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Log canonical thresholds of smooth Fano threefolds

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Abstract. The complex singularity exponent is a local invariant of a holomorphic function determined by the integrability of fractional powers of the function. The log canonical thresholds of effective $\mathbb{Q}$-divisors on normal algebraic varieties are algebraic counterparts of complex singularity exponents. For a Fano variety, these invariants have global analogues. In the former case, it is the so-called α-invariant of Tian; in the latter case, it is the global log canonical threshold of the Fano variety, which is the infimum of log canonical thresholds of all effective $\mathbb{Q}$-divisors numerically equivalent to the anticanonical divisor. An appendix to this paper contains a proof that the global log canonical threshold of a smooth Fano variety coincides with its α-invariant of Tian. The purpose of the paper is to compute the global log canonical thresholds of smooth Fano threefolds (altogether, there are 105 deformation families of such threefolds). The global log canonical thresholds are computed for every smooth threefold in 64 deformation families, and the global log canonical thresholds are computed for a general threefold in 20 deformation families. Some bounds for the global log canonical thresholds are computed for 14 deformation families. Appendix A is due to J.-P. Demailly.

Contents

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1. Introduction

The multiplicity of a polynomial $\varphi \in \mathbb{C}[z_1, z_2, \dots, z_n]$ at the origin $O \in \mathbb{C}^n$ is the number

$$\min\left\{m \in \mathbb{Z}_{\geqslant 0} \;\middle|\; \frac{\partial^m \varphi(z_1, z_2, \dots, z_n)}{\partial^{m_1} z_1\, \partial^{m_2} z_2 \cdots \partial^{m_n} z_n}(O) \neq 0\right\} \in \mathbb{Z}_{\geqslant 0} \cup \{+\infty\}.$$

There is a similar but more subtle invariant $c_0(\varphi) \in \mathbb{Q} \cup \{+\infty\}$ defined by the formula

$$c_0(\varphi) = \sup\{\varepsilon \in \mathbb{Q} \mid \text{the function } |\varphi|^{-2\varepsilon} \text{is integrable in a neighbourhood of } O \in \mathbb{C}^n\},$$

which is called the local singularity exponent of the polynomial φ at the point O.

Example 1.1. Let $m_1, m_2, \dots, m_n$ be positive integers. Then

$$\min\left(1, \sum_{i=1}^{n} \frac{1}{m_i}\right) = c_0\left(\sum_{i=1}^{n} z_i^{m_i}\right) \geqslant c_0\left(\prod_{i=1}^{n} z_i^{m_i}\right) = \min\left(\frac{1}{m_1}, \frac{1}{m_2}, \dots, \frac{1}{m_n}\right).$$

Let Xbe a variety[1] with at most log canonical singularities (see [1]), let $Z \subseteq X$ be a non-empty closed subvariety, and let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on the variety X. Then the number

$$\operatorname{lct}_Z(X, D) = \sup\{\lambda \in \mathbb{Q} \mid \text{the log pair } (X, \lambda D) \text{ is log canonical along } Z\} \in \mathbb{Q} \cup \{+\infty\}$$

is called the log canonical threshold of the divisor D along Z. In follows from [1] that

$$\operatorname{lct}_O\big(\mathbb{C}^n, (\varphi = 0)\big) = c_0(\varphi),$$

so $\operatorname{lct}_Z(X, D)$ is a generalization of the quantity $c_0(\varphi)$. We have

$$\begin{aligned}\operatorname{lct}(X, D) &= \inf\{\operatorname{lct}_P(X, D) \mid P \in X\}\\ &= \sup\{\lambda \in \mathbb{Q} \mid \text{the log pair } (X, \lambda D) \text{ is log canonical}\},\end{aligned}$$

where we have set $\operatorname{lct}(X, D) = \operatorname{lct}_X(X, D)$.

Let X be a Fano variety with at most log terminal singularities (see [2]).

Definition 1.2. The global log canonical threshold of the Fano variety X is the quantity

$$\operatorname{lct}(X) = \inf\{\operatorname{lct}(X, D) \mid D \text{ is an effective } \mathbb{Q}\text{-divisor on } X \text{ such that } D \sim_{\mathbb{Q}} -K_X\} \geqslant 0.$$

The number $\operatorname{lct}(X)$ is an algebraic counterpart of the so-called α-invariant of a variety X introduced in [3]. It can easily be seen that

$$\operatorname{lct}(X) = \sup\{\varepsilon \in \mathbb{Q} \mid \text{the log pair } (X, n^{-1}\varepsilon D) \text{ is log canonical for each divisor } D \in |-nK_X| \text{ for all } n \in \mathbb{Z}_{>0}\}.$$

[1] All varieties are assumed to be projective and normal and are defined over the field $\mathbb{C}$.

The group $\mathrm{Pic}(X)$ is torsion free, because X is rationally connected (see [4]). Hence,

$$\mathrm{lct}(X) = \sup\{\lambda \in \mathbb{Q} \mid \text{the log pair } (X, \lambda D) \text{ is log canonical} \\ \text{for every effective } \mathbb{Q}\text{-divisor } D \equiv -K_X\}.$$

Example 1.3. Let X be a smooth hypersurface in $\mathbb{P}^n$ of degree $m < n$. Then $\mathrm{lct}(X) = 1/(n+1-m)$ (see [5]). In particular, $\mathrm{lct}(\mathbb{P}^n) = 1/(n+1)$.

Example 1.4. Let X be a rational homogeneous space such that $\mathrm{Pic}(X) = \mathbb{Z}[D]$, where D is an ample divisor. Then $\mathrm{lct}(X) = 1/r$ (see [6]), where $-K_X \sim rD$ and $r \in \mathbb{Z}_{>0}$.

In general the number $\mathrm{lct}(X)$ depends on small deformations of the variety X.

Example 1.5. Let X be a smooth hypersurface in $\mathbb{P}(1,1,1,1,3)$ of degree 6. Then

$$\mathrm{lct}(X) \in \left\{\frac{5}{6}, \frac{43}{50}, \frac{13}{15}, \frac{33}{38}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10}, \frac{11}{12}, \frac{13}{14}, \frac{15}{16}, \frac{17}{18}, \frac{19}{20}, \frac{21}{22}, \frac{29}{30}, 1\right\}$$

(see [7] and [8]). All these value of $\mathrm{lct}(X)$ are attained.

Example 1.6. Let X be a smooth hypersurface in $\mathbb{P}^n$ of degree $n \geqslant 2$. Then

$$1 \geqslant \mathrm{lct}(X) \geqslant 1 - 1/n$$

(see [5]). It follows from [7] and [8] that

$$\mathrm{lct}(X) \geqslant \begin{cases} 1 & \text{if } n \geqslant 6, \\ 22/25 & \text{if } n = 5, \\ 16/21 & \text{if } n = 4, \\ 3/4 & \text{if } n = 3, \end{cases}$$

whenever X is general. On the other hand, $\mathrm{lct}(X) = 1 - 1/n$ if X contains a cone of dimension $n-2$.

Example 1.7. Let X be a quasi-smooth hypersurface in $\mathbb{P}(1, a_1, \dots, a_4)$ of degree $\sum_{i=1}^4 a_i$ such that X has at most terminal singularities; suppose $a_1 \leqslant a_2 \leqslant a_3 \leqslant a_4$. Then $-K_X \sim \mathscr{O}_{\mathbb{P}(1,a_1,\dots,a_4)}(1)|_X$, and there are 95 possibilities for the quadruple (a_1, a_2, a_3, a_4) (see [9], [10]). If X is general, then

$$1 \geqslant \mathrm{lct}(X) \geqslant \begin{cases} 16/21 & \text{if } a_1 = a_2 = a_3 = a_4 = 1, \\ 7/9 & \text{if } (a_1, a_2, a_3, a_4) = (1,1,1,2), \\ 4/5 & \text{if } (a_1, a_2, a_3, a_4) = (1,1,2,2), \\ 6/7 & \text{if } (a_1, a_2, a_3, a_4) = (1,1,2,3), \\ 1 & \text{in all other cases} \end{cases}$$

(see [11], [8], [12]). The global log canonical threshold of the hypersurface

$$w^2 = t^3 + z^9 + y^{18} + x^{18} \subset \mathbb{P}(1,1,2,6,9) \cong \mathrm{Proj}(\mathbb{C}[x,y,z,t,w])$$

is equal to 17/18 (see [11]), where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 2$, $\mathrm{wt}(t) = 6$, $\mathrm{wt}(w) = 9$.

Example 1.8. It follows from Lemma 5.1 that $\operatorname{lct}(\mathbb{P}(a_0,a_1,\dots,a_n)) = a_0/\sum_{i=0}^n a_i$, provided that $\mathbb{P}(a_0,a_1,\dots,a_n)$ is well formed (see [9]) and $a_0 \leqslant a_1 \leqslant \dots \leqslant a_n$.

Example 1.9. Let X be a smooth hypersurface in $\mathbb{P}(1^{n+1}, d)$ of degree $2d$. Then $\operatorname{lct}(X) = 1/(n+1-d)$ for $2 \leqslant d \leqslant n-1$ (see [13], Proposition 20).

Example 1.10. Let X be a smooth del Pezzo surface. It follows from [14] that

$$\operatorname{lct}(X) = \begin{cases} 1 & \text{if } K_X^2 = 1 \text{ and } |-K_X| \text{ contains no cuspidal curves,} \\ 5/6 & \text{if } K_X^2 = 1 \text{ and } |-K_X| \text{ contains a cuspidal curve,} \\ 5/6 & \text{if } K_X^2 = 2 \text{ and } |-K_X| \text{ contains no tacnodal curves,} \\ 3/4 & \text{if } K_X^2 = 2 \text{ and } |-K_X| \text{ contains a tacnodal curve,} \\ 3/4 & \text{if } X \text{ is a cubic in } \mathbb{P}^3 \text{ without} \\ & \qquad \text{Eckardt points,} \\ 2/3 & \text{if } X \text{ either is a qubic in } \mathbb{P}^3 \text{ with an Eckardt point,} \\ & \qquad \text{or } K_X^2 = 4, \\ 1/2 & \text{if } X \cong \mathbb{P}^1 \times \mathbb{P}^1 \text{ or } K_X^2 \in \{5,6\}, \\ 1/3 & \text{in all other cases.} \end{cases}$$

It would be interesting to compute the global log canonical thresholds of del Pezzo surfaces with at most canonical singularities and with Picard rank 1 (see [15]).

Example 1.11. Let X be a singular cubic surface in $\mathbb{P}^3$ with at most canonical singularities. The singularities of X are classified in [16]. It follows from [17] that

$$\operatorname{lct}(X) = \begin{cases} 2/3 & \text{if } \operatorname{Sing}(X) = \{\mathbb{A}_1\}, \\ 1/3 & \text{if } \operatorname{Sing}(X) \supseteq \{\mathbb{A}_4\} \text{ or } \operatorname{Sing}(X) = \{\mathbb{D}_4\} \\ & \qquad \text{or } \operatorname{Sing}(X) \supseteq \{\mathbb{A}_2, \mathbb{A}_2\}, \\ 1/4 & \text{if } \operatorname{Sing}(X) \supseteq \{\mathbb{A}_5\} \text{ or } \operatorname{Sing}(X) = \{\mathbb{D}_5\}, \\ 1/6 & \text{if } \operatorname{Sing}(X) = \{\mathbb{E}_6\}, \\ 1/2 & \text{in all other cases.} \end{cases}$$

It is not yet known whether $\operatorname{lct}(X)$ is rational[2] (cf. Question 1 in [18]).

Conjecture 1.12. *There is an effective* $\mathbb{Q}$*-divisor* $D \sim_{\mathbb{Q}} -K_X$ *on the variety* X *such that* $\operatorname{lct}(X) = \operatorname{lct}(X, D) \in \mathbb{Q}$.

Let $G \subset \operatorname{Aut}(X)$ be an arbitrary subgroup.

Definition 1.13. The *global* G*-invariant log canonical threshold* of a Fano variety X is a number (or $+\infty$) defined by the following equality:

$$\operatorname{lct}(X,G) = \sup\{\lambda \in \mathbb{Q} \mid \text{ the log pair } (X, n^{-1}\varepsilon\mathscr{D}) \text{ has log canonical singularities for every } G\text{-invariant linear subsystem } \mathscr{D} \subset |-nK_X|,\ n \in \mathbb{Z}_{>0}\}.$$

[2] It is not even known whether $\operatorname{lct}(X)$ is rational if X is a del Pezzo surface with quotient singularities.

Remark 1.14. In Definitions 1.2 and 1.13 we only need to assume that $|-nK_X| \neq \varnothing$ for some $n \gg 0$. This property is shared, for instance, by toric varieties and weak Fano varieties. However, all the known applications of the numbers $\mathrm{lct}(X)$ and $\mathrm{lct}(X, G)$ are connected with the case when $-K_X$ is ample and G is compact.

It is shown in Appendix A that when X is smooth and G is compact, the equality $\mathrm{lct}(X, G) = \alpha_G(X)$ holds, where $\alpha_G(X)$ is Tian's α-invariant introduced in [3]. We note that

$$\mathrm{lct}(X, G) = \sup\{\lambda \in \mathbb{Q} \mid \text{ the log pair } (X, \lambda D) \text{ has log canonical singularities for every } G\text{-invariant effective } \mathbb{Q}\text{-divisor } D \sim_{\mathbb{Q}} -K_X\}$$

in the case when $|G| < +\infty$. It is clear that $0 \leqslant \mathrm{lct}(X) \leqslant \mathrm{lct}(X, G) \in \mathbb{R} \cup \{+\infty\}$.

Example 1.15. Let X be a smooth del Pezzo surface such that $K_X^2 = 5$. Then we have an isomorphism $\mathrm{Aut}(X) \cong \mathrm{S}_5$ (see [19]) and $\mathrm{lct}(X, \mathrm{S}_5) = \mathrm{lct}(X, \mathrm{A}_5) = 2$ (see [14]).

Example 1.16. Let X be the cubic surface in $\mathbb{P}^3$ given by the equation

$$x^3 + y^3 + z^3 + t^3 = 0 \subset \mathbb{P}^3 \cong \mathrm{Proj}(\mathbb{C}[x, y, z, t]),$$

and let $G = \mathrm{Aut}(X) \cong \mathbb{Z}_3^3 \rtimes \mathrm{S}_4$. Then $\mathrm{lct}(X, G) = 4$ (see [14]).

The following result was proved in [3], [20], [21] (see Appendix A).

Theorem 1.17. *Let X be a Fano variety with at most quotient singularities and assume that G is compact. Assume that the inequality*

$$\mathrm{lct}(X, G) > \frac{\dim X}{\dim X + 1}$$

holds. Then X admits an orbifold Kähler–Einstein metric.

Theorem 1.17 has various applications (see [20] and also Examples 1.6 and 1.7).

Example 1.18. Let X be a Fano variety equal to a blow-up of $\mathbb{P}^3$ along a disjoint union of two lines. Let G be a maximal compact subgroup of $\mathrm{Aut}(X)$. Then $\mathrm{lct}(X, G) \geqslant 1$ by [20]. On the other hand, $\mathrm{lct}(X) = 1/3$ by Theorem 1.46.

If a variety with at most quotient singularities admits an orbifold Kähler–Einstein metric, then its canonical divisor is numerically trivial, or its canonical divisor is ample, or its anticanonical divisor is ample (a Fano variety). Every variety with quotient singularities that has a numerically trivial or ample canonical divisor admits a Kähler–Einstein metric (see [22]–[24]).

There are several known obstructions for a Fano variety X to carry a Kähler–Einstein metric. For example, if the variety X is smooth, then it does not admit a Kähler–Einstein metric if even one of the following conditions is fulfilled:

- the group $\mathrm{Aut}(X)$ is not reductive (see [25]);
- the tangent bundle of X is not polystable with respect to $-K_X$ (see [26]);
- the Futaki character of holomorphic vector fields on X does not identically vanish (see [27]).

Example 1.19. The following varieties have no Kähler–Einstein metric: a blow-up of $\mathbb{P}^2$ at one or two distinct points (see [25]); the smooth Fano threefold $\mathbb{P}\bigl(\mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(1)\bigr)$ (see [28]); the smooth Fano fourfold $\mathbb{P}\bigl(\alpha^*(\mathscr{O}_{\mathbb{P}^1}(1)) \oplus \beta^*(\mathscr{O}_{\mathbb{P}^2}(1))\bigr)$ (see [27]), where $\alpha\colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ and $\beta\colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ are projections.

The problem of the existence of Kähler–Einstein metrics on smooth toric Fano varieties is completely solved. Namely, the following result holds (see [29]–[32]).

Theorem 1.20. *If X is a smooth toric Fano variety, then the following conditions are equivalent*:

(a) *X admits a Kähler–Einstein metric*;
(b) *the Futaki character of holomorphic vector fields on X vanishes*;
(c) *the barycentre of the reflexive polytope of X is at the origin.*

It should be pointed out that Theorem 1.17 gives only a sufficient condition for the existence of a Kähler–Einstein metric on a Fano variety X.

Example 1.21. Let X be a general cubic surface in $\mathbb{P}^3$ with one Eckardt point (see Definition 3.1). Then $\operatorname{lct}(X, \operatorname{Aut}(X)) = 2/3$ (see [14]), while $\operatorname{Aut}(X) \cong \mathbb{Z}_2$ (see [19]). However, every smooth del Pezzo surface with reductive automorphism group admits a Kähler–Einstein metric (see [33]).

Example 1.22. Let X be a general hypersurface in $\mathbb{P}(1^5, 3)$ of degree 6. Then $\operatorname{Aut}(X) \cong \mathbb{Z}_2$ (see [34]) and $\operatorname{lct}(X, \operatorname{Aut}(X)) = 1/2$ (see Example 1.9), but X admits a Kähler–Einstein metric (see [35]).

The numbers $\operatorname{lct}(X)$ and $\operatorname{lct}(X, G)$ play an important role in birational geometry.

Example 1.23. Suppose that there exists a commutative diagram

$$\begin{array}{ccc} V & \overset{\rho}{\dashrightarrow} & \overline{V} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\bar{\pi}} \\ Z & = & Z \end{array},$$

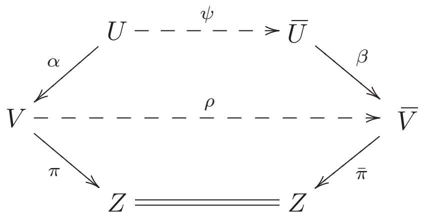
in which V and $\overline{V}$ are varieties with at most terminal and $\mathbb{Q}$-factorial singularities, Z is a smooth curve, π and $\bar{\pi}$ are flat morphisms, and ρ is a birational map that induces an isomorphism $V \setminus X \cong \overline{V} \setminus \overline{X}$, where X and $\overline{X}$ are scheme fibres over a point $O \in Z$ of π and $\bar{\pi}$, respectively. Suppose that the fibres X and $\overline{X}$ are irreducible and reduced, the divisors $-K_V$ and $-K_{\overline{V}}$ are π-ample and $\bar{\pi}$-ample, respectively, the varieties X and $\overline{X}$ have at most log terminal singularities, and ρ is not an isomorphism. Then it follows from [36] and [17] that

$$\operatorname{lct}(X) + \operatorname{lct}(\overline{X}\,) \leqslant 1, \tag{1.1}$$

where X and $\overline{X}$ are Fano varieties by the adjunction formula.

In general the inequality (1.1) is sharp.

Example 1.24. Let $\pi\colon V \to Z$ be a surjective flat morphism from a smooth threefold V to a smooth curve Z such that the divisor $-K_V$ is π-ample, let X be a scheme fibre of the morphism π over a point $O \in Z$ such that X is a smooth cubic surface in $\mathbb{P}^3$ containing lines L_1, L_2, and L_3 intersecting at one point $P \in V$. Then it follows from [37] that there exists a commutative diagram



such that α is a blow-up of P, the map ψ is an antiflip in the proper transforms of the curves L_1, L_2, L_3, and β is a contraction of the proper transform of the fibre X. Then the birational map ρ is not an isomorphism, the threefold $\overline{V}$ has terminal and $\mathbb{Q}$-factorial singularities, the divisor $-K_{\overline{V}}$ is a Cartier $\bar{\pi}$-ample divisor, the map ρ induces an isomorphism $V \setminus X \cong \overline{V} \setminus \overline{X}$, where $\overline{X}$ is a scheme fibre of $\bar{\pi}$ over the point O. In this case $\overline{X}$ is a cubic surface with one singular point of type $\mathbb{D}_4$, and therefore $\operatorname{lct}(X) = 2/3$ and $\operatorname{lct}(\overline{X}) = 1/3$ (see Examples 1.10 and 1.11).

Global log canonical thresholds can be used to prove that some higher-dimensional Fano varieties are non-rational.

Definition 1.25. A Fano variety X is said to be birationally superrigid if the following conditions hold:

(i) $\operatorname{rk}\operatorname{Pic}(X) = 1$;

(ii) X has terminal $\mathbb{Q}$-factorial singularities;

(iii) there is no rational dominant map $\rho\colon X \dashrightarrow Y$ with rationally connected fibres such that $0 \neq \dim Y < \dim X$;

(iv) there is no birational map $\rho\colon X \dashrightarrow Y$ onto a variety Y with terminal $\mathbb{Q}$-factorial singularities such that $\operatorname{rk}\operatorname{Pic}(Y) = 1$;

(v) the groups $\operatorname{Bir}(X)$ and $\operatorname{Aut}(X)$ coincide.

The following result is known as the Noether–Fano inequality (see [38]).

Theorem 1.26. *A variety X is birationally superrigid if and only if* $\operatorname{rk}\operatorname{Pic}(X) = 1$, *$X$ has terminal $\mathbb{Q}$-factorial singularities, and for every linear system $\mathscr{M}$ on X without fixed components the log pair $(X, \mathscr{M})$ has canonical singularities, where $K_X + \lambda\mathscr{M} \equiv 0$.*

Proof. Because one part of the required result is well known (see [38]), we prove only the other part. Suppose that X is birationally superrigid, but there is a linear system $\mathscr{M}$ on X such that $\mathscr{M}$ has no fixed components but the singularities of the log pair $(X, \lambda\mathscr{M})$ are not canonical, where $K_X + \lambda\mathscr{M} \sim_{\mathbb{Q}} 0$.

Let $\pi\colon V \to X$ be a birational morphism such that the variety V is smooth and the proper transform of $\mathscr{M}$ on the variety V has no base points. Let $\mathscr{B}$ be the proper transform of the linear system $\mathscr{M}$ on the variety V. Then

$$K_V + \lambda\mathscr{B} \sim_{\mathbb{Q}} \pi^*(K_X + \lambda\mathscr{M}) + \sum_{i=1}^{r} a_i E_i \sim_{\mathbb{Q}} \sum_{i=1}^{r} a_i E_i,$$

where E_i is an exceptional divisor of π and $a_i \in \mathbb{Q}$.

It follows from [39] that there is a commutative diagram

$$\begin{array}{ccc} & V & \\ {\scriptstyle\rho}\swarrow & & \searrow{\scriptstyle\pi} \\ U & \xrightarrow{\;\varphi\;} & X \end{array}$$

such that ρ is a birational map, the morphism φ is birational, the divisor

$$K_U + \lambda\rho(\mathscr{B}) \sim_{\mathbb{Q}} \varphi^*(K_X + \lambda\mathscr{M}) + \sum_{i=1}^{r} a_i\rho(E_i) \sim_{\mathbb{Q}} \sum_{i=1}^{r} a_i\rho(E_i)$$

is φ-nef, the variety U is $\mathbb{Q}$-factorial, and the log pair, $(U, \lambda\rho(\mathscr{B}))$ has terminal singularities.

Note that φ is not an isomorphism: it follows from [40], § 1.1 that

$$a_i > 0 \implies \dim\bigl(\rho(E_i)\bigr) \leqslant \dim X - 2,$$

and because the singularities of $(X, \lambda\mathscr{M})$ are not canonical by assumption, it follows from the construction of the map ρ that there exists $k \in \{1, \dots, r\}$ such that $a_k < 0$ and the subvariety $\rho(E_k)$ is a divisor on U.

We see that the divisor $K_U + \lambda\rho(\mathscr{B})$ is not pseudo-effective. Then it follows from [39] that there is a diagram

$$\begin{array}{ccc} U & \overset{\psi}{\dashrightarrow} & Y \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\tau} \\ X & & Z \end{array}$$

such that ψ is a birational map, the morphism τ is a Mori fibred space (see [41]), and the divisor $-\bigl(K_Y + \lambda(\psi \circ \rho)(\mathscr{B})\bigr)$ is τ-ample.

The variety Y has terminal $\mathbb{Q}$-factorial singularities and $\operatorname{rk}\operatorname{Pic}(Y/Z) = 1$. Then the map $\psi \circ \rho \circ \pi^{-1}$ is not an isomorphism, because $K_X + \lambda\mathscr{M} \sim_{\mathbb{Q}} 0$, but a general fibre of the morphism τ is rationally connected (see [4]), which contradicts the assumption that X is birationally superrigid. The proof is complete.

Birationally superrigid Fano varieties are non-rational (see [38]). In particular, $\dim(X) \neq 2$ if the variety X is birationally superrigid (cf. [42]).

Example 1.27. A general hypersurface in $\mathbb{P}^n$ of degree $n \geqslant 4$ or in $\mathbb{P}(1^{n+1}, n)$ of degree $2n \geqslant 6$ is birationally superrigid (see [43], [7]).

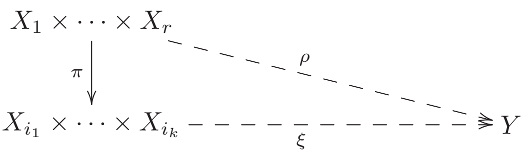
The following result is proved in [7].

Theorem 1.28. *Let $X_1, \dots, X_r$ be birationally superrigid Fano varieties such that* $\operatorname{lct}(X_i) \geqslant 1$, $i = 1, \dots, r$. *Then*

(a) *the variety $X_1 \times \dots \times X_r$ is non-rational and*

$$\operatorname{Bir}(X_1 \times \dots \times X_r) = \operatorname{Aut}(X_1 \times \dots \times X_r),$$

(b) *for every rational dominant map* $\rho\colon X_1 \times \dots \times X_r \dashrightarrow Y$ *whose general fibre is rationally connected there is a commutative diagram*



for some subset $\{i_1, \dots, i_k\} \subseteq \{1, \dots, r\}$, *where* ξ *is a birational map and* π *is the projection.*

Examples 1.6 and 1.27 show that varieties satisfying all the hypotheses of Theorem 1.28 exist. We can construct explicit examples of them.

Example 1.29. Let X be the hypersurface given by

$$w^2 = x^6 + y^6 + z^6 + t^6 + x^2y^2zt \subset \mathbb{P}(1,1,1,1,3) \cong \operatorname{Proj}(\mathbb{C}[x,y,z,t,w]),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = \mathrm{wt}(z) = \mathrm{wt}(t) = 1$ and $\mathrm{wt}(w) = 3$. Then X is smooth and birationally superrigid (see [44]); it follows from [8] that $\mathrm{lct}(X) = 1$.

Suppose in addition that the subgroup $G \subset \mathrm{Aut}(X)$ is finite.

Definition 1.30. A Fano variety X is G-birationally superrigid if

- the G-invariant subgroup of the group $\mathrm{Cl}(X)$ is isomorphic to $\mathbb{Z}$;
- X has terminal singularities;
- there is no dominant G-equivariant rational map $\rho\colon X \dashrightarrow Y$ with rationally connected fibres such that $0 \neq \dim Y < \dim X$;
- there is no G-equivariant non-biregular birational map $\rho\colon X \dashrightarrow Y$ onto a variety Y with terminal singularities such that the G-invariant subgroup of the group $\mathrm{Cl}(Y)$ is isomorphic to $\mathbb{Z}$.

Arguing as in the proof of Theorem 1.26, we obtain the following result.

Theorem 1.31. *The variety* X *is* G*-birationally superrigid if and only if the* G*-invariant subgroup of the group* $\mathrm{Cl}(X)$ *is isomorphic to* $\mathbb{Z}$, X *has terminal singularities, and for every* G*-invariant linear system* $\mathscr{M}$ *on* X *without fixed components the log pair* $(X, \lambda\mathscr{M})$ *is canonical, where* $K_X + \lambda\mathscr{M} \sim_{\mathbb{Q}} 0$.

The proof of Theorem 1.28 implies the following result (see [14]).

Theorem 1.32. *Let* X_i *be a Fano variety and let* $G_i \subset \mathrm{Aut}(X_i)$ *be a finite subgroup such that* X_i *is* G_i*-birationally superrigid and the inequality* $\mathrm{lct}(X_i, G_i) \geqslant 1$ *holds for* $i = 1, \dots, r$. *Then*

(a) *no* $G_1 \times \dots \times G_r$*-equivariant birational map* $\rho\colon X_1 \times \dots \times X_r \dashrightarrow \mathbb{P}^n$ *exists;*

(b) *every* $G_1 \times \dots \times G_r$*-equivariant birational automorphism of* $X_1 \times \dots \times X_r$ *is biregular;*

(c) *a* $G_1 \times \cdots \times G_r$*-equivariant rational dominant map* $\rho\colon X_1 \times \cdots \times X_r \dashrightarrow Y$ *whose general fibre is rationally connected has a commutative diagram*

$$\begin{array}{ccc} X_1 \times \cdots \times X_r & & \\ \downarrow{\scriptstyle \pi} & \overset{\rho}{\searrow} & \\ X_{i_1} \times \cdots \times X_{i_k} & \overset{}{\underset{\xi}{\dashrightarrow}} & Y, \end{array}$$

where ξ *is a birational map,* π *is the natural projection, and* $\{i_1, \dots, i_k\} \subseteq \{1, \dots, r\}$.

Varieties satisfying all hypotheses of Theorem 1.32 do exist (see Example 1.16).

Example 1.33. The simple group A_6 is a group of automorphisms of the sextic

$$10x^3y^3 + 9zx^5 + 9zy^5 + 27z^6 = 45x^2y^2z^2 + 135xyz^4 \subset \mathbb{P}^2 \cong \operatorname{Proj}(\mathbb{C}[x, y, z]),$$

which induces an embedding $\mathrm{A}_6 \subset \operatorname{Aut}(\mathbb{P}^2)$. Then $\mathbb{P}^2$ is A_6-birationally superrigid and $\operatorname{lct}(\mathbb{P}^2, \mathrm{A}_6) = 2$ (see [14]). Hence there exists by Theorem 1.32 an induced embedding $\mathrm{A}_6 \times \mathrm{A}_6 \cong \Omega \subset \operatorname{Bir}(\mathbb{P}^4)$ such that Ω is not conjugate to any subgroup of $\operatorname{Aut}(\mathbb{P}^4)$.

We now consider Fano varieties whose birational geometry is close to that of birationally superrigid Fano varieties.

Definition 1.34. A variety X is birationally rigid if

- the equality $\operatorname{rk}\operatorname{Pic}(X) = 1$ holds;
- X has $\mathbb{Q}$-factorial and terminal singularities;
- there is no rational map $\rho\colon X \dashrightarrow Y$ with rationally connected fibres such that $0 \neq \dim Y < \dim X$;
- there is no birational map $\rho\colon X \dashrightarrow Y$ onto a variety $Y \ncong X$ with terminal $\mathbb{Q}$-factorial singularities such that $\operatorname{rk}\operatorname{Pic}(Y) = 1$.

Arguing as in the proof of Theorem 1.26, we obtain the following result.

Theorem 1.35. *The variety* X *is birationally rigid if and only if* $\operatorname{rk}\operatorname{Pic}(X) = 1$, X *has terminal* $\mathbb{Q}$*-factorial singularities, and for any non-empty linear system* $\mathscr{M}$ *on* X *without fixed components there is a* $\xi \in \operatorname{Bir}(X)$ *such that the log pair* $(X, \lambda\xi(\mathscr{M}))$ *has canonical singularities, where* $K_X + \lambda\xi(\mathscr{M}) \equiv 0$.

Birationally rigid Fano varieties are non-rational (see [38]).

Definition 1.36. Suppose that X is birationally rigid. A subset $\Gamma \subset \operatorname{Bir}(X)$ *untwists* all maximal singularities if for every linear system $\mathscr{M}$ on X without fixed components there is a birational automorphism $\xi \in \Gamma$ such that the log pair $(X, \lambda\xi(\mathscr{M}))$ has canonical singularities, where λ is a rational number such that $K_X + \lambda\xi(\mathscr{M}) \equiv 0$.

If X is birationally rigid and there is a subset $\Gamma \subset \operatorname{Bir}(X)$ that untwists all maximal singularities, then $\operatorname{Bir}(X) = \langle \Gamma, \operatorname{Aut}(X) \rangle$.

Definition 1.37. A variety X is *universally birationally rigid* if for any variety U the variety $X \otimes \operatorname{Spec}(\mathbb{C}(U))$ is birationally rigid over the field $\mathbb{C}(U)$ of rational functions of the variety U.

Definition 1.34 also makes sense for Fano varieties over an arbitrary perfect field (see [42], [19]).

Example 1.38. Let X be a threefold such that there is a finite morphism $\pi\colon X \to Q \subset \mathbb{P}^3$, where Q is a smooth quadric threefold and π is a double cover branched in a smooth surface $S \subset Q$ of degree 8. There exists a one-parameter family of curves

$$\mathscr{C} = \{C \subset X \mid C \text{ is a smooth curve such that } -K_X \cdot C = 1\},$$

and for every curve $C \in \mathscr{C}$ there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad\pi\quad} & Q \\ {\scriptstyle\psi_C}\,\big\downarrow & & \big\downarrow\,{\scriptstyle\varphi_C} \\ \mathbb{P}^2 & =\!=\!=\!= & \mathbb{P}^2 \end{array},$$

where φ_C is the projection from the line $\pi(C)$. The general fibre of the map ψ_C is a smooth elliptic curve. The rational map ψ_C induces an elliptic fibration with a section which induces a birational involution τ_C. It is known that

$$\psi_C \in \operatorname{Aut}(X) \quad \Longleftrightarrow \quad C \subset S$$

and if X is sufficiently general, then S contains no curves in $\mathscr{C}$. It follows from [44] that there exists an exact sequence of groups

$$1 \to \Gamma \to \operatorname{Bir}(X) \to \operatorname{Aut}(X) \to 1,$$

where Γ is a free product of subgroups generated by birational non-biregular involutions τ_C, $C \in \mathscr{C}$. Hence X is universally birationally rigid.

Birationally superrigid Fano manifolds are universally birationally rigid.

Definition 1.39. Suppose that X is universally birationally rigid. A subset $\Gamma \subset \operatorname{Bir}(X)$ universally untwists all maximal singularities if for every variety U the induced subset

$$\Gamma \subset \operatorname{Bir}(X) \subseteq \operatorname{Bir}\bigl(X \otimes \operatorname{Spec}(\mathbb{C}(U))\bigr)$$

untwists all maximal singularities on $X \otimes \operatorname{Spec}(\mathbb{C}(U))$.

It is easy to see that any subset of $\operatorname{Aut}(X)$ universally untwists all maximal singularities if the Fano variety X is birationally superrigid.

Remark 1.40. Let X be a birationally rigid Fano variety. Let $\Gamma \subseteq \operatorname{Bir}(X)$ be an arbitrary subset and assume that $\dim X \neq 1$. Then it follows from [45] that the following conditions are equivalent:

- Γ universally untwists all maximal singularities;
- Γ untwists all maximal singularities, and $\operatorname{Bir}(X)$ is countable.

Example 1.41. In the assumptions of Example 1.7 suppose that X is general. Then

- the hypersurface X is universally birationally rigid (see [46]),

- there are involutions $\tau_1, \dots, \tau_k \in \mathrm{Bir}(X)$ such that the sequence of groups

$$1 \to \langle \tau_1, \dots, \tau_k \rangle \to \mathrm{Bir}(X) \to \mathrm{Aut}(X) \to 1$$

is exact (see [46], [47]), where $\langle \tau_1, \dots, \tau_k \rangle$ is the subgroup generated by $\tau_1, \dots, \tau_k$,
- $\langle \tau_1, \dots, \tau_k \rangle$ universally untwists all maximal singularities (see [46]).

All relations between the involutions $\tau_1, \dots, \tau_k$ are found in [47].

Let $X_1, \dots, X_r$ be Fano varieties that have at most $\mathbb{Q}$-factorial and terminal singularities such that

$$\mathrm{rk}\,\mathrm{Pic}(X_1) = \dots = \mathrm{rk}\,\mathrm{Pic}(X_r) = 1,$$

let

$$\pi_i \colon X_1 \times \dots \times X_{i-1} \times X_i \times X_{i+1} \times \dots \times X_r \to X_1 \times \dots \times X_{i-1} \times \widehat{X_i} \times X_{i+1} \times \dots \times X_r$$

be the natural projection, and let $\mathscr{X}_i$ be the scheme general fibre of π_i; $\mathscr{X}_i$ is defined over $\mathbb{C}(X_1 \times \dots \times X_{i-1} \times \widehat{X_i} \times X_{i+1} \times \dots \times X_r)$.

Remark 1.42. There are natural embeddings of groups

$$\prod_{i=1}^{r} \mathrm{Bir}(X_i) \subseteq \langle \mathrm{Bir}(\mathscr{X}_1), \dots, \mathrm{Bir}(\mathscr{X}_r) \rangle \subseteq \mathrm{Bir}(X_1 \times \dots \times X_r).$$

The following generalization of Theorem 1.28 was proved in [11].

Theorem 1.43. *Suppose that* $X_1, \dots, X_r$ *are universally birationally rigid and that* $\mathrm{lct}(X_i) \geqslant 1$, $i = 1, \dots, r$. *Then*

(a) *the variety* $X_1 \times \dots \times X_r$ *is non-rational and*

$$\mathrm{Bir}(X_1 \times \dots \times X_r) = \langle \mathrm{Bir}(\mathscr{X}_1), \dots, \mathrm{Bir}(\mathscr{X}_r), \mathrm{Aut}(X_1 \times \dots \times X_r) \rangle,$$

(b) *for every rational dominant map* $\rho \colon X_1 \times \dots \times X_r \dashrightarrow Y$ *whose general fibre is rationally connected there exist a subset* $\{i_1, \dots, i_k\} \subseteq \{1, \dots, r\}$ *and a commutative diagram*

$$\begin{array}{ccccc} X_1 \times \dots \times X_r & \overset{\sigma}{\dashrightarrow} & X_1 \times \dots \times X_r & & \\ \downarrow{\scriptstyle \pi} & & & \searrow{\scriptstyle \rho} & \\ X_{i_1} \times \dots \times X_{i_k} & & \underset{\xi}{\dashrightarrow} & & Y \end{array},$$

where π *is the natural projection and* ξ *and* σ *are birational maps.*

Corollary 1.44. *Suppose that there are subgroups* $\Gamma_i \subseteq \mathrm{Bir}(X_i)$ *that universally untwist all maximal singularities, and assume that* $\mathrm{lct}(X_i) \geqslant 1$ *for all* $i = 1, \dots, r$. *Then*

$$\mathrm{Bir}(X_1 \times \dots \times X_r) = \left\langle \prod_{i=1}^{r} \Gamma_i, \mathrm{Aut}(X_1 \times \dots \times X_r) \right\rangle.$$

In particular, the following example is obtained using Examples 1.7 and 1.41.

Example 1.45 (cf. Example 1.41). Let X be a general hypersurface of degree 20 in $\mathbb{P}(1,1,4,5,10)$. Then the sequence of groups

$$1 \to \prod_{i=1}^{m} (\mathbb{Z}_2 * \mathbb{Z}_2) \to \mathrm{Bir}\big(\underbrace{X \times \cdots \times X}_{m \text{ factors}}\big) \to \mathrm{S}_m \to 1$$

is exact, where S_m is the permutation group and $\mathbb{Z}_2 * \mathbb{Z}_2$ is the infinite dihedral group.

Now let X be a smooth Fano threefold (see [2]). Then X lies in one of 105 deformation families (see [48]–[52]). Let

$$\daleth(X) \in \{1.1, 1.2, \ldots, 1.17, 2.1, \ldots, 2.36, 3.1, \ldots, 3.31, 4.1, \ldots, 4.13, 5.1, \ldots, , 5.8\}$$

be the number of the deformation type of the threefold X in the notation of Table 1 (see Appendix B). The main aim of this paper is to prove the following result.

Theorem 1.46. *The following assertions hold*:

(a) $\mathrm{lct}(X) = 1/5$ *if* $\daleth(X) \in \{2.36, 3.29\}$;

(b) $\mathrm{lct}(X) = 1/4$ *if* $\daleth(X) \in \{1.17, 2.28, 2.30, 2.33, 2.35, 3.23, 3.26, 3.30, 4.12\}$;

(c) $\mathrm{lct}(X) = 1/3$ *if* $\daleth(X) \in \{1.16, 2.29, 2.31, 2.34, 3.9, 3.18, 3.19, 3.20, 3.21, 3.22, 3.24, 3.25, 3.28, 3.31, 4.4, 4.8, 4.9, 4.10, 4.11, 5.1, 5.2\}$;

(d) $\mathrm{lct}(X) = 3/7$ *if* $\daleth(X) = 4.5$;

(e) $\mathrm{lct}(X) = 1/2$ *if* $\daleth(X) \in \{1.11, 1.12, 1.13, 1.14, 1.15, 2.1, 2.3, 2.18, 2.25, 2.27, 2.32, 3.4, 3.10, 3.11, 3.12, 3.14, 3.15, 3.16, 3.17, 3.24, 3.27, 4.1, 4.2, 4.3, 4.6, 4.7, 5.3, 5.4, 5.5, 5.6, 5.7, 5.8\}$;

(f) *if X is a general threefold in its deformation family*, *then*

- $\mathrm{lct}(X) = 1/3$ *if* $\daleth(X) = 2.23$,
- $\mathrm{lct}(X) = 1/2$ *if* $\daleth(X) \in \{2.5, 2.8, 2.10, 2.11, 2.14, 2.15, 2.19, 2.24, 2.26, 3.2, 3.5, 3.6, 3.7, 3.8, 4.13\}$,
- $\mathrm{lct}(X) = 2/3$ *if* $\daleth(X) = 3.3$,
- $\mathrm{lct}(X) = 3/4$ *if* $\daleth(X) \in \{2.4, 3.1\}$,
- $\mathrm{lct}(X) = 1$ *if* $\daleth(X) = 1.1$.

The generality condition in Theorem 1.46 cannot be dropped in the general case.

Example 1.47. Let $\daleth(X) = 4.13$. (We note that this deformation class was left out by mistake in [50] but was later discovered in [51].) Then there is a birational morphism $\alpha\colon X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ that contracts a smooth irreducible surface $E \subset X$ to a curve C such that $C \cdot F_1 = C \cdot F_2 = 1$ and $C \cdot F_3 = 3$, where $F_1 \cong F_2 \cong F_3 \cong \mathbb{P}^1 \times \mathbb{P}^1$ are fibres of the three different natural projections $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. Then $\mathrm{lct}(X) = 1/2$ by Theorem 1.46 if X is general. We note that there is a unique surface $G \in |F_1 + F_2|$ such that $C \subset G$. Then $-K_X \sim 2\overline{G} + E + \overline{F}_3$, where $\overline{F}_3 \subset X \supset \overline{G}$ are the proper transforms of F_3 and G, respectively. Furthermore, $\mathrm{lct}(X) \leqslant 1/2$, but $\mathrm{lct}(X) \leqslant \mathrm{lct}(X, 2\overline{G} + E + \overline{F}_3) \leqslant 4/9 < 1/2$ in the case when $|F_3 \cap C| = 1$.

We organize this paper in the following way. In §§ 2–4 we consider auxiliary results used in the proof of Theorem 1.46. In § 5 we compute the global log canonical thresholds of toric Fano varieties. In § 6 we prove Theorem 1.46 for smooth Fano threefolds of index 2, that is, for $\daleth(X) \in \{1.11, 1.12, 1.13, 1.14, 1.15, 2.32, 2.35, 3.27\}$. In § 7 we prove Theorem 1.46 in the case $\operatorname{rk}\operatorname{Pic}(X) = 2$. In § 8 we prove Theorem 1.46 in the case $\operatorname{rk}\operatorname{Pic}(X) = 3$. In § 9 we prove Theorem 1.46 in the case $\operatorname{rk}\operatorname{Pic}(X) \geqslant 4$. In § 10 we find upper bounds for $\operatorname{lct}(X)$ in the case

$$\daleth(X) \in \{1.8, 1.9, 1.10, 2.2, 2.7, 2.9, 2.12, 2.13, 2.16, 2.17, 2.20, 2.21, 2.22, 3.13\}.$$

In Appendix A, written by J.-P. Demailly, the relation between global log canonical thresholds of smooth Fano varieties and the α-invariants of smooth Fano varieties introduced in [3] for the study of the existence of Kähler–Einstein metrics is established. In Appendix B we present Table 1, containing a list of all smooth Fano threefolds together with the known values and bounds for their global log canonical thresholds.

We use the standard notation $D_1 \sim D_2$ (respectively, $D_1 \sim_{\mathbb{Q}} D_2$) for linearly equivalent (respectively, $\mathbb{Q}$-linearly equivalent) divisors (respectively, $\mathbb{Q}$-divisors). If a divisor (a $\mathbb{Q}$-divisor) D is linearly equivalent to a line bundle $\mathscr{L}$ (respectively, $\mathbb{Q}$-linearly equivalent to a divisor linearly equivalent to a line bundle $\mathscr{L}$), then we write $D \sim \mathscr{L}$ (respectively, $D \sim_{\mathbb{Q}} \mathscr{L}$). We note that $\mathbb{Q}$-linear equivalence coincides with numerical equivalence in the case of log terminal Fano varieties. The projectivization $\mathbb{P}_Y(\mathscr{E})$ of a vector bundle $\mathscr{E}$ on a variety Y is the variety of hyperplanes in the fibres of $\mathscr{E}$. The symbol $\mathbb{F}_n$ denotes the Hirzebruch surface $\mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(n))$. We always refer to a smooth Fano threefold X using the number $\daleth(X)$ of the corresponding deformation family introduced in Table 1.

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2. Preliminaries

Let X be a variety with log terminal singularities. Consider a divisor $B_X = \sum_{i=1}^r a_i B_i$, where B_i is a prime Weil divisor on the variety X and a_i is a non-negative rational number. Suppose that B_X is a $\mathbb{Q}$-Cartier divisor such that $B_i \neq B_j$ for $i \neq j$. Let $\pi\colon \overline{X} \to X$ be a birational morphism such that $\overline{X}$ is smooth. We set $B_{\overline{X}} = \sum_{i=1}^r a_i \overline{B}_i$, where $\overline{B}_i$ is the proper transform of B_i on the variety $\overline{X}$. Then

$$K_V + B_{\overline{X}} \equiv \pi^*(K_X + B_X) + \sum_{i=1}^n c_i E_i,$$

where $c_i \in \mathbb{Q}$ and E_i is an exceptional divisor of the morphism π. Suppose additionally that $\bigl(\bigcup_{i=1}^r \overline{B}_i\bigr) \cup \bigl(\bigcup_{i=1}^n E_i\bigr)$ is a divisor with simple normal crossings. We set $B^{\overline{X}} = B_{\overline{X}} - \sum_{i=1}^n c_i E_i$.

