Cox rings of blowing-ups

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Question 1.1. Given a subfield $K \subseteq \mathbb{C}(x_1, \ldots, x_n)$, Hilberth problem 14 asks when

$$S := K \cap \mathbb{C}[x_1, \ldots, x_n]$$

is a finitely generated algebra [Hil90].

If G is a group that acts linearly on $\mathbb{C}(x_1, \ldots, x_n)$ and $K = \mathbb{C}(x_1, \ldots, x_n)^G$, then Hilbert proved finite generation of S when G is a reductive group and conjectured that S would always be finitely generated. Finally, Nagata produced a counterexample [Nag59].

Cox rings of blowing-ups

Nagata in [Nag59] takes $K = \mathbb{C}(x_1,\dots,x_r,y_1,\dots,y_r)^G$, where G is a subgroup of $\mathbb{C}_a^r$ which acts linearly on $\mathbb{C}[x_1,\dots,x_r,y_1,\dots,y_r]$ by

$$(g_1,\dots,g_r)\cdot u = \begin{cases} x_i & \text{if } u = x_i \\ y_i + g_i x_i & \text{if } u = y_i. \end{cases}$$

More precisely G is the kernel of a maximal rank $n\times r$ matrix, with $n<r$, and such that no columns is the zero vector

$$M := \begin{pmatrix} p_{11} & \cdots & p_{1r} \\ \vdots & & \vdots \\ p_{n1} & \cdots & p_{nr} \end{pmatrix}$$

Cox rings of blowing-ups

Nagata shows that there are isomorphisms of $\mathbb{Z}^{r+1}$-graded algebras

$$\mathbb{C}[x_1,\ldots,x_r,y_1,\ldots,y_r]^G \simeq \bigoplus_{(d,m)\in\mathbb{Z}\times\mathbb{Z}^r} R_{(d,m)}$$

$$\simeq \mathcal{R}(Bl_{\{p_1,\ldots,p_r\}}(\mathbb{P}^{n-1}))$$

where $R_{(d,m)} \subseteq \mathbb{C}[z_1,\ldots,z_n]$ is the subspace of degree d homogeneous polynomials with multiplicity $\geq m_i$ at $p_i \in \mathbb{P}^{n-1}$, for any $i = 1,\ldots,r$. He then shows that $\mathcal{R}(Bl_{\{p_1,\ldots,p_r\}}(\mathbb{P}^{n-1}))$ is not a finitely generated algebra. Proving this leads him to formulate his famous conjecture.

Conjecture 1.2 (Nagata). Let $\mathrm{Bl}_r^{\mathrm{gen}}(\mathbb{P}^2)$ be the blowing-up of $\mathbb{P}^2$ at r points in very general position. Let H be the pullback of a line and let $E_1, \ldots, E_r$ be the exceptional divisors. Then the divisor

$$\sqrt{r}H - E_1 - \cdots - E_r$$

is nef.

A generalization of Nagata example is given by Mukai in [Muk04] who constructs an extended Nagata action and shows that its algebra of invariants is isomorphic to the Cox ring of

$$X_{a,b,c} = \mathrm{Bl}_{b+c}(\mathbb{P}^{c-1}_{a-1}),$$

the blow-up of $(\mathbb{P}^{c-1})^{a-1}$ in $b+c$ points in very general position. For these rings Castravet and Tevelev prove the following [CT06].

Theorem 1.3. The following statements are equivalent:

- $\mathcal{R}(X_{a,b,c})$ is a finitely generated algebra.
- $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}>1$.

Definition 1.4. To any normal projective complex variety X with finitely generated divisor class group $\mathrm{Cl}(X)$ one can associate its Cox sheaf and *Cox ring* [ADHL15]

$$\mathcal{R} := \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{O}_X(D), \qquad \mathcal{R}(X) := \Gamma(X, \mathcal{R}).$$

The Cox ring is a $\mathrm{Cl}(X)$-graded algebra over the base field (complex numbers in what follows).

