On a Torelli principle for automorphisms of many Klein hypersurfaces (with V. González, A. Liendo and R. Villaflor)

Pedro Montero

Universidad Técnica Federico Santa María
Valparaíso, Chile

Algebraic Geometry, Lipschitz Geometry and Singularities – Pipa 2023

Polarized integral Hodge structures

A (pure) **polarized Hodge structure** of weight k is a triple

$$(H, \{H^{p,q}\}_{p+q=k}, \langle\cdot,\cdot\rangle)$$

- $H \cong \mathbf{Z}^h$ free module with $\langle\cdot,\cdot\rangle : H \times H \to \mathbf{Z}$ bilinear non-degenerate.
- $H^{p,q}$ are $\mathbf{C}$-subspaces of $H_{\mathbf{C}} := H \otimes_{\mathbf{Z}} \mathbf{C}$ such that

$$\overline{H^{p,q}} = H^{q,p} \text{ and } H_{\mathbf{C}} = \bigoplus_{p+q=k} H^{p,q}$$

- The linear extension of $\langle\cdot,\cdot\rangle$ to $H_{\mathbf{C}}$ makes the Hodge decomposition orthogonal and $(-1)^p\langle\cdot,\cdot\rangle|_{H^{p,q}}$ is positive definite for $p \geq q$.

Main example: $X \subseteq \mathbf{P}^N$ smooth projective of $\dim_{\mathbf{C}}(X) = n$ and

$$H := \mathrm{Im}\left[\mathsf{H}_{2n-k}(X,\mathbf{Z}) \xrightarrow{\int} \mathsf{H}_{\mathsf{dR}}^{2n-k}(X)^* \xrightarrow{\mathrm{PD}^{-1}} \mathsf{H}_{\mathsf{dR}}^{k}(X) \simeq \mathsf{H}^k(X,\mathbf{C})\right].$$

Moreover, when k is even we remove the classes arising from hyperplane sections $[X \cap \mathbf{P}^{N-1}]^{n-\frac{k}{2}}$ and we consider $\mathsf{H}^k(X,\mathbf{C})_{\mathrm{prim}}$ instead.

Classical and Punctual Torelli Theorem

The Precise/Strong Torelli Theorem states that

Torelli Theorem

Let X and Y smooth projective curves of genus g and let

$$f : \mathrm{H}_1(X, \mathbf{Z}) \xrightarrow{\sim} \mathrm{H}_1(Y, \mathbf{Z})$$

be an isomorphism of polarized Hodge structures. Then

1. If X and Y are hyperelliptic, $\exists!$ isomorphism $\varphi : X \xrightarrow{\sim} Y$ s.t. $f = \varphi_*$.
2. Otherwise, $\exists!$ isomorphism $\varphi : X \xrightarrow{\sim} Y$ s.t. $f = \pm\varphi_*$.

Torelli Principles: Assume that X and Y belong to the same connected component of a moduli space. We should expect that:

- **Strong Torelli**: Every isomorphism between Hodge structures is induced, possibly up to involutions, by an isomorphism $X \xrightarrow{\sim} Y$.
- **Punctual Torelli**: $\mathrm{Aut}(H)$ coincide (up to involutions) with $\mathrm{Aut}(X)$.

Some known cases & Hypersurfaces

Known cases for the Strong Torelli Principle:

- Curves (Torelli 1913)
- Abelian varieties
- K3 surfaces (Shapiro-Shafaverich '71)
- Cubic threefolds (Clemens-Griffiths '72, Beauville '82)
- Cubic fourfolds (Voisin '86)
- HyperKähler manifolds (Verbitsky '09)

For **generic** hypersurfaces (Donagi '83, Voisin '20) $X, Y \subseteq \mathbf{P}^{n+1}$, we have that $X \cong Y$ iff they have isomorphic Hodge structures (Global Torelli).

Non-trivial counter-examples

There are surfaces of general type for which the Global Torelli fails:

- (Catanese '80): $p_g(S) = 1$, $q(S) = 0$, $K_S^2 = 1$.
- (Todorov '81): $p_g(S) = 1$, $q(S) = 0$, $2 \leq K_S^2 \leq 8$.

Automorphisms of large prime order

Let $X \subseteq \mathbf{P}^{n+1}$ be a smooth hypersurface. By Lefschetz hyperplane section theorem, $\mathsf{H}^k(X, \mathbf{C})_{\mathrm{prim}} = 0$ for $k \neq n$. We are left to analyze $\mathsf{H}^n(X, \mathbf{C})_{\mathrm{prim}}$.

If $X \subseteq \mathbf{P}^{n+1}$ is generic $\mathrm{Aut}(X) \simeq \{1\}$ (Matsumura-Monsky '64). We should study the Punctual Torelli for **hypersurfaces with many automorphisms**.

Theorem (González–Liendo 2013)

Let $n \geq 3$, $d \geq 3$. Then, a degree d smooth hypersurface $X \subseteq \mathbf{P}^{n+1}$ admits an automorphism of prime order $p > (d-1)^n$ if and only if

$$X \simeq X_K = \{x_0^{d-1}x_1 + x_1^{d-1}x_2 + \ldots + x_{n+1}^{d-1}x_0 = 0\} \quad \text{(Klein hypersurface)}$$

Moreover, $n+2$ is prime and $p = \frac{(d-1)^{n+2}+1}{d}$ is (a Wagstaff) prime.

We say that X is a **Klein hypersurface of Wagstaff type**.