Definition 2.1. The singularities of a log pair (X, B_X) are log canonical (respectively, log terminal) if

- $a_i \leqslant 1$ (respectively, $a_i < 1$) for all $i = 1, \dots, r$,
- $c_j \geqslant -1$ (respectively, $c_j > -1$) for all $j = 1, \dots, n$.

It is known that Definition 2.1 does not depend on the choice of the morphism $\pi\colon \overline{X} \to X$. Let

$$\mathrm{LCS}(X, B_X) = \biggl(\bigcup_{a_i \geqslant 1} B_i\biggr) \cup \biggl(\bigcup_{c_i \leqslant -1} \pi(E_i)\biggr) \subsetneq X;$$

then $\mathrm{LCS}(X, B_X)$ is called the locus of log canonical singularities of the log pair (X, B_X).

Definition 2.2. A proper irreducible subvariety $Y \subsetneq X$ is called a centre of log canonical singularities of a log pair (X, B_X) if

- either the inequality $a_i \geqslant 1$ holds and $Y = B_i$,
- or the inequality $c_i \leqslant -1$ holds and $Y = \pi(E_i)$ for some choice of the birational morphism $\pi\colon \overline{X} \to X$.

Let $\mathbb{LCS}(X, B_X)$ be the set of all centres of log canonical singularities of (X, B_X). Then

$$Y \in \mathbb{LCS}(X, B_X) \Longrightarrow Y \subseteq \mathrm{LCS}(X, B_X)$$

and $\mathbb{LCS}(X, B_X) = \varnothing \iff \mathrm{LCS}(X, B_X) = \varnothing \iff$ the log pair (X, B_X) is log terminal.

Remark 2.3. Let $\mathscr{H}$ be a linear system on X that has no base points, let H be a sufficiently general divisor in the linear system $\mathscr{H}$, and let $Y \subsetneq X$ be an irreducible subvariety. We set $Y|_H = \sum_{i=1}^{m} Z_i$, where $Z_i \subset H$ is an irreducible subvariety. Then it follows from Definition 2.2 (cf. Theorem 2.19) that

$$Y \in \mathbb{LCS}(X, B_X) \iff \{Z_1, \dots, Z_m\} \subseteq \mathbb{LCS}\bigl(H, B_X|_H\bigr).$$

Example 2.4. Let $\alpha\colon V \to X$ be a blow-up of a smooth point $O \in X$. Then $B_V \equiv \alpha^*(B_X) - \mathrm{mult}_O(B_X)E$, where $\mathrm{mult}_O(B_X) \in \mathbb{Q}$, and E is the exceptional divisor of the blow-up α. In this case $\mathrm{mult}_O(B_X) > 1$ if the log pair (X, B_X) is not log canonical at the point O. Let

$$B^V = B_V + \bigl(\mathrm{mult}_O(B_X) - \dim(X) + 1\bigr)E$$

and suppose that $\mathrm{mult}_O(B_X) \geqslant \dim(X) - 1$. Then $O \in \mathbb{LCS}(X, B_X)$ if and only if

- either $E \in \mathbb{LCS}(V, B^V)$, that is, $\mathrm{mult}_O(B_X) \geqslant \dim(X)$,
- or there exists a subvariety $Z \subsetneq E$ such that $Z \in \mathbb{LCS}(V, B^V)$.

The locus $\mathrm{LCS}(X, B_X)$ can be equipped with a scheme structure (see [20], [40]). Let

$$\mathscr{I}(X, B_X) = \pi_*\biggl(\sum_{i=1}^{n} \lceil c_i \rceil E_i - \sum_{i=1}^{r} \lfloor a_i \rfloor \overline{B}_i\biggr),$$

and let $\mathscr{L}(X, B_X)$ be the subscheme corresponding to the ideal sheaf $\mathscr{I}(X, B_X)$.

Definition 2.5. For the log pair (X, B_X) we call $\mathscr{L}(X, B_X)$ the subscheme of log canonical singularities of (X, B_X) and we call the ideal sheaf $\mathscr{I}(X, B_X)$ the multiplier ideal sheaf of (X, B_X).

It follows immediately from the construction of the subscheme $\mathscr{L}(X, B_X)$ that

$$\mathrm{Supp}\big(\mathscr{L}(X, B_X)\big) = \mathrm{LCS}(X, B_X) \subset X.$$

The following result is the Nadel–Shokurov vanishing theorem (see [40] and [53], Theorem 9.4.8).

Theorem 2.6. *Let H be a nef and big $\mathbb{Q}$-divisor on X such that $K_X + B_X + H \equiv D$ for some Cartier divisor D on the variety X. Then for every $i \geqslant 1$,*

$$H^i\big(X, \mathscr{I}(X, B_X) \otimes D\big) = 0.$$

For each Cartier divisor D on X we consider the exact sequence of sheaves

$$0 \to \mathscr{I}(X, B_X) \otimes D \to \mathscr{O}_X(D) \to \mathscr{O}_{\mathscr{L}(X,B_X)}(D) \to 0$$

and the corresponding exact sequence of cohomology groups

$$H^0\big(\mathscr{O}_X(D)\big) \to H^0\big(\mathscr{O}_{\mathscr{L}(X,B_X)}(D)\big) \to H^1\big(\mathscr{I}(X, B_X) \otimes D\big).$$

Theorem 2.7. *Suppose that $-(K_X + B_X)$ is nef and big. Then $\mathrm{LCS}(X, B_X)$ is connected.*

Proof. We set $D = 0$. Then it follows from Theorem 2.6 that the sequence

$$\mathbb{C} = H^0(\mathscr{O}_X) \to H^0(\mathscr{O}_{\mathscr{L}(X,B_X)}) \to H^1\big(\mathscr{I}(X, B_X)\big) = 0$$

is exact if $-(K_X + B_X)$ is nef and big. Thus, the locus

$$\mathrm{LCS}(X, B_X) = \mathrm{Supp}\big(\mathscr{L}(X, B_X)\big)$$

is connected if the divisor $-(K_X + B_X)$ is nef and big.

We consider a few elementary applications of Theorem 2.7 (cf. Example 1.10).

Lemma 2.8. *Suppose that $\mathrm{LCS}(X, B_X) \neq \varnothing$, where $X \cong \mathbb{P}^n$ and $B_X \sim_{\mathbb{Q}} -\lambda K_X$ for some rational number λ with $0 < \lambda < n/(n+1)$. Then $\dim(\mathrm{LCS}(X, B_X)) \geqslant 1$, and the subscheme $\mathscr{L}(X, B_X)$ does not contain isolated zero-dimensional components.*

Proof. Let $O \in X$ be a point such that $\mathrm{LCS}(X, \lambda B_X) = O \cup \Sigma$, where $\Sigma \subset X$ is a (possibly empty) subset such that $O \notin \Sigma$.

Let H be a general line in $X \cong \mathbb{P}^2$. Then the locus $\mathrm{LCS}(X, \lambda B_X + H) = O \cup H \cup \Sigma$ is disconnected. However, the divisor $-(K_X + \lambda B_X + H)$ is ample, which contradicts Theorem 2.7.

Lemma 2.9. *Suppose that $\mathrm{LCS}(X, B_X) \neq \varnothing$, where $X \cong \mathbb{P}^3$ and $B_X \sim_{\mathbb{Q}} -\lambda K_X$ for some rational number $0 < \lambda < 1/2$. Then $\mathbb{LCS}(X, B_X)$ contains a surface.*

Proof. Suppose that $\mathbb{LCS}(X, B_X)$ contains no surfaces. Let S be a general plane in $\mathbb{P}^3$. Then the locus $\mathrm{LCS}(\mathbb{P}^3, B_X + S)$ is connected by Theorem 2.7. Hence $(S, B_X|_S)$ is not log terminal by Remark 2.3. On the other hand, the locus $\mathrm{LCS}(S, B_X|_S)$ consists of finitely many points, which is impossible by Lemma 2.8.

Lemma 2.10. *Suppose that* $\mathrm{LCS}(X, B_X) \neq \varnothing$*, where* X *is a smooth quadric threefold in* $\mathbb{P}^4$ *and* $B_X \sim_{\mathbb{Q}} -\lambda K_X$ *for some rational number* $0 < \lambda < 1/2$*. Then* $\mathbb{LCS}(X, B_X)$ *contains a surface.*

Proof. Let $L \subset X$ be a general line, let $P_1 \in L \ni P_2$ be two general points, let H_1 and H_2 be the hyperplane sections of $X \subset \mathbb{P}^4$ that are tangent to X at the points P_1 and P_2, respectively. Then

$$\mathrm{LCS}\Big(X, \lambda B_X + \frac{3}{4}(H_1 + H_2)\Big) = \mathrm{LCS}(X, \lambda B_X) \cup L$$

is disconnected, which is impossible by Theorem 2.7.

Remark 2.11. One can prove Lemmas 2.9, 2.10 (and 2.28) using another method. Suppose that $B_X \sim_{\mathbb{Q}} -\lambda K_X$ for some $\lambda \in \mathbb{Q}$ such that $0 < \lambda < 1/2$, where X is $\mathbb{P}^3$, or $\mathbb{P}^1 \times \mathbb{P}^2$, or a smooth quadric threefold. Also, suppose that the set $\mathbb{LCS}(X, B_X)$ contains no surfaces. Then $\mathrm{LCS}(X, B_X) \subseteq \Sigma$, where $\Sigma \subset X$ is a (possibly reducible) curve. For a general automorphism $\varphi \in \mathrm{Aut}(X)$ we have $\varphi(\Sigma) \cap \Sigma = \varnothing$, which implies that $\mathrm{LCS}(X, \varphi(B_X)) \cap \mathrm{LCS}(X, B_X) = \varnothing$. We can show that if φ is sufficiently general, then

$$\mathrm{LCS}\big(X, \varphi(B_X) + B_X\big) = \mathrm{LCS}\big(X, \varphi(B_X)\big) \cup \mathrm{LCS}(X, B_X).$$

This contradicts Theorem 2.7 since $\lambda < 1/2$.

Lemma 2.12. *Suppose that* $\mathrm{LCS}(X, B_X) \neq \varnothing$*, where* X *is a blow-up of* $\mathbb{P}^3$ *in one point and* $B_X \sim_{\mathbb{Q}} -\lambda K_X$ *for some rational number* $0 < \lambda < 1/2$*. Then* $\mathbb{LCS}(X, B_X)$ *contains a surface.*

Proof. Suppose that the set $\mathbb{LCS}(X, B_X)$ contains no surfaces. Let $\alpha\colon X \to \mathbb{P}^3$ be the blow-up of a point and let E be the exceptional divisor of α. In the case when $\mathrm{LCS}(X, \lambda B_X) \not\subseteq E$ we can apply Lemma 2.9 to the pair $(\mathbb{P}^3, \alpha(B_X))$ to get a contradiction. Hence we can assume that $\mathrm{LCS}(X, B_X) \subseteq E$.

Let $H \subset \mathbb{P}^3$ be a general hyperplane and let $H_1 \subset \mathbb{P}^3 \supset H_2$ be general hyperplanes passing through $\alpha(E)$. We denote by $\overline{H}$, $\overline{H}_1$, and $\overline{H}_2$ the proper transforms of the hyperplanes H, H_1 and H_2 on X, respectively. Then

$$\mathrm{LCS}\Big(X, B_X + \frac{1}{2}(\overline{H}_1 + \overline{H}_2 + 2\overline{H}\,)\Big)$$

is disconnected, which is impossible by Theorem 2.7.

Lemma 2.13. *Let* X *be a cone in* $\mathbb{P}^4$ *over a smooth quadric surface and suppose that* $B_X \sim_{\mathbb{Q}} -\lambda K_X$ *for some rational number* $0 < \lambda < 1/3$*. Then* $\mathbb{LCS}(X, B_X) = \varnothing$*.*

Proof. Suppose that $\mathbb{LCS}(X, B_X) \neq \varnothing$. Let S be a general hyperplane section of the cone $X \subset \mathbb{P}^4$. Then $\mathrm{LCS}\big(S, B_X|_S\big) = \varnothing$, because $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathrm{lct}(\mathbb{P}^1 \times \mathbb{P}^1) = 1/2$ (see Example 1.10). Thus, $|\,\mathrm{LCS}(X, B_X)| < +\infty$ by Remark 2.3. Then the locus $\mathrm{LCS}(X, B_X + S)$ is disconnected, which contradicts Theorem 2.7.

The following result is a consequence of Theorem 2.6 (see [20], Theorem 4.1).

Lemma 2.14. *If* $-(K_X+B_X)$ *is nef and big and* $\dim(\mathrm{LCS}(X,B_X))=1$, *then*

(a) *the locus* $\mathrm{LCS}(X,B_X)$ *is a connected union of smooth rational curves,*

(b) *every two irreducible components of the locus* $\mathrm{LCS}(X,B_X)$ *meet at at most one point,*

(c) *every pair of intersecting irreducible components of the locus* $\mathrm{LCS}(X,B_X)$ *meet transversally,*

(d) *no three irreducible components of the locus* $\mathrm{LCS}(X,B_X)$ *meet at one point,*

(e) *the locus* $\mathrm{LCS}(X,B_X)$ *contains no cycles of smooth rational curves.*

Proof. Arguing as in the proof of Theorem 2.7, we see that $\mathrm{LCS}(X,B_X)$ is a connected tree of smooth rational curves with simple normal crossings.

Lemma 2.15 [43]. *Let* X *be a smooth hypersurface in* $\mathbb{P}^m$ *and* $B_X\sim_{\mathbb{Q}}\mathscr{O}_{\mathbb{P}^m}(1)|_X$. *Let* $S\subsetneq X$ *be an irreducible subvariety with* $\dim(S)\geqslant k$. *Then* $\mathrm{mult}_S(B_X)\leqslant 1$.

We consider now a simple application of Theorem 2.7 and Lemma 2.15.

Lemma 2.16. *Let* X *be a cubic hypersurface in* $\mathbb{P}^4$ *with at most isolated singularities. Suppose that* $B_X\sim_{\mathbb{Q}}-K_X$, *but there exists a positive rational number* $\lambda<1/2$ *such that* $\mathrm{LCS}(X,\lambda B_X)\neq\varnothing$. *Then* $\mathrm{LCS}(X,\lambda B_X)=L$, *where* L *is a line in* $X\subset\mathbb{P}^4$ *such that* $L\cap\mathrm{Sing}(X)\neq\varnothing$.

Proof. Let S be a general hyperplane section of X. Then

$$S\cup\mathrm{LCS}(X,\lambda B_X)\subseteq\mathrm{LCS}(X,\lambda B_X+S),$$

hence $\dim(\mathrm{LCS}(X,\lambda B_X))\geqslant 1$ by Theorem 2.7. Therefore, $\mathrm{LCS}(S,\lambda B_X|_S)\neq\varnothing$ by Remark 2.3. On the other hand, $\mathrm{LCS}(S,\lambda B_X|_S)$ consists of finitely many points by Lemma 2.15. By Theorem 2.7 there is a unique point $O\in S$ such that $\mathrm{LCS}(S,\lambda B_X|_S)=O$. It now follows by Remark 2.3 that there is a line $L\subset X$ such that $\mathrm{LCS}(X,\lambda B_X)=L$.

Arguing as in the proof of Lemma 2.15, we see that $L\cap\mathrm{Sing}(X)\neq\varnothing$.

The proof of the following result is similar to that of Lemma 2.16.

Lemma 2.17. *Suppose there is a double cover* $\tau\colon X\to\mathbb{P}^3$ *branched over an irreducible reduced quartic surface* $R\subset\mathbb{P}^3$ *that has at most ordinary double points. Assume that the equivalence* $B_X\sim_{\mathbb{Q}}-\lambda K_X$ *holds but* $\mathrm{LCS}(X,B_X)\neq\varnothing$, *where* $\lambda<1/2$. *Then* $\mathrm{Sing}(X)\neq\varnothing$ *and* $\mathrm{LCS}(X,B_X)=L$, *where* L *is an irreducible curve on* X *such that* $-K_X\cdot L=2$ *and* $L\cap\mathrm{Sing}(X)\neq\varnothing$.

Proof. We observe that $-K_X\sim 2H$, where H is a Cartier divisor on X such that $H\sim\tau^*(\mathscr{O}_{\mathbb{P}^3}(1))$. The variety X is a Fano threefold and $H^3=2$. In particular, the set $\mathrm{LCS}(X,B_X+H)$ must be connected by Theorem 2.7. Thus, there is a curve $C\in\mathbb{LCS}(X,B_X)$, which implies that $\mathrm{mult}_C(B_X)\geqslant 1/\lambda>2$.

Let S be a general surface in $|H|$. We set $B_S=B_X|_S$. Then $-\lambda K_S\sim_{\mathbb{Q}}B_S$, but the log pair (S,B_S) is not log canonical at every point of the intersection $S\cap\mathrm{LCS}(X,B_X)$.

The surface H is a smooth hypersurface in $\mathbb{P}(1,1,1,2)$ of degree 4.

Let P be any point in $S\cap \mathrm{LCS}(X,B_X)$. Then there is a birational morphism $\rho\colon S\to\bar{S}$ such that $\bar{S}$ is a cubic surface in $\mathbb{P}^3$ and ρ is an isomorphism in a neighbourhood of P. In particular, the pair $(\bar{S},\rho(B_S))$ is not log terminal at the point $\rho(P)$. Thus, we have $\mathrm{LCS}(\bar{S},\rho(B_S))\neq\varnothing$, but

$$\frac{1}{\lambda}\,\rho(B_S)\sim_{\mathbb{Q}}-K_{\bar{S}}\sim\mathscr{O}_{\mathbb{P}^3}(1)\big|_{\bar{S}}\,,$$

which implies by Lemma 2.15 and Theorem 2.7 that $\mathrm{LCS}(\bar{S},\rho(B_S))$ consists of one point. Then

$$P=S\cap C=S\cap\mathrm{LCS}(X,B_X)$$

if the point P is sufficiently general. Therefore, $\mathrm{LCS}(X,B_X)=C$, the curve C is irreducible, and $-K_X\cdot C=2$. In particular, $\tau(C)\subset\mathbb{P}^3$ is a line.

We suppose that $C\cap\mathrm{Sing}(X)=\varnothing$ and derive a contradiction.

Suppose that $\tau(C)\subset R$. We take a general point $O\in C$. Let $\tau(O)\in\Pi\subset\mathbb{P}^3$ be a plane tangent to R at the point $\tau(O)$. Arguing as in the proof of Lemma 2.15 (see [43]), we see that $R|_{\Pi}$ is reduced along $\tau(C)$, because $\tau(C)\cap\mathrm{Sing}(R)=\varnothing$. We fix a general line $\Gamma\subset\Pi\subset\mathbb{P}^3$ such that $\tau(O)\in\Gamma$. Let $\bar{\Gamma}\subset X$ be an irreducible curve such that $\tau(\bar{\Gamma})=\Gamma$. Then $\bar{\Gamma}\not\subseteq\mathrm{Supp}(B_X)$, because Γ sweeps out a dense subset of $\mathbb{P}^3$ as we vary the point $O\in C$ and the line $\Gamma\subset\Pi$. Note that either $H\cdot\bar{\Gamma}=1$ or $H\cdot\bar{\Gamma}=2$. In the second case $\mathrm{mult}_O(\bar{\Gamma})=2$. Hence

$$H\cdot\bar{\Gamma}>2\lambda H\cdot\bar{\Gamma}=\bar{\Gamma}\cdot B_X\geqslant\mathrm{mult}_O(\bar{\Gamma})\,\mathrm{mult}_C(B_V)\geqslant H\cdot\bar{\Gamma},$$

which is a contradiction. Thus, $\tau(C)\not\subset R$.

There is an irreducible reduced curve $\bar{C}\subset X$ such that $\tau(\bar{C})=\tau(C)\subset\mathbb{P}^3$ but $\bar{C}\neq C$. Let Y be a general surface in $|H|$ that passes through the curves $\bar{C}$ and C. Then Y is smooth because $C\cap\mathrm{Sing}(X)=\varnothing$, and it is easy to see that $\bar{C}\cdot\bar{C}=C\cdot C=-2$ on the surface Y.

Obviously, $Y\not\subset\mathrm{Supp}(B_X)$. We set $B_Y=B_X|_Y$. Then

$$B_Y=\mathrm{mult}_{\bar{C}}(B_X)\bar{C}+\mathrm{mult}_C(B_X)C+\Delta,$$

where Δ is an effective $\mathbb{Q}$-divisor on the surface Y such that

$$\bar{C}\not\subset\mathrm{Supp}(\Delta)\not\supset C.$$

On the other hand, $B_Y\sim_{\mathbb{Q}}2\lambda(\bar{C}+C)$, which implies in particular that

$$\begin{aligned}\big(2\lambda-\mathrm{mult}_C(B_X)\big)C\cdot C&=\big(\mathrm{mult}_{\bar{C}}(B_X)-2\lambda\big)\bar{C}\cdot C+\Delta\cdot C\\&\geqslant\big(\mathrm{mult}_{\bar{C}}(B_X)-2\lambda\big)\bar{C}\cdot C\geqslant 0,\end{aligned}$$

because $\Delta\cdot C\geqslant 0$ and $\bar{C}\cdot C\geqslant 0$. Then $\mathrm{mult}_{\bar{C}}(B_X)\geqslant 2\lambda$ because $C\cdot C<0$. Thus,

$$-\Delta\sim_{\mathbb{Q}}\big(\mathrm{mult}_{\bar{C}}(B_X)-2\lambda\big)\bar{C}+\big(\mathrm{mult}_C(B_X)-2\lambda\big)C,$$

which is impossible, because $\mathrm{mult}_C(B_X)>2\lambda$ and Y is projective.

One can generalize Theorem 2.7 in the following way (see [40], Lemma 5.7).

Theorem 2.18. *Let* $\psi\colon X \to Z$ *be a morphism. Then* $\mathrm{LCS}\big(\overline{X}, B^{\overline{X}}\big)$ *is connected in a neighbourhood of each fibre of the morphism* $\psi\circ\pi\colon X \to Z$ *in the case when*
(a) ψ *is surjective and has connected fibres,*
(b) *the divisor* $-(K_X+B_X)$ *is nef and big with respect to* ψ.

Let us consider one important application of Theorem 2.18 (see [41], Theorem 5.50).

Theorem 2.19. *Suppose that* B_1 *is a Cartier divisor,* $a_1=1$, *and* B_1 *has at most log terminal singularities. Then the following assertions are equivalent*:
(a) *the log pair* (X, B_X) *is log canonical in a neighbourhood of the divisor* B_1;
(b) *the singularities of the log pair* $\big(B_1, \sum_{i=2}^r a_iB_i|_{B_1}\big)$ *are log canonical.*

The simplest application of Theorem 2.19 is the following non-obvious result (see [41], Corollary 5.57).

Lemma 2.20. *Suppose that* $\dim(X)=2$ *and* $a_1\leqslant 1$. *Then* $\big(\sum_{i=2}^r a_iB_i\big)B_1>1$ *whenever* (X,B_X) *is not log canonical at some point* $O\in B_1$ *such that* $O\notin \mathrm{Sing}(X)\cup\mathrm{Sing}(B_1)$.

Proof. Suppose that (X,B_X) is not log canonical at a point $O\in B_1$. By Theorem 2.19 we have

$$\left(\sum_{i=2}^r a_iB_i\right)\cdot B_1 \geqslant \mathrm{mult}_O\left(\sum_{i=2}^r a_iB_i|_{B_1}\right)>1$$

if $O\notin\mathrm{Sing}(X)\cup\mathrm{Sing}(B_1)$ because $\big(X, B_1+\sum_{i=2}^r a_iB_i\big)$ is not log canonical at O.

Let us consider another application of Theorem 2.19 (cf. Lemma 2.29).

Lemma 2.21. *Let* X *be a Fano variety with log terminal singularities. Then* $\mathrm{lct}(\mathbb{P}^1\times X)=\min(1/2,\mathrm{lct}(X))$.

Proof. The inequalities $1/2\geqslant \mathrm{lct}(V\times U)\leqslant \mathrm{lct}(X)$ are evident. We suppose that $\mathrm{lct}(\mathbb{P}^1\times X)<\min(1/2,\mathrm{lct}(X))$ and show that this leads to a contradiction.

There is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}} -K_{\mathbb{P}^1\times X}$ such that the log pair $(\mathbb{P}^1\times X,\lambda D)$ is not log canonical at some point $P\in\mathbb{P}^1\times X$, where $\lambda<\min(1/2,\mathrm{lct}(X))$.

Let F be a fibre of the projection $\mathbb{P}^1\times X\to\mathbb{P}^1$ such that $P\in F$. Then $D=\mu F+\Omega$, where Ω is an effective $\mathbb{Q}$-divisor on $\mathbb{P}^1\times X$ such that $F\not\subset\mathrm{Supp}(\Omega)$.

Let L be a general fibre of the natural projection $\mathbb{P}^1\times X\to X$. Then

$$2=D\cdot L=\mu+\Omega\cdot L\geqslant\mu,$$

which implies that the log pair $(\mathbb{P}^1\times X, F+\lambda\Omega)$ is also not log canonical at P. Then $\big(F,\lambda\Omega|_F\big)$ is not log canonical at P by Theorem 2.19, but $\Omega|_F\sim_{\mathbb{Q}}-K_F$, which is impossible because $X\cong F$ and $\lambda<\mathrm{lct}(X)$.

Let P be a point in X. We consider an effective divisor

$$\Delta=\sum_{i=1}^r\varepsilon_iB_i\sim_{\mathbb{Q}}B_X,$$

where ε_i is a non-negative rational number. Suppose that Δ is a $\mathbb{Q}$-Cartier divisor, the equivalence $\Delta \sim_{\mathbb{Q}} B_X$ holds, and the log pair (X, Δ) is log canonical at the point $P \in X$.

Remark 2.22. Suppose that (X, B_X) is not log canonical at the point $P \in X$. Let $\alpha = \min\{a_i/\varepsilon_i \mid \varepsilon_i \neq 0\}$, where α is well defined because some of the numbers $\varepsilon_1, \dots, \varepsilon_r$ are non-zero. Then $\alpha < 1$, the log pair

$$\biggl(X, \sum_{i=1}^{r} \frac{a_i - \alpha\varepsilon_i}{1-\alpha}\, B_i\biggr)$$

is not log canonical at the point $P \in X$, the equivalence

$$\sum_{i=1}^{r} \frac{a_i - \alpha\varepsilon_i}{1-\alpha}\, B_i \sim_{\mathbb{Q}} B_X \sim_{\mathbb{Q}} \Delta,$$

holds, but at least one irreducible component of the divisor $\operatorname{Supp}(\Delta)$ does not lie in

$$\operatorname{Supp}\biggl(\sum_{i=1}^{r} \frac{a_i - \alpha\varepsilon_i}{1-\alpha}\, B_i\biggr).$$

The assertion of Remark 2.22 is obvious but nevertheless very useful.

Lemma 2.23. *Suppose that $X \cong C_1 \times C_2$, where C_1 and C_2 are smooth curves, and suppose that $B_X \sim_{\mathbb{Q}} \lambda E + \mu F$, where $E \cong C_1$ and $F \cong C_2$ are curves on the surface X such that $E \cdot E = F \cdot F = 0$ and $E \cdot F = 1$, and where λ and μ are non-negative rational numbers. Then*

(a) *the pair (X, B_X) is log terminal if $\lambda < 1$ and $\mu < 1$,*

(b) *the pair (X, B_X) is log canonical if $\lambda \leqslant 1$ and $\mu \leqslant 1$.*

Proof. It suffices to prove (a). Suppose that $\lambda, \mu < 1$, but (X, B_X) is not log terminal at some point $P \in X$. Then $\operatorname{mult}_P(B_X) \geqslant 1$ and by Remark 2.22 we may assume that $E \not\subset \operatorname{Supp}(B_X)$ or $F \not\subset \operatorname{Supp}(B_X)$. On the other hand, $E \cdot B_X = \mu$ and $F \cdot B_X = \lambda$, which leads at once to a contradiction because $\operatorname{mult}_P(B_X) \geqslant 1$.

Let $[B_X]$ be the class of $\mathbb{Q}$-rational equivalence of the divisor B_X. Let

$$\operatorname{lct}(X, [B_X]) = \inf\{\operatorname{lct}(X, D) \mid D \text{ is an effective } \mathbb{Q}\text{-divisor such that } D \sim_{\mathbb{Q}} B_X\} \geqslant 0$$

and put $\operatorname{lct}(X, [B_X]) = +\infty$ if $B_X = 0$. We note that B_X is an effective divisor.

Remark 2.24. The equality $\operatorname{lct}(X, [-K_X]) = \operatorname{lct}(X)$ holds (see Definition 1.2).

Arguing as in the proof of Lemma 2.21, we obtain the following result.

Lemma 2.25. *Let $\varphi\colon X \to Z$ be a surjective morphism with connected fibres such that $\dim Z = 1$. Let F be a fibre of φ that has log terminal singularities. Then either $\operatorname{lct}_F(X, B_X) \geqslant \operatorname{lct}\bigl(F, \bigl[B_X|_F\bigr]\bigr)$, or there is a rational number $0 < \varepsilon < \operatorname{lct}\bigl(F, \bigl[B_X|_F\bigr]\bigr)$ such that $F \subseteq \operatorname{LCS}(X, \varepsilon B_X)$.*

Proof. Suppose that $\operatorname{lct}_F(X, B_X) < \operatorname{lct}\big(F, \big[B_X|_F\big]\big)$. Then there is a rational number $\varepsilon < \operatorname{lct}\big(F, \big[B_X|_F\big]\big)$ such that the log pair $(X, \varepsilon B_X)$ is not log canonical at some point $P \in F$. Let $B_X = \mu F + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor on X such that $F \not\subset \operatorname{Supp}(\Omega)$.

We may assume that $\varepsilon\mu \leqslant 1$. Then the log pair $(X, F + \varepsilon\Omega)$ is not log canonical at P, and $\big(F, \varepsilon\Omega|_F\big)$ is also not log canonical at P by Theorem 2.19. However, $\Omega|_F \sim_{\mathbb{Q}} B_X|_F$, which is a contradiction.

We now present a simple application of Lemma 2.25.

Lemma 2.26. *Let $Q \subset \mathbb{P}^4$ be a cone over a smooth quadric surface and let $\alpha\colon X \to Q$ be a blow-up along a smooth conic $C \subset Q \setminus \operatorname{Sing}(Q)$. Then* $\operatorname{lct}(X) = 1/3$.

Proof. Let H be a general hyperplane section of $Q \subset \mathbb{P}^4$ that contains C, and let $\overline{H}$ be the proper transform of the surface H on the threefold X. Then $-K_X \sim 3\overline{H} + 2E$, where E is the exceptional divisor of α. In particular, the inequality $\operatorname{lct}(X) \leqslant 1/3$ holds.

We suppose that $\operatorname{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/3$. There is a commutative diagram

$$\begin{array}{ccc} & X & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ Q & \overset{}{\underset{\psi}{- - - - - - \dashrightarrow}} & \mathbb{P}^1 \end{array},$$

where β is the morphism given by the linear system $|\overline{H}|$ and ψ is the projection from the two-dimensional linear subspace containing the conic C.

Suppose that $\mathbb{LCS}(X, \lambda D)$ contains a surface $M \subset X$. Then $D = \mu M + \Omega$, where $\mu \geqslant 1/\lambda$ and Ω is an effective $\mathbb{Q}$-divisor such that $M \not\subset \operatorname{Supp}(\Omega)$.

Let F be a general fibre of β. Then $F \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $D|_F \sim_{\mathbb{Q}} -K_F$, which immediately implies that M is a fibre of β, but $\alpha(D) \sim_{\mathbb{Q}} -K_Q \sim 3\alpha(M)$, which is impossible because $\mu \geqslant 1/\lambda > 3$. Thus, the set $\mathbb{LCS}(X, \lambda D)$ contains no surfaces.

There is a fibre S of β such that $S \neq S \cap \operatorname{LCS}(X, \lambda D) \neq \varnothing$, which implies that S is singular by Lemma 2.25, because $\operatorname{lct}(\mathbb{P}^1 \times \mathbb{P}^1) = 1/2$.

Thus, the surface S is an irreducible quadric cone in $\mathbb{P}^3$. Then $\operatorname{LCS}(X, \lambda D) \subseteq S$ by Theorem 2.7. Because $\big(X, S + \frac{2}{3}E\big)$ has log canonical singularities and the equivalence $3S + 2E \sim_{\mathbb{Q}} D$ holds, we may assume that either $S \not\subset \operatorname{Supp}(D)$ or $E \not\subset \operatorname{Supp}(D)$ by Remark 2.22.

Let $\Gamma = E \cap S$. The curve Γ is an irreducible conic. Then $\operatorname{LCS}(X, \lambda D) \subseteq \Gamma$ by Lemma 2.13. Intersecting D with a general ruling of the cone $S \subset \mathbb{P}^3$ and intersecting D with a general fibre of the projection $E \to C$, we see that $\Gamma \not\subseteq \operatorname{LCS}(X, \lambda D)$, which implies that $\operatorname{LCS}(X, \lambda D)$ consists of a single point $O \in \Gamma$ by Theorem 2.7.

Let R be a general surface in $|\alpha^*(H)|$. Then

$$\operatorname{LCS}\left(X, \lambda D + \frac{1}{2}\,(\overline{H} + 2R)\right) = R \cup O,$$

which is impossible by Theorem 2.7, since $-K_X \sim \overline{H} + 2R \sim_{\mathbb{Q}} D$ and $\lambda < 1/3$.

The following generalization of Lemma 2.25 follows from [54], Proposition 5.19 (cf. [6]).

Theorem 2.27. *Let* $\varphi\colon X \to Z$ *be a surjective flat morphism with connected fibres such that* Z *has rational singularities and all the scheme fibres of* φ *have at most canonical Gorenstein singularities. Let* F *be a scheme fibre of* φ. *Then either* $\mathrm{lct}_F(X, B_X) \geqslant \mathrm{lct}\big(F, \big[B_X|_F\big]\big)$ *or there is a positive rational number* $\varepsilon < \mathrm{lct}\big(F, \big[B_X|_F\big]\big)$ *such that* $F \subseteq \mathrm{LCS}(X, \varepsilon B_X)$.

Let us consider an elementary application of Theorem 2.27.

Lemma 2.28. *Suppose that* $\mathrm{LCS}(X, B_X) \neq \varnothing$, *where* $X \cong \mathbb{P}^1 \times \mathbb{P}^2$ *and* $B_X \sim_{\mathbb{Q}} -\lambda K_X$ *for some rational number* $0 < \lambda < 1/2$. *Then* $\mathbb{LCS}(X, B_X)$ *contains a surface.*

Proof. Suppose that $\mathbb{LCS}(X, B_X)$ contains no surfaces. By Theorems 2.7 and 2.27 we have $\mathrm{LCS}(X, B_X) = F$, where F is a fibre of the natural projection $\pi_2\colon X \to \mathbb{P}^2$. Let S be a general surface in $|\pi_1^*(\mathscr{O}_{\mathbb{P}^2}(1))|$, and let M_1 and M_2 be general fibres of the natural projection $\pi_1\colon X \to \mathbb{P}^1$. Then the locus

$$\mathrm{LCS}\bigg(X, \lambda D + \frac{1}{2}\,(M_1 + M_2 + 3S)\bigg) = F \cup S$$

is disconnected, which is impossible by Theorem 2.7.

Lemma 2.29. *Let* V *and* U *be Fano varieties with at most canonical Gorenstein singularities. Then* $\mathrm{lct}(V \times U) = \min(\mathrm{lct}(V), \mathrm{lct}(U))$.

Proof. The inequalities $\mathrm{lct}(V) \geqslant \mathrm{lct}(V \times U) \leqslant \mathrm{lct}(U)$ are obvious. We suppose that $\mathrm{lct}(V \times U) < \min(\mathrm{lct}(V), \mathrm{lct}(U))$ and show that this leads to a contradiction.

There is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_{V\times U}$ such that the log pair $(V \times U, \lambda D)$ is not log canonical at some point $P \in V \times U$, where $\lambda < \min(\mathrm{lct}(V), \mathrm{lct}(U))$.

Let us identify V with a fibre of the projection $V \times U \to U$ that contains the point P. The inequalities

$$\mathrm{lct}(V) > \lambda > \mathrm{lct}_V(V \times U, D) \geqslant \mathrm{lct}\big(V, [D|_V]\big) = \mathrm{lct}\big(V, [-K_V]\big) = \mathrm{lct}(V)$$

are inconsistent, so it follows from Theorem 2.27 that the log pair $(V \times U, \lambda D)$ is not log canonical at every point of $V \subset V \times U$.

Let us identify U with a general fibre of the projection $V \times U \to V$. Then $D|_U \sim_{\mathbb{Q}} -K_U$, and $(U, \lambda D|_U)$ is not log canonical at the point $U \cap V$ by Remark 2.3 (applied $\dim V$ times). This contradicts the inequality $\lambda < \mathrm{lct}(U)$.

We believe that the assertion of Lemma 2.29 holds also for log terminal Fano varieties (cf. Lemma 2.21).

3. Cubic surfaces

Let X be a cubic surface in $\mathbb{P}^3$ that has at most one ordinary double point.

Definition 3.1. A point $O \in X$ is said to be *Eckardt point* if $O \notin \mathrm{Sing}(X)$ and $O = L_1 \cap L_2 \cap L_3$, where L_1, L_2, L_3 are different lines on the surface $X \subset \mathbb{P}^3$.

General cubic surfaces have no Eckardt points. It follows from Examples 1.10 and 1.11 that

$$\mathrm{lct}(X)=\begin{cases}3/4 & \text{when } X \text{ has no Eckardt points and } \ \mathrm{Sing}(X)=\varnothing,\\ 2/3 & \text{when } X \text{ has an Eckardt point or } \ \mathrm{Sing}(X)\neq\varnothing.\end{cases}$$

Let D be an effective $\mathbb{Q}$-divisor on X such that $D\sim_{\mathbb{Q}}-K_X$, and let $\omega\in\mathbb{Q}_{>0}$ be such that $\omega<3/4$. In this section we prove the following result (cf. [5], [14]).

Theorem 3.2. *Suppose that $(X,\omega D)$ is not log canonical. Then* $\mathrm{LCS}(X,\omega D)=O$, *where $O\in X$ is either a singular point or an Eckardt point.*

Suppose that $(X,\omega D)$ is not log canonical. Let P be a point in $\mathrm{LCS}(X,\omega D)$, and suppose that P is neither a singular point nor an Eckardt point of X.

Lemma 3.3. $\mathrm{LCS}(X,\omega D)=P$.

Proof. Suppose that $\mathrm{LCS}(X,\omega D)\neq P$. Then by Theorem 2.7 there is a curve $C\subset X$ such that $P\in C\subseteq\mathrm{LCS}(X,\omega D)$. Hence there is an effective $\mathbb{Q}$-divisor Ω on X such that $C\not\subset\mathrm{Supp}(\Omega)$ and $D=\mu C+\Omega$, where $\mu\geqslant 1/\omega$. Let H be a general hyperplane section of X. Then

$$3=H\cdot D=\mu H\cdot C+H\cdot\Omega\geqslant\mu\deg C,$$

which implies that either $\deg C=1$ or $\deg C=2$.

Suppose that $\deg C=1$. Let Z be a general conic on X such that $-K_X\sim C+Z$. Then

$$2=Z\cdot D=\mu Z\cdot C+Z\cdot\Omega\geqslant\mu Z\cdot C=\begin{cases}2\mu & \text{if } C\cap\mathrm{Sing}(X)=\varnothing,\\ 3\mu/2 & \text{if } C\cap\mathrm{Sing}(X)\neq\varnothing,\end{cases}$$

which implies that $\mu\leqslant 4/3$. But $\mu\geqslant 1/\omega>4/3$, a contradiction.

We see that $\deg C=2$. Let L be a line on X such that $-K_X\sim C+L$. Then $D=\mu C+\lambda L+\Upsilon$, where Υ is an effective $\mathbb{Q}$-divisor such that $C\not\subset\mathrm{Supp}(\Upsilon)\not\supset L$. We have

$$\begin{aligned}1=L\cdot D&=\mu L\cdot C+\lambda L\cdot L+L\cdot\Upsilon\geqslant\mu L\cdot C+\lambda L\cdot L\\ &=\begin{cases}2\mu-\lambda & \text{if } C\cap\mathrm{Sing}(X)=\varnothing,\\ 3\mu/2-\lambda/2 & \text{if } C\cap\mathrm{Sing}(X)\neq\varnothing,\end{cases}\end{aligned}$$

which implies that $\mu\leqslant 7/6<4/3$ because $\lambda\leqslant 4/3$ (see the case $\deg C=1$). But $\mu>4/3$, a contradiction.

Let $\pi\colon U\to X$ be a blow-up of P and let E be the π-exceptional curve. Then $\overline{D}\sim_{\mathbb{Q}}\pi^*(D)+\mathrm{mult}_P(D)E$, where $\mathrm{mult}_P(D)\geqslant 1/\omega$ and $\overline{D}$ is the proper transform of D on the surface U. The log pair $\big(U,\omega\overline{D}+(\omega\,\mathrm{mult}_P(D)-1)E\big)$ is not log canonical at some point $Q\in E$. Then either $\mathrm{mult}_P(D)\geqslant 2/\omega$, or

$$\mathrm{mult}_Q(\overline{D})+\mathrm{mult}_P(D)\geqslant 2/\omega>8/3, \tag{3.1}$$

because the divisor $\omega\overline{D}+(\omega\,\mathrm{mult}_P(D)-1)E$ is effective.

Let T be the unique hyperplane section of X that is singular at P. We may assume by Remark 2.22 that $\mathrm{Supp}(T) \not\subseteq \mathrm{Supp}(D)$, because $(X, \omega T)$ is log canonical. The curve T is reduced. Thus, the following cases are possible: T is an irreducible and reduced cubic curve; T is the union of a line and an irreducible conic; T consists of three different lines.

We note that $\mathrm{mult}_P(T) = 2$ since P is not an Eckardt point. In the rest of the section we shall exclude these cases one by one.

Lemma 3.4. *The curve T is reducible.*

Proof. Suppose that T is an irreducible cubic curve. Then there is a commutative diagram

$$\begin{array}{ccc} & U & \\ {}^{\pi}\swarrow & & \searrow^{\psi} \\ X & \underset{\rho}{\dashrightarrow} & \mathbb{P}^2 \end{array},$$

where ψ is a double cover branched over a quartic curve and ρ is the projection from P.

Let $\overline{T}$ be the proper transform of T on U. Suppose that $Q \in \overline{T}$. Then

$$\begin{aligned} 3 - 2\,\mathrm{mult}_P(D) = \overline{T} \cdot \overline{D} &\geqslant \mathrm{mult}_Q(\overline{T})\,\mathrm{mult}_Q(\overline{D}) \\ &> \mathrm{mult}_Q(\overline{T})\big(8/3 - \mathrm{mult}_P(D)\big) \geqslant 8/3 - \mathrm{mult}_P(D), \end{aligned}$$

which implies that $\mathrm{mult}_P(D) \leqslant 1/3$. This inequality is absurd; thus, $Q \notin \overline{T}$.

Let $\tau \in \mathrm{Aut}(U)$ be the natural involution[3] induced by the double cover ψ. It follows from [42] that

$$\tau^*\big(\pi^*(-K_X)\big) \sim \pi^*(-2K_X) - 3E$$

and $\tau(\overline{T}) = E$. We set $\breve{Q} = \pi \circ \tau(Q)$. Then $\breve{Q} \neq P$, because $Q \notin \overline{T}$.

Let H be the hyperplane section of X that is singular at $\breve{Q}$. Then $T \neq H$, because $P \neq \breve{Q}$ and T is smooth away from P. Hence $P \notin H$, because otherwise

$$3 = H \cdot T \geqslant \mathrm{mult}_P(H)\,\mathrm{mult}_P(T) + \mathrm{mult}_{\breve{Q}}(H)\,\mathrm{mult}_{\breve{Q}}(T) \geqslant 4.$$

Let $\overline{H}$ be the proper transform of H on the surface U. We set $\overline{R} = \tau(\overline{H})$ and $R = \pi(\overline{R})$. Then $\overline{R} \sim \pi^*(-2K_X) - 3E$, and the curve $\overline{R}$ must be singular at the point Q.

Suppose that R irreducible. Taking into account all possible singularities of $\overline{R}$, we see that $\big(X, \frac{3}{8}R\big)$ is log canonical. Thus, by Remark 2.22 we may assume that $R \not\subseteq \mathrm{Supp}(D)$. Then

$$6 - 3\,\mathrm{mult}_P(D) = \overline{R} \cdot \overline{D} \geqslant \mathrm{mult}_Q(\overline{R})\,\mathrm{mult}_Q(\overline{D}) > 2\big(8/3 - \mathrm{mult}_P(D)\big),$$

[3] The involution τ induces an involution in $\mathrm{Bir}(X)$ which is called the Geiser involution.

which implies that $\mathrm{mult}_P(D) < 2/3$. However, this is absurd since $\mathrm{mult}_P(D) > 4/3$. Thus, the curve R must be reducible.

The curves R and H are reducible, so there is a line $L \subset X$ such that $P \notin L \ni \breve{Q}$.

Let $\overline{L}$ be the proper transform of L on U. We set $\overline{Z} = \tau(\overline{L})$. Then $\overline{L} \cdot E = 0$ and $\overline{L} \cdot \overline{T} = \overline{L} \cdot \pi^*(-K_X) = 1$, which implies that $\overline{Z} \cdot E = 1$ and $\overline{Z} \cdot \pi^*(-K_X) = 2$. We have $Q \in \overline{Z}$. Then

$$2 - \mathrm{mult}_P(D) = \overline{Z} \cdot \overline{D} \geqslant \mathrm{mult}_Q(\overline{D}) > 8/3 - \mathrm{mult}_P(D) > 2 - \mathrm{mult}_P(D)$$

in the case when $\overline{Z} \not\subseteq \mathrm{Supp}(\overline{D})$. Hence $\overline{Z} \subseteq \mathrm{Supp}(\overline{D})$.

We put $Z = \pi(\overline{Z})$. Then Z is an irreducible conic such that $P \in Z$ and $-K_X \sim L + Z$, which means that $L \cup Z$ is cut out by the plane in $\mathbb{P}^3$ passing through Z. We set $D = \varepsilon Z + \Upsilon$, where Υ is an effective $\mathbb{Q}$-divisor such that $Z \not\subset \mathrm{Supp}(\Upsilon)$.