A normal projective variety X with finitely generated Cox ring admits a quotient construction similar to the one of projective space [ADHL15]

$$\begin{array}{ccc} \widehat{X} & \subseteq & \overline{X} \\ \big\downarrow{\scriptstyle p_X} & & \\ X & & \end{array}$$

where $\widehat{X} := \operatorname{Spec} \mathcal{R}$, $\overline{X} := \operatorname{Spec} \mathcal{R}(X)$, and p_X is a good quotient by the action of $H_X = \operatorname{Spec} \mathbb{C}[\mathrm{Cl}(X)]$. The ideal $\mathscr{I}_X \subseteq \mathcal{R}(X)$ of $\overline{X} \setminus \widehat{X}$ is called the *irrelevant ideal*.

Example 1.5. A *toric variety* X is quotient of a certain open subset $\widehat{X} \subseteq \mathbb{C}^r$ which is invariant under the diagonal action of a quasitorus $H = (\mathbb{C}^*)^{r-n} \oplus T$, where T is a finite group.

$$\begin{array}{ccccc} (\mathbb{C}^*)^r & \subseteq & \widehat{X} & \subseteq & \mathbb{C}^r \\ \downarrow & & \downarrow{\scriptstyle p_X} & & \\ (\mathbb{C}^*)^n & \subseteq & X & & \end{array}$$

It is known that $\mathrm{Cl}(X) \simeq \mathbb{Z}^{r-n} \oplus T$ and $\mathcal{R}(X) \simeq \mathbb{C}[x_1, \ldots, x_r]$, see [ADHL15, §2].

Let X be a normal projective variety with finitely generated Cox ring R, and let $\mathrm{Bl}_p X$ be the blow-up at a smooth point $p \in X$ (see also [Cut91]).

Theorem 2.1 ([HKL16]). Let $I \subseteq R$ be the ideal of $p_X^{-1}(p)$ in $\overline{X}$. Then

$$R(\mathrm{Bl}_p X) \simeq \bigoplus_{m \in \mathbb{Z}} (I^m : \mathscr{I}_X^\infty) t^{-m} \subseteq R[t^{\pm 1}],$$

where $I^m = R$ for any $m \leq 0$.

As a consequence of the above theorem one can provide the following criterion to decide if a subalgebra is the whole Cox ring [HKL16, HKL18].

Proposition 2.2. Let X, R and $I \subseteq R$ be as in Theorem 2.1 and let $S \subseteq R$ be a finite set of homogeneous elements which generate R. Let

- $f := \prod_{g \in S \setminus I} g$;
- $f_1 t^{-m_1}, \ldots, f_k t^{-m_k} \in R(\mathrm{Bl}_p X)$ homogeneous elements;
- $B_0 := \{t^{m_i} s_i - f_i \, : \, i = 1, \ldots, k\} \subseteq R[s_1, \ldots, s_k, t]$.

Assume that there is a finite set $B_0 \subseteq B \subseteq \langle B_0 \rangle : \langle t \rangle^\infty$ such that

$$\dim(R) = \dim(\langle B \cup \{t\} \rangle) > \dim(\langle B \cup \{t, f\} \rangle).$$

Then $R(\mathrm{Bl}_p X)$ is generated by $S \cup \{t, f_1 t^{-m_1}, \ldots, f_k t^{-m_k}\}$.

Observation 2.3. There are no known necessary and sufficient conditions for the finite generation of the Cox ring of $\mathrm{Bl}_p X$, when X is a toric variety. Partial results and examples are given in [HKL16, HKL18, GL22, GAGK19, Cas18, GAGK23].

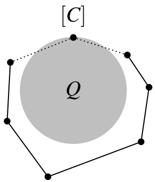
When X is a surface, as a consequence of [HK00], there are two reasons for not having finite generation:

1. $\mathrm{Bl}_p X$ contains a nef class which is not semiample;
2. the effective cone $\mathrm{Eff}(\mathrm{Bl}_p X)$ is not rational polyhedral.

Theorem 3.1 ([CLTU23]). In every characteristic, there exist projective toric surfaces X such that the pseudo-effective cone $\overline{\mathrm{Eff}}(\mathrm{Bl}_p X)$ is not polyhedral.

Idea of proof.