Main result

Theorem (González–Liendo–M.–Villaflor, 2023)

The **Punctual Torelli Principle holds** for Klein hypersurfaces of Wagstaff type $X_d \subseteq \mathbf{P}^{n+1}$ in the following cases:

- $d \mid n+3$.
- $d = 3$ and $n \geq 5$.

Namely, $\mathrm{Aut}(X) \simeq \mathrm{Aut}(H)/\{\pm 1\}$ in these cases.

The proof consists of computing both groups by independent methods:

1. The computation of $\mathrm{Aut}(X)$ is based on a refinement of the so-called Differential Method by Poonen and Oguiso-Yu (cf. Alvaro's talk).
2. The computation of $\mathrm{Aut}(H)/\{\pm 1\}$ requires to extend the notion of extremal ppav to any polarized integral Hodge structure.

Extremal Hodge structures

Let $\sigma \in \mathrm{Aut}(H)$ of order m. Then $\ker(\sigma - \mathrm{Id}_H) \subseteq H$ is a Hodge sub-structure and therefore

$$H_0 := \sum_{i=1}^{m-1} \ker(\sigma^i - \mathrm{Id}_H) \text{ is a Hodge sub-structure.}$$

Thus, σ induces an automorphism of $H' := H/H_0$ such that every power $\sigma^i_{\mathbf{C}} := \sigma^i \otimes \mathrm{Id}_{H_{\mathbf{C}}}$ has no fixed points.

Since the eigenvalues of $\sigma_{\mathbf{C}}|_{H'}$ are primitive m-roots of 1, its characteristic polynomial is $\Phi_m(t)^{\dim(H')/\varphi(m)}$. In particular, $\varphi(m) \mid h - h_0$.

Definition (Extremal polarized Hodge structures)

Let H be a polarized Hodge structure of weight k. We say that H is **extremal** if there is $\sigma \in \mathrm{Aut}(H)$ of prime order p with $\varphi(p) = h - h_0$, i.e.,

$$p = h - h_0 + 1$$

Remark: This extends the notion of **extremal ppav**, where $p = 2g + 1$.

Induced partitions

In the extremal case, there is a spectral decomposition

$$H_{\mathbf{C}} = \bigoplus_{i=0}^{p-1} V(\xi_p^i) \text{ compatible with the Hodge decomposition}$$

with $V(1) = H_0$ and each $V(\xi_p^i) \simeq \mathbf{C}$ for $i = 1, \ldots, p-1$. Thus,

$$H^{r,s} = \bigoplus_{\text{some } i} V(\xi_p^i).$$

In particular, by considering the eigenvalues of the eigenspaces contained in $(H')^{r,s}$, there is a natural partition of the set

$$\{\xi, \xi^2, \ldots, \xi^{p-1}\} = \coprod_{r+s=k} C^{r,s} \text{ such that } \overline{C^{r,s}} = C^{s,r}.$$

Remark: This kind of argument shows that extremal ppav are of CM-type.

Proposition (González–Liendo–M.–Villaflor)

Let $X_d \subseteq \mathbf{P}^{n+1}$ be a Klein hypersurface of Wagstaff type. Then, the primitive cohomology $\mathrm{H}^n(X,\mathbf{Z})_{\mathrm{prim}}$ is an extremal Hodge structure.

It follows from the previous discussion and the Griffiths Basis Theorem that

Theorem (González–Liendo–M.–Villaflor)

Assume $(n,d) \neq (3,3)$ and let $p = \frac{(d-1)^{n+2}+1}{d}$. For $q \in \{0,1,\ldots,n\}$ consider

$$S_q := \left\{ \sum_{i=0}^{n+1} \beta_i (1-d)^i \in \mathbf{F}_p \ : \ 0 \leq \beta_i \leq d-2, \sum_{i=0}^{n+1} \beta_i = d(q+1)-n-2 \right\}.$$

Then X satisfies the **Punctual Torelli Principle** if and only if

$$\{m \in \mathbf{F}_p^\times : m \cdot S_q = S_q, \ \forall q = 0,\ldots,n\} = \langle 1-d \rangle < \mathbf{F}_p^\times$$

We check this for (n,d) with $d \mid n+3$ and $d=3$ and $n \geq 5$. □

A mysterious abelian variety in $\mathcal{A}_{21}$

Deligne (1972), Rapoport (1972): the only smooth hypersurfaces $X \subseteq \mathbf{P}^{n+1}$ of dimension $n \geq 2$ with polarized Hodge structure of weight 1 are:

- $X_3 \subseteq \mathbf{P}^4$ smooth cubic threefold, where $\mathrm{H}_3(X, \mathbf{Z}) \simeq \mathbf{Z}^{10}$.
- $X_4 \subseteq \mathbf{P}^4$ smooth quartic threefold, where $\mathrm{H}_3(X, \mathbf{Z}) \simeq \mathbf{Z}^{60}$.
- $X_3 \subseteq \mathbf{P}^6$ smooth cubic fivefold, where $\mathrm{H}_5(X, \mathbf{Z}) \simeq \mathbf{Z}^{42}$.

Corollary (González–Liendo–M.–Villaflor)

Let X be the Klein quartic threefold (resp. cubic fivefold). Then

$$\mathrm{Aut}(X) \simeq \mathrm{Aut}(J(X), \Theta)/\{\pm 1\}.$$

As $h_{\mathbf{Q}(\sqrt{-43})} = 1$, it follows (Bennama-Bertin '97) that there is a **unique ppav** $(A, \Theta) \in \mathcal{A}_{21}$ such that $\mathrm{Aut}(A, \Theta) \cong \mathrm{PSL}_2(\mathbf{F}_{43})$.

Muito obrigado !