We may assume that $L \not\subseteq \mathrm{Supp}(\Upsilon)$ (see Remark 2.22). Then

$$1 = L \cdot D = \varepsilon Z \cdot L + L \cdot \Upsilon \geqslant \varepsilon Z \cdot L = \begin{cases} 2\varepsilon & \text{if } Z \cap \mathrm{Sing}(X) = \varnothing, \\ 3\varepsilon/2 & \text{if } Z \cap \mathrm{Sing}(X) \neq \varnothing, \end{cases}$$

which implies that $\varepsilon \leqslant 2/3$.

Let $\overline{\Upsilon}$ be the proper transform of Υ on the surface U. Then the log pair $\big(U, \varepsilon\omega\overline{Z} + \omega\overline{\Upsilon} + (\omega\,\mathrm{mult}_P(D) - 1)E\big)$ is not log canonical at $Q \in \overline{Z}$. Hence

$$\omega\overline{\Upsilon} \cdot \overline{Z} + \big(\omega\,\mathrm{mult}_P(D) - 1\big) = \big(\omega\overline{\Upsilon} + \big(\omega\,\mathrm{mult}_P(D) - 1\big)E\big) \cdot \overline{Z} > 1$$

by Lemma 2.20, because $\varepsilon \leqslant 2/3$. In particular, we see that

$$\begin{aligned} 8/3 - \mathrm{mult}_P(D) < \overline{Z} \cdot \overline{\Upsilon} &= 2 - \mathrm{mult}_P(D) - \varepsilon\overline{Z} \cdot \overline{Z} \\ &= \begin{cases} 2 - \mathrm{mult}_P(D) + \varepsilon & \text{if } Z \cap \mathrm{Sing}(X) = \varnothing, \\ 2 - \mathrm{mult}_P(D) + \varepsilon/2 & \text{if } Z \cap \mathrm{Sing}(X) \neq \varnothing, \end{cases} \end{aligned}$$

which implies that $\varepsilon > 2/3$. But we have already shown that $\varepsilon \leqslant 2/3$. This contradiction completes the proof of Lemma 3.4.

Therefore, there is a line $L_1 \subset X$ such that $P \in L_1$. We set $D = m_1 L_1 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_1 \not\subset\in \mathrm{Supp}(\Omega)$. Then

$$4/3 < 1/\omega < \Omega \cdot L_1 = 1 - m_1 L_1 \cdot L_1 = \begin{cases} 1 + m_1 & \text{if } L_1 \cap \mathrm{Sing}(X) = \varnothing, \\ 1 + m_1/2 & \text{if } L_1 \cap \mathrm{Sing}(X) \neq \varnothing. \end{cases}$$

Corollary 3.5. *The following inequality holds*:

$$m_1 > \begin{cases} 1/3 & \text{if } L_1 \cap \mathrm{Sing}(X) = \varnothing, \\ 2/3 & \text{if } L_1 \cap \mathrm{Sing}(X) \neq \varnothing. \end{cases}$$

Remark 3.6. Suppose that X is singular and put $O = \mathrm{Sing}(X)$. It follows from [16] that $O = \Gamma_1 \cap \Gamma_2 \cap \Gamma_3 \cap \Gamma_4 \cap \Gamma_5 \cap \Gamma_6$, where $\Gamma_1, \dots, \Gamma_6$ are different lines on the surface $X \subset \mathbb{P}^3$. Moreover, $-2K_X \sim \sum_{i=1}^{6} \Gamma_i$. Suppose that $L_1 = \Gamma_1$. Let $\Pi_2, \dots, \Pi_6 \subset \mathbb{P}^3$

be planes such that $L_1 \subset \Pi_i \supset \Gamma_i$ and let $\Lambda_2, \ldots, \Lambda_6$ be lines on the surface X such that

$$L_1 \cup \Gamma_i \cup \Lambda_i = \Pi_i \cap X \subset X \subset \mathbb{P}^3,$$

which implies that $-K_X \sim L_1 + \Gamma_i + \Lambda_i$. Then

$$-5K_X \sim 4L_1 + \sum_{i=2}^{6} \Lambda_i + \left(L_1 + \sum_{i=2}^{6} \Gamma_i\right) \sim 4L_1 + \sum_{i=2}^{6} \Lambda_i - 2K_X,$$

which implies that $-3K_X \sim 4L_1 + \sum_{i=2}^{6} \Lambda_i$. On the other hand, the log pair

$$\left(X, L_1 + \frac{\sum_{i=2}^{6} \Lambda_i}{3}\right)$$

is log canonical at the point P. Thus, in completing the proof of Theorem 3.2 we may assume by Remark 2.22 that

$$\mathrm{Supp}\left(\sum_{i=2}^{6} \Lambda_i\right) \not\subseteq \mathrm{Supp}(D),$$

because $L_1 \subseteq \mathrm{Supp}(D)$. Then there is a line Λ_k such that

$$1 = D \cdot \Lambda_k = (m_1 L_1 + \Omega) \cdot \Lambda_k = m_1 + \Omega \cdot \Lambda_k \geqslant m_1,$$

since $O \notin \Lambda_k$. For the completion of the proof of Theorem 3.2 we may assume that $m_1 \leqslant 1$ if $L_1 \cap \mathrm{Sing}(X) \neq \varnothing$.

Arguing as in the proof of Lemma 2.15, we readily see that $m_1 \leqslant 1$ if $L_1 \cap \mathrm{Sing}(X) = \varnothing$.

Lemma 3.7. *There is a line $L_2 \subset X$ such that $L_1 \neq L_2$ and $P \in L_2$.*

Proof. Suppose there is no line $L_2 \subset X$ such that $L_1 \neq L_2$ and $P \in L_2$. Then $T = L_1 + C$, where C is an irreducible conic on X such that $P \in C$.

By Remark 2.22 we may assume that $C \not\subseteq \mathrm{Supp}(\Omega)$, since $m_1 \neq 0$.

Let $\bar{L}_1$ and $\overline{C}$ be the proper transforms of L_1 and C on the surface U, respectively. Then

$$\overline{D} \sim_{\mathbb{Q}} m_1 \bar{L}_1 + \overline{\Omega} \sim_{\mathbb{Q}} \pi^*(m_1 L_1 + \Omega) - \big(m_1 + \mathrm{mult}_P(\Omega)\big)E \sim_{\mathbb{Q}} \pi^*(D) - \mathrm{mult}_P(D)E,$$

where $\overline{\Omega}$ is the proper transform of the divisor Ω on the surface U. We have

$$\begin{aligned} 0 \leqslant \overline{C} \cdot \overline{\Omega} = 2 - \mathrm{mult}_P(D) + m_1 \overline{C} \cdot \bar{L} &< 2/3 - m_1 \overline{C} \cdot \bar{L}_1 \\ &= \begin{cases} 2/3 - m_1 & \text{if } L_1 \cap \mathrm{Sing}(X) = \varnothing, \\ 2/3 - m_1/2 & \text{if } L_1 \cap \mathrm{Sing}(X) \neq \varnothing, \end{cases} \end{aligned}$$

which implies that $m_1 < 2/3$ if $L_1 \cap \mathrm{Sing}(X) = \varnothing$. It follows from the inequality (3.1) that

$$\mathrm{mult}_Q(\overline{\Omega}) > 8/3 - \mathrm{mult}_P(\Omega) - m_1\big(1 + \mathrm{mult}_Q(\bar{L}_1)\big).$$

Suppose that $Q \in \bar{L}_1$. Then by Lemma 2.20

$$8/3 < \bar{L}_1 \cdot \big(\bar{\Omega} + \big(\operatorname{mult}_P(\Omega) + m_1\big)E\big) = 1 - m_1 \bar{L}_1 \cdot \bar{L}_1 = \begin{cases} 1 + 2m_1 & \text{if } L_1 \cap \operatorname{Sing}(X) = \varnothing, \\ 1 + 3m_1/2 & \text{if } L_1 \cap \operatorname{Sing}(X) \neq \varnothing, \end{cases}$$

which is impossible, because $m_1 \leqslant 1$ if $L_1 \cap \operatorname{Sing}(X) \neq \varnothing$ (see Remark 3.6).

We see that $Q \notin \bar{L}_1$. Suppose that $Q \in \bar{C}$. Then

$$2 - \operatorname{mult}_P(\Omega) - m_1 - m_1 \bar{C} \cdot \bar{L}1 = \bar{C} \cdot \bar{\Omega} > 8/3 - \operatorname{mult}_P(\Omega) - m_1,$$

which is impossible, because $m_1 \bar{C} \cdot \bar{L}_1 \geqslant 0$. Hence, we see that $Q \notin \bar{C}$.

There is a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\ \zeta\ } & W \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\psi} \\ X & \overset{}{\underset{\rho}{\dashrightarrow}} & \mathbb{P}^2 \end{array},$$

where ζ is the birational morphism contracting the curve $\bar{L}_1$, the morphism ψ is a double cover branched over a plane quartic curve, and the rational map ρ is the linear projection from the point $P \in X$.

Let τ be the birational involution of U induced by ψ. Then

- τ is biregular $\iff$ $L_1 \cap \operatorname{Sing}(X) = \varnothing$,
- if $L_1 \cap \operatorname{Sing}(X) \neq \varnothing$, then τ acts biregularly on $U \setminus \bar{L}_1$,
- it follows from the construction of τ that $\tau(E) = \bar{C}$,
- if $L_1 \cap \operatorname{Sing}(X) = \varnothing$, then

$$\tau^*(\bar{L}_1) \sim \bar{L}_1, \qquad \tau^*(E) \sim \bar{C}, \qquad \tau^*\big(\pi^*(-K_X)\big) \sim \pi^*(-2K_X) - 3E - \bar{L}_1.$$

Let H be a hyperplane section of the cubic surface X such that H is singular at $\pi \circ \tau(Q) \in C$. Then $P \notin H$ because C is smooth. Let $\bar{H}$ be the proper transform of H on the surface U. Then $\bar{L}_1 \not\subseteq \operatorname{Supp}(\bar{H}) \not\supseteq \bar{C}$.

We put $\bar{R} = \tau(\bar{H})$ and $R = \pi(\bar{R})$. Then $\bar{R}$ is singular at the point Q, and

$$\bar{R} \sim \pi^*(-2K_X) - 3E - \bar{L}_1,$$

because R does not pass through a singular point of the surface X for $\operatorname{Sing}(X) \neq \varnothing$.

Suppose that R is irreducible. Then $R + L_1 \sim -2K_X$, but the log pair $\big(X, \frac{3}{8}(R + L_1)\big)$ is log canonical. Thus (see Remark 2.22), we may assume that $R \not\subseteq \operatorname{Supp}(D)$. Then

$$5 - 2\big(m_1 + \operatorname{mult}_P(\Omega)\big) + m_1(1 + \bar{L}_1 \cdot \bar{L}_1) = \bar{R} \cdot \bar{\Omega} \geqslant 2 \operatorname{mult}_Q(\bar{\Omega}) > 2\big(8/3 - m_1 - \operatorname{mult}_P(\Omega)\big),$$

which implies that $m_1 < 0$, a contradiction. We have shown that R must be reducible.

It follows immediately from the reducibility of R that there is a line $L \subset X$ such that $P \notin L$ and $\pi \circ \tau(Q) \in L$. Then $L \cap L_1 = \varnothing$, because $\pi \circ \tau(Q) \in C$ and $(C + L_1) \cdot L = T \cdot L = 1$. Thus, there is a unique conic $Z \subset X$ such that $-K_X \sim L + Z$ and $P \in Z$. Then Z is irreducible and $P = Z \cap L_1$, because $(L + Z) \cdot L_1 = 1$.

Let $\overline{L}$ and $\overline{Z}$ be the proper transforms of the curves L and Z on U, respectively. Then

$$\overline{L} \cdot \overline{C} = \overline{Z} \cdot E = 1, \qquad \overline{L}_1 \cdot \overline{Z} = \overline{L} \cdot E = \overline{L} \cdot \overline{L}_1 = 0,$$
$$\overline{Z} \cdot \overline{Z} = 1 - \overline{L} \cdot \overline{Z}, \qquad \overline{L} \cdot \overline{Z} = \begin{cases} 2 & \text{if } L \cap \operatorname{Sing}(X) = \varnothing, \\ 3/2 & \text{if } L \cap \operatorname{Sing}(X) \neq \varnothing. \end{cases}$$

By construction $\tau(\overline{Z}) = \overline{L}$. Then $Q \in \overline{Z}$. Suppose that $\overline{Z} \not\subseteq \operatorname{Supp}(\overline{\Omega})$. Then

$$2 - m_1 - \operatorname{mult}_P(\Omega) = \overline{Z} \cdot \overline{\Omega} > 8/3 - m_1 - \operatorname{mult}_P(\Omega),$$

which is a contradiction. Thus, $\overline{Z} \subseteq \operatorname{Supp}(\overline{\Omega})$. The log pair $(X, \omega(L + Z))$ is log canonical at the point P. Hence we may assume that $\overline{L} \not\subseteq \operatorname{Supp}(\overline{\Omega})$ (see Remark 2.22). We put $D = \varepsilon Z + m_1 L_1 + \Upsilon$, where Υ is an effective $\mathbb{Q}$-divisor such that $Z \not\subset \operatorname{Supp}(\Upsilon) \not\supset L_1$. Then

$$1 = L \cdot D = \varepsilon L \cdot Z + m_1 L \cdot L_1 + L \cdot \Upsilon = \varepsilon L \cdot Z + L \cdot \Upsilon \geqslant \varepsilon L \cdot Z$$
$$= \begin{cases} 2\varepsilon & \text{if } L \cap \operatorname{Sing}(X) = \varnothing, \\ 3\varepsilon/2 & \text{if } L \cap \operatorname{Sing}(X) \neq \varnothing, \end{cases}$$

which implies that $\varepsilon \leqslant 2/3$. However, $\overline{Z} \cap \overline{L}_1 = \varnothing$. Hence it follows from Lemma 2.20 that

$$2 - \operatorname{mult}_P(D) - \varepsilon \overline{Z} \cdot \overline{Z} = \overline{Z} \cdot \overline{\Upsilon} > 8/3 - \operatorname{mult}_P(D),$$

where $\overline{\Upsilon}$ is a proper transform of Υ on the surface U. We conclude that $\varepsilon > 2/3$; however, $\varepsilon \leqslant 2/3$. This contradiction completes the proof of Lemma 3.7.

We see therefore that $T = L_1 + L_2 + L_3$, where L_3 is a line such that $P \notin L_3$. We put $D = m_1 L_1 + m_2 L_2 + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor such that $L_1 \not\subseteq \operatorname{Supp}(\Delta) \not\supseteq L_2$.

We point out that $m_1 > 1/3$ and $m_2 > 1/3$ by Corollary 3.5. Hence we may assume by Remark 2.22 that $L_3 \not\subseteq \operatorname{Supp}(\Delta)$. If L_1 or L_2 contains a singular point of X, then we may assume without loss of generality that it lies in L_1. Then $L_3 \cdot L_2 = 1$ and $L_3 \cdot L_1 = 1/2$ if $L_1 \cap \operatorname{Sing}(X) \neq \varnothing$. Similarly, we see that $L_3 \cdot L_2 = L_3 \cdot L_1 = 1$ in the case $L_1 \cap \operatorname{Sing}(X) = \varnothing$. Then $1 - m_1 L_1 \cdot L_3 - m_2 = L_3 \cdot \Delta \geqslant 0$.

Let $\overline{L}_1$ and $\overline{L}_3$ be the proper transforms of L_1 and L_2 on U, respectively. Then

$$m_1 \overline{L}_1 + m_2 \overline{L}_2 + \overline{\Delta} \sim_{\mathbb{Q}} \pi^*(m_1 L_1 + m_2 L_2 + \Delta) - \big(m_1 + m_2 + \operatorname{mult}_P(\Delta)\big) E,$$

where $\overline{\Delta}$ is the proper transform of Δ on U. The inequality (3.1) implies that

$$\operatorname{mult}_Q(\overline{\Delta}) > 8/3 - \operatorname{mult}_P(\Delta) - m_1\big(1 + \operatorname{mult}_Q(\overline{L}_1)\big) - m_1\big(1 + \operatorname{mult}_Q(\overline{L}_2)\big). \quad (3.2)$$

Lemma 3.8. *The curve $\overline{L}_2$ does not contain the point Q.*

Proof. Suppose that $Q \in \overline{L}_2$. Then

$$1 - \mathrm{mult}_P(\Delta) - m_1 + m_2 = \overline{L}_2 \cdot \overline{\Delta} > 8/3 - \mathrm{mult}_P(\Delta) - m_1 - m_2$$

by Lemma 2.20. Hence $m_2 > 5/6$. On the other hand, it follows from Lemma 2.20 that

$$1 - m_2 - m_1 L_1 \cdot L_1 = \Delta \cdot L_1 > 4/3 - m_2.$$

However, $L_1 \cdot L_1 = -1$ if $L_1 \cap \mathrm{Sing}(X) = \varnothing$ and $L_1 \cdot L_1 = -1/2$ if $L_1 \cap \mathrm{Sing}(X) \neq \varnothing$. Then

$$m_1 > \begin{cases} 1/3 & \text{if } L_1 \cap \mathrm{Sing}(X) = \varnothing, \\ 2/3 & \text{if } L_1 \cap \mathrm{Sing}(X) \neq \varnothing, \end{cases}$$

by Corollary 3.5, which is impossible because $m_2 > 5/6$ and $1 > m_1 L_1 \cdot L_3 + m_2$.

Lemma 3.9. *The curve $\overline{L}_1$ does not contain the point Q.*

Proof. Suppose that $Q \in \overline{L}_1$. Arguing as in the proof of Lemma 3.8, we see that $L_1 \cap \mathrm{Sing}(X) \neq \varnothing$, which implies that $\overline{L}_1 \cdot \overline{L}_1 = -1/2$. Then $m_1 > 10/9$, because

$$1 + 3m_1/2 = \overline{L}_2 \cdot \big(\overline{\Delta} + \big(\mathrm{mult}_P(\Delta) - m_1 - m_2\big)E\big) > 8/3$$

by Lemma 2.20. On the other hand, $m_1 \leqslant 1$ by Remark 3.6. This contradiction completes the proof.

We see therefore that $\overline{L}_1 \not\ni Q \notin \overline{L}_2$. There is a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\ \zeta\ } & W \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\psi} \\ X & \dashrightarrow_{\rho} & \mathbb{P}^2 \end{array},$$

where ζ is a birational morphism contracting the curves $\overline{L}_1$ and $\overline{L}_2$, the morphism ψ is a double cover branched over a plane quartic curve, and ρ is the projection from the point P.

Let τ be the birational involution of U induced by ψ. Then

- τ is biregular $\iff$ $L_1 \cap \mathrm{Sing}(X) = \varnothing$,
- τ acts biregularly on $U \setminus \overline{L}_1$ if $L_1 \cap \mathrm{Sing}(X) \neq \varnothing$,
- the construction of τ shows that $\tau(\overline{L}_2) = \overline{L}_2$,
- if $L_1 \cap \mathrm{Sing}(X) = \varnothing$, then $\tau(\overline{L}_1) = \overline{L}_1$ and

$$\tau^*\big(\pi^*(-K_X)\big) \sim \pi^*(-2K_X) - 3E - \overline{L}_1 - \overline{L}_2.$$

Let $\bar{L}_3$ be the proper transform of L_3 on the surface U. Then $\tau(E) = \bar{L}_3$ and

$$L_1 \cup L_2 \not\ni \pi \circ \tau(Q) \in L_3.$$

Lemma 3.10. *The line L_3 is the only line on X that passes through the point $\pi \circ \tau(Q)$.*

Proof. Suppose there is a line $L \subset X$ such that $L \neq L_3$ and $\pi \circ \tau(Q) \in L$. Then $L \cap L_1 = L \cap L_2 = \varnothing$, because $\pi \circ \tau(Q) \in L_3$ and $(L_1 + L_2 + L_3) \cdot L = 1$. Thus, there is a unique conic $Z \subset X$ such that $-K_X \sim L + Z$ and $P \in Z$. Then Z is irreducible, since $P \notin L$ and P is not an Eckardt point.

Let $\bar{L}$ and $\bar{Z}$ be the proper transforms of L and Z on U, respectively. Then

$$\bar{L} \cdot \bar{L}_3 = \bar{Z} \cdot E = 1, \quad \bar{Z} \cdot \bar{Z} = 1 - \bar{L} \cdot \bar{Z}, \quad \bar{L} \cdot \bar{Z} = \begin{cases} 2 & \text{if } L \cap \mathrm{Sing}(X) = \varnothing, \\ 3/2 & \text{if } L \cap \mathrm{Sing}(X) \neq \varnothing, \end{cases}$$

and $\bar{L}_1 \cdot \bar{Z} = \bar{L}_2 \cdot \bar{Z} = \bar{L} \cdot E = \bar{L} \cdot \bar{L}_1 = \bar{L} \cdot \bar{L}_2 = 0$. By the construction of τ we have $\tau(\bar{Z}) = \bar{L}$. Then $Q \in \bar{Z}$, which implies that $\bar{Z} \subseteq \mathrm{Supp}(\bar{\Delta})$, because

$$2 - \mathrm{mult}_P(\Delta) - m_1 - m_2 = \bar{Z} \cdot \bar{\Omega} > 8/3 - \mathrm{mult}_P(\Delta) - m_1 - m_2$$

in the case when $\bar{Z} \not\subseteq \mathrm{Supp}(\bar{\Delta})$. On the other hand, the log pair $(X, \omega(L + Z))$ is log canonical at the point P. Hence by Remark 2.22 we may assume that $\bar{L} \not\subseteq \mathrm{Supp}(\bar{\Delta})$. Let $D = \varepsilon Z + m_1 L_1 + m_2 L_2 + \Upsilon$, where Υ is an effective $\mathbb{Q}$-divisor such that $Z \not\subseteq \mathrm{Supp}(\Upsilon)$. Then

$$\begin{aligned} 1 = L \cdot D = \varepsilon L \cdot Z + m_1 L \cdot L_1 + L \cdot \Upsilon = \varepsilon L \cdot Z + L \cdot \Upsilon \geqslant \varepsilon L \cdot Z \\ = \begin{cases} 2\varepsilon & \text{if } L \cap \mathrm{Sing}(X) = \varnothing, \\ 3\varepsilon/2 & \text{if } L \cap \mathrm{Sing}(X) \neq \varnothing, \end{cases} \end{aligned}$$

which implies that $\varepsilon \leqslant 2/3$. On the other hand, $\bar{Z} \cap \bar{L}_1 = \varnothing$. Hence it follows from Lemma 2.20 that

$$2 - \mathrm{mult}_P(D) - \varepsilon \bar{Z} \cdot \bar{Z} = \bar{Z} \cdot \bar{\Upsilon} > 8/3 - \mathrm{mult}_P(D),$$

where $\bar{\Upsilon}$ is the proper transform of Υ on U. We deduce that $\varepsilon > 2/3$, but we have already shown that $\varepsilon \leqslant 2/3$: a contradiction which completes the proof.

Therefore, there is a unique irreducible conic $C \subset X$ such that $-K_X \sim L_3 + C$ and $\pi \circ \tau(Q) \in C$. Then $C + L_3$ is a hyperplane section of X which is singular at $\pi \circ \tau(Q)$. Let $\bar{C}$ be the proper transform of C on U. We set $\bar{Z} = \tau(\bar{C})$ and $Z = \pi(\bar{Z})$.

Lemma 3.11. $L_1 \cap \mathrm{Sing}(X) \neq \varnothing$.

Proof. Suppose that $L_1 \cap \mathrm{Sing}(X) = \varnothing$. Then $C \cap L_1 = C \cap L_2 = \varnothing$, because $(L_1 + L_2 + L_3) \cdot C = L_3 \cdot C = 2$. One can easily check that $\bar{Z} \sim \pi^*(-2K_X) - 4E - \bar{L}_1 - \bar{L}_2$, and Z is singular at P. Then $-2K_X \sim Z + L_1 + L_2$, but the log pair $\big(U, \frac{1}{2}(Z + L_1 + L_2)\big)$ is log canonical at P. Thus (see Remark 2.22), we may assume that $Z \not\subseteq \mathrm{Supp}(D)$. By construction, $Q \in \overline{Z}$ and $\overline{Z} \cdot E = 2$. Then it follows from the inequality (3.1) that

$$4 - 2\,\mathrm{mult}_P(D) = \overline{Z} \cdot \overline{D} \geqslant \mathrm{mult}_Q(\overline{D}\,) > 8/3 - \mathrm{mult}_P(D),$$

which implies that $\mathrm{mult}_P(D) < 4/3$. However, this is impossible since $\mathrm{mult}_P(D) > 4/3$. The proof is complete.

Thus, $L_1 \cap L_3 = \mathrm{Sing}(X) \neq \varnothing$. Then $L_1 \cap L_2 \in C$, which implies that

$$\overline{Z} \sim \pi^*(-2K_X) - 4E - 2\overline{L}_1 - \overline{L}_2,$$

and Z is a smooth rational cubic. Then $-2K_X \sim Z + 2L_1 + L_2$, but the log pair $\bigl(U, \frac{1}{2}(Z+2L_1+L_2)\bigr)$ is log canonical at P. Thus, we may assume that $Z \not\subseteq \mathrm{Supp}(D)$ by Remark 2.22. We have $Q \in \overline{Z}$ and $\overline{Z} \cdot E = \overline{L}_1 = 1$. Then it follows from the inequality (3.1) that

$$3 - \mathrm{mult}_P(\Delta) - 2m_1 - m_2 = \overline{Z} \cdot \overline{\Delta} \geqslant \mathrm{mult}_Q(\,\overline{\Delta}\,) > 8/3 - \mathrm{mult}_P(\Delta) - m_1 - m_2,$$

which implies that $m_1 < 1/3$. On the other hand, $m_1 > 2/3$ by Corollary 3.5. This contradiction completes the proof of Theorem 3.2.

4. Del Pezzo surfaces

Let X be a del Pezzo surface that has at most canonical singularities, let O be a point of X, and let B_X be an effective $\mathbb{Q}$-divisor on X. Suppose that O is a smooth or an ordinary double point of X and that X is smooth away from $O \in X$.

Lemma 4.1. *Let* $\mathrm{Sing}(X) = O$ *and* $K_X^2 = 2$*, and suppose that* $B_X \sim_{\mathbb{Q}} -\mu K_X$*, where* $0 < \mu < 2/3$*. Then* $\mathbb{LCS}(X, \mu B_X) = \varnothing$*.*

Proof. Suppose that $\mathbb{LCS}(X, \mu B_X) \neq \varnothing$. Then there is a curve L with $\mathbb{P}^1 \cong L \subset X$ such that $\mathrm{LCS}(X, \mu B_X) \not\subseteq L$, the equality $L \cdot L = -1$ holds, and $L \cap \mathrm{Sing}(X) = \varnothing$. Therefore, there is a birational morphism $\pi\colon X \to S$ that contracts the curve L. Then $\mathbb{LCS}(S, \mu\pi(B_X)) \neq \varnothing$ due to the choice of the curve $L \subset X$. On the other hand, $-K_S \sim_{\mathbb{Q}} \pi(B_X)$, and S is a cubic surface in $\mathbb{P}^3$ that has at most one ordinary double point, which is impossible (see Examples 1.11 and 1.10).

Lemma 4.2. *Suppose that* $\mathrm{Sing}(X) = \varnothing$, $K_X^2 = 5$*, and* $B_X \sim_{\mathbb{Q}} -\mu K_X$*, where* $\mu \in \mathbb{Q}$ *is such that* $0 < \mu < 2/3$*. Assume that* $\mathbb{LCS}(X, B_X) \neq \varnothing$*. Then either the set* $\mathbb{LCS}(X, B_X)$ *contains a curve, or there exist a curve* L *with* $\mathbb{P}^1 \cong L \subset X$ *and a point* $P \in L$ *such that* $L \cdot L = -1$ *and* $\mathrm{LCS}(X, B_X) = P$*.*

Proof. Suppose that $\mathbb{LCS}(X, B_X)$ contains no curves. Then it follows from Theorem 2.7 that $\mathrm{LCS}(X, B_X) = P$ for some point $P \in X$. We may assume that P does not lie on any curve L with $\mathbb{P}^1 \cong L \subset X$ such that $L \cdot L = -1$. Then there is a birational morphism $\varphi\colon X \to \mathbb{P}^2$ that is an isomorphism in a neighbourhood of the point P. We note that $\varphi(P) \in \mathrm{LCS}(\mathbb{P}^2, \varphi(B_X))$, the set $\mathbb{LCS}(\mathbb{P}^2, \varphi(B_X))$ contains no curves, and $\varphi(B_X) \sim_{\mathbb{Q}} -\mu K_{\mathbb{P}^2}$. Since $\mu < 2/3$, the latter is impossible by Lemma 2.8, .

Example 4.3. Suppose that $O = \mathrm{Sing}(X)$ and $K_X^2 = 5$. Let $\alpha\colon V \to X$ be a blow-up of O and let E be the exceptional divisor of α. Then there is a birational morphism $\omega\colon V \to \mathbb{P}^2$ such that the morphism ω contracts the curves E_1, E_2, E_3, E_4, and the curve $\omega(E)$ is a line in $\mathbb{P}^2$ that contains $\omega(E_1)$, $\omega(E_2)$, and $\omega(E_3)$, but $\omega(E) \not\ni \omega(E_4)$.

Let Z be a line in $\mathbb{P}^2$ such that $\omega(E_1) \in Z \ni \omega(E_4)$. Then

$$2E + \overline{Z} + 2E_1 + E_2 + E_3 \sim -K_V,$$

where $\overline{Z}$ is the proper transform of Z on V. One has

$$\mathrm{lct}\big(X, \alpha(\overline{Z}\,) + 2\alpha(E_1) + \alpha(E_2) + \alpha(E_3)\big) = 1/2,$$

which implies that $\mathrm{lct}(X) \leqslant 1/2$. Suppose that $-K_X \sim_{\mathbb{Q}} 2B_X$, but (X, B_X) is not log canonical. Then

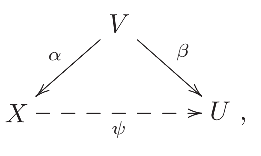
$$K_V + B_V + mE \sim_{\mathbb{Q}} \alpha^*(K_X + B_X)$$

for some $m \geqslant 0$, where B_V is the proper transform of B_X on the surface V. Then the log pair $(V, B_V + mE)$ is not log canonical at some point $P \in V$. There is a birational morphism $\pi\colon V \to U$ such that π is an isomorphism in a neighbourhood of $P \in X$ and U is a smooth del Pezzo surface with $K_U^2 = 6$. This implies that $\big(U, \pi(B_V) + m\pi(E)\big)$ is not log canonical at $\pi(P)$. On the other hand, $\pi(B_V) + m\pi(E) \sim_{\mathbb{Q}} -(1/2)K_U$, which is impossible because $\mathrm{lct}(U) = 1/2$ (see Example 1.10). Thus, $\mathrm{lct}(X) = 1/2$.

Example 4.4. Suppose that $K_X^2 = 4$. Arguing as in Example 4.3, we see that

$$\mathrm{lct}(X) = \begin{cases} 1/2 & \text{if } O = \mathrm{Sing}(X), \\ 2/3 & \text{if } \mathrm{Sing}(X) = \varnothing. \end{cases}$$

Suppose that $B_X \sim_{\mathbb{Q}} -K_X$ but the log pair $(X, \lambda B_X)$ is not log canonical at some point $P \in X \setminus O$. There is a commutative diagram



where U is a cubic surface in $\mathbb{P}^3$ that has canonical singularities, the morphism α is a blow-up of the point P, the morphism β is birational, and ψ is the projection from the point $P \in X$. Then

$$K_V + \lambda B_V + \big(\lambda\, \mathrm{mult}_P(B_X) - 1\big)E \sim_{\mathbb{Q}} \alpha^*(K_X + \lambda B_X),$$

where E is the exceptional divisor of α and B_V is the proper transform of B_X on V. We note that

$$\big(V, \lambda B_V + (\lambda\, \mathrm{mult}_P(B_X) - 1)E\big)$$

is not log canonical at some point $Q \in E$, and $\mathrm{mult}_P(B_X) > 1/\lambda$. Then the log pair

$$\big(V, \lambda B_V + (\lambda\, \mathrm{mult}_P(B_X) - \lambda)E\big)$$

is also not log canonical at the point $Q \in E$, but

$$B_V + \big(\mathrm{mult}_P(B_X) - 1\big)E \sim_{\mathbb{Q}} -K_V + \alpha^*(K_X + B_X) \sim_{\mathbb{Q}} -K_V.$$

Suppose that P is not contained in any line on the surface X. Then

- the morphism $\beta \colon V \to U$ is an isomorphism,
- is cubic surface is smooth away from $\psi(O)$,
- the point $\psi(O)$ is an ordinary double point of the surface U,

which implies that $\lambda > 2/3$ (see Example 1.11).

Let $\lambda = 3/4$. Then $\psi(Q) \in U \subset \mathbb{P}^3$ must be an Eckardt point of the surface U by Theorem 3.2 (see Definition 3.1). On the other hand, $\beta(E) \subset U$ is a line, so X contains two irreducible conics $C_1 \neq C_2$ such that $P = C_1 \cap C_2$ and $C_1 + C_2 \sim -K_X$.

Lemma 4.5. *Suppose that* $O = \mathrm{Sing}(X)$, $K_X^2 = 6$, *and there is a diagram*

$$\begin{array}{ccc} & V & \\ {\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\beta} \\ X & & \mathbb{P}^2 \end{array},$$

where β is a blow-up of three points $P_1, P_2, P_3 \in \mathbb{P}^2$ lying on a line $L \subset \mathbb{P}^2$, and α is a birational morphism contracting an irreducible curve $\bar{L}$ to the point O such that $\beta(\bar{L}) = L$. Then $\mathrm{LCS}(X, \lambda B_X) = O$ *in the case when* $\mathrm{LCS}(X, \lambda B_X) \neq \varnothing$, $B_X \sim_{\mathbb{Q}} -K_X$, *and* $\lambda < 1/2$.

Proof. Suppose that $\varnothing \neq \mathrm{LCS}(X, \lambda B_X) \neq O$ but $B_X \sim_{\mathbb{Q}} -K_X$. Let M be a general line in $\mathbb{P}^2$ and let $\overline{M}$ be its proper transform on V. Then $-K_X \sim 2\alpha(\overline{M}\,)$ and $O \in \alpha(\overline{M}\,)$. Thus, the set $\mathbb{LCS}(X, \lambda B_X)$ contains a curve, because otherwise the locus $\mathrm{LCS}(X, \lambda B_X + \alpha(\overline{M}\,))$ would be disconnected, which is impossible by Theorem 2.7.

Let C be an irreducible curve on X such that $C \subseteq \mathrm{LCS}(X, \lambda B_X)$. Then $B_X = \varepsilon C + \Omega$, where $\varepsilon > 2$ and Ω is an effective $\mathbb{Q}$-divisor such that $C \not\subset \mathrm{Supp}(\Omega)$.

Let Γ_i be a proper transform on X of a sufficiently general line in $\mathbb{P}^2$ that passes through P_i. Then $O \notin \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $-K_X \cdot \Gamma_1 = -K_X \cdot \Gamma_2 = -K_X \cdot \Gamma_3 = 2$. On the other hand, $-K_X \sim_{\mathbb{Q}} \Gamma_1 + \Gamma_2 + \Gamma_3$, which implies that there is an $m \in \{1, 2, 3\}$ such that $C \cdot \Gamma_m \neq 0$. Then

$$2 = B_X \cdot \Gamma_m = (\varepsilon C + \Omega) \cdot \Gamma_m \geqslant \varepsilon C \cdot \Gamma_m \geqslant \varepsilon > 2,$$

because $\Gamma_m \not\subset \mathrm{Supp}(B_X)$. This contradiction completes the proof.

Remark 4.6. Suppose that $O = \mathrm{Sing}(X)$ and $K_X^2 = 6$. Let $\alpha \colon V \to X$ be a blow-up of the point $O \in X$, and let E be the exceptional divisor of α. Then

$$K_V + B_V + mE \sim_{\mathbb{Q}} \alpha^*(K_X + B_X)$$

for some $m \geqslant 0$, where B_V is the proper transform of B_X on V. We note that $\mathrm{lct}(X) \leqslant 1/3$. Suppose that $\mathrm{lct}(X) < 1/3$, that is, there exists an effective $\mathbb{Q}$-divisor $B_X \sim_{\mathbb{Q}} -K_X$ such that the log pair $\big(X, \frac{1}{3}B_X\big)$ is not log canonical. Then the log pair $\big(V, \frac{1}{3}(B_V + mE)\big)$ is not log canonical at some point $P \in V$. There is a birational morphism $\pi\colon V \to U$ such that either $U \cong \mathbb{F}_1$ or $U \cong \mathbb{P}^1 \times \mathbb{P}^1$, and the birational morphism π is an isomorphism in a neighbourhood of $P \in X$. Then the log pair $\big(U, \frac{1}{3}(\pi(B_V)+m\pi(E))\big)$ is not log canonical at the point $\pi(P)$. On the other hand, $-K_U \sim_{\mathbb{Q}} \pi(B_V)+m\pi(E)$, which immediately yields a contradiction to Example 1.10. Hence $\mathrm{lct}(X) = 1/3$.

Lemma 4.7. *Suppose that* $X \cong \mathbb{P}(1,1,2)$ *and* $B_X \sim_{\mathbb{Q}} -K_X$*, but there is a point* $P \in X$ *such that* $O \neq P \in \mathrm{LCS}(X, \lambda B_X)$ *for some non-negative rational* $\lambda < 1/2$*. Let* L *be the unique curve in the linear system* $|\mathscr{O}_{\mathbb{P}(1,1,2)}(1)|$ *such that* $P \in L$*. Then* $L \subseteq \mathrm{LCS}(X, \lambda B_X)$*.*

Proof. Suppose there is a curve $\Gamma \in \mathrm{LCS}(X, \lambda B_X)$ such that $P \in \Gamma \neq L$. Then $B_X = \mu\Gamma + \Omega$, where $\mu > 2$ and Ω is an effective $\mathbb{Q}$-divisor such that $\Gamma \not\subset \mathrm{Supp}(\Omega)$. Hence $\mu\Gamma + \Omega \sim_{\mathbb{Q}} 4L$ and $\Gamma \sim mL$, where $m \in \mathbb{Z}_{>0}$. On the other hand, we have $P \in \Gamma \neq L$, and therefore $m \geqslant 2$, which yields a contradiction.

Suppose that $L \not\subseteq \mathrm{LCS}(X, \lambda B_X)$. Then it follows from Theorem 2.7 that $\mathrm{LCS}(X, \lambda B_X) = P$, because we have proved that $\mathbb{LCS}(X, \lambda B_X)$ contains no curves passing through P.

Let C be a general curve in the linear system $|\mathscr{O}_{\mathbb{P}(1,1,2)}(1)|$. Then $\mathrm{LCS}(X, \lambda B_X + C) = P \cup C$, which is impossible by Theorem 2.7.

Lemma 4.8. *Suppose that* $X \cong \mathbb{F}_1$*. Then there are* $0 \leqslant \mu \in \mathbb{Q} \ni \lambda \geqslant 0$ *such that* $B_X \sim_{\mathbb{Q}} \mu C + \lambda L$*, where* C *and* L *are irreducible curves on* X *such that* $C \cdot C = -1$*,* $C \cdot L = 1$*, and* $L \cdot L = 0$*. Suppose that* $\mu < 1$ *and* $\lambda < 1$*. Then* $\mathbb{LCS}(X, B_X) = \varnothing$*.*

Proof. Obviously, the set $\mathbb{LCS}(X, B_X)$ contains no curves, because L and C generate the cone of effective divisors of the surface X. Suppose that $\mathbb{LCS}(X, B_X)$ contains a point $O \in X$. Then

$$K_X + B_X + \big((1-\mu)C + (2-\lambda)L\big) \sim_{\mathbb{Q}} -(L + C),$$

because $-K_X \sim_{\mathbb{Q}} 2C + 3L$. On the other hand, it follows from Theorem 2.6 that the map

$$0 = H^0\big(\mathscr{O}_X(-L-C)\big) \to H^0(\mathscr{O}_{\mathscr{L}(X,\,B_X)}) \neq 0$$

is surjective, because the divisor $(1-\mu)C + (2-\lambda)L$ is ample: a contradiction.

Lemma 4.9. *Suppose that* $\mathrm{Sing}(X) = \varnothing$ *and* $K_X^2 = 7$*. Then*

$$L_1 \cdot L_1 = L_2 \cdot L_2 = L_3 \cdot L_3 = -1, \qquad L_1 \cdot L_2 = L_2 \cdot L_3 = 1, \qquad L_1 \cdot L_3 = 0,$$

where L_1*,* L_2*,* L_3 *are exceptional curves on* X*. Suppose that* $\mathrm{LCS}(X, B_X) \neq \varnothing$ *but* $B_X \sim_{\mathbb{Q}} -\mu K_X$*, where* $\mu < 1/2$*. Then* $\mathrm{LCS}(X, B_X) = L_2$*.*

Proof. Let P be a point in $\mathrm{LCS}(X, B_X)$. Then $P \in L_2$, because $\mathrm{lct}(\mathbb{P}^1 \times \mathbb{P}^1) = 1/2$ and there is a birational morphism $X \to \mathbb{P}^1 \times \mathbb{P}^1$ that contracts only the curve L_2.

Suppose that $\mathrm{LCS}(X, B_X) \neq L_2$. Then $\mathrm{LCS}(X, B_X) = P$ by Theorem 2.7.

We may assume that $P \notin L_3$. Then there is a birational morphism $\varphi\colon X \to \mathbb{P}^2$ that contracts the curves L_1 and L_3. Let C_1 and C_3 be the proper transforms on X of sufficiently general lines in $\mathbb{P}^2$ that pass through the points $\varphi(L_1)$ and $\varphi(L_3)$, respectively. Then $-K_X \sim C_1 + 2C_3 + L_3$ but $C_1 \not\ni P \notin C_3$. We see that

$$C_3 \cup P \subseteq \mathrm{LCS}\biggl(X, \lambda D + \frac{1}{2}\,(C_1 + 2C_3 + L_3)\biggr) \subseteq C_3 \cup P \cup L_3,$$

which is impossible by Theorem 2.7, because $P \notin L_3$.

Lemma 4.10. *Suppose that* $O = \mathrm{Sing}(X)$, $K_X^2 = 7$, *and* $B_X \sim_{\mathbb{Q}} C + (4/3)L$, *where* $L \cong \mathbb{P}^1 \cong C$ *are curves on the surface* X *such that* $L \cdot L = -1/2$, $C \cdot C = -1$, *and* $C \cdot L = 1$, *but the log pair* (X, B_X) *is not log canonical at some point* $P \in C$. *Then* $P \in L$.

Proof. Let S be a quadratic cone in $\mathbb{P}^3$. Then $S \cong \mathbb{P}(1,1,2)$ and there is a birational morphism $\varphi\colon X \to S \subset \mathbb{P}^3$ that contracts the curve C to a smooth point $Q \in S$. Then $Q \in \varphi(L) \in |\mathscr{O}_{\mathbb{P}(1,1,2)}(1)|$.

Suppose that $P \notin L$. Then it follows from Remark 2.22 that to complete the proof we may assume that either $C \not\subset \mathrm{Supp}(B_X)$ or $L \not\subset \mathrm{Supp}(B_X)$, because the log pair $(X, C + (4/3)L)$ is log canonical at the point $P \in X$. Suppose that $C \not\subset \mathrm{Supp}(B_X)$. Then $1/3 = B_X C \geqslant \mathrm{mult}_P(B_X) > 1$, which is impossible. Therefore, $C \subset \mathrm{Supp}(B_X)$. Hence we may assume that $L \not\subset \mathrm{Supp}(B_X)$.

We put $B_X = \varepsilon C + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $C \not\subset \mathrm{Supp}(\Omega)$. Then $1/3 = B_X \cdot L = \varepsilon + \Omega \cdot L \geqslant \varepsilon$, which implies that $\varepsilon \leqslant 1/3$. Then $1 < \Omega \cdot C = 1/3 + \varepsilon \leqslant 2/3$ by Lemma 2.20, a contradiction. The proof is complete.

5. Toric varieties

The aim of his section is to prove Lemma 5.1 (cf. [30], [55]).

Let $N = \mathbb{Z}^n$ be a lattice of rank n and $M = \mathrm{Hom}(N, \mathbb{Z})$ the dual lattice. Let $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. Let X be a toric variety defined by a complete fan $\Sigma \subset N_{\mathbb{R}}$; let $\Delta_1 = \{v_1, \dots, v_m\}$ be the set of generators of one-dimensional cones of the fan Σ. We put

$$\Delta = \{w \in M \mid \langle w, v_i \rangle \geqslant -1 \text{ for all } i = 1, \dots, m\}.$$

Let $T = (\mathbb{C}^*)^n \subset \mathrm{Aut}(X)$, let $\mathscr{N}$ be the normalizer of T in $\mathrm{Aut}(X)$ and $\mathscr{W} = \mathscr{N}/T$.

Lemma 5.1. *Let* $G \subset \mathscr{W}$ *be a subgroup. Suppose that* X *is* $\mathbb{Q}$*-factorial. Then*

$$\mathrm{lct}(X, G) = \frac{1}{1 + \max\{\langle w, v \rangle \mid w \in \Delta^G,\ v \in \Delta_1\}}\,,$$

where Δ^G *is the set of points in* Δ *that are fixed by the group* G.

Proof. We put $\mu = 1 + \max\{\langle w, v \rangle \mid w \in \Delta^G,\ v \in \Delta_1\}$. Then $\mu \in \mathbb{Q}$ is the largest number such that $-K_X \sim_{\mathbb{Q}} \mu R + H$, where R is a $T \rtimes G$-invariant effective Weil divisor and H is an effective $\mathbb{Q}$-divisor. Hence $\mathrm{lct}(X, G) \leqslant 1/\mu$.

Suppose that $\mathrm{lct}(X, G) < 1/\mu$. Then there is a G-invariant effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/\mu$.

There exists a family $\{D_t \mid t \in \mathbb{C}\}$ of G-invariant effective $\mathbb{Q}$-divisors such that

- $D_t \sim_{\mathbb{Q}} D$ for every $t \in \mathbb{C}$,
- $D_1 = D$,
- for every $t \neq 0$ there is a $\varphi_t \in \operatorname{Aut}(X)$ such that $D_t = \varphi_t(D) \cong D$,
- the divisor D_0 is T-invariant,

which implies that $(X, \lambda D_0)$ is not log canonical (see [21]).