- Riemann-Roch theorem implies that $\overline{\mathrm{Eff}}(\mathrm{Bl}_p X)$ contains a circular cone Q.
- Using intersection theory and elliptic curve defined over $\mathbb{Q}$ one constructs an extremal ray $\mathbb{R}_{\geq 0} \cdot [C]$ of $\overline{\mathrm{Eff}}(\mathrm{Bl}_p X)$ which lies in Q.

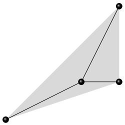


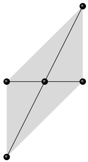
Thus, as shown in the picture, $\overline{\mathrm{Eff}}(\mathrm{Bl}_p X)$ cannot be polyhedral. □

Using Theorem 3.1 and a construction in [CT15] we could prove the following.

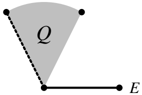
Theorem 3.2 ([CLTU23]). The cone $\overline{\mathrm{Eff}}(\overline{M}_{0,n})$ is not polyhedral for $n \geq 10$, both in characteristic 0 and in characteristic p, for all primes p.

Definition 3.3. A projective toric surface X is *minimal* if it does not contain curves of negative self-intersection. These are quotients of $\mathbb{P}^2$ or of $\mathbb{P}^1 \times \mathbb{P}^1$ by the action of a finite abelian group of the maximal torus $(\mathbb{C}^*)^2$.

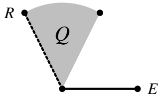




Theorem 3.4. There are minimal toric surfaces X of Picard rank one, such that $\mathrm{Eff}(\mathrm{Bl}_p X)$ is a two dimensional cone open on one side [GAGK23].



Open problem. Is there such an example where the slope of the line is not a rational number?



The quotient morphism $\mathbb{P}^2 \to X$, with fiber $\{q_1, \ldots, q_r\}$ over p, induces a surjection $\pi \colon \mathrm{Bl}_r(\mathbb{P}^2) \to \mathrm{Bl}_p(X)$. So

$$\begin{aligned} R \in \overline{\mathrm{Eff}}(\mathrm{Bl}_p(X)) \setminus \mathrm{Eff}_p(X) &\Rightarrow R \text{ is nef} \\ &\Rightarrow \pi^* R \text{ is nef on } \mathrm{Bl}_r(\mathbb{P}^2) \\ &\Rightarrow \sqrt{r} H - E_1 - \cdots - E_r \text{ is nef on } \mathrm{Bl}_r(\mathbb{P}^2) \\ &\Rightarrow \sqrt{r} H - E_1 - \cdots - E_r \text{ is nef on } \mathrm{Bl}_r^{\mathrm{gen}}(\mathbb{P}^2) \end{aligned}$$

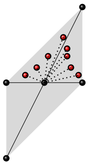
Theorem 3.5 ([LU23]). Let X be a minimal toric surface of Picard rank two. Then the Cox ring of $\mathrm{Bl}_p X$ is finitely generated.

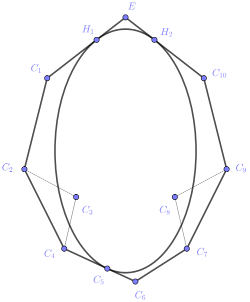
Cox rings of blowing-ups

3. Recent results

Idea of proof.

- The effective cone is generated by the subset $\mathcal{S} \subseteq \mathrm{Eff}(\mathrm{Bl}_p X)$ of classes of strict transforms of one-parameters subgroups parametrized by the Hilbert bases of the four cones.





- $\mathcal{S}$ satisfies the hypothesis of Lemma 3.6.

Lemma 3.6. Let X be a normal projective $\mathbb{Q}$-factorial surface with polyhedral effective cone $\mathrm{Eff}(X)$. Then the following are equivalent:

1. The Cox ring of X is finitely generated.
2. There exists a finite subset $\mathcal{S} \subseteq \mathrm{Eff}(X)$ such that $\mathrm{Eff}(X) = \mathrm{Cone}(\mathcal{S})$ and for any facet F of $\mathrm{Eff}(X)$, the ray $R \in \mathrm{Nef}(X)$, orthogonal to F, satisfies the following

$$R \in \bigcap_{C \in \mathrm{Rays}(F)} \mathrm{Cone}(\mathcal{S} \setminus C).$$

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