(S \cap R)$, and let $\widehat{Z} = \psi(Z \cap R)$. Then $\widehat{S}$ is a divisor on the complete intersection $R \subset \mathbb{P}^n$. However, $\operatorname{mult}_{\widehat{Z}}(\widehat{S}) > r$ and $\widehat{S} \sim \mathcal{O}_{\mathbb{P}^n}(r)|_R$, which contradicts Proposition 1.3.12.

The arguments used in the proof of Theorem 3.5.3 imply the following result.

Theorem 3.5.4. *Let $d = 3$, $m = 2$, and $n \geqslant 8$. Then X is birationally superrigid.*

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