On the other hand, the divisor D_0 does not have components with multiplicity greater than μ, which implies that $(X, \lambda D_0)$ is log canonical (see [56]). This is a contradiction.

Corollary 5.2. *Let $X = \mathbb{P}\big(\mathscr{O}_{\mathbb{P}^n} \oplus \mathscr{O}_{\mathbb{P}^n}(-a_1) \oplus \cdots \oplus \mathscr{O}_{\mathbb{P}^n}(-a_k)\big)$, where $a_i \geqslant 0$ for $i = 1, \dots, k$. Then*

$$\operatorname{lct}(X) = \frac{1}{1 + \max\big\{k, n + \sum_{i=1}^{k} a_i\big\}}.$$

Proof. We note that X is a toric variety and Δ_1 consists of the following vectors:

$$\begin{gathered}\big(\overbrace{1,0,\dots,0}^{k},\overbrace{0,0,\dots,0}^{n}\big),\dots,(0,\dots,0,1,0,0,\dots,0),\\ (-1,\dots,-1,0,0,\dots,0),\\ (0,0,\dots,0,1,0,\dots,0),\dots,(0,0,\dots,0,0,\dots,0,1),\\ (-a_1,\dots,-a_k,-1,\dots,-1),\end{gathered}$$

which implies the required assertion by Lemma 5.1.

Applying Corollary 5.2, we obtain the following result.

Corollary 5.3. *In the notation of* § 1 *one has* $\operatorname{lct}(X) = 1/4$ *if* $\daleth(X) \in \{2.33, 2.35\}$, *and one has* $\operatorname{lct}(X) = 1/5$ *if* $\daleth(X) = 2.36$.

Straightforward calculations using Lemma 5.1 yield the following result.

Corollary 5.4. *In the notation of* § 1,

$$\operatorname{lct}(X) = \begin{cases} 1/3 & \textit{if } \daleth(X) \in \{3.25, 3.31, 4.9, 4.11, 5.2\},\\ 1/4 & \textit{if } \daleth(X) \in \{3.26, 3.30, 4.12\},\\ 1/5 & \textit{if } \daleth(X) = 3.29.\end{cases}$$

Remark 5.5. Suppose that the toric variety X is symmetric, that is, $\Delta^{\mathscr{W}} = \{0\}$ (see, for instance, [30]). Then it follows from Lemma 5.1 that the global log canonical threshold $\operatorname{lct}(X, \mathscr{W})$ is equal to 1. We note that this equality was proved in [30] and [55] under the additional assumption that X is smooth.

6. Del Pezzo threefolds

Throughout this section we use the assumptions and the notation from § 1. Suppose that $-K_X \sim 2H$, where H is a Cartier divisor that is indivisible in $\operatorname{Pic}(X)$. The aim of this section is to prove the following result.

Theorem 6.1. *The equality* $\operatorname{lct}(X) = 1/2$ *holds unless* $\daleth(X) = 2.35$, *when* $\operatorname{lct}(X) = 1/4$.

It follows from Theorems 3.1.14 and 3.3.1 in [2] that $\gimel(X) \in \{1.11, \ldots, 1.15, 2.32, 2.35, 3.27\}$. By [5] and [13] (see also Lemma 2.17) one has $\mathrm{lct}(X) = 1/2$ if $\gimel(X) \in \{1.12, 1.13\}$. It follows from Lemma 2.29 that $\mathrm{lct}(X) = 1/2$ when $\gimel(X) = 3.27$. Lemma 5.1 implies that $\mathrm{lct}(X) = 1/4$ if $\gimel(X) = 2.35$.

The remaining cases are: $\gimel(X) \in \{1.11, 1.14, 1.15, 2.32\}$, and the inequality $\mathrm{lct}(X) \leqslant 1/2$ is obvious here, because the linear system $|H|$ is non-empty.

Lemma 6.2. *If* $\gimel(X) = 2.32$, *then* $\mathrm{lct}(X) = 1/2$.

Proof. We may suppose that $\mathrm{lct}(X) < 1/2$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} H$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1$.

The threefold X is a divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,1)$. There are two natural $\mathbb{P}^1$-bundles $\pi_1 \colon X \to \mathbb{P}^2$ and $\pi_2 \colon X \to \mathbb{P}^2$; applying Theorem 2.27 to them, we immediately obtain a contradiction.

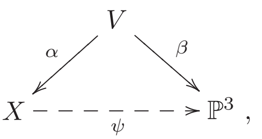
Remark 6.3. Suppose that $\mathrm{Pic}(X) = \mathbb{Z}[H]$ and there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} H$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1$. We put $D = \varepsilon S + \Omega \sim_{\mathbb{Q}} H$, where S is an (irreducible) surface and Ω is an effective $\mathbb{Q}$-divisor such that $\mathrm{Supp}(\Omega) \not\supset S$. Then $\varepsilon \leqslant 1$ because $\mathrm{Pic}(X) = \mathbb{Z}[H]$, which implies that the set $\mathbb{LCS}(X, \lambda D)$ contains no surfaces. Moreover, for any choice of $H \in |H|$ the locus $\mathrm{LCS}(X, \lambda D + H)$ is connected by Theorem 2.7. Let H be a general surface in the linear system $|H|$. Since $\mathrm{LCS}(X, \lambda D + H)$ is connected, it follows that $\mathrm{LCS}(X, \lambda D + H)$ has no isolated zero-dimensional components outside the base locus of $|H|$. Furthermore, $|H|$ has no base points except in the case $\gimel(X) = 1.11$, when the base locus of $|H|$ consists of a single point O. We note that in the last case $O \notin \mathrm{LCS}(X, \lambda D)$, since X is covered by the curves of anticanonical degree 2 passing through O. Hence the locus $\mathrm{LCS}(X, \lambda D)$ never has isolated zero-dimensional components; in particular, it contains an (irreducible) curve C. We put $D|_H = \overline{D}$. Then $-K_H \sim_{\mathbb{Q}} \overline{D}$, but $(H, \lambda\overline{D})$ is not log canonical at every point of the intersection $H \cap C$. The locus $\mathrm{LCS}(H, \lambda\overline{D})$ is connected by Theorem 2.7. But the scheme $\mathscr{L}(H, \lambda\overline{D})$ is zero dimensional, so $H \cdot C = |H \cap C| = 1$ and the locus $\mathrm{LCS}(X, \lambda D)$ contains no curves besides C.

Lemma 6.4. *If* $\gimel(X) = 1.14$, *then* $\mathrm{lct}(X) = 1/2$.

Proof. We may suppose that $\mathrm{lct}(X) < 1/2$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} H$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1$.

The linear system $|H|$ induces an embedding $X \subset \mathbb{P}^5$ such that X is a complete intersection of two quadrics. Then $\mathrm{LCS}(X, \lambda D)$ consists of a single line $C \subset X$ by Remark 6.3.

It follows from Proposition 3.4.1 in [2] that there is a commutative diagram



where ψ is the projection from C, the morphism α is a blow-up of the line C, and β is a blow-up of a smooth curve $Z \subset \mathbb{P}^3$ of degree 5 and genus 2.

Let S be the exceptional divisor of β and let L be a fibre of the morphism β over a general point of the curve Z. We put $\bar{S}=\alpha(S)$ and $\bar{L}=\alpha(L)$. Then $\bar{S}\sim 2H$, the curve $\bar{L}$ is a line, and $\mathrm{mult}_C(\bar{S})=3$. Here the log pair $(X,(1/2)\bar{S})$ is log canonical, so we may assume (see Remark 2.22) that $\mathrm{Supp}(D)\not\supset\bar{S}$. Then $1=\bar{L}\cdot D\geqslant\mathrm{mult}_C(D)>1$, a contradiction.

Remark 6.5. Let $V\subset\mathbb{P}^5$ be a complete intersection of two quadric hypersurfaces that has isolated singularities, and let B_V be an effective $\mathbb{Q}$-divisor on V such that $B_V\sim_{\mathbb{Q}}-K_V$ and $\mathrm{LCS}(V,\mu B_V)\neq\varnothing$, where $\mu<1/2$. Arguing as in the proof of Lemma 6.4, we see that $\mathrm{LCS}(V,\mu B_V)\subseteq L$, where $L\subset V$ is a line such that $L\cap\mathrm{Sing}(V)\neq\varnothing$.

Lemma 6.6. *If* $\daleth(X)=1.15$, *then* $\mathrm{lct}(X)=1/2$.

Proof. This is analogous to the proof of Lemma 6.4.

Lemma 6.7. *If* $\daleth(X)=1.11$, *then* $\mathrm{lct}(X)=1/2$.

Proof. We may suppose that $\mathrm{lct}(X)<1/2$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}H$ such that the log pair $(X,\lambda D)$ is not log canonical for some $\lambda<1$.

Recall that the threefold X can be given by an equation

$$w^2=t^3+t^2f_2(x,y,z)+tf_4(x,y,z)+f_6(x,y,z)\subset\mathbb{P}(1,1,1,2,3)\cong\mathrm{Proj}(\mathbb{C}[x,y,z,t,w]),$$

where $\mathrm{wt}(x)=\mathrm{wt}(y)=\mathrm{wt}(z)=1$, $\mathrm{wt}(t)=2$, $\mathrm{wt}(w)=3$, and f_i is a polynomial of degree i.

By Remark 6.3 the locus $\mathrm{LCS}(X,\lambda D)$ consists of a single curve $C\subset X$ such that $H\cdot C=1$.

Let $\psi\colon X\dashrightarrow\mathbb{P}^2$ be the natural projection. Then ψ is not defined at the point O cut out by $x=y=z=0$. The curve C does not contain the point O, because otherwise

$$1=\Gamma\cdot D\geqslant\mathrm{mult}_O(D)\,\mathrm{mult}_O(\Gamma)\geqslant\mathrm{mult}_C(D)>1/\lambda>1,$$

where Γ is a general fibre of the projection ψ. Thus, we see that $\psi(C)\subset\mathbb{P}^2$ is a line.

Let S be the (unique) surface in $|H|$ such that $C\subset S$. Let L be a general fibre of the rational map ψ that intersects the curve C. Then $L\subset\mathrm{Supp}(D)$ since otherwise $1=D\cdot L\geqslant\mathrm{mult}_C(D)>1/\lambda>1$.

We may assume that $D=S$ by Remark 2.22. Then S has a cuspidal singularity along C. We may assume that the surface S is cut out on X by the equation $x=0$, and the curve C is given by the equations $w=t=x=0$. Then S is given by

$$w^2=t^3+t^2f_2(0,y,z)+tf_4(0,y,z)\subset\mathbb{P}(1,1,2,3)\cong\mathrm{Proj}(\mathbb{C}[y,z,t,w]),$$

and $f_6(x,y,z)=xf_5(x,y,z)$, where $f_5(x,y,z)$ is a homogeneous polynomial of degree 5.

Since the surface S is singular along C, it follows that $f_4(x,y,z) = xf_3(x,y,z)$, where $f_3(x,y,z)$ is a homogeneous polynomial of degree 3. Then every point of the set

$$x = f_5(x,y,z) = t = w = 0 \subset \mathbb{P}(1,1,1,2,3)$$

must be singular on X, which is a contradiction because X is smooth.

The proof of Theorem 6.1 is complete.

7. Threefolds with Picard number $\rho = 2$

We use the assumptions and notation introduced in § 1.

Lemma 7.1. *If* $\gimel(X) = 2.1$ *or* 2.3, *then* $\operatorname{lct}(X) = 1/2$.

Proof. There is a birational morphism $\alpha\colon X \to V$ that contracts a surface $E \subset X$ to a smooth elliptic curve $C \subset V$, where V is one of the following Fano threefolds: a smooth hypersurface in $\mathbb{P}(1,1,1,2,3)$ of degree 6; a smooth hypersurface in $\mathbb{P}(1,1,1,1,2)$ of degree 4.

The curve C lies in a surface $H \subset V$ such that $\operatorname{Pic}(V) = \mathbb{Z}[H]$ and $-K_X \sim 2H$. Then C is a complete intersection of two surfaces in $|H|$, and $-K_X \sim 2\overline{H} + E$, where E is the exceptional divisor of the birational morphism α, and $\overline{H}$ is the proper transform of the surface H on the threefold X. In particular, the inequality $\operatorname{lct}(X) \leqslant 1/2$ holds.

Suppose that $\operatorname{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$. Then $\operatorname{LCS}(X, \lambda D) \subseteq E$ since $\operatorname{lct}(V) = 1/2$ by Theorem 6.1 and $\alpha(D) \sim_{\mathbb{Q}} 2H \sim -K_V$.

We put $k = H \cdot C$. Then $k = H^3 \in \{1, 2\}$. We note that

$$\mathscr{N}_{C/V} \cong \mathscr{O}_C\bigl(H|_C\bigr) \oplus \mathscr{O}_C\bigl(H|_C\bigr),$$

which implies that $E \cong C \times \mathbb{P}^1$. Let $Z \cong C$ and $L \cong \mathbb{P}^1$ be curves on E such that $Z \cdot Z = L \cdot L = 0$ and $Z \cdot L = 1$. Then $\alpha^*(H)|_E \sim kL$, and since

$$-2Z \sim K_E \sim (K_X + E)|_E \sim \bigl(2E - 2\alpha^*(H)\bigr)\big|_E \sim -2kL + 2E|_E,$$

we see that $E|_E \sim -Z + kL$. We put $D = \mu E + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor on X such that $E \not\subset \operatorname{Supp}(\Omega)$. The pair $(X, E + \lambda\Omega)$ is not log canonical in a neighbourhood of E. Hence the pair $(E, \lambda\Omega|_E)$ is also not log canonical by Theorem 2.19. But

$$\Omega|_E \sim_{\mathbb{Q}} (-K_X - \mu E)|_E \sim_{\mathbb{Q}} \bigl(2\alpha^*(H) - (1+\mu)E\bigr)\big|_E \sim_{\mathbb{Q}} (1+\mu)Z + k(1-\mu)L,$$

and $0 \leqslant \lambda k(1-\mu) \leqslant 1$, which contradicts Lemma 2.23.

Lemma 7.2. *If* $\gimel(X) = 2.4$ *and* X *is general, then* $\operatorname{lct}(X) = 3/4$.

Proof. There is a commutative diagram

$$\begin{array}{ccc} & X & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ \mathbb{P}^3 & \overset{\psi}{\dashrightarrow} & \mathbb{P}^1 \end{array},$$

where ψ is a rational map, α is a blow-up of a smooth curve $C \subset \mathbb{P}^3$ such that $C = H_1 \cdot H_2$ for some $H_1, H_2 \in |\mathscr{O}_{\mathbb{P}^3}(3)|$, and β is a fibration into cubic surfaces.

Let $\mathscr{P}$ be the pencil in $|\mathscr{O}_{\mathbb{P}^3}(3)|$ generated by H_1 and H_2. Then ψ is given by $\mathscr{P}$.

We assume that X satisfies the following generality conditions: every surface in $\mathscr{P}$ has at most one ordinary double point; the curve C contains no Eckardt points[4] (see Definition 3.1) of any surface in $\mathscr{P}$.

Let E be the exceptional divisor of the blow-up α. Then

$$\frac{4}{3}\overline{H}_1 + \frac{1}{3}E \sim_{\mathbb{Q}} \frac{4}{3}\overline{H}_2 + \frac{1}{3}E \sim_{\mathbb{Q}} -K_X,$$

where $\overline{H}_i$ is the proper transform of H_i on the threefold X. In particular, we see that $\operatorname{lct}(X) \leqslant 3/4$.

Suppose that $\operatorname{lct}(X) < 3/4$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 3/4$.

Suppose that the set $\mathbb{LCS}(X, \lambda D)$ contains an (irreducible) surface $S \subset X$. Then $D = \varepsilon S + \Delta$, where $\varepsilon \geqslant 1/\lambda$ and Δ is an effective $\mathbb{Q}$-divisor such that $S \not\subset \operatorname{Supp}(\Delta)$. By Remark 2.3, in this case the log pair $\big(\overline{H}_1, D|_{\overline{H}_1}\big)$ is not log canonical if $S \cap \overline{H}_1 \neq \varnothing$. But $D|_{\overline{H}_1} \sim_{\mathbb{Q}} -K_{\overline{H}_1}$. We can choose $\overline{H}_1$ to be a smooth cubic surface in $\mathbb{P}^3$. Thus, it follows from Theorem 3.2 that $S \cap \overline{H}_1 = \varnothing$, which implies that $S \sim \overline{H}_1$. Thus, $\alpha(S)$ is a surface in $\mathscr{P}$. Then $\varepsilon\alpha(S) + \alpha(\Delta) \sim_{\mathbb{Q}} \mathscr{O}_{\mathbb{P}^3}(4)$, which is impossible because $\varepsilon \geqslant 1/\lambda > 4/3$.

Let F be a fibre of β such that $F \cap \operatorname{LCS}(X, \lambda D) \neq \varnothing$. We set $D = \mu F + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $F \not\subset \operatorname{Supp}(\Omega)$. Then the log pair $\big(F, \lambda\Omega|_F\big)$ is not log canonical by Theorem 2.19, because $\lambda\mu < 1$. It follows from Theorem 3.2 that $\operatorname{LCS}\big(F, \lambda\Omega|_F\big) = O$, where O is either an Eckardt point of the surface F or a singular point of F. By Theorem 2.7

$$\operatorname{LCS}(X, \lambda D) = \operatorname{LCS}(X, \lambda\mu F + \lambda\Omega D) = O,$$

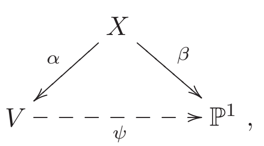
because it follows from Theorem 2.19 that $(X, F + \lambda\Omega D)$ is not log canonical at O but is log canonical in a punctured neighbourhood of O. But $O \notin E$ by our generality assumptions. Hence

$$\alpha(O) \subset \operatorname{LCS}\big(\mathbb{P}^3, \lambda\alpha(D)\big) \subseteq \alpha(O) \cup C,$$

where $\alpha(O) \notin C$. On the other hand, $\lambda < 3/4$, which contradicts Lemma 2.8.

Lemma 7.3. *If* $\daleth(X) \in \{2.5, 2.10, 2.14\}$ *and X is general, then* $\operatorname{lct}(X) = 1/2$.

Proof. There is a commutative diagram



[4]We note that C also does not contain singular points of surfaces in $\mathscr{P}$, since C is a complete intersection of two surfaces in $\mathscr{P}$.

where V is a smooth Fano threefold such that $-K_V \sim 2H$ for some $H \in \operatorname{Pic}(V)$ and $\gimel(V) \in \{1.13, 1.14, 1.15\}$, the morphism α is a blow-up of a smooth curve $C \subset V$ such that $C = H_1 \cdot H_2$ for some $H_1, H_2 \in |H|$ with $H_1 \neq H_2$, the morphism β is a del Pezzo fibration, and ψ is the projection from C.

Let E be the exceptional divisor of the blow-up α. Then $2\overline{H}_1 + E \sim 2\overline{H}_2 + E \sim -K_X$, where $\overline{H}_i$ is the proper transform of H_i on the threefold X. In particular, $\operatorname{lct}(X) \leqslant 1/2$.

Suppose that $\operatorname{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$. Then $\operatorname{LCS}(X, \lambda D) \subseteq E$, because $\alpha(D) \sim_{\mathbb{Q}} -K_V$ and $\operatorname{lct}(V) = 1/2$ by Theorem 6.1.

We assume that the threefold X satisfies the following generality condition: every fibre of the fibration β has at most one singular point, which is an ordinary double point.

Let F be a fibre of β such that $F \cap \operatorname{LCS}(X, \lambda D) \neq \varnothing$. We put $D = \mu F + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor on X such that $F \not\subset \operatorname{Supp}(\Omega)$. Then

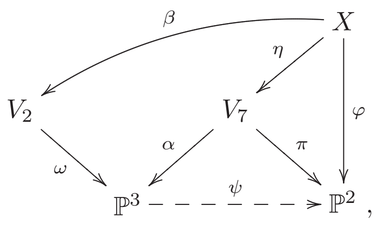
$$\alpha(D) = \mu\alpha(F) + \alpha(\Omega) \sim_{\mathbb{Q}} 2\alpha(F) \sim_{\mathbb{Q}} -K_V,$$

which implies that $\mu \leqslant 2$. We note that the pair $\bigl(F, \lambda\Omega|_F\bigr)$ is not log canonical by Theorem 2.19. However, $\Omega|_F \sim_{\mathbb{Q}} -K_F$, which implies that $\operatorname{lct}(F) \leqslant \lambda < 1/2$. On the other hand, F has at most one ordinary double point and $K_F^2 = H^3 \leqslant 5$, which implies that $\operatorname{lct}(F) \geqslant 1/2$ (see Examples 1.10, 1.11, 4.3, and 4.4), which is a contradiction.

Lemma 7.4. *If* $\gimel(X) = 2.8$ *and* X *is general, then* $\operatorname{lct}(X) = 1/2$.

Proof. Let $O \in \mathbb{P}^3$ and let $\alpha\colon V_7 \to \mathbb{P}^3$ be a blow-up of the point O. Then $V_7 \cong \mathbb{P}(\mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(1))$ and there is a $\mathbb{P}^1$-bundle $\pi\colon V_7 \to \mathbb{P}^2$. Let E be the exceptional divisor of the birational morphism α. Then E is a section of π.

There is a quartic surface $R \subset \mathbb{P}^3$ such that $\operatorname{Sing}(R) = O$, the point O is an isolated double point of the surface R, and there is a commutative diagram



where ω is a double cover branched in R, the morphism η is a double cover branched in the proper transform of R, β is a birational morphism that contracts a surface $\overline{E}$ with $\eta(\overline{E}) = E$ to the singular point of V_2, $\omega(\operatorname{Sing}(V_2)) = O$, the map ψ is the projection from the point O, and φ is a conic bundle.

We assume that X satisfies the following generality condition: the point O is an *ordinary* double point of the surface R. Then $\overline{E} \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Let $\overline{H}$ be the proper transform on X of the general plane in $\mathbb{P}^3$ passing through O. Then $-K_X \sim 2\overline{H} + \overline{E}$, which implies that $\operatorname{lct}(X) \leqslant 1/2$.

Suppose that $\operatorname{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$.

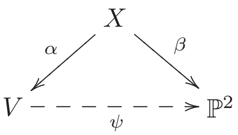
It follows from Lemma 2.17 that $\mathrm{LCS}(X, D) \cap \overline{E} \neq \varnothing$. Put $D = \mu\overline{E} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor on X such that $\overline{E} \not\subset \mathrm{Supp}(\Omega)$. Then

$$2 = D \cdot \Gamma = (\mu\overline{E} + \Omega) \cdot \Gamma = 2\mu + \Omega \cdot \Gamma \geqslant 2\mu,$$

where Γ is a general fibre of the bundle φ. Hence the log pair $\bigl(\overline{E}, \lambda\Omega|_{\overline{E}}\bigr)$ is not log canonical by Theorem 2.19, because $\mathrm{LCS}(X, D) \cap \overline{E} \neq \varnothing$. Furthermore, $\Omega|_{\overline{E}} \sim_{\mathbb{Q}} -((1+\mu)/2)K_{\overline{E}}$, which is impossible by Lemma 2.23.

Lemma 7.5. *If* $\daleth(X) = 2.11$ *and* X *is general, then* $\mathrm{lct}(X) = 1/2$.

Proof. Let V be a cubic hypersurface in $\mathbb{P}^4$. Then there is a commutative diagram



such that α contracts a surface $E \subset X$ to a line $L \subset V$, the map ψ is a projection from the line L, and the morphism β is a conic bundle.

We assume that X satisfies the following generality condition: the normal bundle $\mathscr{N}_{L/V}$ to the line L on the variety V is isomorphic to $\mathscr{O}_L \oplus \mathscr{O}_L$.

Let H be a hyperplane section of V such that $L \subset H$. Then $-K_X \sim 2\overline{H} + E$, where $\overline{H} \subset X$ is the proper transform of the surface H. In particular, $\mathrm{lct}(X) \leqslant 1/2$.

Suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$. Then $\mathrm{LCS}(X, \lambda D) \subseteq E$ since $\mathrm{lct}(V) = 1/2$ and $\alpha(D) \sim_{\mathbb{Q}} -K_V$. We note that $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ by the generality condition.

Let $F \subset E$ be a fibre of the induced projection $E \to L$, and let $Z \subset E$ be a section of this projection such that $Z \cdot Z = 0$. Then $\alpha^*(H)|_E \sim F$ and $E|_E \sim -Z$, because

$$-2Z - 2F \sim K_E \sim (K_X + E)|_E \sim 2\bigl(E - \alpha^*(H)\bigr)\big|_E \sim -2F + 2E|_E.$$

We put $D = \mu E + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor on X and $E \not\subset \mathrm{Supp}(\Omega)$. Then

$$2 = D \cdot \Gamma = \mu E \cdot \Gamma + \Omega \cdot \Gamma \geqslant \mu E \cdot \Gamma = 2\mu,$$

where Γ is a general fibre of the conic bundle β. Thus, we see that $\mu \leqslant 1$. The log pair $\bigl(E, \lambda\Omega|_E\bigr)$ is not log canonical by Theorem 2.19. But

$$\Omega|_E \sim_{\mathbb{Q}} (-K_X - \mu E)|_E \sim_{\mathbb{Q}} (1+\mu)Z + 2F,$$

which contradicts Lemma 2.23, because $\mu \leqslant 1$ and $\lambda < 1/2$.

Lemma 7.6. *If* $\daleth(X) = 2.15$ *and* X *is general, then* $\mathrm{lct}(X) = 1/2$.

Proof. There is a birational morphism $\alpha\colon X \to \mathbb{P}^3$ that contracts a surface $E \subset X$ to a smooth curve $C \subset \mathbb{P}^3$ that is the complete intersection of an (irreducible but possibly singular) quadric $Q \subset \mathbb{P}^3$ and a cubic $F \subset \mathbb{P}^3$.

We assume that X satisfies the following generality condition: the quadric Q is smooth.

Let $\overline{Q}$ be the proper transform of Q on the threefold X. Then there is a commutative diagram

$$\begin{array}{ccc} & X & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ \mathbb{P}^3 & \overset{\gamma}{\dashleftarrow} & V \end{array},$$

where V is a cubic in $\mathbb{P}^4$ that has one ordinary double point $P \in V$, the morphism β contracts the surface $\overline{Q}$ to the point P, and γ is the projection from the point P.

Let E be the exceptional divisor of the birational morphism α. Then $-K_X \sim 2\overline{Q}+E$ and $\beta(E) \subset V$ is a surface containing all the lines on V that pass through P. In particular, $\operatorname{lct}(X) \leqslant 1/2$.

Suppose that $\operatorname{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$.

It follows from Lemma 2.16 that either $\operatorname{LCS}(X, \lambda D) \subseteq \overline{Q}$, or the set $\operatorname{LCS}(X, \lambda D)$ contains a fibre of the natural projection $E \to C$. We have $\operatorname{LCS}(X, \lambda D) \cap \overline{Q} \neq \varnothing$ in both cases.

We have $\overline{Q} \cong \mathbb{P}^1 \times \mathbb{P}^1$. Put $D = \mu\overline{Q} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor on X such that $\overline{Q} \not\subset \operatorname{Supp}(\Omega)$. Then $\alpha(D) \sim_{\mathbb{Q}} \mu Q + \alpha(\Omega) \sim_{\mathbb{Q}} -K_{\mathbb{P}^3}$, which gives $\mu \leqslant 2$. The log pair $\bigl(\overline{Q}, \lambda\Omega|_{\overline{Q}}\bigr)$ is not log canonical by Theorem 2.19. But $\Omega|_{\overline{Q}} \sim_{\mathbb{Q}} -((1+\mu)/2)K_{\overline{Q}}$, which implies by Lemma 2.23 that $\mu > 1$.

By Remark 2.22 we may assume that $E \not\subset \operatorname{Supp}(D)$. Then

$$1 = D \cdot F = \mu\overline{Q} \cdot F + \Omega \cdot F = \mu + \Omega \cdot F \geqslant \mu,$$

where F is a general fibre of the natural projection $E \to C$. But $\mu > 1$, which is a contradiction.

Lemma 7.7. *If* $\daleth(X) = 2.18$, *then* $\operatorname{lct}(X) = 1/2$.

Proof. There is a smooth divisor $B \subset \mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(2, 2)$ such that the diagram

$$\begin{array}{ccccc} & & X & & \\ & {}^{\varphi_1}\swarrow & \downarrow{\scriptstyle\pi} & \searrow^{\varphi_2} & \\ \mathbb{P}^1 & \xleftarrow{\;\pi_1\;} & \mathbb{P}^1 \times \mathbb{P}^2 & \xrightarrow{\;\pi_2\;} & \mathbb{P}^2 \end{array}$$

is commutative, where π is a double cover branched in B, the morphisms π_1 and π_2 are the natural projections, φ_1 is a quadric fibration, and φ_2 is a conic bundle.

Let H_1 be a general fibre of π_1, and let H_2 be a general surface in $|\pi_2^*(\mathscr{O}_{\mathbb{P}^2}(1))|$. Then $B \sim 2H_1 + 2H_2$.

Let $\overline{H}_1$ be a general fibre of φ_1, and let $\overline{H}_2$ be a general surface in the linear system $|\varphi_2^*(\mathscr{O}_{\mathbb{P}^2}(1))|$. Then $-K_X \sim \overline{H}_1 + 2\overline{H}_2$, which implies that $\operatorname{lct}(X) \leqslant 1/2$.

Suppose that $\operatorname{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/2$.

Applying Lemma 2.25 to the fibration φ_1, we see that $\operatorname{LCS}(X, \lambda D) \subseteq Q$, where Q is a singular fibre of φ_1. Moreover, applying Theorem 2.27 to the fibration φ_2, we see that $\mathrm{LCS}(X, \lambda D) \subseteq Q \cap R$, where $R \subset X$ is an irreducible surface swept out by singular fibres of φ_2. In particular, the set $\mathbb{LCS}(X, \lambda D)$ contains no surfaces.

Suppose that $\mathrm{LCS}(X, \lambda D)$ is zero-dimensional. Then

$$\mathrm{LCS}\Big(X, \lambda D + \frac{1}{2}(\overline{H}_1 + 2\overline{H}_2)\Big) = \mathrm{LCS}(X, \lambda D) \cup \overline{H}_2,$$

which is impossible by Theorem 2.7.

We see that the set $\mathbb{LCS}(X, \lambda D)$ contains a curve $\Gamma \subset Q \cap R$. Let $D = \mu Q + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $Q \not\subseteq \mathrm{Supp}(\Omega)$. Then the log pair $(Q, \lambda\Omega|_Q)$ is also not log canonical along Γ by Theorem 2.19. But $\Omega|_Q \sim_{\mathbb{Q}} -K_Q$, which implies (see Lemma 4.7) that Γ is a ruling of the cone $Q \subset \mathbb{P}^3$. Then $\varphi_2(\Gamma) \subset \mathbb{P}^2$ is a line and $\varphi_2(\Gamma) \subseteq \varphi_2(R)$. But $\varphi_2(R) \subset \mathbb{P}^2$ is a curve of degree 4. Thus, we see that $\varphi_2(R) = \varphi_2(\Gamma) \cup Z$, where $Z \subset \mathbb{P}^2$ is a reduced cubic curve. Then φ_2 induces a double cover of $\varphi_2(\Gamma) \setminus (\varphi_2(\Gamma) \cap Z)$ that must be unramified (see [57]). But the curve $\varphi_2(R)$ has at most ordinary double points (see [57]), therefore $|\varphi_2(\Gamma) \cap Z| = 3$, which is impossible because $\varphi_2(\Gamma) \cong \mathbb{P}^1$.

Lemma 7.8. *If* $\daleth(X) = 2.19$ *and* X *is general, then* $\mathrm{lct}(X) = 1/2$.

Proof. It follows from Proposition 3.4.1 in [2] that there is a commutative diagram

$$\begin{array}{ccc} & X & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ V & \overset{}{\underset{\psi}{\dashrightarrow}} & \mathbb{P}^3 \end{array},$$

where V is a complete intersection of two quadric fourfolds in $\mathbb{P}^5$, the morphism α is a blow-up of a line $L \subset V$, the morphism β is a blow-up of a smooth curve $C \subset \mathbb{P}^3$ of degree 5 and genus 2, and the map ψ is a projection from the line L.

Let E and R be the exceptional divisors of α and β, respectively. Then the surface $\beta(E) \subset \mathbb{P}^3$ is an irreducible quadric and the surface $\alpha(R) \subset V$ is swept out by lines in V that intersect the line L.

We assume that X satisfies the following generality condition: the surface $\beta(E)$ is smooth.

Let H be a hyperplane section of $V \subset \mathbb{P}^5$ such that $L \subset H$. Then $2\overline{H} + E \sim R + 2E \sim -K_X$, where $\overline{H}$ is the proper transform of H on the threefold X. In particular, $\mathrm{lct}(X) \leqslant 1/2$.

Suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$. We note that $\mathrm{LCS}(X, \lambda D) \subseteq E \cong \mathbb{P}^1 \times \mathbb{P}^1$, because $\alpha(D) \sim_{\mathbb{Q}} -K_V$ and $\mathrm{lct}(V) = 1/2$ by Theorem 6.1.

Let F be a fibre of the projection $E \to L$ and let Z be a section of this projection such that $Z \cdot Z = 0$. Then $\alpha^*(H)|_E \sim F$ and $E|_E \sim -Z$, because

$$-2Z - 2F \sim K_E \sim (K_X + E)|_E \sim 2\big(E - \alpha^*(H)\big)\big|_E \sim 2E|_E - 2F.$$

By Remark 2.22 we may assume that either $E \not\subseteq \mathrm{Supp}(D)$ or $R \not\subseteq \mathrm{Supp}(D)$, because the log pair $(X, \lambda(R + 2E))$ is log canonical and $-K_X \sim R + 2E$. We put $D = \mu E + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor on X such that $E \not\subseteq \mathrm{Supp}(\Omega)$.

Suppose that $\mu \leqslant 1$. Then $(X, E+\lambda\Omega)$ is not log canonical, which implies that $\big(E, \lambda\Omega|_E\big)$ is also not log canonical by Theorem 2.19. But

$$\Omega|_E \sim_{\mathbb{Q}} (-K_X - \mu E)|_E \sim_{\mathbb{Q}} (1+\mu)Z + 2F,$$

which contradicts Lemma 2.23, because $\mu \leqslant 1$ and $\lambda < 1/2$.

Thus, $\mu > 1$. Hence we may assume that $R \not\subset \mathrm{Supp}(D)$.

Let Γ be a general fibre of the projection $R \to C$. Then $\Gamma \not\subset \mathrm{Supp}(D)$ and

$$1 = -K_X \cdot \Gamma = \mu E \cdot \Gamma + \Omega \cdot \Gamma = \mu + \Omega \cdot \Gamma \geqslant \mu,$$

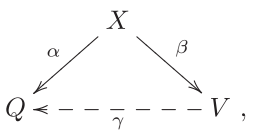
a contradiction.

Lemma 7.9. *If* $\daleth(X) = 2.23$ *and* X *is general, then* $\mathrm{lct}(X) = 1/3$.

Proof. There is a birational morphism $\alpha\colon X \to Q$ with $Q \subset \mathbb{P}^4$ a smooth quadric threefold that contracts a surface $E \subset X$ to a smooth curve $C \subset Q$ that is a complete intersection of a hyperplane section $H \subset Q$ and a divisor $F \in |\mathscr{O}_Q(2)|$.

We assume that X satisfies the following generality condition: the quadric surface H is smooth.

Let $\overline{H}$ be a proper transform of H on X. Then there is a commutative diagram



where V is a complete intersection of two quadrics in $\mathbb{P}^5$ such that V has one ordinary double point $P \in V$, the morphism β contracts $\overline{H}$ to the point P, and γ is a projection from P.

Let E be the exceptional divisor of α. Then $-K_X \sim 3\overline{H} + 2E$ and $\beta(E) \subset V$ is a surface containing all the lines in V that pass through P. In particular, $\mathrm{lct}(X) \leqslant 1/3$.

Suppose that $\mathrm{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/3$.

It follows from Remark 6.5 that either $\mathrm{LCS}(X, \lambda D) \subseteq \overline{H}$ or the set $\mathrm{LCS}(X, \lambda D)$ contains a fibre of the natural projection $E \to C$. In both cases $\mathrm{LCS}(X, \lambda D) \cap \overline{H} \neq \varnothing$.

We have $\overline{H} \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $D = \mu\overline{H} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor on X such that $\overline{H} \not\subset \mathrm{Supp}(\Omega)$. Then $\alpha(D) \sim_{\mathbb{Q}} \mu H + \alpha(\Omega) \sim_{\mathbb{Q}} -K_Q$, which gives $\mu \leqslant 3$. The log pair $(\overline{H}, \lambda\Omega|_{\overline{H}})$ is not log canonical by Theorem 2.19. But $\Omega|_{\overline{H}} \sim_{\mathbb{Q}} -((1+\mu)/2)K_{\overline{H}}$, which implies that $\mu > 1$ by Lemma 2.23. By Remark 2.22 we may assume that $E \not\subset \mathrm{Supp}(D)$, because the log pair $(X, \lambda(3\overline{H} + 2E))$ is log canonical. Let F be a general fibre of the natural projection $E \to C$. Then

$$1 = D \cdot F = \mu\overline{H} \cdot F + \Omega \cdot F = \mu + \Omega \cdot F \geqslant \mu,$$

which is a contradiction because $\mu > 1$.

Lemma 7.10. *If* $\daleth(X) = 2.24$ *and* X *is general, then* $\mathrm{lct}(X) = 1/2$.

Proof. The threefold X is a divisor on $\mathbb{P}^2\times\mathbb{P}^2$ of bidegree $(1,2)$. Let H_i be a surface in $|\pi_i^*(\mathscr{O}_{\mathbb{P}^2}(1))|$, where $\pi_i\colon X\to\mathbb{P}^2$ is the projection of X onto the ith factor of $\mathbb{P}^2\times\mathbb{P}^2$, $i\in\{1,2\}$. Then $-K_X\sim 2H_1+H_2$, which implies that $\mathrm{lct}(X)\leqslant 1/2$. We note that π_1 is a conic bundle and π_2 is a $\mathbb{P}^1$-bundle. Let $\Delta\subset\mathbb{P}^2$ be the degeneration curve of the conic bundle π_1. Then Δ is a cubic curve.

We suppose that X satisfies the following generality condition: the curve Δ is irreducible.

Assume that $\mathrm{lct}(X)<1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the log pair $(X,\lambda D)$ is not log canonical for some $\lambda<1/2$.

Suppose that the set $\mathbb{LCS}(X,\lambda D)$ contains a surface $S\subset X$. We set $D=\mu S+\Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $S\not\subset\mathrm{Supp}(\Omega)$ and $\mu>1/\lambda$. Let F_i be a general fibre of π_i, $i\in\{1,2\}$. Then

$$2=D\cdot F_i=\mu S\cdot F_i+\Omega\cdot F_i\geqslant\mu S\cdot F_i,$$

but either $S\cdot F_1\geqslant 1$ or $S\cdot F_2\geqslant 1$. Thus, we see that $\mu\leqslant 2$, a contradiction.

By Theorem 2.27 and Theorem 2.7 there is a fibre Γ_2 of the $\mathbb{P}^1$-bundle π_2 such that $\mathrm{LCS}(X,\lambda D)=\Gamma_2$, because the set $\mathbb{LCS}(X,\lambda D)$ contains no surfaces.

Applying Theorem 2.27 to the conic bundle π_1, we see that $\pi_1(\Gamma_2)\subset\Delta$, which is impossible, because $\Delta\subset\mathbb{P}^2$ is an irreducible cubic curve and $\pi_1(\Gamma_2)\subset\mathbb{P}^2$ is a line.

Lemma 7.11. *If* $\daleth(X)=2.25$, *then* $\mathrm{lct}(X)=1/2$.

Proof. We recall that X is a blow-up $\alpha\colon X\to\mathbb{P}^3$ along a normal elliptic curve C of degree 4.

Let $Q\subset\mathbb{P}^3$ be a general quadric containing C and $\overline{Q}\subset X$ the proper transform of Q. Then $-K_X\sim 2\overline{Q}+E$, where E is the exceptional divisor of α. In particular, $\mathrm{lct}(X)\leqslant 1/2$.

Suppose that $\mathrm{lct}(X)<1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the log pair $(X,\lambda D)$ is not log canonical for some $\lambda<1/2$.

We note that the linear system $|\overline{Q}|$ defines a quadric fibration $\varphi\colon X\to\mathbb{P}^1$ with irreducible fibres. Moreover, by Theorem 2.27 the log pair $(X,\lambda D)$ is log canonical along every non-singular fibre $\widetilde{Q}$ of the fibration φ since $\mathrm{lct}(\widetilde{Q})=1/2$ (see Example 1.10).

The locus $\mathrm{LCS}(X,\lambda D)$ does not contain any fibre of φ, because $\alpha(D)\sim_{\mathbb{Q}}2Q$ and every fibre of φ is irreducible. Therefore, $\dim(\mathrm{LCS}(X,\lambda D))\leqslant 1$.

Let $Z\in\mathbb{LCS}(X,\lambda D)$. Then there is a singular fibre $\overline{Q}_1$ of φ such that $Z\subset\overline{Q}_1$. Note that φ has 4 singular fibres and each of them is the proper transform of a quadric cone in $\mathbb{P}^3$ with vertex outside C.

Let $\overline{Q}_2$ be a singular fibre of φ different from $\overline{Q}_1$; let $\overline{H}$ be the proper transform of a general plane in $\mathbb{P}^3$ that is tangent to the cone $\alpha(\overline{Q}_2)\subset\mathbb{P}^3$ along one of its rulings $L\subset\alpha(\overline{Q}_2)$; and let $\overline{R}$ be the proper transform of a sufficiently general plane in $\mathbb{P}^3$. We put

$$\Delta=\lambda D+\frac{1}{2}\big((1+\varepsilon)\overline{Q}_2+(2-\varepsilon)\overline{H}+3\varepsilon R\big)$$

for some positive rational number $\varepsilon<1-2\lambda$. Then

$$\Delta \sim_{\mathbb{Q}} -\Big(\lambda + \frac{1}{2}(1+\varepsilon)\Big)K_X \sim_{\mathbb{Q}} -\frac{1+\varepsilon+2\lambda}{2}K_X,$$

which implies that $-(K_X+\Delta)$ is ample.

Let $\bar{L}$ be the proper transform on X of the line L. Then

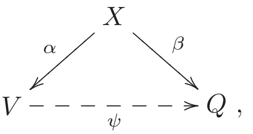
$$Z \cup \bar{L} \subset \mathrm{LCS}(X,\Delta) \subset \bar{Q}_1 \cup \bar{Q}_2,$$

which is impossible by Theorem 2.7, because $-(K_X+\Delta)$ is ample.

Lemma 7.12. *If* $\gimel(X) = 2.26$ *and* X *is general, then* $\mathrm{lct}(X) = 1/2$.

Proof. Let V be a smooth Fano threefold such that $-K_V \sim 2H$ and $\mathrm{Pic}(V) = \mathbb{Z}[H]$, where H is a Cartier divisor such that $H^3 = 5$ (that is, $\gimel(V) = 1.15$). Then the linear system $|H|$ induces an embedding $X \subset \mathbb{P}^6$.

It follows from Proposition 3.4.1 in [2] that there is a line $L \subset V \subset \mathbb{P}^6$ such that there is a commutative diagram



where Q is a smooth quadric in $\mathbb{P}^4$, the morphism α is a blow-up of the line $L \subset V$, the morphism β is a blow-up of a twisted cubic curve C with $\mathbb{P}^1 \cong C \subset Q$, and ψ is the projection from the line L.

Let S be the exceptional divisor of the morphism β. We set $\bar{S} = \alpha(S)$. Then $\bar{S} \sim H$ and $\bar{S}$ is singular along the line L. Let E be the exceptional divisor of the blow-up α. Then $\beta(E) \sim \mathscr{O}_{\mathbb{P}^4}(1)|_Q$, which implies that $\beta(E)$ is an irreducible quadric surface.

Suppose that X satisfies the following generality condition: the surface $\beta(E)$ is smooth.

We note that $-K_X \sim 2S + 3E$. Moreover, the log pair $\big(X, (1/3)(2S+3E)\big)$ is log canonical but not log terminal. Thus, $\mathrm{lct}(X) \leqslant 1/3$.

Suppose that $\mathrm{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/3$. Then $\mathrm{LCS}(X, \lambda D) \subseteq E$, because $\alpha(D) \sim_{\mathbb{Q}} -K_V$ and $\mathrm{lct}(V) = 1/2$ by Theorem 6.1.

We note that $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ by our generality condition. Let F be a fibre of the projection $E \to L$, and let Z be a section of this projection such that $Z \cdot Z = 0$. Then $\alpha^*(H)|_E \sim F$ and $E|_E \sim -Z$, because

$$-2Z - 2F \sim K_E \sim (K_X + E)|_E \sim 2\big(E - \alpha^*(H)\big)\big|_E \sim 2E|_E - 2F.$$

By Remark 2.22 we may assume that either $E \not\subset \mathrm{Supp}(D)$ or $S \not\subset \mathrm{Supp}(D)$. We put $D = \mu E + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor on X such that $E \not\subset \mathrm{Supp}(\Omega)$.

Suppose that $\mu \leqslant 2$. Then $(X, E + \lambda\Omega)$ is not log canonical, which implies that $\big(E, \lambda\Omega|_E\big)$ is also not log canonical by Theorem 2.19. But

$$\Omega|_E \sim_{\mathbb{Q}} (-K_X - \mu E)|_E \sim_{\mathbb{Q}} (1+\mu)Z + 2F,$$

which contradicts Lemma 2.23, because $\mu \leqslant 2$ and $\lambda < 1/3$.

Thus $\mu > 2$, so we may assume that $S \not\subset \mathrm{Supp}(D)$.

Let Γ be a general fibre of the projection $S \to C$. Then $\Gamma \not\subset \mathrm{Supp}(D)$ and

$$1 = -K_X \cdot \Gamma = \mu E \cdot \Gamma + \Omega \cdot \Gamma = \mu + \Omega \cdot \Gamma \geqslant \mu,$$

which is a contradiction.

Lemma 7.13. *If* $\gimel(X) = 2.27$, *then* $\mathrm{lct}(X) = 1/2$.

Proof. There is a birational morphism $\alpha\colon X \to \mathbb{P}^3$ contracting a surface E to a twisted cubic curve $C \subset \mathbb{P}^3$, and $X \cong \mathbb{P}(\mathscr{E})$, where $\mathscr{E}$ is a stable rank-2 vector bundle on $\mathbb{P}^2$ with $c_1(\mathscr{E}) = 0$ and $c_2(\mathscr{E}) = 2$ such that the sequence

$$0 \to \mathscr{O}_{\mathbb{P}_2}(-1) \oplus \mathscr{O}_{\mathbb{P}_2}(-1) \to \mathscr{O}_{\mathbb{P}_2} \oplus \mathscr{O}_{\mathbb{P}_2} \oplus \mathscr{O}_{\mathbb{P}_2} \oplus \mathscr{O}_{\mathbb{P}_2} \to \mathscr{E} \otimes \mathscr{O}_{\mathbb{P}_2}(1) \to 0$$

is exact (see [58], Application 1). Let $Q \subset \mathbb{P}^3$ be a general quadric containing C, and let $\overline{Q} \subset X$ be the proper transform of Q. Then $-K_X \sim 2\overline{Q} + E$, where E is the exceptional divisor of α. Hence $\mathrm{lct}(X) \leqslant 1/2$.

Suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$.

Assume that the set $\mathbb{LCS}(X, \lambda D)$ contains a surface $S \subset X$. We put $D = \mu F + \Omega$, where $\mu \geqslant 1/\lambda$, and Ω is an effective $\mathbb{Q}$-divisor such that $F \not\subset \mathrm{Supp}(\Omega)$.

Let $\varphi\colon X \to \mathbb{P}^2$ be the natural $\mathbb{P}^1$-bundle. Then

$$2 = D \cdot \Gamma = \mu F \cdot \Gamma + \Omega \cdot \Gamma = \mu F \cdot \Gamma + \Omega \cdot F \geqslant \mu F \cdot \Gamma,$$

where Γ is a general fibre of φ. Thus, F is swept out by the fibres of φ. Then $\alpha(F) \sim \mathscr{O}_{\mathbb{P}^3}(d)$, where $d \geqslant 2$. However, $\alpha(D) \sim_{\mathbb{Q}} \mu\alpha(F) + \alpha(\Omega) \sim_{\mathbb{Q}} \mathscr{O}_{\mathbb{P}^3}(4)$, which is a contradiction.

We see that the locus $\mathrm{LCS}(X, \lambda D)$ contains no surfaces. Applying Theorem 2.27 to $(X, \lambda D)$ and φ, we see that $L \subseteq \mathrm{LCS}(X, \lambda D)$, where L is a fibre of φ. Then $\alpha(L)$ is a secant line of the twisted cubic $C \subset \mathbb{P}^3$. One has

$$\alpha(L) \subseteq \mathrm{LCS}\big(\mathbb{P}^3, \lambda\alpha(D)\big) \subseteq \alpha\big(\mathrm{LCS}(X, \lambda D)\big) \cup C,$$

which is impossible by Lemma 2.9.

Lemma 7.14. *If* $\gimel(X) = 2.28$, *then* $\mathrm{lct}(X) = 1/4$.

Proof. We recall that there exists a blow-up $\alpha\colon X \to \mathbb{P}^3$ along a plane cubic curve $C \subset \mathbb{P}^3$, and one has $-K_X \sim 4G + 3E$, where E is the exceptional divisor of α and G is the proper transform of the plane in $\mathbb{P}^3$ which contains the curve C. In particular, $\mathrm{lct}(X) \leqslant 1/4$.

Suppose that $\mathrm{lct}(X) < 1/4$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/4$. Therefore, $\mathrm{LCS}(X, \lambda D) \subseteq E$ since $\mathrm{lct}(\mathbb{P}^4) = 1/4$. Computing the intersections with the proper transform of a general line in $\mathbb{P}^3$ intersecting the curve C, we get that $\mathrm{LCS}(X, \lambda D)$ does not contain the divisor E. Moreover, every curve $\Gamma \in \mathbb{LCS}(X, \lambda D)$ must be a fibre of the natural projection $\psi\colon E \to C$ by Lemma 2.14. Therefore, we see from Theorem 2.7 that either the locus $\mathrm{LCS}(X,\lambda D)$ consists of a single point or it consists of a single fibre of the projection ψ.

Let R be a sufficiently general cone in $\mathbb{P}^3$ over the curve C and H a sufficiently general plane in $\mathbb{P}^3$ which passes through the point $\mathrm{Sing}(R)$. Then

$$\mathrm{LCS}\biggl(X,\lambda D+\frac{3}{4}\,(\overline{R}+\overline{H}\,)\biggr)=\mathrm{LCS}(X,\lambda D)\cup\mathrm{Sing}(\overline{R}\,),$$

where $\overline{R}$ and $\overline{H}$ are the proper transforms of R and H on the threefold X. Then the divisor

$$-\biggl(K_X+\lambda D+\frac{3}{4}\,(\overline{R}+\overline{H}\,)\biggr)\sim_{\mathbb{Q}}\biggl(\lambda-\frac{1}{4}\biggr)K_X$$

is ample, which contradicts Theorem 2.7.

Lemma 7.15. *If* $\daleth(X)=2.29$, *then* $\mathrm{lct}(X)=1/3$.

Proof. We recall that there is a blow-up $\alpha\colon X\to Q$ of a smooth quadric hypersurface Q along a conic $C\subset Q$.

Let H be a general hyperplane section of $Q\subset\mathbb{P}^4$ that contains C, and let $\overline{H}$ be the proper transform of the surface H on the threefold X. Then $-K_X\sim 3\overline{H}+2E$, where E is the exceptional divisor of α. In particular, $\mathrm{lct}(X)\leqslant 1/3$.

Suppose that $\mathrm{lct}(X)<1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the log pair $(X,\lambda D)$ is not log canonical for some $\lambda<1/3$. And then $\mathrm{LCS}(X,\lambda D)\subseteq E$ since $\mathrm{lct}(Q)=1/3$ (see Example 1.3) and $\alpha(D)\sim_{\mathbb{Q}}-K_Q$.

The linear system $|\overline{H}|$ has no base points and defines a morphism $\beta\colon X\to\mathbb{P}^1$, whose general fibre is a smooth quadric surface. Then the log pair $(X,\lambda D)$ is log canonical along the smooth fibres of β by Theorem 2.27 (see Example 1.10).

It follows from Theorem 2.7 that there is a singular fibre $S\sim\overline{H}$ of the morphism β such that $\mathrm{LCS}(X,\lambda D)\subsetneq E\cap S$ and $\alpha(S)\subset\mathbb{P}^3$ is a quadric cone. We put $\Gamma=E\cap S$. Then Γ is an irreducible conic, the log pair $\bigl(X,S+(2/3)E\bigr)$ has log canonical singularities, and $3S+2E\sim_{\mathbb{Q}}D$. Therefore, it follows from Remark 2.22 that to complete the proof we may assume that either $S\not\subset\mathrm{Supp}(D)$ or $E\not\subset\mathrm{Supp}(D)$.

Intersecting the divisor D with the proper transform of a general ruling of the cone $\alpha(S)\subset\mathbb{P}^3$ and with a general fibre of the projection $E\to C$, we see that $\Gamma\not\subseteq\mathrm{LCS}(X,\lambda D)$, which implies that $\mathrm{LCS}(X,\lambda D)$ consists of a single point $O\in\Gamma$ by Theorem 2.7.

Let R be a general (not passing through O) surface in $|\alpha^*(H)|$. Then

$$\mathrm{LCS}\biggl(X,\lambda D+\frac{1}{2}\,(\overline{H}+2R)\biggr)=R\cup O,$$

which is impossible by Theorem 2.7 since $-K_X\sim\overline{H}+2R\sim_{\mathbb{Q}}D$ and $\lambda<1/3$.

Lemma 7.16. *If* $\daleth(X)=2.30$, *then* $\mathrm{lct}(X)=1/4$.

Proof. There is a commutative diagram

$$\begin{array}{ccc} & X & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ \mathbb{P}^3 & \underset{\gamma}{\longleftarrow\!-\!-\!-} & Q \end{array},$$

where Q is a smooth quadric threefold in $\mathbb{P}^4$, the morphism α is a blow-up of a smooth conic $C \subset \mathbb{P}^3$, the morphism β is a blow-up of a point, and γ is a projection from a point.

Let G be the proper transform on X of the unique plane in $\mathbb{P}^3$ containing the conic C. Then the surface G is contracted by the morphism β, and $-K_X \sim 4G+3E$, where E is the exceptional divisor of the blow-up α. Thus, $\mathrm{lct}(X) \leqslant 1/4$.

Suppose that $\mathrm{lct}(X) < 1/4$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/4$. Then $\mathrm{LCS}(X, \lambda D) \subseteq E \cap G$, because $\mathrm{lct}(\mathbb{P}^4) = 1/4$ and $\mathrm{lct}(Q) = 1/3$.

By Remark 2.22 we may assume that either $G \not\subset \mathrm{Supp}(X)$ or $E \not\subset \mathrm{Supp}(X)$.

Intersecting D with lines in $G \cong \mathbb{P}^2$ and with fibres of the projection $E \to C$, we see that $\mathrm{LCS}(X, \lambda D) \subsetneq E \cap G$, which implies that there is a point $O \in E \cap G$ such that $\mathrm{LCS}(X, \lambda D) = O$ by Theorem 2.7.

Let R be a general surface in $|\alpha^*(H)|$ and F a general surface in $|\alpha^*(2H) - E|$. Then

$$\mathrm{LCS}\left(X, \lambda D + \frac{1}{2}(F + 2R)\right) = R \cup O,$$

which is impossible by Theorem 2.7 since $-K_X \sim F + 2R \sim_{\mathbb{Q}} D$ and $\lambda < 1/4$.

Lemma 7.17. *If* $\gimel(X) = 2.31$, *then* $\mathrm{lct}(X) = 1/3$.

Proof. There is a blow-up $\alpha \colon X \to Q$ of a smooth quadric Q along a line $L \subset Q$.

Let H be a sufficiently general hyperplane section of the quadric $Q \subset \mathbb{P}^4$ that passes through the line L, and let $\overline{H}$ be a proper transform of the surface H on X. Then $-K_X \sim 3\overline{H} + 2E$, where E is the exceptional divisor of α. In particular, $\mathrm{lct}(X) \leqslant 1/3$.

Suppose that $\mathrm{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/3$. Then $\mathrm{LCS}(X, \lambda D) \subseteq E$ since $\mathrm{lct}(Q) = 1/3$ and $\alpha(D) \sim_{\mathbb{Q}} -K_Q$.

The linear system $|\overline{H}|$ defines a $\mathbb{P}^1$-bundle $\varphi \colon X \to \mathbb{P}^2$ such that the induced morphism $E \cong \mathbb{F}_1 \to \mathbb{P}^2$ contracts an irreducible curve $Z \subset E$. Then $\mathrm{LCS}(X, \lambda D) = Z$ by Theorem 2.27. We put $D = \mu E + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor on X such that $E \not\subset \mathrm{Supp}(\Omega)$. Then

$$2 = D \cdot F = \mu E \cdot F + \Omega \cdot F = \mu + \Omega \cdot F \geqslant \mu,$$

where F is a general fibre of φ. Note that the log pair $(X, E + \lambda\Omega)$ is not log canonical because $\lambda < 1/3$. Then $\big(E, \lambda\Omega|_E\big)$ is also not log canonical by Theorem 2.19.

Let C be a fibre of the natural projection $E \to L$. Then $\Omega|_E \sim_{\mathbb{Q}} 3C + (1+\mu)Z$, which implies that $\big(E, \lambda\Omega|_E\big)$ is log canonical by Lemma 4.8, and this is a contradiction.

8. Fano threefolds with $\rho = 3$

In this section we use the assumptions and notation introduced in § 1.

Lemma 8.1. *If* $\daleth(X) = 3.1$ *and* X *is general, then* $\operatorname{lct}(X) = 3/4$.

Proof. There is a double cover $\omega\colon X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched over a divisor of tridegree $(2,2,2)$. The projection $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ onto the ith factor induces a morphism $\pi_i\colon X \to \mathbb{P}^1$, whose fibres are del Pezzo surfaces of degree 4.

Let R_1 be a singular fibre of the fibration π_1, let Q be a singular point of R_1, and let R_2 and R_3 be fibres of π_2 and π_3 such that $R_2 \ni Q \in R_3$. Then $\operatorname{mult}_Q(R_1+R_2+R_3) = 4$, which implies that the log pair $(X, (3/4)(R_1+R_2+R_3))$ is not log terminal at Q. We have $-K_X \sim R_1 + R_2 + R_3$, therefore $\operatorname{lct}(X) \leqslant 3/4$.

Suppose that $\operatorname{lct}(X) < 3/4$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical at some point $P \in X$ for some $\lambda < 3/4$.

Let S_i be the fibre of π_i such that $P \in S_i$. Since X is general, we may assume (after a possible renumbering) that

- the surface S_1 is smooth at the point P,
- the singularities of S_1 consist of at most an ordinary double point (or S_1 is smooth).
- for every smooth curve $L \subset S_1$ such that $-K_{S_1} \cdot L = 1$ we have $P \notin L$,
- for any smooth curves $C_1 \subset S_1 \supset C_2$ such that $-K_{S_1} \cdot C_1 = -K_{S_1} \cdot C_2 = 2$ and $C_1 + C_2 \sim -K_{S_1}$ we have $P \neq C_1 \cap C_2$.

The surface S_1 is a del Pezzo surface of degree 4. We have $D = \mu S_1 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor on X such that $S_1 \not\subset \operatorname{Supp}(\Omega)$.

Let $\varphi\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the conic bundle induced by the linear system $|S_2 + S_3|$, and let Γ be a general fibre of φ. Then

$$2 = D \cdot \Gamma = \mu S_1 \cdot \Gamma + \Omega \cdot \Gamma = 2\mu + \Omega \cdot \Gamma \geqslant 2\mu,$$

which implies that $\mu \leqslant 1$. Then $(X, S_1 + \lambda\Omega)$ is not log canonical at P. Hence $(S_1, \lambda\Omega|_{S_1})$ is not log canonical at P by Theorem 2.19. But $\Omega|_{S_1} \sim_{\mathbb{Q}} -K_{S_k}$, which is impossible (cf. Example 4.4).

Lemma 8.2. *If* $\daleth(X) = 3.2$ *and* X *is general, then* $\operatorname{lct}(X) = 1/2$.

Proof. We recall that X is a primitive Fano threefold (see [52], Definition 1.3). Let

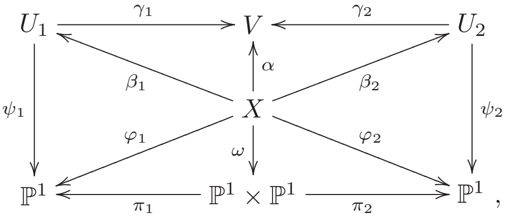
$$U = \mathbb{P}\bigl(\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1) \oplus \mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)\bigr),$$

let $\pi\colon U \to \mathbb{P}^1 \times \mathbb{P}^1$ be the natural projection, and let L be the tautological line bundle on U. Then $X \in |2L + \pi^*(\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,3))|$.

Let us show that $\operatorname{lct}(X) \leqslant 1/2$. Let E_1 and E_2 be divisors on X such that $\pi(E_1)$ and $\pi(E_2)$ are divisors on $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(1,0)$ and $(0,1)$, respectively. Then $-K_X \sim L|_X + 2E_1 + E_2$, which implies that $\operatorname{lct}(X) \leqslant 1/2$.

Suppose that $\operatorname{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical at some point $P \in X$ for some $\lambda < 1/2$.

It follows from [59] (Proposition 3.8) that there is a commutative diagram



where V is a Fano threefold with one ordinary double point $O \in V$ such that $\mathrm{Pic}(V) = \mathbb{Z}[-K_V]$ and $-K_V^3 = 16$, the morphism α contracts a unique surface S with $\mathbb{P}^1 \times \mathbb{P}^1 \cong S \subset X$ and $S \sim L|_X$ to the point $O \in V$, the morphism β_i contracts S to a smooth rational curve, the morphism γ_i contracts the curve $\beta_i(S)$ to the point $O \in V$ so that the rational map $\gamma_2 \circ \gamma_1^{-1} \colon U_1 \dashrightarrow U_2$ is a flop in $\beta_1(S) \cong \mathbb{P}^1$, the morphism ψ_2 is a quadric fibration, and the morphisms ψ_1, φ_1, and φ_2 are fibrations whose fibres are del Pezzo surfaces of degrees 4, 3, and 6, respectively. The morphisms π_1 and π_2 are the natural projections, and $\omega = \pi|_X$. We note that $\mathrm{Cl}(V) = \mathbb{Z}[\alpha(E_1)] \oplus \mathbb{Z}[\alpha(E_2)]$ and ω is a conic bundle. The curve $\beta_1(S)$ is a section of ψ_1, and $\beta_2(S)$ is a 2-section of ψ_2.

We assume that the threefold X satisfies the following generality condition: any singular fibre of the fibration φ_2 has at most $\mathbb{A}_1$ singularities.

Applying Lemma 2.25 to the fibration φ_1, we see that $\mathrm{LCS}(X, \lambda D) \subseteq S_1$, where S_1 is a singular fibre of φ_1, because the global log canonical threshold of a smooth del Pezzo surface of degree 6 is equal to 1/2 (see Example 1.10).

Applying Lemma 2.25 to φ_2, we obtain a contradiction to Example 1.11.

Lemma 8.3. *If* $\daleth(X) = 3.3$ *and* X *is general, then* $\mathrm{lct}(X) = 2/3$.

Proof. The threefold X is a divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of tridegree $(1, 1, 2)$. In particular, $-K_X \sim \pi_1^*(\mathscr{O}_{\mathbb{P}^1}(1)) + \pi_2^*(\mathscr{O}_{\mathbb{P}^1}(1)) + \varphi^*(\mathscr{O}_{\mathbb{P}^2}(1))$, where $\pi_1 \colon X \to \mathbb{P}^1$ and $\pi_2 \colon X \to \mathbb{P}^1$ are fibrations by del Pezzo surfaces of degree 4 induced by the projections of the variety $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ onto the first and the second factor, respectively, and $\varphi \colon X \to \mathbb{P}^2$ is the conic bundle induced by the projection $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$.

Let $\alpha_2 \colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ be a birational morphism induced by the linear system $|\pi_2^*(\mathscr{O}_{\mathbb{P}^1}(1)) + \varphi^*(\mathscr{O}_{\mathbb{P}^2}(1))|$, and let $H_i \in |\pi_i^*(\mathscr{O}_{\mathbb{P}^1}(1))|$ and $R \in |\varphi^*(\mathscr{O}_{\mathbb{P}^2}(1))|$ be general surfaces. Then $H_1 \sim H_2 + 2R - E_2$, where E_2 is the exceptional divisor of the birational morphism α_2. Hence

$$-K_X \sim H_1 + H_2 + R \sim_{\mathbb{Q}} \frac{3}{2} H_1 + \frac{1}{2} H_2 + \frac{1}{2} E_2,$$

which implies that $\mathrm{lct}(X) \leqslant 2/3$.

Suppose that $\mathrm{lct}(X) < 2/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical at some point $P \in X$ for some $\lambda < 2/3$.

Let S_i be a fibre of π_i such that $P \in S_i$. Since X is general, we may assume (after a possible renumbering) that

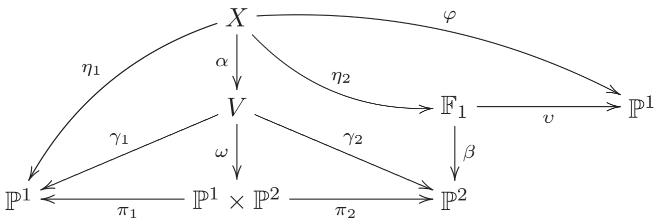
- the surface S_1 is smooth at the point P,

- the singularities of S_1 consist of at most one ordinary double point (or S_1 is smooth),
- for every smooth curve $L \subset S_1$ such that $-K_{S_1} \cdot L = 1$ we have $P \notin L$ if $\operatorname{Sing}(S_1) \neq \varnothing$.

We put $D = \mu S_1 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $S_1 \not\subset \operatorname{Supp}(\Omega)$. Then $\big(H_2, \lambda\mu S_1|_{H_2} + \lambda\Omega|_{H_2}\big)$ is not log canonical because $\operatorname{lct}(H_2) = 2/3$. Hence $\mu \leqslant 1/\lambda$, and the log pair $\big(S_1, \lambda\Omega|_{S_1}\big)$ is not log canonical at the point P by Theorem 2.19. But $\Omega|_{S_1} \sim_{\mathbb{Q}} -K_{S_1}$, which is impossible (see Example 4.4).

Lemma 8.4. *If* $\daleth(X) = 3.4$, *then* $\operatorname{lct}(X) = 1/2$.

Proof. Let O be a point in $\mathbb{P}^2$. Then there is a commutative diagram



such that π_i and υ are the natural projections, ω is a double cover branched over a divisor $B \subset \mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(2,2)$, the morphism γ_1 is a fibration into quadrics, γ_2 and η_2 are conic bundles, β is a blow-up of the point O, the morphism α is a blow-up of the smooth curve that is the fibre of γ_2 over O, the morphism η_1 is a fibration into del Pezzo surfaces of degree 6, and φ is a fibration into del Pezzo surfaces of degree 4.

Let H be a general fibre of η_1 and let S be a general fibre of φ. Then $-K_X \sim H + 2S + E$, where E is the exceptional divisor of α. In particular, $\operatorname{lct}(X) \leqslant 1/2$.

Suppose that $\operatorname{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$. Then $\operatorname{LCS}(X, \lambda D) \subseteq E$, because $\alpha(D) \sim_{\mathbb{Q}} -K_V$ and $\operatorname{lct}(V) = 1/2$ by Lemma 7.7.

Let Γ be a fibre of η_2 such that $\Gamma \cap \operatorname{LCS}(X, \lambda D) \neq \varnothing$. Then $\Gamma \subseteq \operatorname{LCS}(X, \lambda D) \subseteq E$ by Theorem 2.27. Hence $\big(H, \lambda D|_H\big)$ is not log canonical at points in $H \cap \Gamma$. But $D|_H \sim_{\mathbb{Q}} -K_H$ and $\operatorname{lct}(H) = 1/2$, because H is a del Pezzo surface of degree 6, which is a contradiction.

Lemma 8.5. *If* $\daleth(X) = 3.5$ *and* X *is general, then* $\operatorname{lct}(X) = 1/2$.

Proof. There is a birational morphism $\alpha\colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ that contracts a surface $E \subset X$ to a curve $C \subset \mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(5,2)$. Let $\pi_1\colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ and $\pi_2\colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ be the natural projections. There is a divisor $Q \in |\pi_2^*(\mathscr{O}_{\mathbb{P}^1}(2))|$ such that $C \subset Q$. Let H_1 be a general fibre of π_1 and let H_2 be a surface in the linear system $|\pi_2^*(\mathscr{O}_{\mathbb{P}^1}(1))|$. We have $-K_X \sim 2\overline{H}_1 + \overline{H}_2 + \overline{Q}$, where $\overline{H}_1, \overline{H}_2, \overline{Q} \subset X$ are the proper transforms of H_1, H_2, Q, respectively. In particular, $\operatorname{lct}(X) \leqslant 1/2$.

We suppose that X satisfies the following generality condition: every fibre F of $\pi_1 \circ \alpha$ is singular at at most one ordinary double point.

Assume that $\operatorname{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$.

Let $S \subset X$ be an irreducible surface. We put $D = \mu S + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that S does not lie in $\mathrm{Supp}(\Omega)$. Then $\big(\overline{H}_1, (1/2)(\mu S + \Omega)|_{\overline{H}_1}\big)$ is log canonical (see Example 1.10). Thus, either $\mu \leqslant 2$ or S is a fibre of $\pi_1 \circ \alpha$.

Let $\Gamma \cong \mathbb{P}^1$ be a general fibre of the conic bundle $\pi_2 \circ \alpha$. Then

$$2 = D \cdot \Gamma = \mu S \cdot \Gamma + \Omega \cdot \Gamma \geqslant \mu S \cdot \Gamma,$$

which implies that $\mu \leqslant 2$ in the case when S is a fibre of $\pi_1 \circ \alpha$.

We see that the set $\mathbb{LCS}(X, \lambda D)$ contains no surfaces. Applying Lemma 2.25 now to $\pi_1 \circ \alpha$, we obtain a contradiction to Example 4.4.

Lemma 8.6. *If* $\gimel(X) = 3.6$ *and* X *is general, then* $\mathrm{lct}(X) = 1/2$.

Proof. Let $\varepsilon\colon V \to \mathbb{P}^3$ be a blow-up of a line $L \subset \mathbb{P}^3$. Then

$$V \cong \mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1))$$

and there is a natural $\mathbb{P}^2$-bundle $\eta\colon V \to \mathbb{P}^1$. There is a smooth elliptic curve $C \subset \mathbb{P}^3$ of degree 4 such that $L \cap C = \varnothing$ and there is a commutative diagram

$$\begin{array}{ccccc} Y & \xleftarrow{\ \gamma\ } & X & \xrightarrow{\ \varphi\ } & \mathbb{P}^1 \\ \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\beta} & \nearrow{\scriptstyle\eta} & \\ \mathbb{P}^3 & \xleftarrow[\ \delta\ ]{} & V & & \end{array},$$

where δ is a blow-up of C, β is a blow-up of the proper transform of the line L, γ is a blow-up of the proper transform of the curve C, and φ is a fibration into del Pezzo surfaces of degree 5.

We suppose that X satisfies the following generality condition: every fibre F of φ has at most one singular point which is an ordinary double point of F.

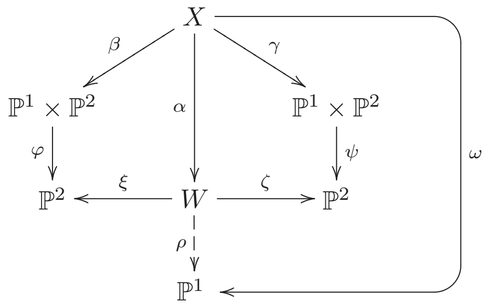
Let E and G be the exceptional surfaces of β and γ, respectively; let $H \subset \mathbb{P}^3$ be a general plane that passes through L, and let $Q \subset \mathbb{P}^3$ be a quadric surface that passes through C. Then $-K_X \sim 2\overline{H} + \overline{Q} + E$, where $\overline{H} \subset X \supset \overline{Q}$ are the proper transforms of H and Q, respectively. In particular, $\mathrm{lct}(X) \leqslant 1/2$.

Suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$.

It follows from Lemma 7.11 that $\mathrm{lct}(V) = 1/2$. Therefore, $\mathrm{LCS}(X, \lambda D) \subseteq G$. Note that every fibre of φ is a del Pezzo surface of degree 5 which has at most one ordinary double point. Thus, applying Lemma 2.25 to φ, we obtain a contradiction to Example 4.3.

Lemma 8.7. *If* $\gimel(X) = 3.7$ *and* X *is general, then* $\mathrm{lct}(X) = 1/2$.

Proof. Let W be a divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,1)$. Then $-K_W \sim 2H$, where H is a Cartier divisor on W. There is a commutative diagram



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where φ and ψ are the natural projections, α is a blow-up of a smooth curve $C \subset W$ such that

$$C = H_1 \cap H_2,$$

where $H_1 \neq H_2$ are surfaces in $|H|$, the map ρ is induced by the pencil generated by H_1 and H_2, ω is a del Pezzo fibration of degree 6, the morphisms ζ and ξ are $\mathbb{P}^1$-bundles, while β and γ contract surfaces $\overline{M}_1 \subset X \supset \overline{M}_2$ such that $\varphi \circ \beta(\overline{M}_1) = \xi(C)$ and $\psi \circ \gamma(\overline{M}_2) = \zeta(C)$.

We note that $\operatorname{lct}(X) \leqslant 1/2$ because $-K_X \sim 2\overline{H}_1 + E$, where $\overline{H}_1 \subset X$ is the proper transform of H_1 and E is the exceptional surface of α.

We suppose that X satisfies the following generality condition: all singular fibres of the fibration ω satisfy the hypotheses of Lemma 4.5.

Assume that $\operatorname{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$. Then $\operatorname{LCS}(X, \lambda D) \subseteq E$, because $\operatorname{lct}(W) = 1/2$ by Theorem 6.1. Using Lemma 2.25, we see that $\operatorname{LCS}(X, \lambda D) \subseteq E \cap F$, where F is a singular fibre of ω. Recall that F is a del Pezzo surface of degree 6. We put $D = \mu F + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $F \not\subset \operatorname{Supp}(\Omega)$. Then $\Omega|_F \sim_{\mathbb{Q}} -K_F$ and the surface F is smooth along the curve $E \cap F$. But the log pair $\bigl(F, \lambda\Omega|_F\bigr)$ is not log canonical at some point $P \in E \cap F$ by Theorem 2.19, and this is impossible by Lemma 4.5.

Remark 8.8. Let us use the notation and the assumptions of Lemma 8.7. Then we have

$$\operatorname{LCS}(X, \lambda D) \subseteq E \cap F,$$

where F is a singular fibre of the fibration ω. Applying Theorem 2.27 to φ and ψ and using Lemma 2.28, we see that $\operatorname{LCS}(X, \lambda D) \subseteq E \cap F \cap \overline{M}_1 \cap \overline{M}_2$. Regardless of how singular F is, if the threefold X is sufficiently general, then $E \cap F \cap \overline{M}_1 \cap \overline{M}_2 = \varnothing$, which implies that an alternative generality condition can be used in Lemma 8.7.

Lemma 8.9. *If* $\daleth(X) = 3.8$ *and* X *is general, then* $\operatorname{lct}(X) = 1/2$.

Proof. Let $\pi_1 \colon \mathscr{F}_1 \times \mathbb{P}^2 \to \mathscr{F}_1$ and $\pi_2 \colon \mathscr{F}_1 \times \mathbb{P}^2 \to \mathbb{P}^2$ be the natural projections. Then $X \in |(\alpha \circ \pi_1)^*(\mathscr{O}_{\mathbb{P}^2}(1)) \otimes \pi_2^*(\mathscr{O}_{\mathbb{P}^2}(2))|$, where $\alpha \colon \mathscr{F}_1 \to \mathbb{P}^2$ is a blow-up of a point. Let H be a surface in $|\pi_2^*(\mathscr{O}_{\mathbb{P}^2}(1))|$. Then $-K_X \sim E + 2L + H$, where $E \subset X \supset L$ are irreducible surfaces such that $\pi_1(E) \subset \mathbb{F}_1$ is the exceptional curve of α and $\pi_1(L) \subset \mathbb{F}_1$ is a fibre of the natural projection $\mathbb{F}_1 \to \mathbb{P}^2$. In particular, $\mathrm{lct}(X) \leqslant 1/2$.

The projection π_1 induces a fibration $\varphi\colon X \to \mathbb{P}^1$ into del Pezzo surfaces of degree 5.

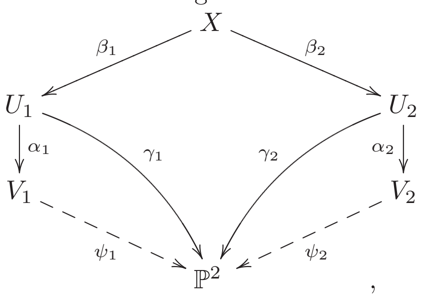
We suppose that X satisfies the following generality condition: every fibre F of φ has at most one singular point which is an ordinary double point of F.

Assume that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$.

Applying Lemma 2.25 to the morphism φ we obtain a contradiction to Example 4.3.

Lemma 8.10. *If* $\gimel(X) = 3.9$, *then* $\mathrm{lct}(X) = 1/3$.

Proof. Let O_i be a singular point of $V_i \cong \mathbb{P}(1,1,1,2)$, $i = 1, 2$. Let S_1 with $O_1 \notin S_1 \in |\mathscr{O}_{\mathbb{P}(1,1,1,2)}(2)|$ be a smooth surface and let $C_1 \subset S_1 \cong \mathbb{P}^2$ be a smooth quartic curve. Then there is a commutative diagram



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where ψ_i is the natural projection, α_i is a (weighted) blow-up of the point O_i with weights $(1,1,1)$, the morphism γ_i is a $\mathbb{P}^1$-bundle, and β_i is a birational morphism that contracts a surface $\mathbb{P}^1 \times C_1 \cong G_i \subset X$ to a smooth curve $C_1 \cong C_i \subset U_i$.

Let $E_i \subset X$ be the proper transform of the exceptional divisor of α_i. Then the divisors

$$S_1 = \alpha_1 \circ \beta_1(E_2) \subset V_1 \cong \mathbb{P}(1,1,1,2) \cong V_2 \supset \alpha_2 \circ \beta_2(E_1)$$

are surfaces in $|\mathscr{O}_{\mathbb{P}(1,1,1,2)}(2)|$ that contain the curves C_1 and C_2, respectively. On the other hand,

$$\alpha_1 \circ \beta_1(G_2) \subset V_1 \cong \mathbb{P}(1,1,1,2) \cong V_2 \supset \alpha_2 \circ \beta_2(G_1)$$

are surfaces in $|\mathscr{O}_{\mathbb{P}(1,1,1,2)}(4)|$ that contain $O_1 \cup C_1$ and $O_2 \cup C_2$, respectively.

Let $\overline{H} \subset X$ be the proper transform of a general surface in $|\mathscr{O}_{\mathbb{P}(1,1,1,2)}(1)|$. Then $-K_X \sim 3\overline{H} + E_2 + E_1$, which yields $\mathrm{lct}(X) \leqslant 1/3$.

Suppose that $\mathrm{lct}(X) < 1/3$. Then there is an effective $\mathbb{Q}$-divisor

$$D \sim_{\mathbb{Q}} -K_X \sim_{\mathbb{Q}} \frac{5}{2}\,(G_1 + G_2) - 5(E_1 + E_2)$$

such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/3$. We put $D = \mu_1 E_1 + \mu_2 E_2 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor on X such that $E_1 \not\subseteq \mathrm{Supp}(\Omega) \not\supseteq E_2$.

Let Γ be a general fibre of the conic bundle $\gamma_1 \circ \beta_1$. Then

$$2 = \Gamma \cdot D = \Gamma \cdot (\mu_1 E_1 + \mu_2 E_2 + \Omega) = \mu_1 + \mu_2 + \Gamma \cdot \Omega \geqslant \mu_1 + \mu_2,$$

and we may assume without loss of generality that $\mu_1 \leqslant \mu_2$. Then $\mu_1 \leqslant 1$.

Suppose that there is a surface $S \in \mathbb{LCS}(X, \lambda D)$. Then $S \neq E_1$ and $S \neq G_1$, because $\alpha_2 \circ \beta_2(G_1) \in |\mathscr{O}_{\mathbb{P}(1,1,1,2)}(4)|$ and $\alpha_2 \circ \beta_2(D) \in |\mathscr{O}_{\mathbb{P}(1,1,1,2)}(5)|$. Hence $S \cap E_1 \neq \varnothing$. But

$$-\frac{1}{3} K_{E_1} \sim_{\mathbb{Q}} D|_{E_1} = -\frac{2\mu_1}{3} K_{E_1} + \Omega|_{E_1}$$

and $E_1 \cong \mathbb{P}^2$, which is impossible by Theorem 2.19, because $\lambda < 1/3 = \operatorname{lct}(\mathbb{P}^2)$.

We see that the set $\mathbb{LCS}(X, \lambda D)$ contains no surfaces. Let $P \in \operatorname{LCS}(X, \lambda D)$ be a point. Suppose that $P \notin G_1$. Let Z be the fibre of γ_1 containing $\beta_1(P)$. Then $Z \subseteq \operatorname{LCS}(U_1, \lambda\beta_1(D))$ by Theorem 2.27. We put $\overline{E}_1 = \beta_1(E_1)$. Then we have $Z \cap \overline{E}_1 \in \operatorname{LCS}\big(\overline{E}_1, \lambda\Omega|_{\overline{E}_1}\big)$ by Theorem 2.19, which is impossible by Lemma 2.8, because $\mu_1 \leqslant 1$. Hence $\operatorname{LCS}(X, \lambda D) \subsetneq G_1$.

Suppose that $\operatorname{LCS}(X, \lambda D) \subseteq G_1 \cap G_2$. Then $|\operatorname{LCS}(X, \lambda D)| = 1$ by Lemma 2.14 and Theorem 2.7. We have

$$\operatorname{LCS}(X, \lambda D) \cup \overline{H} \subseteq \operatorname{LCS}\bigg(X, \lambda D + \frac{1}{3}(E_2 + E_2) + \overline{H}\bigg) \subset \operatorname{LCS}(X, \lambda D) \cup \overline{H} \cup E_1 \cup E_1,$$

which contradicts Theorem 2.7, because $\overline{H}$ is a general surface in $|(\beta_1 \circ \gamma_1)^*(\mathscr{O}_{\mathbb{P}^2}(1))|$ and

$$\lambda D + \frac{1}{3}(E_2 + E_2) + \overline{H} \sim_{\mathbb{Q}} \bigg(\lambda - \frac{1}{3}\bigg) K_X.$$

Thus, we see that $G_1 \supsetneq \mathbb{LCS}(X, \lambda D) \not\subseteq G_1 \cap G_2$. Then

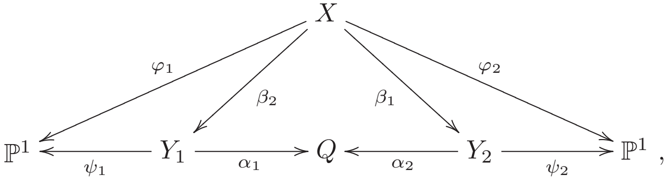
$$\varnothing \neq \operatorname{LCS}\big(U_2, \lambda\beta_2(D)\big) \subsetneq \beta_2(G_1),$$

and it follows from Theorems 2.7 and 2.27 that there is a fibre L of γ_2 such that $\operatorname{LCS}(U_2, \lambda\beta_2(D)) = L$.

Let B be a general surface in $|\alpha_2^*(\mathscr{O}_{\mathbb{P}(1,1,1,2)}(2))|$. Then $\beta_2(D)|_B \sim_{\mathbb{Q}} \mathscr{O}_{\mathbb{P}^2}(5)$ and $B \cong \mathbb{P}^2$. But $\operatorname{LCS}\big(B, \lambda\beta_2(D)|_B\big) = L \cap B$ and $|L \cap B| = 1$, which is impossible by Lemma 2.8.

Lemma 8.11. *If* $\daleth(X) = 3.10$, *then* $\operatorname{lct}(X) = 1/2$.

Proof. Let $Q \subset \mathbb{P}^4$ be a smooth quadric hypersurface. Let $C_1 \subset Q \supset C_2$ be disjoint (irreducible) conics. Then there is a commutative diagram



where α_i is a blow-up along the conic C_i, the morphism β_i is a blow-up along the proper transform of the conic C_i, the morphism ψ_i is a fibration into quadric surfaces, and φ_i is a del Pezzo fibration of degree 6.

Let E_i be the exceptional divisor of β_i, and let H_i be a sufficiently general hyperplane section of the quadric Q that passes through the conic C_i. Then $-K_X \sim \overline{H}_1 + 2\overline{H}_2 + E_2$, where $\overline{H}_i \subset X$ is the proper transform of the divisor H_i. In particular, $\mathrm{lct}(X) \leqslant 1/2$.

Suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$.

Using Example 1.10 and Lemma 2.25, we see that $\mathrm{LCS}(X, \lambda D) \subseteq S_1 \cap S_2$, where S_i is a singular fibre of φ_i. Hence the set $\mathbb{LCS}(X, \lambda D)$ contains no surfaces.

It follows from Theorem 2.7 that either $\mathrm{LCS}(X, \lambda D)$ is a point in $E_1 \cup E_2$ or

$$\mathrm{LCS}(X, \lambda D) \cap \big(X \setminus (E_1 \cup E_2)\big) \neq \varnothing,$$

which implies that we may assume that $\mathrm{LCS}(X, \lambda D)$ is a point E_1 by Lemma 2.10.

Since β_2 is an isomorphism on $X \setminus E_2$, we see that

$$P \in \mathrm{LCS}\big(Y_1, \lambda\beta_2(D)\big) \subset P \cup \beta_2(E_2)$$

for some point $P \in E_1$. Then $\mathrm{LCS}(Y_1, \lambda\beta_2(D)) = P$ by Theorem 2.7, because $P \notin \beta_2(E_2)$.

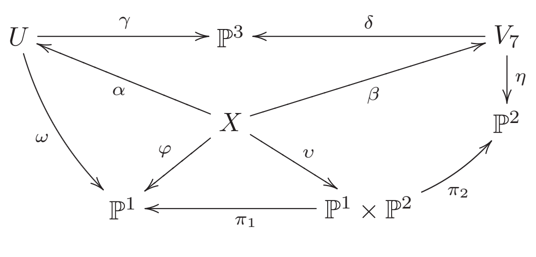
Let H be a general hyperplane section of the quadric Q. Then $-K_{Y_1} \sim \widetilde{H}_1 + 2\widetilde{H} \sim_{\mathbb{Q}} \beta_2(D)$, where $\widetilde{H} \subset Y_1 \supset \widetilde{H}_1$ are the proper transforms of H and H_1, respectively, and we have

$$\mathrm{LCS}\Big(Y_1, \lambda\beta_2(D) + \frac{1}{2}\,(\widetilde{H}_1 + 2\widetilde{H})\Big) = P \cup \widetilde{H},$$

which is impossible by Theorem 2.7 because $\lambda < 1/2$.

Lemma 8.12. *If* $\gimel(X) = 3.11$, *then* $\mathrm{lct}(X) = 1/2$.

Proof. Let $O \in \mathbb{P}^3$ be a point, let $\delta\colon V_7 \to \mathbb{P}^3$ be a blow-up of the point O, and let E be the exceptional divisor of δ. Then $V_7 \cong \mathbb{P}(\mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(1))$, there is a natural $\mathbb{P}^1$-bundle $\eta\colon V_7 \to \mathbb{P}^2$, and E is a section of η. There is a normal elliptic curve C with $O \in C \subset \mathbb{P}^3$ of degree 4 such that the diagram



is commutative, where π_1 and π_2 are the natural projections, the morphism γ contracts a surface

$$C \times \mathbb{P}^1 \cong G \subset U$$

to the curve C, the morphism α is a blow-up of the fibre of the morphism γ over the point $O \in \mathbb{P}^3$, the morphism β is a blow-up of the proper transform of C, the morphism ω is a fibration into quadric surfaces, φ is a fibration into del Pezzo surfaces of degree 7, and υ contracts a surface

$$C \times \mathbb{P}^1 \cong F \subset X$$

to an elliptic curve $Z \subset \mathbb{P}^1 \times \mathbb{P}^2$ such that $-K_{\mathbb{P}^1\times\mathbb{P}^2} \cdot Z = 13$ and $Z \cong C$.

Let H_1 be a general fibre of φ, and let H_2 be a general surface in $|(\eta\circ\beta)^*(\mathscr{O}_{\mathbb{P}^2}(1))|$. Then $-K_X \sim H_1 + 2H_2$, which implies that $\operatorname{lct}(X) \leqslant 1/2$.

Suppose that $\operatorname{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$. Note that $\operatorname{LCS}(X, \lambda D) \subseteq \overline{E}$, where $\overline{E}$ is the exceptional divisor of α because $\operatorname{lct}(U) = 1/2$ by Lemma 7.11.

Let $\Gamma \cong \mathbb{P}^2$ be a general fibre of $\pi_2 \circ \upsilon$. Then

$$2 = -K_X \cdot \Gamma = D \cdot \Gamma = 2\overline{E} \cdot \Gamma,$$

which implies that $\overline{E} \not\subset \operatorname{LCS}(X, \lambda D)$. Applying Lemma 2.25 to the log pair $(V_7, \lambda\beta(D))$, we have $\operatorname{LCS}(X, \lambda D) \subseteq \overline{E} \cap G$. Applying Lemma 2.28 to the log pair $(\mathbb{P}^1 \times \mathbb{P}^2, \lambda\upsilon(D))$, we see that $\operatorname{LCS}(X, \lambda D) = \overline{E} \cap F \cap G$, where $|\overline{E} \cap F \cap G| = 1$. Hence

$$\operatorname{LCS}(X, \lambda D + H_2) = \operatorname{LCS}(X, \lambda D) \cup H_2$$

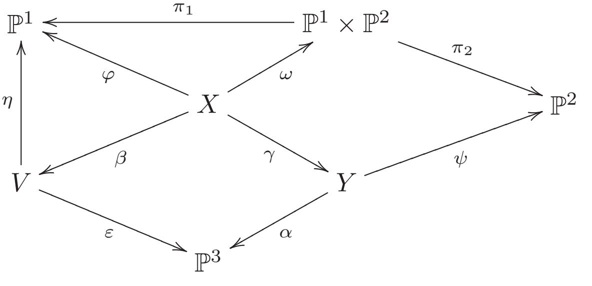
and $H_2 \cap \operatorname{LCS}(X, \lambda D) = \varnothing$. But the divisor

$$-(K_X + \lambda D + H_2) = \left(\lambda - \frac{1}{2}\right)K_X + \frac{1}{2}H_1$$

is ample, which is impossible by Theorem 2.7.

Lemma 8.13. *If* $\daleth(X) = 3.12$, *then* $\operatorname{lct}(X) = 1/2$.

Proof. Let $\varepsilon\colon V \to \mathbb{P}^3$ be a blow-up of a line $L \subset \mathbb{P}^3$. There is a natural $\mathbb{P}^2$-bundle $\eta\colon V \to \mathbb{P}^1$ and there is a twisted cubic $C \subset \mathbb{P}^3$ disjoint from L such that the diagram



is commutative, where α and β are blow-ups of C and its proper transform, respectively, γ is a blow-up of the proper transform of L, the morphism ψ is a $\mathbb{P}^1$-bundle, the morphism ω is a contraction to a curve of a surface $F \subset X$ such that $\alpha \circ \gamma(F)$ contains $C \cup L$ and consists of secant lines of $C \subset \mathbb{P}^3$ that intersect L; the morphism φ is a fibration into del Pezzo surfaces of degree 6, and the morphisms π_1 and π_2 are the natural projections.

Let E and G be the exceptional divisors of β and γ, respectively, let $Q \subset \mathbb{P}^3$ be a general quadric surface passing through C, and let $H \subset \mathbb{P}^3$ be a general plane passing through L. Then $-K_X \sim \overline{Q} + 2\overline{H} + G$, where $\overline{Q} \subset X \supset \overline{H}$ are the proper transforms of $Q \subset \mathbb{P}^3 \supset H$, respectively. In particular, $\mathrm{lct}(X) \leqslant 1/2$.

Suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$. Note that $\mathrm{LCS}(X, \lambda D) \subset G$ since $\mathrm{lct}(Y) = 1/2$ by Lemma 7.13. Applying Theorem 2.27 to φ, we see that $\mathrm{LCS}(X, \lambda D) \subset G \cap S_\varphi$, where S_φ is a singular fibre of the fibration φ (see Example 1.10). Then $\mathrm{LCS}(X, \lambda D) \subset G \cap S_\varphi \cap F$ by Theorem 2.27 applied to the log pair $(\mathbb{P}^1 \times \mathbb{P}^2, \lambda\omega(D))$ and the $\mathbb{P}^1$-bundle π_2.

Let $Z_1 \cong \mathbb{P}^1$ be a section of the natural projection $\mathbb{P}^1 \times \mathbb{P}^1 \cong G \to L \cong \mathbb{P}^1$ such that $Z_1 \cdot Z_1 = 0$, and let Z_2 be a fibre of this projection. Then $F|_G \sim Z_1 + 3Z_2$ and $S_\varphi|_G \sim Z_1$. The curve $F \cap G$ is irreducible. Thus, $|G \cap F \cap S_\varphi| < +\infty$, which implies by Theorem 2.7 that the set $\mathrm{LCS}(X, \lambda D)$ consists of a single point $P \in G$.

The log pair $(V, \lambda\beta(D))$ is not log canonical. Since β is an isomorphism on $X \setminus E$, we have

$$\beta(P) \in \mathrm{LCS}\big(V, \lambda\beta(D)\big) \subseteq \beta(P) \cup \beta(E),$$

which implies by Theorem 2.7 that $\mathrm{LCS}(V, \lambda\beta(D)) = \beta(P)$. Let $H \subset \mathbb{P}^3$ be a general plane. Then

$$\mathrm{LCS}\bigg(V, \lambda\beta(D) + \frac{1}{2}\,(\widetilde{H}_1 + 3\widetilde{H})\bigg) = \beta(P) \cup \widetilde{H},$$

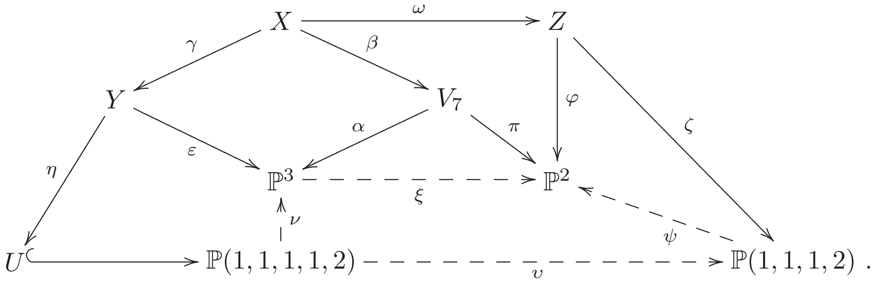
where $\widetilde{H} \subset V \supset \widetilde{H}_1$ are the proper transforms of $H \subset \mathbb{P}^3 \supset H_1$, respectively, and we have $-K_V \sim \widetilde{H}_1 + 3\widetilde{H} \sim_{\mathbb{Q}} \beta(D)$, which contradicts Theorem 2.7 because $\lambda < 1/2$.

Lemma 8.14. *If* $\daleth(X) = 3.14$, *then* $\mathrm{lct}(X) = 1/2$.

Proof. Let $P \in \mathbb{P}^3$ be a point and let $\alpha\colon V_7 \to \mathbb{P}^3$ be a blow-up of P. Then there is a natural $\mathbb{P}^1$-bundle $\pi\colon V_7 \to \mathbb{P}^2$.

Let $\zeta\colon Z \to \mathbb{P}(1,1,1,2)$ be a blow-up of the singular point of $\mathbb{P}(1,1,1,2)$. Then $Z \cong \mathbb{P}(\mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(2))$ and there is a natural $\mathbb{P}^1$-bundle $\varphi\colon Z \to \mathbb{P}^2$.

There is a plane $\Pi \subset \mathbb{P}^3$ and a smooth cubic curve $C \subset \Pi$ such that $P \notin \Pi$ and there is a commutative diagram (see [28], Example 3.6)



Here we use the following notation: the morphism ε is a blow-up of the curve C; the threefold U is a cubic hypersurface in $\mathbb{P}(1,1,1,1,2)$; the rational map ξ is a projection from the point P; the morphism γ is a blow-up of the point dominating P; the morphism β is a blow-up of the proper transform of the curve C; the morphism η contracts the proper transform of Π to the point $\mathrm{Sing}(U)$, the morphism ω contracts to a curve a surface $R \subset X$ such that $\beta \circ \alpha(R)$ is a cone over C with vertex at P; the rational maps ψ and ν are the natural projections; the rational map υ is a linear projection from a point.

Let E and G be the exceptional divisors of γ and β, respectively, and let $\overline{H} \subset X$ be the proper transform of a general plane in $\mathbb{P}^3$ passing through the point P. Then $-K_X \sim \overline{\Pi} + 3\overline{H} + G$, where $\overline{\Pi} \subset X$ is the proper transform of the plane Π. Thus, $\mathrm{lct}(X) \leqslant 1/3$.

Suppose that $\mathrm{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some positive rational number $\lambda < 1/3$.

Let $\overline{L} \subset X$ be the proper transform of the general line in $\mathbb{P}^3$ that intersects C. Then

$$D \cdot \overline{L} = \overline{\Pi} \cdot \overline{L} + 3\overline{H} \cdot \overline{L} + G \cdot \overline{L} = 3\overline{H} \cdot \overline{L} = 3,$$

which implies that $\mathbb{LCS}(X, \lambda D)$ contains no surfaces with the possible exception of $\overline{\Pi}$ and E.

Let Γ be a general fibre of $\pi \circ \beta$. Then

$$D \cdot \Gamma = \overline{\Pi} \cdot \Gamma + 3\overline{H} \cdot \Gamma + G \cdot \Gamma = \overline{\Pi} \cdot \Gamma + G \cdot \Gamma = 2,$$

which implies that $\mathbb{LCS}(X, \lambda D)$ does not contain $\overline{\Pi}$ or E. Thus, by Lemma 2.9 we obtain $\mathrm{LCS}(X, \lambda D) \subsetneq E \cup G$.

Suppose that $\mathrm{LCS}(X, \lambda D) \subseteq E$. Then $\varnothing \neq \mathrm{LCS}(V_7, \lambda\beta(D)) \subseteq \beta(E)$, which contradicts Theorem 2.27, because $\beta(E)$ is a section of π. Hence $\mathrm{LCS}(X, \lambda D) \subsetneq G$.

Applying Theorem 2.27 to $(Z, \lambda\omega(D))$ and φ and applying Theorem 2.7 to $(X, \lambda D)$, we see that $\mathrm{LCS}(X, \lambda D) \subseteq F$, where F is a fibre of the natural projection $G \to \beta(G)$. Hence $\varnothing \neq \mathrm{LCS}(Y, \lambda\gamma(D)) \subseteq \gamma(F)$, where $\gamma(F)$ is the fibre of the blow-up ε over a point of the curve C.

Let $S \subset \mathbb{P}^3$ be a general cone over the curve C and let $O \in C$ be an inflection point such that $\varepsilon \circ \gamma(F) \neq O$. Let $L \subset S$ be the line passing through the point O, and let $H \subset \mathbb{P}^3$ be the plane tangent to the cone S along the line L. Since O is an inflection point of C, it follows that $\mathrm{mult}_L(S \cdot H) = 3$. Let $\breve{S}$, $\breve{H}$, and $\breve{L}$ be the proper transforms of S, H, and L on the threefold Y. Then

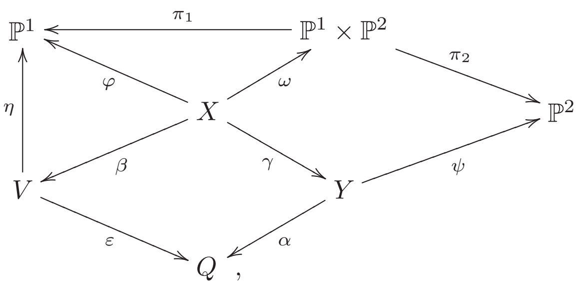
$$\mathrm{LCS}\Big(Y, \lambda\gamma(D) + \frac{2}{3}\,(\breve{S} + \breve{H})\Big) = \mathrm{LCS}\big(Y, \lambda\gamma(D)\big) \cup \breve{L}$$

due to the generality in the choice of S. But $-K_Y \sim \breve{S} + \breve{H}$, which is impossible be Theorem 2.7.

Lemma 8.15. *If* $\daleth(X) = 3.15$, *then* $\mathrm{lct}(X) = 1/2$.

Proof. Let $Q \subset \mathbb{P}^4$ be a smooth quadric hypersurface, let $C \subset Q$ be a smooth conic, and let $\varepsilon\colon V \to Q$ be a blow-up of the conic $C \subset Q$. Then there is a natural morphism $\eta\colon V \to \mathbb{P}^1$ induced by the projection $Q \dashrightarrow \mathbb{P}^1$ from the two-dimensional linear subspace of $\mathbb{P}^4$ that contains C. Then a general fibre of η is a smooth quadric surface in $\mathbb{P}^3$.

Take a line $L \subset Q$ such that $L \cap C = \varnothing$. Then there is a commutative diagram



where α and β are blow-ups of the line $L \subset Q$ and its proper transform, respectively, γ is a blow-up of the proper transform of the conic C, the morphism ψ is a $\mathbb{P}^1$-bundle, ω is a birational contraction to a curve of a surface $F \subset X$ such that $C \cup L \subset \alpha \circ \gamma(F) \subset Q$, $\alpha \circ \gamma(F)$ consists of all the lines in $Q \subset \mathbb{P}^4$ that intersect L and C, the morphism φ is a fibration into del Pezzo surfaces of degree 7, and the morphisms π_1 and π_2 are the natural projections.

Let E_1 and E_2 be the exceptional surfaces of β and γ, respectively, let $H_1, H_2 \subset Q$ be general hyperplane sections that pass through L and C, respectively. We have $-K_X \sim \overline{H}_1 + 2\overline{H}_2 + E_2 \sim \overline{H}_2 + 2\overline{H}_1 + E_1$, where $\overline{H}_1 \subset X \supset \overline{H}_2$ are the proper transforms of $H_1 \subset Q \supset H_2$, respectively. In particular, $\mathrm{lct}(X) \leqslant 1/2$.

Suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$.

Let $S \subset X$ be an irreducible surface. We put $D = \mu S + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $S \not\subset \mathrm{Supp}(\Omega)$. Then

$$\mathrm{LCS}\left(\overline{H}_2, \frac{1}{2}\,(\mu S + \Omega)|_{\overline{H}_2}\right) \subset E_1 \cap \overline{H}_2$$

by Lemma 4.9. Thus, if $\mu \leqslant 2$ then either $S = E_1$ or S is a fibre of φ.

Let $\Gamma \cong \mathbb{P}^1$ be a general fibre of the conic bundle $\psi \circ \gamma$. Then

$$2 = D \cdot \Gamma = \mu S \cdot \Gamma + \Omega \cdot \Gamma \geqslant \mu S \cdot \Gamma,$$

which implies that $\mu \leqslant 2$ in the case when either $S = E_1$ or S is a fibre of φ.

Therefore, we see that $\mathbb{LCS}(X, \lambda D)$ does not contain surfaces.

Application of Theorem 2.27 to the log pair $(Y, \lambda\gamma(D))$ and ψ gives us that $\mathrm{LCS}(X, \lambda D) \subsetneq E_2 \cup \bar{L}$, where $\mathbb{P}^1 \cong \bar{L} \subset X$ is a curve such that $\gamma(\bar{L})$ is a fibre of the conic bundle ψ.

Suppose that $\bar{L} \not\subset E_1$ and $\bar{L} \subset \mathrm{LCS}(X, \lambda D)$. Then

$$\alpha \circ \gamma(\bar{L}) \subseteq \mathrm{LCS}\big(Q, \lambda\alpha \circ \gamma(D)\big) \subseteq \alpha \circ \gamma(\bar{L}) \cup C \cup L,$$

which is impossible by Lemma 2.10. Hence by Theorem 2.7 either $\mathrm{LCS}(X, \lambda D) \subsetneq E_2$ or $\mathrm{LCS}(X, \lambda D) \subseteq \bar{L}$ and $\bar{L} \subset E_1$.

We may assume that $\bar{L} \subset E_1$. Note that $E_1 \cong \mathbb{F}_1$. Hence $\bar{L} \cdot \bar{L} = -1$ on the surface E_1.

Applying Lemma 2.28 to the log pair $(\mathbb{P}^1 \times \mathbb{P}^2, \lambda\omega(D))$, we see that

$$\mathrm{LCS}(X, \lambda D) \subset F,$$

because $\omega(D) \sim_{\mathbb{Q}} -K_{\mathbb{P}^1\times\mathbb{P}^2}$ and $\lambda < 1/2$. Applying Lemma 2.25 to the log pair $(V, \lambda\beta(D))$ and the fibration η, we see that $\mathrm{LCS}(X, \lambda D) \subsetneq E_1 \cup S_\varphi$, where S_φ is a singular fibre of φ, because $\mathrm{lct}(\mathbb{P}^1 \times \mathbb{P}^1) = 1/2$ (see Example 1.10).

We have $F \cap \bar{L} = \varnothing$ and $|F \cap \bar{S}_\varphi \cap E_2| < +\infty$. Thus, there is a point $P \in E_2$ such that $\mathrm{LCS}(X, \lambda D) = P \in E_2$ by Theorem 2.7, and we have $\beta(E_1) \cap \beta(P) = \varnothing$. Thus, it follows from Theorem 2.7 that $\mathrm{LCS}(V, \lambda\beta(D)) = \beta(P)$.

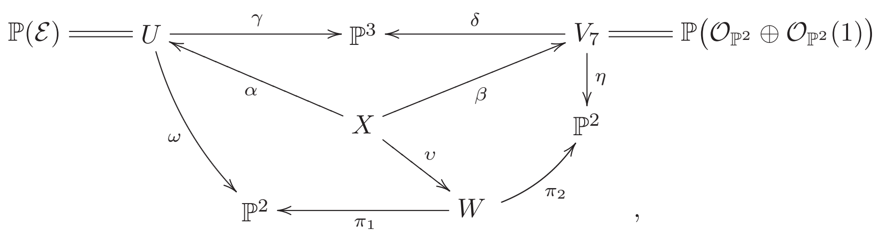
Let $\widetilde{H}_1 \subset V \supset \widetilde{H}_2$ be the proper transforms of the divisors $H_1 \subset Q \supset H_2$, respectively. Then $-K_V \sim \widetilde{H}_2 + 2\widetilde{H}_1 \sim_{\mathbb{Q}} \beta(D)$. It follows from the generality of H_1 and H_2 that

$$\mathrm{LCS}\Big(V, \lambda\beta(D) + \frac{1}{2}\,(\widetilde{H}_2 + 2\widetilde{H}_1)\Big) = \beta(P) \cup \widetilde{H}_1,$$

which is impossible by Theorem 2.7 because $\lambda < 1/2$.

Lemma 8.16. *If* $\gimel(X) = 3.16$, *then* $\mathrm{lct}(X) = 1/2$.

Proof. Let $\mathbb{P}^1 \cong C \subset \mathbb{P}^3$ be a twisted cubic curve and let $O \in C$ be a point. There is a commutative diagram



where $\mathscr{E}$ is a stable rank-2 vector bundle on $\mathbb{P}^2$ (see the proof of Lemma 7.13). Here we use the following notation: the morphism δ is a blow-up of the point O; the morphism γ contracts a surface $G \subset U$ to the curve $C \subset \mathbb{P}^3$; the morphism α contracts a surface $E \cong \mathbb{F}_1$ to the fibre of γ over the point $O \in \mathbb{P}^3$; the morphism β is a blow-up of the proper transform of the curve C; the variety W is a smooth divisor of bidegree $(1,1)$ on $\mathbb{P}^2 \times \mathbb{P}^2$; the morphisms π_1 and π_2 are the natural projections; the morphisms ω and η are natural $\mathbb{P}^1$-bundles; the morphism υ contracts a surface $F \subset X$ to a curve Z with $\mathbb{P}^1 \cong Z \subset W$ such that $\omega \circ \alpha(E) = \pi_1(Z)$ and $\eta \circ \beta(G) = \pi_2(Z)$.

We take general divisors $H_1 \in |(\omega \circ \alpha)^*(\mathscr{O}_{\mathbb{P}^2}(1))|$ and $H_2 \in |(\eta \circ \beta)^*(\mathscr{O}_{\mathbb{P}^2}(1))|$. Then $-K_X \sim H_1 + 2H_2$, which implies that $\mathrm{lct}(X) \leqslant 1/2$.

Suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$. Note that $\mathrm{LCS}(X, \lambda D) \subseteq E \cap F$, because $\mathrm{lct}(U) = 1/2$ by Lemma 7.11 and $\mathrm{lct}(W) = 1/2$ by Theorem 6.1.

Applying Lemma 2.12 to the log pair $(V_7, \lambda\beta(D))$ we see that $\mathrm{LCS}(X, \lambda D) = E \cap F \cap G$, where $|E \cap F \cap G| = 1$. Thus,

$$\mathrm{LCS}(X, \lambda D + H_2) = \mathrm{LCS}(X, \lambda D) \cup H_2,$$

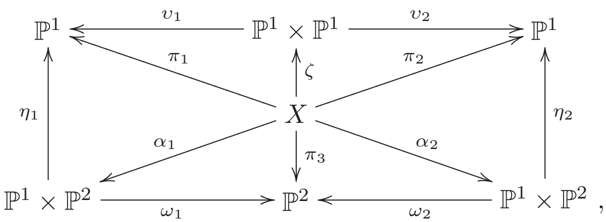
where $H_2 \cap \mathrm{LCS}(X, \lambda D) = \varnothing$. But the divisor

$$-(K_X + \lambda D + H_2) \sim_{\mathbb{Q}} \left(\lambda - \frac{1}{2}\right) K_X + \frac{1}{2} H_1$$

is ample, which is impossible by Theorem 2.7.

Lemma 8.17. *If* $\daleth(X) = 3.17$, *then* $\mathrm{lct}(X) = 1/2$.

Proof. The threefold X is a divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of tridegree $(1, 1, 1)$. We take general surfaces $H_1 \in |\pi_1^*(\mathscr{O}_{\mathbb{P}^1}(1))|$, $H_2 \in |\pi_2^*(\mathscr{O}_{\mathbb{P}^1}(1))|$, $H_3 \in |\pi_3^*(\mathscr{O}_{\mathbb{P}^2}(1))|$, where π_i is the projection of X onto the ith factor of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$. Then $-K_X \sim H_1 + H_2 + 2H_3$, which implies that $\mathrm{lct}(X) \leqslant 1/2$. There is a commutative diagram



where ω_i, η_i, and υ_i are the natural projections, ζ is a $\mathbb{P}^1$-bundle, and α_i is a birational morphism contracting a surface $E_i \subset X$ to a smooth curve $C_i \subset \mathbb{P}^1 \times \mathbb{P}^2$ such that $\omega_1(C_1) = \omega_2(C_2)$ is an (irreducible) conic.

Note that $E_2 \sim H_1 + H_3 - H_2$ and $E_1 \sim H_2 + H_3 - H_1$.

Suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$.

Suppose that the set $\mathbb{LCS}(X, \lambda D)$ contains an (irreducible) surface $S \subset X$. We put $D = \mu S + \Omega$, where $\mu \geqslant 1/\lambda$ and Ω is an effective $\mathbb{Q}$-divisor such that $S \not\subset \mathrm{Supp}(\Omega)$. Then

$$2 = D \cdot \Gamma = \mu S \cdot \Gamma + \Omega \cdot \Gamma \geqslant \mu S \cdot \Gamma,$$

where $\Gamma \cong \mathbb{P}^1$ is a general fibre of ζ. Hence $S \cdot \Gamma = 0$, which implies that $E_2 \neq S \neq E_1$. We also have

$$2 = D \cdot \Delta = \mu S \cdot \Delta + \Omega \cdot \Delta \geqslant \mu S \cdot \Delta,$$

where $\Delta \cong \mathbb{P}^1$ is a general fibre of the conic bundle π_2. Hence $S \cdot \Delta = 0$, which immediately implies that $S \in |\pi_3^*(\mathscr{O}_{\mathbb{P}^2}(m))|$ for some $m \in \mathbb{Z}_{>0}$, because $E_2 \neq S \neq E_1$ and S is an irreducible surface. In particular, $0 = S \cdot \Gamma = m \neq 0$, which is a contradiction. Hence the set $\mathbb{LCS}(X, \lambda D)$ contains no surfaces.

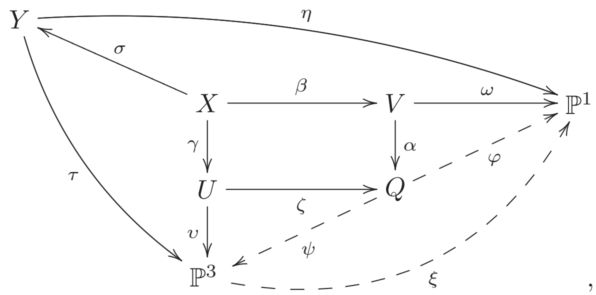
Applying Theorem 2.27 to ζ and using Theorem 2.7, we have $\mathrm{LCS}(X, \lambda D) = F$, where F is a fibre of the $\mathbb{P}^1$-bundle ζ. Applying Theorem 2.27 to the conic bundle π_3, we see that every fibre of π_3 that intersects F must be reducible. This means that

$$\pi_3(F) \subset \omega_1(C_1) = \omega_2(C_2),$$

which is impossible, because $\pi_3(F)$ is a line and $\omega_1(C_1) = \omega_2(C_2)$ is an irreducible conic.

Lemma 8.18. *If* $\daleth(X) = 3.18$, *then* $\operatorname{lct}(X) = 1/3$.

Proof. Let $Q \subset \mathbb{P}^4$ be a smooth quartic hypersurface, $C \subset Q$ an irreducible conic, and $O \in C$ a point. Then there is a commutative diagram



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where ζ is a blow-up of the point O, the morphisms α and γ are blow-ups of the conic C and its proper transform, respectively, β is a blow-up of the fibre of α over the point O, the map ψ is the projection from O, the map φ is induced by the projection from the two-dimensional linear subspace of $\mathbb{P}^4$ containing the conic C, the morphism τ is a blow-up of the line $\psi(C)$, the morphism υ is a blow-up of an irreducible conic $Z \subset \mathbb{P}^3$ such that $\psi(C) \cap Z \neq \varnothing$ and Z and $\psi(C)$ are not coplanar, the morphism σ is a blow-up of the proper transform of Z, the map ξ is a projection from $\psi(C)$, the morphism η is a $\mathbb{P}^1$-bundle, and ω is a fibration into quadric surfaces.

Let $\overline{H}$ be a general fibre of $\omega \circ \beta$. Then $\overline{H}$ is a del Pezzo surface such that $K^2_{\overline{H}} = 7$ and $-K_X \sim 3\overline{H} + 2E + G$, where G and E are the exceptional divisors of β and γ, respectively. In particular, $\operatorname{lct}(X) \leqslant 1/3$.

Suppose that $\operatorname{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/3$. Note that $\operatorname{LCS}(X, \lambda D) \subseteq G$, since $\operatorname{lct}(V) = 1/3$ by Lemma 7.15 and $\beta(D) \sim_{\mathbb{Q}} -K_V$.

Applying Lemma 2.25 to the del Pezzo fibration $\omega \circ \beta$ and using Theorem 2.7, we see that there is a unique singular fibre S of the fibration $\omega \circ \beta$ such that $\operatorname{LCS}(X, \lambda D) \subseteq G \cap S$, because $\operatorname{lct}(\overline{H}\,) = 1/3$ (see Example 1.10).

Let $P \in G \cap S$ be an arbitrary point in the locus $\operatorname{LCS}(X, \lambda D)$. We put $D = \mu S + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $S \not\subset \operatorname{Supp}(\Omega)$. Then $P \in \operatorname{LCS}(S, \lambda\Omega|_S)$ by Theorem 2.19.

We can identify the surface $\beta(S)$ with a quadric cone in $\mathbb{P}^3$. Note that $G \cap S$ is an exceptional curve on S, that is, there exists a unique ruling of the cone $\beta(S)$ intersecting the curve $\beta(G)$. Let $L \subset S$ be the proper transform of this ruling. Then $L \cap G \neq \varnothing$ (moreover, $|L \cap G| = 1$), while $L \cap E = \varnothing$. Hence $P = L \cap G$ by Lemma 4.10. In particular, $\operatorname{LCS}(X, \lambda D) = P$. Hence

$$\overline{H} \cup P \subseteq \operatorname{LCS}\Big(X, \lambda D + \overline{H} + \frac{2}{3}\, E\Big) \subseteq \overline{H} \cup P \cup E,$$

because $\overline{H}$ is a sufficiently general fibre of the fibration $\omega \circ \beta$. Therefore, the locus $\mathrm{LCS}\big(X, \lambda D + \overline{H} + \frac{2}{3}E\big)$ must be disconnected, because $P \notin \overline{H}$ and $P \notin E$. But

$$-\left(K_X + \lambda D + \overline{H} + \frac{2}{3}\,E\right) \sim_{\mathbb{Q}} \overline{H} + \frac{2}{3}\,(E+G) + \left(\lambda - \frac{1}{3}\right)K_X$$

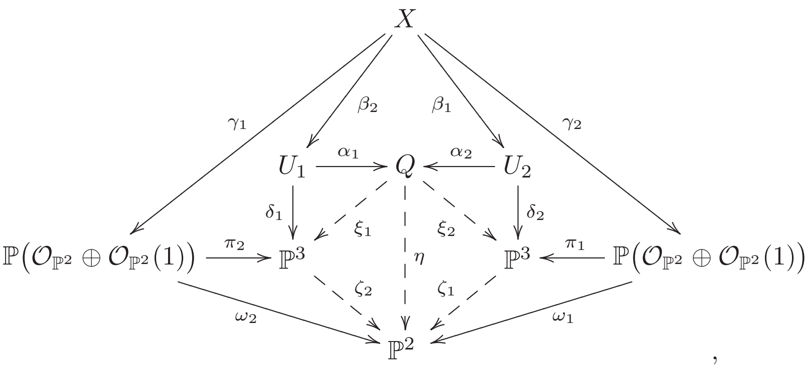
is an ample divisor, which is impossible by Theorem 2.7.

The proof of Lemma 8.18 implies the following corollary.

Corollary 8.19. *If* $\daleth(X) = 4.4$ *or* 5.1, *then* $\mathrm{lct}(X) = 1/3$.

Lemma 8.20. *If* $\daleth(X) = 3.19$, *then* $\mathrm{lct}(X) = 1/3$.

Proof. Let $Q \subset \mathbb{P}^4$ be a smooth quadric and let $L \subset \mathbb{P}^4$ be a line such that $L \cap Q = P_1 \cup P_2$, where P_1 and P_2 are different points. Let $\eta\colon Q \dashrightarrow \mathbb{P}^2$ be the projection from L. There exists a commutative diagram



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where α_i is a blow-up of the point P_i, the morphism β_i contracts a surface $\mathbb{P}^2 \cong E_i \subset X$ to the point dominating $P_i \in Q$, the map ξ_i is the projection from P_i, the map ζ_i is the projection from the image of P_i, the morphism δ_i is a contraction of a surface $\mathbb{F}_2 \cong G_i \subset U_i$ to a conic $C_i \subset \mathbb{P}^3$, the morphism π_i is a blow-up of the image of P_i, the morphism γ_i contracts the proper transform of G_i to the proper transform of C_i, and ω_i is the natural projection.

The map $\gamma_1 \circ \gamma_2^{-1}$ is an elementary transformation of a conic bundle (see [57]) and $\delta_1 \circ \beta_2(E_1) \subset \mathbb{P}^3 \supset \delta_2 \circ \beta_1(E_2)$ are the planes containing the conics C_1 and C_2, respectively.

Let H be a general hyperplane section of Q such that $P_1 \in H \ni P_2$. Then $-K_X \sim 3\overline{H} + E_1 + E_2$, where $\overline{H}$ is the proper transform of H on the threefold X. In particular, $\mathrm{lct}(X) \leqslant 1/3$.

Suppose that $\mathrm{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/3$. Then $\mathrm{LCS}(X, \lambda D) \subseteq E_1 \cup E_2$ because $\mathrm{lct}(Q) = 1/3$. By Theorem 2.7 we may assume that, $\mathrm{LCS}(X, \lambda D) \subseteq E_1$.

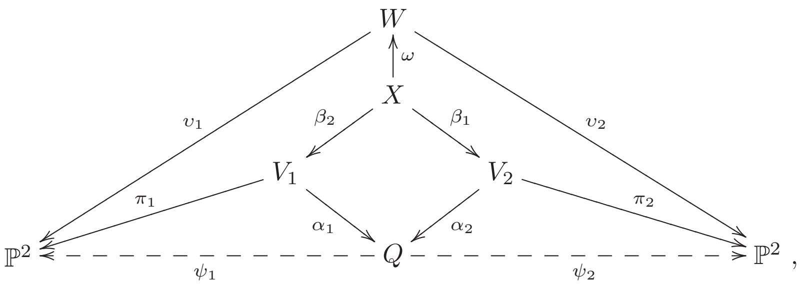
Let $\overline{G}_2 \subset X$ be the proper transform of G_2. Then $\overline{G}_2 \cap E_1 = \varnothing$, because $\alpha_2(G_2) \subset Q$ is a quadric cone with vertex at the point P_2, and the line L does not lie in Q. Hence

$$\varnothing \neq \mathrm{LCS}\big(\mathbb{P}\big(\mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(1)\big), \lambda\gamma_2(D)\big) \subseteq \gamma_2(E_1),$$

where $\gamma_2(E_1)$ is a section of ω_1. Applying Theorem 2.27 to ω_1 we obtain a contradiction.

Lemma 8.21. *If* $\gimel(X) = 3.20$, *then* $\mathrm{lct}(X) = 1/3$.

Proof. Let $Q \subset \mathbb{P}^4$ be a smooth quadric threefold and let W be a smooth divisor of bidegree $(1,1)$ on $\mathbb{P}^2 \times \mathbb{P}^2$. Let $L_1 \subset Q \supset L_2$ be disjoint lines. Then there exists a commutative diagram



where α_i and β_i are blow-ups of the lines L_i and their proper transforms, respectively, ω is a blow-up of a smooth curve $C \subset W$ of bidegree $(1,1)$, the morphisms υ_i and π_i are natural $\mathbb{P}^1$-bundles, and the map ψ_i is a linear projection from the line L_i.

Let $\overline{H}$ be the exceptional divisor of ω and let E_i be the exceptional divisor of β_i. Then $-K_X \sim 3\overline{H} + 2E_1 + 2E_2$, because $\alpha_2 \circ \beta_1(\overline{H}\,) \subset Q$ is a hyperplane section that contains L_1 and L_2. In particular, $\mathrm{lct}(X) \leqslant 1/3$.

Suppose that $\mathrm{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/3$. Then $\mathrm{LCS}(X, \lambda D) \subseteq E_1 \cap E_2 \cap \overline{H} = \varnothing$, because $\mathrm{lct}(V_1) = \mathrm{lct}(V_2) = 1/3$ by Lemma 7.17 and $\mathrm{lct}(W) = 1/2$ by Theorem 6.1, which gives a contradiction.

Lemma 8.22. *If* $\gimel(X) = 3.21$, *then* $\mathrm{lct}(X) = 1/3$.

Proof. Let $\pi_1 \colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ and $\pi_2 \colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ be the natural projections. There is a morphism $\alpha \colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ contracting a surface E to a curve C such that $\pi_1^*(\mathscr{O}_{\mathbb{P}^1}(1)) \cdot C = 2$ and $\pi_2^*(\mathscr{O}_{\mathbb{P}^2}(1)) \cdot C = 1$.

The curve $\pi_2(C) \subset \mathbb{P}^2$ is a line. Therefore, there is a unique surface $H_2 \in |\pi_2^*(\mathscr{O}_{\mathbb{P}^2}(1))|$ such that $C \subset H_2$. Let H_1 be a fibre of the $\mathbb{P}^2$-bundle π_1. Then $-K_X \sim 2\overline{H}_1 + 3\overline{H}_2 + 2E$, where $\overline{H}_i \subset X$ is the proper transform of the surface H_i. In particular $\mathrm{lct}(X) \leqslant 1/3$.

Suppose that $\mathrm{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/3$. Note that $\mathrm{LCS}(X, \lambda D) \subseteq E$, because $\mathrm{lct}(\mathbb{P}^1 \times \mathbb{P}^2) = 1/3$ by Lemma 2.21. There is a commutative diagram

$$\begin{array}{ccccc} & & V & & \\ & \overset{\delta_1}{\nearrow} & \uparrow{\scriptstyle\gamma} & \overset{\delta_2}{\nwarrow} & \\ U_1 & \xleftarrow{\ \beta_1\ } & X & \xrightarrow{\ \beta_2\ } & U_2 \\ {\scriptstyle\omega_1}\downarrow & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\omega_2} \\ \mathbb{P}^1 & \xleftarrow{\ \pi_1\ } & \mathbb{P}^1\times\mathbb{P}^2 & \xrightarrow{\ \pi_2\ } & \mathbb{P}^2 \end{array},$$

where V is a Fano threefold of index 2 with one ordinary double point $O \in V$ such that $-K_V^3 = 40$, the birational morphism β_i is a contraction of the surface $\overline{H}_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ to a smooth rational curve, δ_i contracts the curve $\beta_i(\overline{H}_2)$ to the point $O \in V$ so that the rational map $\delta_2 \circ \delta_1^{-1} \colon U_1 \dashrightarrow U_2$ is a standard flop in $\beta_1(\overline{H}_2) \cong \mathbb{P}^1$, the morphism ω_1 is a fibration whose general fibre is $\mathbb{P}^1 \times \mathbb{P}^1$, the morphism ω_2 is a $\mathbb{P}^1$-bundle, and γ contracts the surface $\gamma(\overline{H}_2)$ to $O \in V$.

The variety V is a section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ by a linear subspace of codimension 3. We have $-K_V \sim 2(\gamma(\overline{H}_1) + \gamma(E))$, and the divisor $\gamma(\overline{H}_1) + \gamma(E)$ is very ample. There is a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\ \ \gamma\ \ } & V & \xhookrightarrow{\ \ \zeta\ \ } & \mathbb{P}^6 \\ {\scriptstyle\alpha}\downarrow & & & & \vdots\,{\scriptstyle\xi} \\ \mathbb{P}^1\times\mathbb{P}^2 & & \xhookrightarrow{\qquad\eta\qquad} & & \mathbb{P}^5 \end{array}$$

such that the embedding ζ is given by the linear system $|\gamma(\overline{H}_1) + \gamma(E)|$, the map ξ is the projection from the point O, and the embedding η is given by the linear system $|H_1 + H_2|$.

It follows from Theorem 3.6 in [60] (see also [59], Theorem 3.13) that $U_2 \cong \mathbb{P}(\mathscr{E})$, where $\mathscr{E}$ is a stable rank-2 vector bundle on $\mathbb{P}^2$ such that the sequence

$$0 \to \mathscr{O}_{\mathbb{P}_2} \to \mathscr{E} \otimes \mathscr{O}_{\mathbb{P}_2}(1) \to \mathscr{I} \otimes \mathscr{O}_{\mathbb{P}_2}(1) \to 0$$

is exact, where $\mathscr{I}$ is the ideal sheaf of two general points in $\mathbb{P}^2$. We have $c_1(\mathscr{E}) = -1$ and $c_2(\mathscr{E}) = 2$. It follows from Theorem 3.5 in [60] that

$$U_1 \subset \mathbb{P}\bigl(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1)\bigr)$$

and $U_1 \in |2T - F|$, where T is the tautological line bundle on $\mathbb{P}\bigl(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1)\bigr)$ and F is the fibre of the projection $\mathbb{P}\bigl(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(1)\bigr) \to \mathbb{P}^1$.

Note that because $H_1 \cdot C = 2$, either $\overline{H}_1$ is a smooth del Pezzo surface with $K_{\overline{H}_1}^2 = 7$, or $|H_1 \cap C| = 1$. Applying Lemma 2.25 to the morphism $\omega_1 \circ \beta_1$ and the surface $\overline{H}_1$, we see that either $|H_1 \cap C| = 1$ or $H_1 \cap \mathrm{LCS}(X, \lambda D) = \varnothing$, because $\mathrm{lct}(\overline{H}_1) = 1/3$ if $\overline{H}_1$ is smooth. So there is a fibre L of the projection $E \to C$ such that $\mathrm{LCS}(X, \lambda D) \subseteq L$ by Theorem 2.7. We put $\overline{C} = \overline{H}_2 \cap E$ and $P = L \cap \overline{C}$. Applying Theorem 2.27 to ω_2 and $(U_2, \lambda\beta_2(D))$, we see from Theorem 2.7 that either $\mathrm{LCS}(X, \lambda D) = P$ or $\mathrm{LCS}(X, \lambda D) = L$.

Suppose that $\mathrm{LCS}(X, \lambda D) = L$. Then

$$\mathrm{LCS}\big(V, \lambda\gamma(D)\big) = \gamma(L),$$

where $\gamma(L) \subset V \subset \mathbb{P}^6$ is a line, because $-K_V \cdot \gamma(L) = 2$ and $-K_V \sim_{\mathbb{Q}} \gamma(D)$. We have $\mathrm{Sing}(V) = O \in \gamma(L)$.

Let $S \subset V$ be a general hyperplane section of $V \subset \mathbb{P}^6$ such that $\gamma(L) \subset S$. Then the surface S is a del Pezzo surface such that $K_S^2 = 5$, O is an ordinary double point of the surface S, S is smooth away from $O \in \gamma(L)$, the equivalence $K_S \sim \mathscr{O}_{\mathbb{P}^6}(1)|_S$ holds, and hence S contains finitely many lines which intersect the line $\gamma(L)$.

Let $H \subset V$ be a general hyperplane section of $V \subset \mathbb{P}^6$. We put $Q = \gamma(L) \cap H$. Then $\mathrm{LCS}(H, \lambda\gamma(D)|_H) = Q$ by Remark 2.3, which contradicts Lemma 4.2 because $\lambda < 1/3$.

Thus, $\mathrm{LCS}(X, \lambda D) = P \in \overline{C}$. Let F_1 be a general fibre of π_1. Then

$$F_1 \cap C = P_1 \cup P_2 \not\ni \alpha(P),$$

where P_1 and P_2 are different points. We have $P_1 \cup P_2 \subset H_2 \cap F_1$ because $C \subset H_2$. Let Z be a general line in $F_1 \cong \mathbb{P}^2$ containing P_1. Then there is a surface $F_2 \in |\pi_2^*(\mathscr{O}_{\mathbb{P}^2}(1))|$ such that $Z \subset F_2$. Let $\overline{F}_1 \subset X \supset \overline{F}_2$ be the proper transforms of F_1 and F_2, respectively. Then $P \notin \overline{F}_1 \cup \overline{F}_2$.

Let $\overline{Z} \subset X$ be the proper transform of the curve Z. Then $-K_X \cdot \overline{Z} = 2$ and $\overline{Z} \subset \overline{F}_1 \cap \overline{F}_2$, but $\overline{Z} \cap \overline{H}_2 = \varnothing$. Thus, the curve $\gamma(\overline{Z})$ is a line on $V \subset \mathbb{P}^6$ such that $\mathrm{Sing}(V) = O \notin \gamma(\overline{Z})$.

Let T be a general hyperplane section of the threefold $V \subset \mathbb{P}^6$ such that $\gamma(\overline{Z}) \subset T$. Then

$$\overline{T} \sim 2\overline{H}_2 + \overline{H}_1 + E \sim 2\overline{H}_2 + \overline{F}_1 + E \sim 2\overline{F}_2 + \overline{F}_1 - E,$$

where $\overline{T}$ is the proper transform of the surface T on the threefold X. Thus,

$$\overline{F}_1 + \overline{F}_2 + \overline{T} \sim 3\overline{F}_2 + 2\overline{F}_1 - E \sim 2\overline{H}_2 + 2\overline{H}_1 + 2E \sim -K_X,$$

and applying Theorem 2.7, we see that the locus

$$P \cup \overline{Z} = \mathrm{LCS}\bigg(X, \lambda D + \frac{2}{3}(\overline{F}_1 + \overline{F}_2 + \overline{T})\bigg)$$

must be connected. But $P \notin \overline{Z}$, a contradiction.

Lemma 8.23. *If* $\daleth(X) = 3.22$, *then* $\mathrm{lct}(X) = 1/3$.

Proof. Let $\pi_1\colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$ and $\pi_2\colon \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ be the natural projections. There is a morphism $\alpha\colon X \to \mathbb{P}^1 \times \mathbb{P}^2$ contracting a surface E to the curve C lying in a fibre H_1 of π_1 such that the curve $\pi_2(C)$ is a conic in $\mathbb{P}^2$.

We have $E \cong \mathbb{F}_2$. Let H_2 be a general surface in $|\pi_2^*(\mathscr{O}_{\mathbb{P}^2}(1))|$. We have the equivalence $-K_X \sim 2\overline{H}_1 + 3\overline{H}_2 + E$, where $\overline{H}_i \subset X$ is the proper transform of the surface H_i. Hence $\mathrm{lct}(X) \leqslant 1/3$.

Suppose that $\mathrm{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/3$. Note that $\mathrm{LCS}(X, \lambda D) \subseteq E$, since $\mathrm{lct}(\mathbb{P}^1 \times \mathbb{P}^2) = 1/3$ by Lemma 2.21.

Let Q be the unique surface in $|\pi_2^*(\mathscr{O}_{\mathbb{P}^2}(2))|$ containing C and let $\overline{Q} \subset X$ be the proper transform of Q. Then $\overline{Q} \cap \overline{H}_1 = \varnothing$ and there is a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\ \beta\ } & \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2)\big) & & \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\varphi} & \searrow^{\gamma} & \\ \mathbb{P}^1 \times \mathbb{P}^2 & \xrightarrow{\ \pi_2\ } & \mathbb{P}_2 & \overset{\psi}{\dashleftarrow} & \mathbb{P}(1,1,1,2) \end{array}$$

such that β is a contraction of $\overline{Q}$ to a curve, γ is a contraction of $\beta(\overline{H}_1)$ to a point, the morphism φ is a natural $\mathbb{P}^1$-bundle, and the map ψ is the natural projection. We have

$$\gamma \circ \beta(D) \sim_{\mathbb{Q}} \frac{5\gamma \circ \beta(E)}{2} \sim_{\mathbb{Q}} -K_{\mathbb{P}(1,1,1,2)} \sim_{\mathbb{Q}} \mathscr{O}_{\mathbb{P}(1,1,1,2)}(5),$$

which implies that $E \not\subseteq \mathrm{LCS}(X, \lambda D)$ because $\lambda < 1/3$.

Applying Theorem 2.27 to φ, we see that there is a fibre F of the projection $E \to C$ such that $\mathrm{LCS}(X, \lambda D) \subseteq (E \cap \overline{Q}) \cup F$, including the possibility that $\mathrm{LCS}(X, \lambda D) \subset E \cap \overline{Q}$.

Suppose that $\mathrm{LCS}(X, \lambda D) \subset E \cap \overline{Q}$. Let $M \subset \mathbb{P}^1 \times \mathbb{P}^2$ be a general surface in $|H_1 + H_2|$ and let $\overline{M} \subset X$ be the proper transform of the surface M. Then $\overline{M} \cap \overline{H}_1 = L$, where L is a line on $\overline{H}_1 \cong \mathbb{P}^2$. Let R be the unique surface in $|\pi_2^*(\mathscr{O}_{\mathbb{P}^2}(1))|$ containing $\alpha(L)$ and let $\overline{R}$ be the proper transform of R on the threefold X. Then

$$\mathrm{LCS}(X, \lambda D) \cup L \subseteq \mathrm{LCS}\Big(X, \lambda D + \frac{2}{3}(\overline{M} + \overline{H}_1 + \overline{R} + \overline{H}_2)\Big) \subseteq \mathrm{LCS}(X, \lambda D) \cup L \cup \overline{H}_1,$$

but $L \cap E \cap \overline{Q} = \overline{Q} \cap \overline{H}_1 = \varnothing$ and $-K_X \sim \overline{M} + \overline{H}_1 + \overline{R} + \overline{H}_2$, which contradicts Theorem 2.7.

Therefore, $F \subseteq \mathrm{LCS}(X, \lambda D)$. We put $\breve{F} = \gamma \circ \beta(F)$ and $\breve{D} = \gamma \circ \beta(D)$. Then

$$\breve{F} \subseteq \mathrm{LCS}\big(\mathbb{P}(1,1,1,2), \lambda \breve{D}\big) \subseteq \breve{C} \cup \breve{F},$$

where $\breve{C} = \gamma \circ \beta(\overline{Q}) \subset \mathbb{P}(1,1,1,2)$ is a curve such that $\psi(\breve{C}) = \pi_2(C)$.

Let S be a general surface in $|\mathscr{O}_{\mathbb{P}(1,1,1,2)}(2)|$. Then $S \cong \mathbb{P}^2$ and

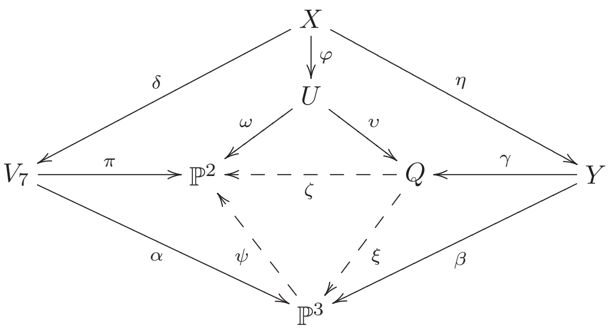
$$\breve{F} \cap S \subseteq \mathrm{LCS}\big(S, \lambda \breve{D}|_S\big) \subseteq (\breve{C} \cup \breve{F}) \cap S;$$

but $3D|_S \sim_{\mathbb{Q}} -5K_S$, which is impossible by Lemma 2.8.

Lemma 8.24. *If* $\daleth(X) = 3.23$, *then* $\mathrm{lct}(X) = 1/4$.

Proof. Let $O \in \mathbb{P}^3$ be a point, let $C \subset \mathbb{P}^3$ be a conic such that $O \in C$; let $\Pi \subset \mathbb{P}^3$ be the unique plane containing C, and let $Q \subset \mathbb{P}^4$ be a smooth quadric threefold.

Then the diagram



is commutative, where we use the following notation: the morphism α is a blow-up of the point O with exceptional divisor E; the morphism π is the natural $\mathbb{P}^1$-bundle; the morphisms β and δ are blow-ups of C and its proper transform, respectively; the morphism γ contracts the proper transform of the plane Π to a point; the morphism φ contracts the proper transform of the plane Π to a curve; the morphism η contracts the proper transform of E to a curve $L \subset Y$ such that $\gamma(\Pi) \in \gamma(L) \subset Q \subset \mathbb{P}^4$ and $\gamma(L)$ is a line in $\mathbb{P}^4$; the morphism ω is a natural $\mathbb{P}^1$-bundle; the morphism υ is a blow-up of the line $\gamma(L)$; the maps ψ, ξ, and ζ are projections from O, $\gamma(\Pi)$, and $\gamma(L)$, respectively. Note that E is a section of π.

Let $\overline{\Pi} \subset X$ be a proper transform of the plane $\Pi \subset \mathbb{P}^3$. Then $\mathrm{lct}(X) \leqslant 1/4$, because $-K_X \sim 4\overline{\Pi} + 2\overline{E} + 3G$, where $\overline{E}$ and G are the exceptional surfaces of η and δ, respectively.

Suppose that $\mathrm{lct}(X) < 1/4$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/4$. We note that

$$\varnothing \neq \mathrm{LCS}(X, \lambda D) \subseteq \overline{E} \cap \overline{\Pi} \cap G,$$

because $\mathrm{lct}(V_7) = 1/4$ by Theorem 6.1, $\mathrm{lct}(Y) = 1/4$ by Lemma 7.16, and $\mathrm{lct}(U) = 1/3$ by Lemma 7.17.

Let $R \subset \mathbb{P}^3$ be a general cone over C with vertex $P \in \mathbb{P}^3$, let $H_1 \subset \mathbb{P}^3$ be a general plane passing through O and P, and let $H_2 \subset \mathbb{P}^3$ be a general plane passing through P. Then

$$\overline{R} \sim (\alpha \circ \delta)^*(R) - \overline{E} - G, \qquad \overline{H}_1 \sim (\alpha \circ \delta)^*(H_1) - \overline{E}, \qquad \overline{H}_2 \sim (\alpha \circ \delta)^*(H_2),$$

where $\overline{R}$, $\overline{H}_1$, and $\overline{H}_2$ are the proper transforms of R, H_1, and H_2 on the threefold X, respectively. We have $-K_X \sim \overline{Q} + \overline{H}_1 + \overline{H}_2$, but it follows from the generality of R, H_1, and H_2 that the locus

$$\mathrm{LCS}\Big(X, \lambda D + \frac{3}{4}\,(\overline{Q} + \overline{H}_1 + \overline{H}_2)\Big) = \mathrm{LCS}(X, \lambda D) \cup P$$

is disconnected, which is impossible by Lemma 2.7.

Lemma 8.25. *If* $\gimel(X) = 3.24$, *then* $\mathrm{lct}(X) = 1/3$.

Proof. Let W be a divisor of bidegree $(1,1)$ on $\mathbb{P}^2 \times \mathbb{P}^2$. There is a commutative diagram

$$\begin{array}{ccccc} & & X & \xrightarrow{\ \alpha\ } & W \\ & \swarrow^{\zeta} & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \omega_1} \\ \mathbb{P}^1 & \xleftarrow[\ \xi\ ]{} & \mathbb{F}_1 & \xrightarrow{\ \gamma\ } & \mathbb{P}^2\ , \end{array}$$

where ω_1 is a natural $\mathbb{P}^1$-bundle, the morphism α contracts a surface $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ to a fibre L of ω_1, γ is a blow-up of the point $\omega_1(L)$, the morphism ξ is a $\mathbb{P}^1$-bundle, and ζ is an $\mathbb{F}_1$-bundle.

Let $\omega_2 \colon X \to \mathbb{P}^2$ be a natural $\mathbb{P}^1$-bundle distinct from ω_1. Then there is a surface $G \in |\omega_2^*(\mathscr{O}_{\mathbb{P}^2}(1))|$ such that $L \subset G$, because $\omega_2(L)$ is a line in $\mathbb{P}^2$. Let $\overline{G} \subset X$ be the proper transform of G. Then $-K_X \sim 2F + 2\overline{G} + 3E$, where E is the exceptional divisor of α and F is a fibre of ζ. We see that $\operatorname{lct}(X) \leqslant 1/3$.

Suppose that $\operatorname{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/3$. Note that $\operatorname{LCS}(X, \lambda D) \subseteq E$ since $\operatorname{lct}(W) = 1/2$ by Theorem 6.1. We may assume that $F \cap \operatorname{LCS}(X, \lambda D) \neq \varnothing$. Then

$$\mathbb{F}_1 \cong F \subseteq \operatorname{LCS}(X, \lambda D) \subseteq E \cong \mathbb{P}^1 \times \mathbb{P}^1$$

by Lemma 2.25 because $\operatorname{lct}(F) = 1/3$ (see Example 1.10), and this is a contradiction.

9. Fano threefolds with $\rho \geqslant 4$

Throughout this section we use the assumptions and notation introduced in § 1.

Lemma 9.1. *If* $\daleth(X) = 4.1$, *then* $\operatorname{lct}(X) = 1/2$.

Proof. The threefold X is a divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of multidegree $(1,1,1,1)$. Let $[(x_1 : y_1), (x_2 : y_2), (x_3 : y_3), (x_4 : y_4)]$ be coordinates on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Then X is given by an equation $F(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4) = 0$, where F is a form of multidegree $(1,1,1,1)$. Let $\pi_1 \colon X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be the projection given by

$$[(x_1 : y_1), (x_2 : y_2), (x_3 : y_3), (x_4 : y_4)] \mapsto [(x_2 : y_2), (x_3 : y_3), (x_4 : y_4)] \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1,$$

and let π_2, π_3, and $\pi_4 \colon X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ be projections defined in a similar way. We put

$$F = x_1 G(x_2, y_2, x_3, y_3, x_4, y_4) + y_1 H(x_2, y_2, x_3, y_3, x_4, y_4),$$

where $G(x_2, y_2, x_3, y_3, x_4, y_4)$ and $H(x_2, y_2, x_3, y_3, x_4, y_4)$ are multilinear forms independent of x_1 and y_1. Then π_1 is a blow-up of the curve $C_1 \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ given by the equations

$$G(x_2, y_2, x_3, y_3, x_4, y_4) = H(x_2, y_2, x_3, y_3, x_4, y_4) = 0,$$

which also define a surface $E_1 \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ which is contracted by π_1. The equations $x_1 = H(x_2, y_2, x_3, y_3, x_4, y_4) = 0$ define a divisor $H_1 \subset X$ such that $-K_X \sim 2H_1 + E_1$, which implies that $\operatorname{lct}(X) \leqslant 1/2$.

Suppose that $\operatorname{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$.

Let E_2, E_3, and E_4 be surfaces in X analogous to E_1. Then

$$\varnothing \neq \operatorname{LCS}(X, \lambda D) \subseteq E_1 \cap E_2 \cap E_3 \cap E_4,$$

because $\operatorname{lct}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = 1/2$ by Lemma 2.21. But $E_i \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is given by the equations

$$\frac{\partial F(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)}{\partial x_i} = \frac{\partial F(x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4)}{\partial y_i} = 0;$$

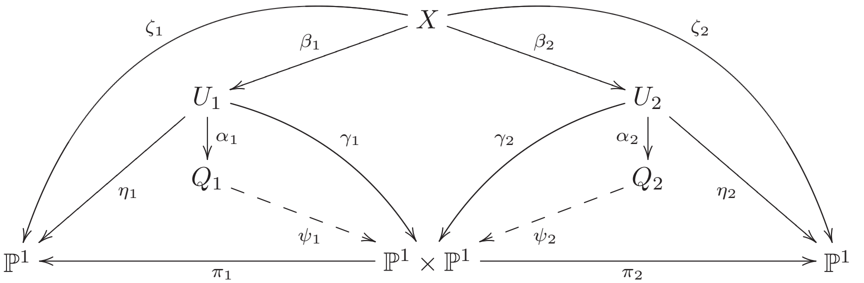
which implies that the intersection $E_1 \cap E_2 \cap E_3 \cap E_4$ is given by the equations

$$\frac{\partial F}{\partial x_1} = \frac{\partial F}{\partial y_1} = \frac{\partial F}{\partial x_2} = \frac{\partial F}{\partial y_2} = \frac{\partial F}{\partial x_3} = \frac{\partial F}{\partial y_3} = \frac{\partial F}{\partial x_4} = \frac{\partial F}{\partial y_4} = 0.$$

Hence $E_1 \cap E_2 \cap E_3 \cap E_4 = \operatorname{Sing}(X) = \varnothing$ and $\operatorname{LCS}(X, \lambda D) = \varnothing$.

Lemma 9.2. *If* $\gimel(X) = 4.2$, *then* $\operatorname{lct}(X) = 1/2$.

Proof. Let $Q_1 \subset \mathbb{P}^4 \supset Q_2$ be quadric cones with vertices $O_1 \in \mathbb{P}^4 \ni O_2$, respectively. Let $O_1 \notin S_1 \subset Q_1 \subset \mathbb{P}^4$ be a hyperplane section of Q_1. Then there exists a smooth elliptic curve $C_1 \subset |-K_{S_1}|$ such that the diagram



is commutative, where $\pi_1 \neq \pi_2$ are the natural projections, the map ψ_i is the projection from $O_i \in Q_i \subset \mathbb{P}^4$, the morphism α_i is a blow-up of the vertex O_i of Q_i, the morphism β_i contracts a surface

$$\mathbb{P}^1 \times C_1 \cong G_i \subset X$$

to a curve $C_1 \cong C_i \subset U_i$, the morphism η_i is an $\mathbb{F}_1$-bundle, γ_i is a $\mathbb{P}^1$-bundle, and ζ_i is a fibration into del Pezzo surfaces of degree 6 which has 4 singular fibres.

Let $E_i \subset X$ be the proper transform of the exceptional divisor of α_i. Then

$$S_1 = \alpha_1 \circ \beta_1(E_2) \subset Q_1 \subset \mathbb{P}^4 \supset Q_2 \supset \alpha_2 \circ \beta_2(E_1)$$

are hyperplane sections of Q_1 and Q_2 containing C_1 and C_2, respectively. It is also easy to see that $\alpha_1 \circ \beta_1(G_2)$ and $\alpha_2 \circ \beta_2(G_1)$ are the cones in $\mathbb{P}^4$ over the curves C_1 and C_2, respectively.

Let $\overline{H}\subset X$ be the proper transform of a hyperplane section of $Q_1 \subset \mathbb{P}^4$ which contains O_1. Then $-K_X \sim 2\overline{H} + E_2 + E_1$, which yields $\mathrm{lct}(X) \leqslant 1/2$.

Suppose that $\mathrm{lct}(X) < 1/2$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$. We put $D = \mu_1 E_1 + \mu_2 E_2 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor on X such that $E_1 \not\subseteq \mathrm{Supp}(\Omega) \not\supseteq E_2$.

Let Γ be a general fibre of the conic bundle $\gamma_1 \circ \beta_1$. Then

$$2 = \Gamma \cdot D = \Gamma \cdot (\mu_1 E_1 + \mu_2 E_2 + \Omega) = \mu_1 + \mu_2 + \Gamma \cdot \Omega \geqslant \mu_1 + \mu_2,$$

and we may assume without loss of generality that $\mu_1 \leqslant \mu_2$. Then $\mu_1 \leqslant 1$.

Suppose that there is a surface $S \in \mathbb{LCS}(X, \lambda D)$. Then $S \neq E_1$. Moreover, $S \neq G_1$, because $\alpha_2 \circ \beta_2(G_1)$ is a quadric surface and $\lambda < 1/2$. Hence $S \cap E_1 \neq \varnothing$. But $-(1/2)K_{E_1} \sim_{\mathbb{Q}} D|_{E_1}$ and $E_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$, which is impossible by Theorem 2.19 and Lemma 2.23. We see that the set $\mathbb{LCS}(X, \lambda D)$ contains no surfaces.

Let $P \in \mathrm{LCS}(X, \lambda D)$. Suppose that $P \notin G_1$. Let Z be a fibre of γ_1 such that $\beta_1(P) \in Z$. Then $Z \subseteq \mathrm{LCS}(U_1, \lambda\beta_1(D))$ by Theorem 2.27. We put $\overline{E}_1 = \beta_1(E_1)$. Then $Z \cap \overline{E}_1 \in \mathrm{LCS}\big(\overline{E}_1, \lambda\Omega|_{\overline{E}_1}\big)$ by Theorem 2.19, which is impossible by Lemma 2.23, because $\mu_1 \leqslant 1$.

Thus, $P \in G_1$. Let $F_1 \subset X \supset F_2$ be fibres of ζ_1 and ζ_2 passing through the point P. Then either F_1 or F_2 is smooth, because $\alpha_1(P) \in C_1$. But $\mathrm{lct}(F_i) = 1/2$ if F_i is smooth (see Example 1.10), which contradicts Lemma 2.25.

Lemma 9.3. *If* $\gimel(X) = 4.3$, *then* $\mathrm{lct}(X) = 1/2$.

Proof. Let $F_1 \cong F_2 \cong F_3 \cong \mathbb{P}^1 \times \mathbb{P}^1$ be fibres of the three different projections $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. There is a contraction $\alpha\colon X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of a surface $E \subset X$ to a curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ such that $C \cdot F_1 = C \cdot F_2 = 1$ and $C \cdot F_3 = 2$. There is a smooth surface $G \in |F_1 + F_2|$ containing C. In particular, $-K_X \sim 2\overline{G} + E + \overline{F}_3$, where $\overline{F}_3$ and $\overline{G}$ are the proper transforms of F_3 and G, respectively. Hence $\mathrm{lct}(X) \leqslant 1/2$.

Suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$. We note that $\mathrm{LCS}(X, \lambda D) \subseteq E$, because $\mathrm{lct}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = 1/2$ and $\alpha(D) \sim_{\mathbb{Q}} -K_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}$.

Let $H \in |3F_1 + F_3|$ be a smooth surface such that $C = G \cap H$, and let $\overline{H}$ be the proper transform of H on the threefold X. Then $\overline{H} \cap \overline{G} = \varnothing$ and there is a commutative diagram

$$\begin{array}{ccccc}
U & \xleftarrow{\;\gamma\;} & X & \xrightarrow{\;\beta\;} & V \\
\downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\pi} \\
\mathbb{P}^1 \times \mathbb{P}^1 & \xleftarrow{\;\zeta\;} & \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 & \xrightarrow{\;\xi\;} & \mathbb{P}^1 \times \mathbb{P}^1
\end{array}$$

such that β and γ are contractions of the surfaces $\overline{G}$ and $\overline{H}$ to smooth curves, the morphisms π and φ are $\mathbb{P}^1$-bundles, ζ and ξ are the projections given by the linear systems $|F_1 + F_2|$ and $|F_1 + F_3|$, respectively.

It follows from $\overline{H} \cap \overline{G} = \varnothing$ that either $(V, \lambda\beta(D))$ or $(U, \lambda\gamma(D))$ is not log canonical.

Applying Theorem 2.27 to $(V, \lambda\beta(D))$ or $(U, \lambda\gamma(D))$ (and the fibrations π or φ, respectively) and using Theorem 2.7, we see that $\mathrm{LCS}(X, \lambda D) = \Gamma$, where Γ is a fibre of the natural projection $E \to C$.

We may assume that $\alpha(\Gamma) \in F_3$. Let $\overline{F}_3 \subset X$ be the proper transform of the surface F_3. We put $D = \mu\overline{F}_3 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor on X such that $\overline{F}_3 \not\subset \mathrm{Supp}(\Omega)$. Then

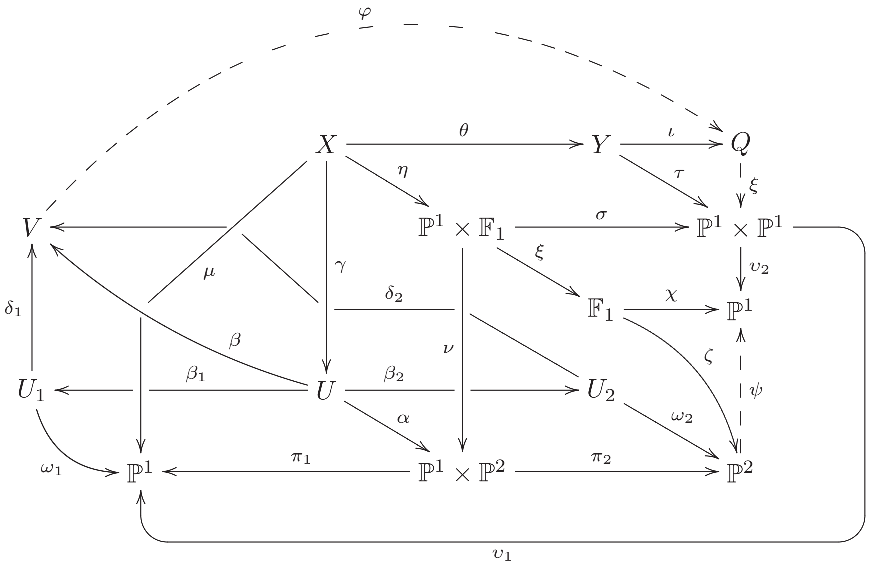
$$\mu F_3 + \alpha(\Omega) \sim_{\mathbb{Q}} 2(F_1 + F_2 + F_3),$$

which yields $\mu \leqslant 2$. Hence the log pair $(\overline{F}_3, \lambda\Omega|_{\overline{F}_3})$ is not log canonical along the curve $\Gamma \subset \overline{F}_3$ by Theorem 2.19. But $\Omega|_{\overline{F}_3} \sim_{\mathbb{Q}} -K_{\overline{F}_3}$ and $\overline{F}_3$ is a del Pezzo surface such that $K^2_{\overline{F}_3} = 6$, and either $\overline{F}_3$ is smooth and $|C \cap F_3| = 2$, or $\overline{F}_3$ has one ordinary double point and $|C \cap F_3| = 1$.

We have $\mathrm{lct}(\overline{F}_3) \leqslant \lambda$. Then $\overline{F}_3$ is singular by Example 1.10. It follows from Lemma 4.5 that $\mathrm{LCS}\big(\overline{F}_3, \lambda\Omega|_{\overline{F}_3}\big) = \mathrm{Sing}(\overline{F}_3)$, but the log pair $\big(\overline{F}_3, \lambda\Omega|_{\overline{F}_3}\big)$ is not log canonical along the whole of $\Gamma \subset \overline{F}_3$, which is a contradiction.

Lemma 9.4. *If* $\beth(X) = 4.5$, *then* $\mathrm{lct}(X) = 3/7$.

Proof. Let $Q \subset \mathbb{P}^4$ be a quadric cone and let $V \subset \mathbb{P}^6$ be a section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ by a linear subspace of dimension 6 such that V has one ordinary double point. Then the diagram



is commutative (cf. [61], Lemma 2.6), where we use the following notation:

- the morphisms π_i, υ_i, ξ, and χ are the natural projections;
- the morphism α contracts a surface $\mathbb{F}_3 \cong E \subset U$ to a curve C such that

$$\pi_1^*\big(\mathscr{O}_{\mathbb{P}^1}(1)\big) \cdot C = 2, \qquad \pi_2^*\big(\mathscr{O}_{\mathbb{P}^2}(1)\big) \cdot C = 1;$$

- the morphism β contracts a surface $\mathbb{P}^1 \times \mathbb{P}^1 \cong \overline{H}_2 \subset U$ to the singular point of V;

- the morphism β_i contracts the surface $\overline{H}_2$ to a smooth rational curve;
- the morphism δ_i contracts the curve $\beta_i(\overline{H}_2)$ to the singular point of V so that the map $\delta_2\circ\delta_1^{-1}\colon U_1 \dashrightarrow U_2$ is a standard flop in the curve $\beta_1(\overline{H}_2)\cong\mathbb{P}^1$;
- the morphism ω_1 is a fibration with general fibre $\mathbb{P}^1\times\mathbb{P}^1$;
- the morphisms ω_2, π_2, ξ, σ, and τ are $\mathbb{P}^1$-bundles;
- the morphism ζ is a blow-up of a point $O\in\mathbb{P}^2$ such that $O\notin\pi_2(C)$;
- the map ψ is a linear projection from the point $O\in\mathbb{P}^2$;
- the morphism ν contracts a surface $G\cong\mathbb{P}^1\times\mathbb{P}^1$ to a curve L such that $\pi_2(L)=O$;
- the morphism γ contracts a surface $\breve{G}$ to a curve $\bar{L}$ such that $\alpha(\bar{L})=L\subset\mathbb{P}^1\times\mathbb{P}^2$ and the curve $\beta(\bar{L})$ is a line in $V\subset\mathbb{P}^6$ such that $\beta(\bar{L})\cap\mathrm{Sing}(V)=\varnothing$;
- η contracts to a curve a surface $\breve{E}$ such that $\nu\circ\eta(\breve{E})=C\subset\mathbb{P}^1\times\mathbb{P}^2$;
- the morphism θ contracts to a curve a surface $\breve{R}\neq\breve{E}$ such that $\tau\circ\theta(\breve{R})=\sigma\circ\eta(\breve{E})$;
- the morphism μ is a fibration into del Pezzo surfaces of degree 6;
- the morphism ι contracts the surface $\theta(\breve{H}_2)$ to the singular point of the quadric Q;
- the map φ is the linear projection from the line $\beta(\bar{L})\subset V\subset\mathbb{P}^6$.

The curve $\pi_2(C)\subset\mathbb{P}^2$ is a line. Hence $\alpha(\overline{H}_2)\in|\pi_2^*(\mathscr{O}_{\mathbb{P}^2}(1))|$ and $C\subset\alpha(\overline{H}_2)$.

The morphism π_1 induces a double cover $C\to\mathbb{P}^1$ branched in two points $Q_1\in C\ni Q_2$. Let T_i be the unique surface in $|\pi_1^*(\mathscr{O}_{\mathbb{P}^1}(1))|$ passing through Q_i. Let $\overline{T}_i\subset U$ be the proper transform of T_i. Then the surface $\overline{T}_i$

- has one ordinary double point,
- is tangent to the surface E along the curve $E\cap\overline{T}_i$,
- is a del Pezzo surface such that $K^2_{\overline{T}_i}=7$.

Let $Z_i\subset\mathbb{P}^2$ be the unique line passing through the points O and $\pi_2\circ\alpha(Q_i)$. Then there is a unique surface $\overline{R}_i\in|(\pi_2\circ\alpha)^*(\mathscr{O}_{\mathbb{P}^2}(1))|$ such that $Z_i\subset\pi_2\circ\alpha(\overline{R}_i)$. We have $\bar{L}\subset\overline{R}_i$ and $-K_U\sim 2\overline{H}_2+\overline{R}_i+2\overline{T}_i+E$.

Let Γ_i be the fibre of the projection $E\to C$ over the point Q_i. Then $\Gamma_i=E\cap\overline{T}_i$ and

$$\Gamma_i\subset\mathrm{LCS}\Big(U,\frac{3}{7}\,(2\overline{H}_2+\overline{R}_i+2\overline{T}_i+E)\Big).$$

Let $\breve{R}_i$ and $\breve{T}_i$ be the proper transforms of $\overline{R}_i$ and $\overline{T}_i$ on the threefold X, respectively. Then $-K_X\sim 2\breve{H}_2+\breve{R}_i+2\breve{T}_i+\breve{E}$, because $\bar{L}\subset\overline{R}_i$. Let $\breve{\Gamma}_i\subset X$ be the proper transform of the curve Γ_i. Then the log pair

$$\Big(X,\frac{3}{7}\,(2\breve{H}_2+\breve{R}_i+2\breve{T}_i+\breve{E})\Big)$$

is log canonical but not log terminal. Thus, $\mathrm{lct}(X)\leqslant 3/7$.

Suppose that $\mathrm{lct}(X)<3/7$. Then there exists an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the log pair $(X,\lambda D)$ is not log canonical for some $\lambda<3/7$.

The surfaces $\breve{T}_1$ and $\breve{T}_2$ are the only singular fibres of the fibration $\mu\colon X\to\mathbb{P}^1$. Then

$$\breve{T}_i\not\subseteq\mathrm{LCS}(X,\lambda D)\subsetneq\breve{T}_1\cup\breve{T}_2$$

by Lemma 2.25, because $D\cdot Z=\breve{T}_1=2$, where Z is a general fibre of $\pi_2\circ\alpha\circ\gamma$.

By Theorem 2.7 we may assume that $\mathrm{LCS}(X, \lambda D) \subseteq \breve{T}_1$.

Applying Theorem 2.27 to the log pair $(\mathbb{P}^1 \times \mathbb{F}_1, \lambda\eta(D))$, we see that

$$\mathrm{LCS}(X, \lambda D) \neq \breve{T}_1 \cap \breve{G},$$

because $G = \eta(\breve{G})$ is a section of the $\mathbb{P}^1$-bundle σ.

Applying Theorem 2.27 to the log pair $(\mathbb{P}^1 \times \mathbb{P}^2, \lambda\alpha \circ \gamma(D))$, we see that

$$\varnothing \neq \mathrm{LCS}(X, \lambda D) \subseteq \breve{T}_1 \cap \breve{E} = \breve{\Gamma}_1$$

by Theorem 2.7, because $\breve{G} \cap \breve{E} = \varnothing$ and T_1 is a section of π_2.

Applying Theorem 2.27 to the log pairs $(Y, \lambda\theta(D))$ and $(U_2, \lambda\beta_2 \circ \gamma(D))$ (and the fibrations τ and ω_2, respectively), we see that $\mathrm{LCS}(X, \lambda D) = \breve{\Gamma}_1$ because $\breve{R} \cap \breve{H}_2 = \varnothing$. Let $\overline{D} = \gamma(D)$. Then $\mathrm{LCS}(U, \lambda\overline{D}) = \Gamma_1$. We put $\overline{D} = \varepsilon\overline{H}_2 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $\overline{H}_2 \not\subseteq \mathrm{Supp}(\Omega)$. Then

$$\Omega|_{\overline{H}_2} \sim_{\mathbb{Q}} -\frac{(1+\varepsilon)}{2} K_{\overline{H}_2}$$

and the log pair $\bigl(\overline{H}_2, \lambda\Omega|_{\overline{H}_2}\bigr)$ is not log canonical by Theorem 2.19. The latter implies that

$$\frac{3}{7}\,\frac{1+\varepsilon}{2} > \lambda\frac{1+\varepsilon}{2} > \frac{1}{2}$$

by Lemma 2.23, so that $\varepsilon > 4/3$.

We may assume (see Remark 2.22) that either $E \not\subseteq \mathrm{Supp}(\overline{D})$ or $\overline{T}_1 \not\subseteq \mathrm{Supp}(\overline{D})$.

Suppose that $E \not\subseteq \mathrm{Supp}(\overline{D})$. Let Z be a general fibre of the projection $E \to C$. Then

$$1 = -K_U \cdot Z = \overline{D} \cdot Z = \varepsilon + \Omega \cdot Z \geqslant \varepsilon,$$

which is a contradiction because $\varepsilon > 4/3$. Thus, $\overline{T}_1 \not\subseteq \mathrm{Supp}(\overline{D})$.

Let $\overline{\Delta} \subset \overline{T}_1$ be the proper transform of a general line in $T_1 \cong \mathbb{P}^2$ passing through Q_1. Then

$$2 = -K_U \cdot \overline{\Delta} = \overline{D} \cdot \overline{\Delta} \geqslant \mathrm{mult}_{\Gamma_1}(\overline{D}) \geqslant \frac{1}{\lambda} > \frac{7}{3},$$

because $\overline{\Delta} \not\subset \mathrm{Supp}(\overline{D})$ and $\overline{\Delta} \cap \Gamma_1 \neq \varnothing$. This contradiction completes the proof.

Lemma 9.5. *If* $\daleth(X) = 4.6$, *then* $\mathrm{lct}(X) = 1/2$.

Proof. There is a birational morphism $\alpha\colon X \to \mathbb{P}^3$ that blows up three disjoint lines L_1, L_2, and L_3.

Let H_i be the proper transform on X of a general plane in $\mathbb{P}^3$ containing L_i. Then

$$-K_X \sim 2H_1 + E_1 + H_2 + H_3 \sim 2H_2 + E_2 + H_1 + H_3 \sim 2H_3 + E_3 + H_1 + H_2,$$

where E_i is the exceptional divisor of α such that $\alpha(E_i) = L_i$. In particular, $\mathrm{lct}(X) \leqslant 1/2$.

Suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$.

The surface H_i is a smooth del Pezzo surface such that $K_{H_i}^2 = 7$, the linear system $|H_i|$ has no base points and induces a morphism $\varphi_i\colon X \to \mathbb{P}^1$ whose fibres are isomorphic to H_i.

Suppose that $|\operatorname{LCS}(X, \lambda D)| < +\infty$. We may assume that $\operatorname{LCS}(X, \lambda D) \not\subseteq E_1$. Then the set

$$\operatorname{LCS}\Big(X, \lambda D + H_1 + \frac{1}{2}\, E_1\Big)$$

is disconnected, which is impossible by Theorem 2.7, because $H_2+H_3+(\lambda-1/2)K_X$ is ample.

We may assume that $H_1 \cap \operatorname{LCS}(X, \lambda D) \neq \varnothing$. Then

$$\varnothing \neq H_1 \cap \operatorname{LCS}(X, \lambda D) \subseteq \operatorname{LCS}\big(H_1, \lambda D|_{H_1}\big)$$

by Remark 2.3. We put $C_2 = E_2|_{H_1}$ and $C_3 = E_3|_{H_1}$. Then $C_2 \cdot C_2 = C_3 \cdot C_3 = -1$ and there is a unique curve C with $\mathbb{P}^1 \cong C \subset H_1$ such that $C \cdot C_2 = C \cdot C_3 = 1$ and $C \cdot C = -1$. Note that $\operatorname{LCS}\big(H_1, \lambda D|_{H_1}\big) = C$ by Lemma 4.9.

There is a unique smooth quadric $Q \subset \mathbb{P}^3$ that contains L_1, L_2, and L_3. Note that $\overline{Q} \cap H_1 = C$, where $\overline{Q} \subset X$ is the proper transform of the quadric Q.

There is a birational morphism $\sigma\colon X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ contracting $\overline{Q}$ to a curve of tridegree $(1,1,1)$. Since $\overline{Q} \cap H_1 = C$, it follows (see Remark 2.3) that $\operatorname{LCS}(X, \lambda D) \supset \overline{Q}$, and hence $\operatorname{LCS}(X, \lambda D) = \overline{Q}$ because $\operatorname{lct}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = 1/2$. We put $D = \mu \overline{Q} + \Omega$, where $\mu \geqslant 1/\lambda > 2$ and Ω is an effective $\mathbb{Q}$-divisor such that $\overline{Q} \not\subset \operatorname{Supp}(\Omega)$. Then $\alpha(D) = \mu Q + \alpha\big(\Omega\big)$, which is impossible because $\alpha(D) \sim_{\mathbb{Q}} 2Q \sim -K_{\mathbb{P}^3}$ and $\mu > 2$.

Lemma 9.6. *If* $\daleth(X) = 4.7$, *then* $\operatorname{lct}(X) = 1/2$.

Proof. There is a blow-up morphism $\alpha\colon X \to W$ such that the variety W is a smooth divisor of bidegree $(1,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$, the morphism α contracts two (irreducible) surfaces $E_1 \neq E_2$ to two disjoint curves L_1 and L_2, and the curves L_i are fibres of one natural $\mathbb{P}^1$-bundle $W \to \mathbb{P}^2$.

There is a surface $H \subset W$ such that $-K_X \sim 2H$ and $L_1 \subset H \supset L_2$. We have $-K_X \sim 2\overline{H} + E_1 + E_2$, where $\overline{H}$ is the proper transform of H on the threefold X. Then $\operatorname{lct}(X) \leqslant 1/2$.

Suppose that $\operatorname{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$. Note that $\operatorname{LCS}(X, \lambda D) \subseteq E_1 \cup E_2$ since $\operatorname{lct}(W) = 1/2$ by Theorem 6.1.

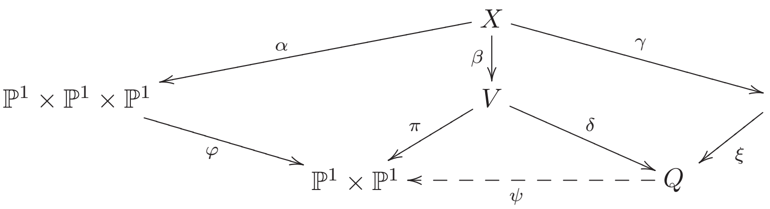
We may assume that $\operatorname{LCS}(X, \lambda D) \cap E_1 \neq \varnothing$. Let $\beta\colon X \to Y$ be a contraction of E_2. Then $\mathbb{LCS}(Y, \lambda\beta(D)) \neq \varnothing$ and $\beta(D) \sim_{\mathbb{Q}} -K_Y$, which contradicts Lemma 8.25.

Lemma 9.7. *If* $\daleth(X) = 4.8$, *then* $\operatorname{lct}(X) = 1/3$.

Proof. There is a blow-up $\alpha\colon X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of a curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ such that $C \subset F_1$ and $C \cdot F_2 = C \cdot F_3 = 1$, where F_i is a fibre of the projection $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ onto the ith factor. There is a surface $G \in |F_2 + F_3|$ containing the curve C. Let E be the exceptional divisor of α. Then $-K_X \sim 2\overline{F}_1 + 2\overline{G} + 3E$, where $\overline{F}_1$ and $\overline{G}$ are the proper transforms of F_1 and G, respectively. In particular, $\operatorname{lct}(X) \leqslant 1/3$.

Suppose that $\mathrm{lct}(X) < 1/3$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/3$. Note that $\mathrm{LCS}(X, \lambda D) \subseteq E$, because $\mathrm{lct}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = 1/2$ and $\alpha(D) \sim_{\mathbb{Q}} -K_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}$.

Let Q be a quadric cone in $\mathbb{P}^4$. Then there is a commutative diagram



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where we use the following notation: V is a variety with $\gimel(V) = 3.31$; the morphism β is a contraction of the surface $\overline{G}$ to a curve; the morphism γ is a contraction of $\overline{F}_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$ to an ordinary double point; the morphism δ is a blow-up of the vertex of the quadric cone $Q \subset \mathbb{P}^4$; the morphism ξ is a blow-up of a smooth conic in Q; the map ψ is the projection from the vertex of the cone Q; the morphism φ is induced by $|F_2 + F_3|$, that is, is the projection of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ onto the product of the last two factors; the morphism π is a natural $\mathbb{P}^1$-bundle.

It follows from Corollary 5.4 that $\mathrm{lct}(V) = 1/3$. On the other hand, $\mathrm{lct}(U) = 1/3$ by Lemma 2.26. Hence $\mathrm{LCS}(X, \lambda D) \subseteq E \cap \overline{G} \cap \overline{F}_1 = \varnothing$, a contradiction.

The following result is implied by Corollaries 5.4 and 8.19, Lemma 2.29, and Example 1.10.

Corollary 9.8. *Suppose that* $\rho \geqslant 5$. *Then* $\mathrm{lct}(X) = 1/3$ *if* $\gimel(X) \in \{5.1, 5.2\}$, *and* $\mathrm{lct}(X) = 1/2$ *otherwise.*

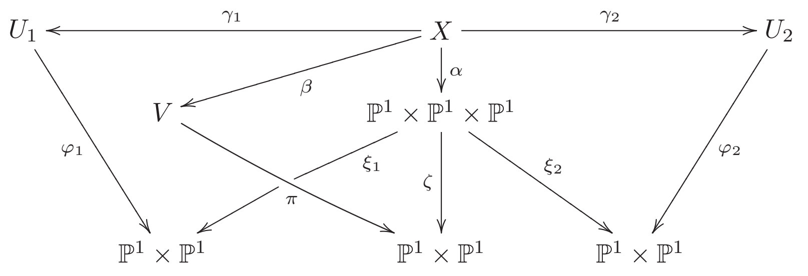
Lemma 9.9. *If* $\gimel(X) = 4.13$ *and* X *is general, then* $\mathrm{lct}(X) = 1/2$.

Proof. Let $F_1 \cong F_2 \cong F_3 \cong \mathbb{P}^1 \times \mathbb{P}^1$ be fibres of the three different projections $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. There is a contraction $\alpha\colon X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of a surface $E \subset X$ to a curve $C \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ such that $C \cdot F_1 = C \cdot F_2 = 1$ and $C \cdot F_3 = 3$. Then there is a smooth surface $G \in |F_1 + F_2|$ containing C. In particular, we see that $-K_X \sim 2\overline{G} + E + 2\overline{F}_3$, where $\overline{F}_3$ and $\overline{G}$ are the proper transforms of the divisors F_3 and G, respectively. Hence $\mathrm{lct}(X) \leqslant 1/2$.

Suppose that $\mathrm{lct}(X) < 1/2$. Then there exists an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$. Then $\mathrm{LCS}(X, \lambda D) \subseteq E \cong \mathbb{F}_4$, because $\mathrm{lct}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1) = 1/2$ and $\alpha(D) \sim_{\mathbb{Q}} -K_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1}$.

There are smooth surfaces $H_1 \in |3F_1 + F_3|$ and $H_2 \in |3F_2 + F_3|$ such that $C = G \cdot H_1 = G \cdot H_2$ and $H_1 \cong H_2 \cong \mathbb{P}^1 \times \mathbb{P}^1$. Let $\overline{H}_i$ be the proper transform of H_i on the threefold X. Then $\overline{H}_1 \cap \overline{G} = \overline{H}_2 \cap \overline{G} = \varnothing$.

There is a commutative diagram



such that β and γ_i are contractions of the surfaces $\overline{G}$ and $\overline{H}_i$ to smooth curves, the morphisms π and φ_i are $\mathbb{P}^1$-bundles, and the morphisms ζ and ξ_i are the projections given by the linear systems $|F_1+F_2|$ and $|F_i+F_3|$, respectively.

It follows from $\overline{H}_1 \cap \overline{G} = \varnothing$ that either $(V, \lambda\beta(D))$ or $(U_1, \lambda\gamma_1(D))$ is not log canonical.

Applying Theorem 2.27 to $(V, \lambda\beta(D))$ or $(U_1, \lambda\gamma_1(D))$ (and the fibration π or φ_1, respectively) and using Theorem 2.7, we see that $\mathrm{LCS}(X, \lambda D) = \Gamma$, where Γ is a fibre of the natural projection $E \to C$.

We may assume that $\alpha(\Gamma) \in F_3$. We put $D = \mu\overline{F}_3 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor on X such that $\overline{F}_3 \not\subset \mathrm{Supp}(\Omega)$. Then $\mu F_3 + \alpha(\Omega) \sim_{\mathbb{Q}} 2(F_1+F_2+F_3)$, which yields $\mu \leqslant 2$. The log pair $\big(\overline{F}_3, \lambda\Omega|_{\overline{F}_3}\big)$ is not log canonical along $\Gamma \subset \overline{F}_3$ by Theorem 2.19. We have $\Omega|_{\overline{F}_3} \sim_{\mathbb{Q}} -K_{\overline{F}_3}$ and $\overline{F}_3$ is a del Pezzo surface such that $K^2_{\overline{F}_3} = 5$. Note that $\overline{F}_3$ can be singular. Namely, we have

$$\mathrm{Sing}(\overline{F}_3) = \varnothing \iff |C \cap F_3| = F_3 \cdot C = 3,$$

and $\mathrm{Sing}(\overline{F}_3) \subset \Gamma$. The following cases are possible:

- the surface $\overline{F}_3$ is smooth and $|C \cap F_3| = 3$;
- the surface $\overline{F}_3$ has one ordinary double point and $|C \cap F_3| = 2$;
- the surface $\overline{F}_3$ has a singular point of type $\mathbb{A}_2$ and $|C \cap F_3| = 1$.

We have $\mathrm{lct}(\overline{F}_3) \leqslant \lambda < 1/2$. Thus, it follows from Examples 1.10 and 4.3 that $|C \cap F_3| = 1$, which is impossible if the threefold X is sufficiently general.

10. Upper bounds

We use the assumptions and the notation introduced in § 1. The main aim of this section is to find upper bounds for the global log canonical thresholds of the varieties X in several cases not covered by Theorem 1.46.

Lemma 10.1. *If* $\daleth(X) = 1.8$, *then* $\mathrm{lct}(X) \leqslant 6/7$.

Proof. The linear system $|-K_X|$ does not have base points and induces an embedding $X \subset \mathbb{P}^{10}$, and the threefold X contains a line $L \subset X$ (see [62]).

It follows from Theorem 4.3.3 in [2] that there is a commutative diagram

$$\begin{array}{ccc} U & \overset{\rho}{\dashrightarrow} & W \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ X & \underset{\psi}{\dashrightarrow} & \mathbb{P}^3 \end{array},$$

where α is a blow-up of the line L, the map ρ is a composition of flops, the morphism β is a blow-up of a smooth curve of degree 7 and genus 3, and ψ is a double projection from the line L.

Let $S \subset X$ be the proper transform of the exceptional surface of β. Then $\mathrm{mult}_L(S) = 7$ and $S \sim -3K_X$, which implies that $\mathrm{lct}(X) \leqslant 6/7$.

Lemma 10.2. *If* $\daleth(X) = 1.9$, *then* $\mathrm{lct}(X) \leqslant 4/5$.

Proof. The linear system $|-K_X|$ does not have base points and induces an embedding $X \subset \mathbb{P}^{11}$, and the threefold X contains a line $L \subset X$ (see [62]).

It follows from Theorem 4.3.3 in [2] that there is a commutative diagram

$$\begin{array}{ccc} U & \overset{\rho}{\dashrightarrow} & W \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ X & \underset{\psi}{\dashrightarrow} & Q \end{array},$$

where $Q \subset \mathbb{P}^4$ is a smooth quadric threefold, α is a blow-up of the line L, the map ρ is a composition of flops, the morphism β is a blow-up along a smooth curve of degree 7 and genus 2, and ψ is a double projection from the line L.

Let $S \subset X$ be the proper transform of the exceptional surface of β. Then $\mathrm{mult}_L(S) = 5$ and $S \sim -2K_X$, which implies that $\mathrm{lct}(X) \leqslant 4/5$.

Lemma 10.3. *If* $\daleth(X) = 1.10$, *then* $\mathrm{lct}(X) \leqslant 2/3$.

Proof. The linear system $|-K_X|$ does not have base points and induces an embedding $X \subset \mathbb{P}^{13}$, and the threefold X contains a line $L \subset X$ (see [62]).

It follows from Theorem 4.3.3 in [2] that the diagram

$$\begin{array}{ccc} U & \overset{\rho}{\dashrightarrow} & W \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ X & \underset{\psi}{\dashrightarrow} & V_5 \end{array}$$

is commutative, where V_5 is a smooth section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ by a linear subspace of dimension 6, the morphism α is a blow-up of the line L, the map ρ is a composition of flops, the morphism β is a normal rational curve of degree 5, and ψ is a double projection from L.

Let $S \subset X$ be the proper transform of the exceptional surface of β. Then $\mathrm{mult}_L(S) = 3$ and $S \sim -K_X$, which implies that $\mathrm{lct}(X) \leqslant 2/3$.

Lemma 10.4. *If* $\daleth(X) = 2.2$, *then* $\mathrm{lct}(X) \leqslant 13/14$.

Proof. There is a smooth divisor $B \subset \mathbb{P}^1 \times \mathbb{P}^2$ of bidegree $(2,4)$ such that the diagram

$$\begin{array}{ccccc} & & X & & \\ & {}^{\varphi_1}\swarrow & \downarrow{\scriptstyle \pi} & \searrow^{\varphi_2} & \\ \mathbb{P}^1 & \xleftarrow{\ \pi_1\ } & \mathbb{P}^1 \times \mathbb{P}^2 & \xrightarrow{\ \pi_2\ } & \mathbb{P}^2 \end{array}$$

is commutative, where π is a double cover branched along B, the morphisms π_1 and π_2 are the natural projections, φ_1 is a fibration into del Pezzo surfaces of degree 2, and φ_2 is a conic bundle.

Let H_1 be a general fibre of φ_1. We put $\overline{H}_1 = \pi(H_1)$. Then the intersection

$$C = \overline{H}_1 \cap B \subset \overline{H}_1 \cong \mathbb{P}^2$$

is a smooth quartic curve 4.

There is a point $P \in C$ such that $\mathrm{mult}_P(C \cdot L) \geqslant 3$, where $L \subset \overline{H}_1 \cong \mathbb{P}^2$ is the line tangent to C at P.

The curve $\pi_2(L)$ is a line. Thus, there is a unique surface $H_2 \in |\varphi_2^*(\mathscr{O}_{\mathbb{P}^2}(1))|$ such that $\varphi_2(H_2) = \pi_2(L)$. Hence $-K_X \sim H_1 + H_2$.

Let us show that $\mathrm{lct}(X, H_1 + H_2) \leqslant 13/14$. We put $\overline{H}_2 = \pi(H_2)$. Then

$$\mathbb{LCS}\Big(X, \frac{13}{14}(H_1 + H_2)\Big) \neq \varnothing \iff \mathbb{LCS}\Big(\mathbb{P}^1 \times \mathbb{P}^2, \frac{1}{2}B + \frac{13}{14}(\overline{H}_1 + \overline{H}_2)\Big) \neq \varnothing$$

by [1], Proposition 3.16. Let $\alpha\colon V \to \mathbb{P}^1 \times \mathbb{P}^2$ be a blow-up of the curve C. Then

$$K_V + \frac{1}{2}\widetilde{B} + \frac{13}{14}(\widetilde{H}_1 + \widetilde{H}_2) + \frac{3}{7}E \sim_{\mathbb{Q}} \alpha^*\Big(K_{\mathbb{P}^1 \times \mathbb{P}^2} + \frac{1}{2}B + \frac{13}{14}(\overline{H}_1 + \overline{H}_2)\Big),$$

where $\widetilde{B}, \widetilde{H}_1, \widetilde{H}_2 \subset V$ are the proper transforms of B, $\overline{H}_1$, $\overline{H}_2$, respectively. But the log pair $(V, (13/14)\widetilde{H}_2 + (3/7)E)$ is not log terminal along the fibre Γ of the morphism α such that $\alpha(\Gamma) = P$, because

$$\mathrm{mult}_\Gamma(\widetilde{H}_2 \cdot E) = \mathrm{mult}_P(C \cdot \overline{H}_2) \geqslant \mathrm{mult}_P(C \cdot L) \geqslant 3$$

due to the generality of the fibre H_1. We see that

$$\Gamma \subseteq \mathrm{LCS}\Big(V, \frac{13}{14}\widetilde{H}_2 + \frac{3}{7}E\Big) \subseteq \mathrm{LCS}\Big(V, \frac{1}{2}\widetilde{B} + \frac{13}{14}(\widetilde{H}_1 + \widetilde{H}_2) + \frac{3}{7}E\Big),$$

which implies that $\mathrm{lct}(X, H_1 + H_2) \leqslant 13/14$. Hence $\mathrm{lct}(X) \leqslant 13/14$.

Remark 10.5. It follows from Lemmas 2.25 and 4.1 that $\mathrm{lct}(X) \geqslant 2/3$ if $\gimel(X) = 2.2$ and the threefold X satisfies the following generality condition: any fibre of φ_1 satisfies the hypotheses of Lemma 4.1.

Lemma 10.6. *If* $\gimel(X) = 2.7$, *then* $\mathrm{lct}(X) \leqslant 2/3$.

Proof. There is a commutative diagram

$$\begin{array}{ccc} & X & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ Q & \overset{}{\underset{\psi}{\dashrightarrow}} & \mathbb{P}^1 \end{array},$$

where $Q \subset \mathbb{P}^4$ is a smooth quadric threefold, α is a blow-up of a smooth curve that is the complete intersection of two divisors $S_1, S_2 \in \left|\mathscr{O}_{\mathbb{P}^4}(2)|_Q\right|$, the morphism β is a fibration into del Pezzo surfaces of degree 4, and ψ is the rational map induced by the pencil generated by the surfaces S_1 and S_2. Then $\operatorname{lct}(X) \leqslant 2/3$ because $-K_X \sim_{\mathbb{Q}} (3/2)\overline{S}_1 + (1/2)E$, where $\overline{S}_1 \subset X$ is the proper transform of the surface S_1 and E is the exceptional divisor of α.

Lemma 10.7. *If* $\gimel(X) = 2.9$, *then* $\operatorname{lct}(X) \leqslant 3/4$.

Proof. There is a commutative diagram

$$\begin{array}{ccc} & X & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ \mathbb{P}^3 & \underset{\psi}{\dashrightarrow} & \mathbb{P}^2 \end{array},$$

where α is a blow-up of a smooth curve $C \subset \mathbb{P}^3$ of degree 7 and genus 5 that is an intersection of cubic surfaces in $\mathbb{P}^3$, the morphism β is a conic bundle, and ψ is a rational map given by the linear system of cubics containing C. We have $-K_X \sim_{\mathbb{Q}} (4/3)S + (1/3)E$, where $S \in |\beta^*(\mathscr{O}_{\mathbb{P}^2}(1))|$ and E is the exceptional divisor of α. Hence $\operatorname{lct}(X) \leqslant 3/4$.

Lemma 10.8. *If* $\gimel(X) = 2.12$, *then* $\operatorname{lct}(X) \leqslant 3/4$.

Proof. There is a commutative diagram

$$\begin{array}{ccc} & X & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ \mathbb{P}^3 & \underset{\psi}{\dashrightarrow} & \mathbb{P}^3 \end{array},$$

where α and β are blow-ups of smooth curves $C \subset \mathbb{P}^3$ and $Z \subset \mathbb{P}^3$ of degree 6 and genus 3 that are intersections of cubic surfaces in $\mathbb{P}^3$, and ψ is a birational map given by the linear system of cubic surfaces containing C. Then $-K_X \sim_{\mathbb{Q}} (4/3)S + (1/3)E$, where $S \in |\beta^*(\mathscr{O}_{\mathbb{P}^3}(1))|$ and E is the exceptional divisor of α. Consequently, $\operatorname{lct}(X) \leqslant 3/4$.

Lemma 10.9. *If* $\gimel(X) = 2.13$, *then* $\operatorname{lct}(X) \leqslant 2/3$.

Proof. There is a commutative diagram

$$\begin{array}{ccc} & X & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ Q & \underset{\psi}{\dashrightarrow} & \mathbb{P}^2 \end{array},$$

where $Q \subset \mathbb{P}^4$ is a smooth quadric threefold, α is a blow-up of a smooth curve $C \subset Q$ of degree 6 and genus 2, the morphism β is a conic bundle, and ψ is the rational map given by the linear system of surfaces in $\left|\mathscr{O}_{\mathbb{P}^4}(2)|_Q\right|$ containing the curve C. We have $-K_X \sim_{\mathbb{Q}} (3/2)S + (1/2)E$, where $S \in |\beta^*(\mathscr{O}_{\mathbb{P}^2}(1))|$ and E is the exceptional divisor of α. Hence $\mathrm{lct}(X) \leqslant 2/3$.

Lemma 10.10. *If* $\gimel(X) = 2.16$, *then* $\mathrm{lct}(X) \leqslant 1/2$.

Proof. There is a commutative diagram

$$\begin{array}{ccc} & X & \\ {\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\beta} \\ V_4 & \overset{}{\underset{\psi}{\dashrightarrow}} & \mathbb{P}^2 \end{array},$$

where $V_4 \subset \mathbb{P}^5$ is the smooth complete intersection of two quadric hypersurfaces, α is a blow-up of an irreducible conic $C \subset V_4$, the morphism β is a conic bundle, and ψ is a rational map given by the linear system of surfaces in $\left|\mathscr{O}_{\mathbb{P}^5}(1)|_{V_4}\right|$ containing C. We have $-K_X \sim 2S + E$, where $S \in |\beta^*(\mathscr{O}_{\mathbb{P}^2}(1))|$, and E is the exceptional divisor of α. Hence $\mathrm{lct}(X) \leqslant 1/2$.

Lemma 10.11. *If* $\gimel(X) = 2.17$, *then* $\mathrm{lct}(X) \leqslant 2/3$.

Proof. There is a commutative diagram

$$\begin{array}{ccc} & X & \\ {\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\beta} \\ Q & \underset{\psi}{\dashrightarrow} & \mathbb{P}^3 \end{array},$$

where $Q \subset \mathbb{P}^4$ is a smooth quadric threefold, the morphisms α and β are blow-ups of smooth elliptic curves $C \subset Q$ and $Z \subset \mathbb{P}^3$ of degree 5, respectively, and the map ψ is given by the linear system of surfaces in $\left|\mathscr{O}_{\mathbb{P}^4}(2)|_Q\right|$ that contain C. We have $-K_X \sim_{\mathbb{Q}} (3/2)S + (1/2)E$, where $S \in |\beta^*(\mathscr{O}_{\mathbb{P}^3}(1))|$ and E is the exceptional divisor of α. Hence $\mathrm{lct}(X) \leqslant 2/3$.

Lemma 10.12. *If* $\gimel(X) = 2.20$, *then* $\mathrm{lct}(X) \leqslant 1/2$.

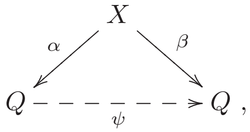
Proof. There is a commutative diagram

$$\begin{array}{ccc} & X & \\ {\scriptstyle\alpha}\swarrow & & \searrow{\scriptstyle\beta} \\ V_5 & \underset{\psi}{\dashrightarrow} & \mathbb{P}^2 \end{array},$$

where $V_5 \subset \mathbb{P}^6$ is a smooth intersection of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ with a linear subspace of dimension 6, the morphism α is a blow-up of a twisted cubic $\mathbb{P}^1 \cong C \subset V_5$, the morphism β is a conic bundle, and the map ψ is given by the linear system of surfaces in $\left|\mathscr{O}_{\mathbb{P}^6}(1)|_{V_5}\right|$ that contain the curve C. We have $-K_X \sim 2S + E$, where $S \in |\beta^*(\mathscr{O}_{\mathbb{P}^2}(1))|$ and E is the exceptional divisor of α. We see that $\mathrm{lct}(X) \leqslant 1/2$.

Lemma 10.13. *If* $\daleth(X)=2.21$, *then* $\operatorname{lct}(X)\leqslant 2/3$.

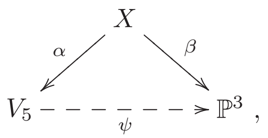
Proof. There is a commutative diagram



where $Q\subset\mathbb{P}^4$ is a smooth quadric threefold, α and β are blow-ups of smooth rational curves $C\subset Q$ and $Z\subset Q$ of degree 4, and ψ is the birational map given by the linear system of surfaces in $\bigl|\mathscr{O}_{\mathbb{P}^4}(2)|_Q\bigr|$ that contain C. We have $-K_X\sim_{\mathbb{Q}}(3/2)S+(1/2)E$, where $S\in\bigl|\beta^*(\mathscr{O}_{\mathbb{P}^4}(1))|_Q\bigr|$ and E is the exceptional divisor of α. Hence $\operatorname{lct}(X)\leqslant 2/3$.

Lemma 10.14. *If* $\daleth(X)=2.22$, *then* $\operatorname{lct}(X)\leqslant 1/2$.

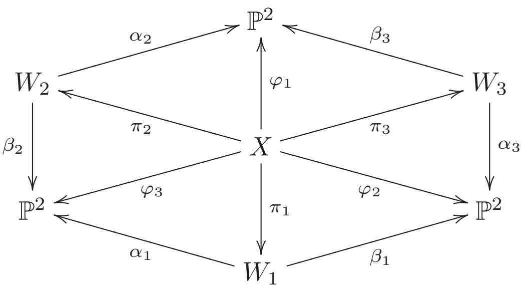
Proof. There is a commutative diagram



where $V_5\subset\mathbb{P}^6$ is a smooth intersection of $\operatorname{Gr}(2,5)\subset\mathbb{P}^9$ with a linear subspace of dimension 6, the morphisms α and β are blow-ups of the conic $C\subset V_5$ and a rational (not linearly normal) quartic $Z\subset\mathbb{P}^3$, respectively, and ψ is given by the linear system of surfaces in $\bigl|\mathscr{O}_{\mathbb{P}^6}(1)|_{V_5}\bigr|$ that contain C. We have $-K_X\sim 2S+E$, where $S\in|\beta^*(\mathscr{O}_{\mathbb{P}^3}(1))|$ and E is the exceptional divisor of α. Then $\operatorname{lct}(X)\leqslant 1/2$.

Lemma 10.15. *If* $\daleth(X)=3.13$, *then* $\operatorname{lct}(X)\leqslant 1/2$.

Proof. There is a commutative diagram



such that $W_i\subset\mathbb{P}^2\times\mathbb{P}^2$ is a divisor of bidegree $(1,1)$, the morphisms α_i and β_i are $\mathbb{P}^1$-bundles, π_i is a blow-up of a smooth curve $C_i\subset W_i$ of bidegree $(2,2)$ such that $\alpha_i(C_i)$ and $\beta_i(C_i)$ are irreducible conics in $\mathbb{P}^2$, and φ_i is a conic bundle. Let E_i be the exceptional divisor of π_i. Then

$$-K_X\sim 2H_1+E_1\sim 2H_2+E_2\sim 2H_3+E_3\sim E_1+E_2+E_3,$$

where $H_i\in|\varphi_i^*(\mathscr{O}_{\mathbb{P}^2}(1))|$. Hence $\operatorname{lct}(X)\leqslant 1/2$.

Remark 10.16. We shall use the notation in the proof of Lemma 10.15 and assume that $\mathrm{lct}(X) < 1/2$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical for some $\lambda < 1/2$. Since $\mathrm{lct}(W_i) = 1/2$ by Theorem 6.1, it follows that

$$\varnothing \neq \mathrm{LCS}(X, \lambda D) \subset E_1 \cap E_2 \cap E_3.$$

In particular, by Theorem 2.7 the locus $\mathrm{LCS}(X, \lambda D)$ consists of a single point P; note that P is the intersection $P = F_1 \cap F_2 \cap F_3$ of three curves F_i such that $F_2 \cup F_3$ (respectively, $F_1 \cup F_3$, $F_1 \cup F_2$) is a reducible fibre of the conic bundle φ_1 (respectively, φ_2, φ_3).

Appendix A.
J.-P. Demailly. On Tian's invariant and log canonical thresholds

The goal of this appendix is to relate log canonical thresholds with the α-invariant introduced by Tian [3] for the study of the existence of Kähler–Einstein metrics. The approximation technique of closed positive $(1,1)$-currents introduced in [63] is used to show that the α-invariant of a smooth Fano variety actually coincides with the log canonical threshold.

Algebraic geometers have been aware of this fact since [21] appeared, and several papers have used it implicitly in recent years (see, for instance, [64] and [65]). However, it turns out that the required result is stated only in a local analytic form in [21], in a language which may not be easily recognizable by algebraically minded people. Therefore, we will repair here the lack of a proper reference by stating and proving the existence of Kähler–Einstein metrics on Fano varieties and Fano orbifolds.

Usually only the case of the anticanonical line bundle $L = -K_X$ is considered in these applications. Here we will consider more generally the case of an arbitrary line bundle L (or $\mathbb{Q}$-line bundle L) on a complex manifold X, with some additional restrictions which will be stated later.

Assume that L is equipped with a *singular Hermitian metric* h (see, for instance, [66]). Locally, L admits trivializations $\theta\colon L|_U \simeq U \times \mathbb{C}$ and on U the metric h is given by a weight function φ such that

$$\|\xi\|_h^2 = |\xi|^2 e^{-2\varphi(z)} \quad \text{for all } z \in U,\ \xi \in L_z,$$

where $\xi \in L_z$ is identified with a complex number. We are interested in the case where φ is (at the very least) a locally integrable function for the Lebesgue measure, since it is then possible to compute the curvature form

$$\Theta_{L,h} = \frac{i}{\pi}\,\partial\bar{\partial}\varphi$$

in the sense of distributions. We have $\Theta_{L,h} \geqslant 0$ as a $(1,1)$-current if and only if the weights φ are plurisubharmonic functions. In the sequel we will be interested only in that case.

Let us first introduce the concept of complex singularity exponent for singular Hermitian metrics, following, for example, [67]–[69] and [21].

Definition A.1. If K is a compact subset of X, we define the complex singularity exponent $c_K(h)$ of the metric h, written locally as $h = e^{-2\varphi}$, to be the supremum of all positive numbers c such that $h^c = e^{-2c\varphi}$ is integrable in a neighbourhood of every point $z_0 \in K$, with respect to the Lebesgue measure in holomorphic coordinates centred at z_0.

Now, we introduce a generalized version of Tian's invariant α, as defined in [3] (see also [70]).

Definition A.2. Assume that X is a compact manifold and that L is a pseudo-effective line distribution, that is, L admits a singular Hermitian metric h_0 with $\Theta_{L,h_0} \geqslant 0$. If K is a compact subset of X, we put

$$\alpha_K(L) = \inf_{\{h,\Theta_{L,h}\geqslant 0\}} c_K(h),$$

where h runs over all singular Hermitian metrics on L such that $\Theta_{L,h} \geqslant 0$.

In algebraic geometry it is more usual to look instead at linear systems defined by a family of linearly independent sections $\sigma_0, \sigma_1, \dots, \sigma_N \in H^0(X, L^{\otimes m})$. We denote by Σ the vector subspace generated by these sections and by

$$|\Sigma| := P(\Sigma) \subset |mL| := P\bigl(H^0(X, L^{\otimes m})\bigr)$$

the corresponding linear system (not necessarily complete). Such an $(N+1)$-tuple $\sigma = (\sigma_j)_{0\leqslant j\leqslant N}$ of sections defines a singular Hermitian metric h on L by putting in any trivialization

$$\|\xi\|_h^2 = \frac{|\xi|^2}{\bigl(\sum_j |\sigma_j(z)|^2\bigr)^{1/m}} = \frac{|\xi|^2}{|\sigma(z)|^{2/m}} \quad \text{for } \xi \in L_z;$$

hence $h(z) = |\sigma(z)|^{-2/m}$ with

$$\varphi(z) = \frac{1}{m}\log|\sigma(z)| = \frac{1}{2m}\log\sum_j |\sigma_j(z)|^2.$$

as the associated weight function. Therefore, we are interested in the number $c_K(|\sigma|^{-2/m})$. In the case of a single section σ_0 (corresponding to a linear system containing a single divisor) this is the same as the log canonical threshold $\mathrm{lct}_K\bigl(X, m^{-1}D\bigr)$, where D is a divisor corresponding to σ_0. We will also use the formal notation $\mathrm{lct}_K\bigl(X, m^{-1}|\Sigma|\bigr)$ in the case of a higher-dimensional linear system $|\Sigma| \subset |mL|$.

Now, recall that the line bundle L is said to be *big* if the Kodaira–Iitaka dimension $\kappa(L)$ equals $n = \dim_{\mathbb{C}}(X)$. The main result of this appendix is the following theorem.

Theorem A.3. *Let L be a big bundle on a compact complex manifold X. Then for every compact set K in X we have*

$$\alpha_K(L) = \inf_{\{h,\Theta_{L,h}\geqslant 0\}} c_K(h) = \inf_{m\in\mathbb{Z}_{>0}} \inf_{D\in|mL|} \mathrm{lct}_K\biggl(X, \frac{1}{m}D\biggr).$$

Observe that the inequality

$$\inf_{m\in\mathbb{Z}_{>0}}\ \inf_{D\in|mL|}\mathrm{lct}_K\Big(X,\frac{1}{m}D\Big)\geqslant\inf_{\{h,\Theta_{L,h}\geqslant0\}}c_K(h),$$

is trivial since any divisor $D\in|mL|$ gives rise to a singular Hermitian metric h. The converse inequality will follow from the approximation technique of [63] and some elementary analysis. The proof is parallel to the proof of Theorem 4.2 in [21], although the language used there was somewhat different. In any case, we use again the crucial concept of multiplier ideal sheaves: if h is a singular Hermitian metric with local plurisubharmonic weight φ, the multiplier ideal sheaf $\mathscr{I}(h)\subset\mathscr{O}_X$ (also denoted by $\mathscr{I}(\varphi)$) is the ideal sheaf defined by

$$\mathscr{I}(h)_z=\Big\{f\in\mathscr{O}_{X,z}\mid \text{there exists a neighbourhood } V\ni z \\ \text{such that } \int_V|f(x)|^2e^{-2\varphi(x)}\,d\lambda(x)<+\infty\Big\},$$

where λ is the Lebesgue measure. By Nadel (see [20]), this is a coherent analytic sheaf on X. Theorem A.3 has a more precise version which can be stated as follows.

Theorem A.4. *Let L be a line bundle on a compact complex manifold X possessing a singular Hermitian metric h with $\Theta_{L,h}\geqslant\varepsilon\omega$ for some $\varepsilon>0$ and some smooth positive-definite Hermitian $(1,1)$-form ω on X. For every real number $m>0$, consider the space $\mathscr{H}_m=H^0(X,L^{\otimes m}\otimes\mathscr{I}(h^m))$ of holomorphic sections σ of $L^{\otimes m}$ on X such that*

$$\int_X\|\sigma\|^2_{h^m}\,dV_\omega=\int_X|\sigma|^2e^{-2m\varphi}\,dV_\omega<+\infty,$$

where $dV_\omega=(m!)^{-1}\omega^m$ is the Hermitian volume form. Then for $m\gg1$, $\mathscr{H}_m$ is a non-zero finite-dimensional Hilbert space, and one can consider the closed positive $(1,1)$-current

$$T_m=\frac{i}{2\pi}\partial\overline{\partial}\Big(\frac{1}{2m}\log\sum_k|g_{m,k}|^2\Big)=\frac{i}{2\pi}\partial\overline{\partial}\Big(\frac{1}{2m}\log\sum_k\|g_{m,k}\|^2_h\Big)+\Theta_{L,h},$$

where $(g_{m,k})_{1\leqslant k\leqslant N(m)}$ is an orthonormal basis of $\mathscr{H}_m$. The following statements hold.

(i) *For every trivialization $L|_U\simeq U\times\mathbb{C}$ on a coordinate open set U of X and every compact set $K\subset U$ there are constants $C_1,C_2>0$ independent of m and φ such that*

$$\varphi(z)-\frac{C_1}{m}\leqslant\psi_m(z):=\frac{1}{2m}\log\sum_k|g_{m,k}(z)|^2\leqslant\sup_{|x-z|<r}\varphi(x)+\frac{1}{m}\log\frac{C_2}{r^n}$$

for every $z\in K$ and $r\leqslant(1/2)\,d(K,\partial U)$. In particular, ψ_m converges to φ pointwise and in the L^1_{loc}-topology on Ω as $m\to+\infty$; hence T_m converges weakly to $T=\Theta_{L,h}$.

(ii) *The Lelong numbers $\nu(T,z)=\nu(\varphi,z)$ and $\nu(T_m,z)=\nu(\psi_m,z)$ are related by*

$$\nu(T,z)-n/m\leqslant\nu(T_m,z)\leqslant\nu(T,z)\quad\textit{for every } z\in X.$$

(iii) *For every compact set $K \subset X$ the complex singularity exponents of the metrics given locally by $h = e^{-2\varphi}$ and $h_m = e^{-2\psi_m}$ satisfy*

$$c_K(h)^{-1} - m^{-1} \leqslant c_K(h_m)^{-1} \leqslant c_K(h)^{-1}.$$

Proof. The major part of the proof is a small variation of the arguments already explained in [63] (see also [21], Theorem 4.2). We give them here in detail for the convenience of the reader.

(i) We note that $\sum |g_{m,k}(z)|^2$ is the square of the norm of the evaluation linear form $\sigma \mapsto \sigma(z)$ on $\mathscr{H}_m$, hence

$$\psi_m(z) = \sup_{\sigma \in B(1)} \frac{1}{m} \log|\sigma(z)|,$$

where $B(1)$ is the unit ball of $\mathscr{H}_m$. For $r \leqslant (1/2)\, d(K, \partial\Omega)$ the mean value inequality applied to the plurisubharmonic function $|\sigma|^2$ implies that

$$\begin{aligned} |\sigma(z)|^2 &\leqslant \frac{1}{\pi^n r^{2n}/n!} \int_{|x-z|<r} |\sigma(x)|^2\, d\lambda(x) \\ &\leqslant \frac{1}{\pi^n r^{2n}/n!} \exp\bigl(2m \sup_{|x-z|<r} \varphi(x)\bigr) \int_\Omega |\sigma|^2 e^{-2m\varphi}\, d\lambda. \end{aligned}$$

If we take the supremum over all $\sigma \in B(1)$, then we get that

$$\psi_m(z) \leqslant \sup_{|x-z|<r} \varphi(x) + \frac{1}{2m} \log \frac{1}{\pi^n r^{2n}/n!}\,,$$

and the right-hand inequality in (i) is proved. Conversely, the Ohsawa–Takegoshi extension theorem [71], [72] applied to the 0-dimensional subvariety $\{z\} \subset U$ shows that for any $a \in \mathbb{C}$ there is a holomorphic function f on U such that $f(z) = a$ and

$$\int_U |f|^2 e^{-2m\varphi}\, d\lambda \leqslant C_3 |a|^2 e^{-2m\varphi(z)},$$

where C_3 depends only on n and $\operatorname{diam}(U)$. Now if a remains in a compact set $K \subset U$, we can use a cut-off function θ with support in U and equal to 1 in a neighbourhood of a, and solve the $\overline{\partial}$-equation in the L^2 space associated with the weight $2m\varphi + 2(n+1)\log|z-a|$, that is, the singular Hermitian metric $h(z)^m |z-a|^{-2(n+1)}$ on $L^{\otimes m}$. For this, we apply the standard Andreotti–Vesentini–Hörmander L^2 estimates (see, for instance, [73] for the required version). This is possible for $m \geqslant m_0$ thanks to the hypothesis that $\Theta_{L,h} \geqslant \varepsilon\omega > 0$ even if X is non-Kähler (X is in any event a Moishezon variety from our assumptions). The bound m_0 depends only on ε and the geometry of a finite covering of X by compact sets $K_j \subset U_j$, where the U_j are coordinate balls (say); it is independent of the point a and even of the metric h. It follows that $g(a) = 0$, and therefore $\sigma = \theta f - g$ is a holomorphic section of $L^{\otimes m}$ such that

$$\int_X \|\sigma\|^2_{h^m}\, dV_\omega = \int_X |\sigma|^2 e^{-2m\varphi}\, dV_\omega \leqslant C_4 \int_U |f|^2 e^{-2m\varphi}\, dV_\omega \leqslant C_5 |a|^2 e^{-2m\varphi(z)},$$

in particular, $\sigma \in \mathscr{H}_m = H^0(X, L^{\otimes m} \otimes \mathscr{I}(h^m))$. We fix a such that the right-hand side of the latter inequality is 1. This gives the inequality

$$\psi_m(z) \geqslant \frac{1}{m} \log |a| = \varphi(z) - \frac{\log C_5}{2m},$$

which is the left-hand part of statement (i).

(ii) The first inequality in (i) implies that $\nu(\psi_m, z) \leqslant \nu(\varphi, z)$. In the opposite direction, we find that

$$\sup_{|x-z|<r} \psi_m(x) \leqslant \sup_{|x-z|<2r} \varphi(x) + \frac{1}{m} \log \frac{C_2}{r^n}.$$

We divide by $\log r < 0$ and take the limit as r tends to 0. The quotient by $\log r$ of the supremum of a plurisubharmonic function over $B(x, r)$ tends to the Lelong number at x. Thus we obtain

$$\nu(\psi_m, x) \geqslant \nu(\varphi, x) - \frac{n}{m}.$$

(iii) Again, the first inequality in (i) immediately yields $h_m \leqslant C_6 h$, hence $c_K(h_m) \geqslant c_K(h)$. Since we have $c_{\bigcup K_j}(h) = \min_j c_{K_j}(h)$, for the converse inequality we can assume without loss of generality that K is contained in a trivializing open patch U of L. Let us take $c < c_K(\psi_m)$. Then by definition, if $V \subset X$ is a sufficiently small open neighbourhood of K, then the Hölder inequality for the conjugate exponents $p = 1 + mc^{-1}$ and $q = 1 + m^{-1}c$ implies, thanks to the equality $\frac{1}{p} = \frac{c}{mq}$, that

$$\begin{aligned}
\int_V e^{-2(m/p)\varphi} \, dV_\omega &= \int_V \Bigg(\sum_{1 \leqslant k \leqslant N(m)} |g_{m,k}|^2 e^{-2m\varphi} \Bigg)^{1/p} \\
&\quad \times \Bigg(\sum_{1 \leqslant k \leqslant N(m)} |g_{m,k}|^2 \Bigg)^{-c/(mq)} dV_\omega \\
&\leqslant \Bigg(\int_X \sum_{1 \leqslant k \leqslant N(m)} |g_{m,k}|^2 e^{-2m\varphi} \, dV_\omega \Bigg)^{1/p} \\
&\quad \times \Bigg(\int_V \Bigg(\sum_{1 \leqslant k \leqslant N(m)} |g_{m,k}|^2 \Bigg)^{-c/m} dV_\omega \Bigg)^{1/q} \\
&= N(m)^{1/p} \Bigg(\int_V \Bigg(\sum_{1 \leqslant k \leqslant N(m)} |g_{m,k}|^2 \Bigg)^{-c/m} dV_\omega \Bigg)^{1/q} < +\infty.
\end{aligned}$$

From this we infer $c_K(h) \geqslant m/p$, that is, $c_K(h)^{-1} \leqslant p/m = 1/m + c^{-1}$. Since $c < c_K(\psi_m)$ was arbitrary, we get that $c_K(h)^{-1} \leqslant 1/m + c_K(h_m)^{-1}$, and the inequalities of (iii) are proved.

Proof of Theorem A.3. Given a big line bundle L on X, there exists a modification $\mu \colon \tilde{X} \to X$ of X such that $\tilde{X}$ is projective and

$$\mu^*(L) \sim A + E,$$

where A is an ample divisor and E an effective divisor with rational coefficients. By pushing forward by μ a smooth metric h_A with positive curvature on A, we get a singular Hermitian metric h_1 on L such that

$$\Theta_{L,h_1} \geqslant \mu_*\Theta_{A,h_A} \geqslant \varepsilon\omega$$

on X. Then for any $\delta > 0$ and any singular Hermitian metric h on L with $\Theta_{L,h} \geqslant 0$, the interpolated metric $h_\delta = h_1^\delta h^{1-\delta}$ satisfies $\Theta_{L,h_\delta} \geqslant \delta\varepsilon\omega$. Since h_1 is bounded away from 0, it follows that $c_K(h) \geqslant (1-\delta)c_K(h_\delta)$ by monotonicity. By Theorem A.4 (iii) applied to h_δ we infer that

$$c_K(h_\delta) = \lim_{m\to+\infty} c_K(h_{\delta,m}),$$

and we also have

$$c_K(h_{\delta,m}) \geqslant \operatorname{lct}_K\biggl(\frac{1}{m}D_{\delta,m}\biggr)$$

for any divisor $D_{\delta,m}$ associated with a section $\sigma \in H^0\bigl(X, L^{\otimes m}\otimes\mathscr{I}(h_\delta^m)\bigr)$, since the metric $h_{\delta,m}$ is given by

$$h_{\delta,m} = \biggl(\sum_k |g_{m,k}|^2\biggr)^{-1/m}$$

for an orthonormal basis of such sections. This clearly implies that

$$c_K(h) \geqslant \liminf_{\delta\to 0}\ \liminf_{m\to+\infty} \operatorname{lct}_K\biggl(\frac{1}{m}D_{\delta,m}\biggr) \geqslant \inf_{m\in\mathbb{Z}_{>0}}\ \inf_{D\in|mL|} \operatorname{lct}_K\biggl(\frac{1}{m}D\biggr),$$

and Theorem A.3 is proved.

In the applications, it is frequent to have a finite or compact group G of automorphisms of X and to look at G-invariant objects, namely, G-equivariant metrics on G-equivariant line bundles L; in the case of a reductive algebraic group G we simply consider a compact real form $G^{\mathbb{R}}$ instead of G itself.

One then gets an α invariant $\alpha_{G,K}(L)$ by looking only at G-equivariant metrics in Definition A.2. All constructions made are then G-equivariant, in particular, $\mathscr{H}_m \subset |mL|$ is a G-invariant linear system. For every G-invariant compact set K in X, we thus infer that

$$\begin{aligned}\alpha_{G,K}(L) &= \inf_{\{h \text{ is } G\text{-equvariant},\ \Theta_{L,h}\geqslant 0\}} c_K(h)\\ &= \inf_{m\in\mathbb{Z}_{>0}}\ \inf_{|\Sigma|\subset|mL|,\Sigma^G=\Sigma} \operatorname{lct}_K\biggl(\frac{1}{m}|\Sigma|\biggr).\end{aligned} \tag{A.1}$$

When G is a finite group, one can pick for large enough m a G-invariant divisor $D_{\delta,m}$ associated with a G-invariant section σ, possibly after multiplying m by the order of G. One then gets the slightly simpler equality

$$\alpha_{G,K}(L) = \inf_{m\in\mathbb{Z}_{>0}}\ \inf_{D\in|mL|^G} \operatorname{lct}_K\biggl(\frac{1}{m}D\biggr). \tag{A.2}$$

In a similar manner, one can work on an orbifold X rather than on a non-singular variety. The L^2 techniques work in this setting with almost no change (L^2 estimates are essentially insensitive to singularities, since one can just use an orbifold metric on the open set of regular points).

Appendix B. The big table

This appendix contains the list of non-singular Fano threefolds. We follow the notation and the numbering of [2], [50], and [51]. We also assume the following conventions. The symbol V_i denotes a smooth Fano threefold such that $-K_X \sim 2H$ and $\mathrm{Pic}(V_i) = \mathbb{Z}[H]$, where H is a Cartier divisor on V_i and $H^3 = 8i \in \{8, 16, \ldots, 40\}$. The symbol W denotes a (smooth) divisor of bidegree $(1,1)$ on $\mathbb{P}^2 \times \mathbb{P}^2$ (or, which is the same, the variety $\mathbb{P}(T_{\mathbb{P}^2})$). The symbol V_7 denotes a blow-up of $\mathbb{P}^3$ at a point (or, which is the same, the variety $\mathbb{P}(\mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(1))$). The symbol Q denotes a smooth quadric threefold. The symbol S_i denotes a smooth del Pezzo surface such that $K_{S_i}^2 = i \in \{1, \ldots, 8\}$, where $S_8 \ncong \mathbb{P}^1 \times \mathbb{P}^1$.

The fourth column of Table 1 contains the values of the global log canonical thresholds of the corresponding Fano varieties. The symbol $\star$ near a number means that $\mathrm{lct}(X)$ is calculated for a general X with given deformation type. If we know only an upper bound $\mathrm{lct}(X) \leqslant \alpha$, then we write $\leqslant \alpha$ instead of the exact value of $\mathrm{lct}(X)$, and the symbol '?' means that we do not know any reasonable upper bound (apart from the trivial $\mathrm{lct}(X) \leqslant 1$).

Table 1: Smooth Fano threefolds

$\beth(X)$	$-K_X^3$	Brief description	$\mathrm{lct}(X)$
1.1	2	a hypersurface of degree 6 in $\mathbb{P}(1,1,1,1,3)$	$1\star$
1.2	4	a hypersurface of degree 4 in $\mathbb{P}^4$ or a double cover of a quadric in $\mathbb{P}^4$ branched over a surface of degree 8	?
1.3	6	a complete intersection of a quadric and a cubic in $\mathbb{P}^5$	?
1.4	8	a complete intersection of three quadrics in $\mathbb{P}^6$	?
1.5	10	a section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ by a quadric and a linear subspace of dimension 7	?
1.6	12	a section of the Hermitian symmetric space $M = G/P \subset \mathbb{P}^{15}$ of type DIII by a linear subspace of dimension 8	?
1.7	14	a section of $\mathrm{Gr}(2,6) \subset \mathbb{P}^{14}$ by a linear subspace of codimension 5	?
1.8	16	a section of the Hermitian symmetric space $M = G/P \subset \mathbb{P}^{19}$ of type CI by a linear subspace of dimension 10	$\leqslant 6/7$
1.9	18	a section of the 5-dimensional rational homogeneous contact manifold $G_2/P \subset \mathbb{P}^{13}$ by a linear subspace of dimension 11	$\leqslant 4/5$
1.10	22	the zero locus of three sections of the rank-3 vector bundle $\bigwedge^2 \mathscr{Q}$, where $\mathscr{Q}$ is the universal quotient bundle on $\mathrm{Gr}(7,3)$	$\leqslant 2/3$
1.11	8	V_1, that is, a hypersurface of degree 6 in $\mathbb{P}(1,1,1,2,3)$	1/2
1.12	16	V_2, that is, a hypersurface of degree 4 in $\mathbb{P}(1,1,1,1,2)$	1/2
1.13	24	V_3, that is, a hypersurface of degree 3 in $\mathbb{P}^4$	1/2
1.14	32	V_4, that is, a complete intersection of two quadrics in $\mathbb{P}^5$	1/2
1.15	40	V_5, that is, a section of $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ by a linear subspace of codimension 3	1/2
1.16	54	Q, that is, a hypersurface of degree 2 in $\mathbb{P}^4$	1/3
1.17	64	$\mathbb{P}^3$	1/4

2.1	4	a blow-up of the Fano threefold V_1 along an elliptic curve that is an intersection of two divisors from $\left|-\frac{1}{2}K_{V_1}\right|$	1/2
2.2	6	a double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ whose branch locus is a divisor of bidegree $(2,4)$	$\leqslant 13/14$
2.3	8	the blow-up of the Fano threefold V_2 along an elliptic curve that is an intersection of two divisors from $\left|-\frac{1}{2}K_{V_2}\right|$	1/2
2.4	10	the blow-up of $\mathbb{P}^3$ along an intersection of two cubics	$3/4\star$
2.5	12	the blow-up of $V_3 \subset \mathbb{P}^4$ along a plane cubic	$1/2\star$
2.6	12	a divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(2,2)$ or a double cover of W whose branch locus is a surface in $\lvert -K_W\rvert$	?
2.7	14	the blow-up of Q along the intersection of two divisors from $\lvert\mathscr{O}_Q(2)\rvert$	$\leqslant 2/3$
2.8	14	a double cover of V_7 whose branch locus is a surface in $\lvert -K_{V_7}\rvert$	$1/2\star$
2.9	16	the blow-up of $\mathbb{P}^3$ along a curve of degree 7 and genus 5 which is an intersection of cubics	$\leqslant 3/4$
2.10	16	the blow-up of $V_4 \subset \mathbb{P}^5$ along an elliptic curve which is an intersection of two hyperplane sections	$1/2\star$
2.11	18	the blow-up of V_3 along a line	$1/2\star$
2.12	20	the blow-up of $\mathbb{P}^3$ along a curve of degree 6 and genus 3 which is an intersection of cubics	$\leqslant 3/4$
2.13	20	the blow-up of $Q \subset \mathbb{P}^4$ along a curve of degree 6 and genus 2	$\leqslant 2/3$
2.14	20	the blow-up of $V_5 \subset \mathbb{P}^6$ along an elliptic curve which is an intersection of two hyperplane sections	$1/2\star$
2.15	22	the blow-up of $\mathbb{P}^3$ along the intersection of a quadric and a cubic section	$1/2\star$
2.16	22	the blow-up of $V_4 \subset \mathbb{P}^5$ along a conic	$\leqslant 1/2$
2.17	24	the blow-up of $Q \subset \mathbb{P}^4$ along a normal elliptic curve of degree 5	$\leqslant 2/3$
2.18	24	a double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ whose branch locus is a divisor of bidegree $(2,2)$	1/2
2.19	26	the blow-up of $V_4 \subset \mathbb{P}^5$ along a line	$1/2\star$
2.20	26	the blow-up of $V_5 \subset \mathbb{P}^6$ along a twisted cubic	$\leqslant 1/2$
2.21	28	the blow-up of $Q \subset \mathbb{P}^4$ along a normal rational quartic	$\leqslant 2/3$
2.22	30	the blow-up of $V_5 \subset \mathbb{P}^6$ along a conic	$\leqslant 1/2$
2.23	30	the blow-up of $Q \subset \mathbb{P}^4$ along a curve of degree 4 that is an intersection of a surface in $\left\lvert\mathscr{O}_{\mathbb{P}^4}(1)\rvert_Q\right\rvert$ and a surface in $\left\lvert\mathscr{O}_{\mathbb{P}^4}(2)\rvert_Q\right\rvert$	$1/3\star$
2.24	30	a divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,2)$	$1/2\star$
2.25	32	the blow-up of $\mathbb{P}^3$ along an elliptic curve which is an intersection of two quadrics	1/2
2.26	34	the blow-up of $V_5 \subset \mathbb{P}^6$ along a line	$1/2\star$
2.27	38	the blow-up of $\mathbb{P}^3$ along a twisted cubic	1/2
2.28	40	the blow-up of $\mathbb{P}^3$ along a plane cubic	1/4

2.29	40	the blow-up of $Q \subset \mathbb{P}^4$ along a conic	1/3
2.30	46	the blow-up of $\mathbb{P}^3$ along a conic	1/4
2.31	46	the blow-up of $Q \subset \mathbb{P}^4$ along a line	1/3
2.32	48	W, that is, a divisor on $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1,1)$	1/2
2.33	54	the blow-up of $\mathbb{P}^3$ along a line	1/4
2.34	54	$\mathbb{P}^1 \times \mathbb{P}^2$	1/3
2.35	56	$V_7 \cong \mathbb{P}(\mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(1))$	1/4
2.36	62	$\mathbb{P}(\mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{O}_{\mathbb{P}^2}(2))$	1/5
3.1	12	a double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched in a divisor of tridegree $(2,2,2)$	3/4⋆
3.2	14	a divisor in the $\mathbb{P}^2$-bundle $\mathbb{P}(\mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1) \oplus \mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1))$ such that $X \in \lvert L^{\otimes 2} \otimes \mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,3)\rvert$, where L is the tautological line bundle	1/2⋆
3.3	18	a divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of tridegree $(1,1,2)$	2/3⋆
3.4	18	the blow-up of the Fano threefold Y with $\daleth(Y) = 2.18$ along a smooth fibre of the composition $Y \to \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ of the double cover and the projection	1/2
3.5	20	the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along a curve C of bidegree $(5,2)$ such that the composition $C \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ is an embedding	1/2⋆
3.6	22	the blow-up of $\mathbb{P}^3$ along a disjoint union of a line and a normal elliptic curve of degree 4	1/2⋆
3.7	24	the blow-up of the threefold W along an elliptic curve that is an intersection of two divisors from $\lvert -\frac{1}{2} K_W \rvert$	1/2⋆
3.8	24	a divisor in $\lvert (\alpha \circ \pi_1)^*(\mathscr{O}_{\mathbb{P}^2}(1)) \otimes \pi_2^*(\mathscr{O}_{\mathbb{P}^2}(2)) \rvert$, where $\pi_1 \colon \mathscr{F}_1 \times \mathbb{P}^2 \to \mathscr{F}_1$ and $\pi_2 \colon \mathscr{F}_1 \times \mathbb{P}^2 \to \mathbb{P}^2$ are projections and $\alpha \colon \mathscr{F}_1 \to \mathbb{P}^2$ is a blow-up of a point	1/2⋆
3.9	26	the blow-up of a cone $W_4 \subset \mathbb{P}^6$ over the Veronese surface $R_4 \subset \mathbb{P}^5$ with centre in the disjoint union of the vertex and a quartic in $R_4 \cong \mathbb{P}^2$	1/3
3.10	26	the blow-up of $Q \subset \mathbb{P}^4$ along a disjoint union of two conics	1/2
3.11	28	the blow-up of the threefold V_7 along an elliptic curve that is an intersection of two divisors from $\lvert -\frac{1}{2} K_{V_7} \rvert$	1/2
3.12	28	the blow-up of $\mathbb{P}^3$ along a disjoint union of a line and a twisted cubic	1/2
3.13	30	the blow-up of $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ along a curve C of bidegree $(2,2)$ such that $\pi_1(C) \subset \mathbb{P}^2$ and $\pi_2(C) \subset \mathbb{P}^2$ are irreducible conics, where $\pi_1 \colon W \to \mathbb{P}^2$ and $\pi_2 \colon W \to \mathbb{P}^2$ are the natural projections $\leqslant 1/2$	
3.14	32	the blow-up of $\mathbb{P}^3$ along a disjoint union of a plane cubic curve lying in a plane $\Pi \subset \mathbb{P}^3$ and a point outside Π	1/2
3.15	32	the blow-up of $Q \subset \mathbb{P}^4$ along a disjoint union of a line and a conic	1/2
3.16	34	the blow-up of V_7 along a proper transform via the blow-up $\alpha \colon V_7 \to \mathbb{P}^3$ of a twisted cubic passing through the centre of the blow-up α	1/2

3.17	36	a divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ of tridegree $(1,1,1)$	1/2
3.18	36	the blow-up of $\mathbb{P}^3$ along a disjoint union of a line and a conic	1/3
3.19	38	the blow-up of $Q \subset \mathbb{P}^4$ at two non-collinear points	1/3
3.20	38	the blow-up of $Q \subset \mathbb{P}^4$ along a disjoint union of two lines	1/3
3.21	38	the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along a curve of bidegree $(2,1)$	1/3
3.22	40	the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along a conic in a fibre of the projection $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^1$	1/3
3.23	42	the blow-up of V_7 along a proper transform via the blow-up $\alpha\colon V_7 \to \mathbb{P}^3$ of an irreducible conic passing through the centre of the blow-up α	1/4
3.24	42	$W \times_{\mathbb{P}^2} \mathbb{F}_1$,where $W \to \mathbb{P}^2$ is a $\mathbb{P}^1$-bundle and $\mathbb{F}_1 \to \mathbb{P}^2$ is a blow-up of a point	1/3
3.25	44	the blow-up of $\mathbb{P}^3$ along a disjoint union of two lines	1/3
3.26	46	the blow-up of $\mathbb{P}^3$ along a disjoint union of a point and a line	1/4
3.27	48	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	1/2
3.28	48	$\mathbb{P}^1 \times \mathbb{F}_1$	1/3
3.29	50	the blow-up of the Fano threefold V_7 along a line in $E \cong \mathbb{P}^2$, where E is the exceptional divisor of the blow-up $V_7 \to \mathbb{P}^3$	1/5
3.30	50	the blow-up of V_7 along the proper transform via the blow-up $\alpha\colon V_7 \to \mathbb{P}^3$ of a line passing through the centre of the blow-up α	1/4
3.31	52	the blow-up of a cone over a smooth quadric in $\mathbb{P}^3$ at the vertex	1/3
4.1	24	a divisor on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of multidegree $(1,1,1,1)$	1/2
4.2	28	the blow-up of the cone over a smooth quadric $S \subset \mathbb{P}^3$ along a disjoint union of the vertex and an elliptic curve in S	1/2
4.3	30	the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a curve of tridegree $(1,1,2)$	1/2
4.4	32	the blow-up of the smooth Fano threefold Y with $\beth(Y) = 3.19$ along the proper transform of a conic on the quadric $Q \subset \mathbb{P}^4$ that passes through both centres of the blow-up $Y \to Q$	1/3
4.5	32	the blow-up of $\mathbb{P}^1 \times \mathbb{P}^2$ along a disjoint union of two irreducible curves of bidegree $(2,1)$ and $(1,0)$	3/7
4.6	34	the blow-up of $\mathbb{P}^3$ along a disjoint union of three lines	1/2
4.7	36	the blow-up of $W \subset \mathbb{P}^2 \times \mathbb{P}^2$ along a disjoint union of two curves of bidegrees $(0,1)$ and $(1,0)$	1/2
4.8	38	the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a curve of tridegree $(0,1,1)$	1/3
4.9	40	the blow-up of the smooth Fano threefold Y with $\beth(Y) = 3.25$ along a curve contracted by the blow-up $Y \to \mathbb{P}^3$	1/3
4.10	42	$\mathbb{P}^1 \times S_7$	1/3
4.11	44	the blow-up of $\mathbb{P}^1 \times \mathbb{F}_1$ along a curve $C \cong \mathbb{P}^1$ such that C lies in a fibre $F \cong \mathbb{F}_1$ of the projection $\mathbb{P}^1 \times \mathbb{F}_1 \to \mathbb{P}^1$ and $C \cdot C = -1$ on F	1/3
4.12	46	the blow-up of the smooth Fano threefold Y with $\beth(Y) = 2.33$ along two curves that are contracted by the blow-up $Y \to \mathbb{P}^3$	1/4

4.13	26	the blow-up of $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ along a curve of tridegree $(1,1,3)$	$1/2\star$
5.1	28	the blow-up of the smooth Fano threefold Y with $\daleth(Y)=2.29$ along three curves contracted by the blow-up $Y\to Q$	$1/3$
5.2	36	the blow-up of the smooth Fano threefold Y with $\daleth(Y)=3.25$ along two curves $C_1\neq C_2$ contracted by the blow-up $\varphi\colon Y\to\mathbb{P}^3$ and lying in the same exceptional divisor of the blow-up φ	$1/3$
5.3	36	$\mathbb{P}^1\times S_6$	$1/2$
5.4	30	$\mathbb{P}^1\times S_5$	$1/2$
5.5	24	$\mathbb{P}^1\times S_4$	$1/2$
5.6	18	$\mathbb{P}^1\times S_3$	$1/2$
5.7	12	$\mathbb{P}^1\times S_2$	$1/2$
5.8	6	$\mathbb{P}^1\times S_1$	$1/2$

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