The proof of theorem 2 is obtained from the following

THEOREM 3: *Let $f: X \to Y$ be a surjective morphism of non-singular projective varieties with connected fibers, D a divisor with normal crossing on Y, such that*:

(1) *When we write $Y_0 = Y - D$ and $X_0 = f^{-1}(Y_0)$, the restriction $f_0 = f|_{X_0}$ is smooth,*

(2) *The local monodromies around D acting on $R^n f_* \mathbf{C}_{X_0}$ are unipotent, where $n = \dim X - \dim Y$.*

Then, the sheaf $f_* K_{X/Y}$ is locally free and semi-positive.

DEFINITION: Let $\pi: V \to X$ be a vector bundle over X. V is said to be *semi-positive*, if for any morphism $\varphi: C \to X$ from any curve C and for any quotient line bundle Q of $\varphi^* V$, we have $\deg Q \geq 0$.

FACT 1 (Fujita): Let V be a semi-positive vector bundle on X and let L be an ample line bundle on X. Then $V \otimes L$ is an ample vector bundle.

FACT 2 (Kodaira): Let Y be a non-singular projective variety and let L be an ample line bundle on Y. If $\kappa(Y) = \dim Y$, then there exists a positive integer m such that $H^0(Y, mK - L) \neq 0$.

The proof of theorem 3 uses the main result of Schmid: Variation of Hodge structures, Inv. Math. 22.

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CANONICAL 3-FOLDS

Miles Reid

It seems to me that one cannot get a good view of the sky carrying a platter on one's head.

Ssu-Ma Ch'ien

§-∞. Introduction

This is a report on the theory of canonical models of 3-folds of f.g. general type, aiming to generalise both the theory of Du Val surface singularities and some theoretical aspects of the global theory of canonical models of surfaces. The heart of the approach is the definition of *canonical singularities*, which generalises the adjunction-theoretic characterisation of Du Val singularities.

The introductory §0 discusses some of the ways in which canonical 3-fold singularities differ from canonical surface singularities – these are the points which any eventual theory will have to cover. §1 discusses some of the formal consequences of the definition of canonical singularities. §2 outlines a theory which gives some hold on the classification of canonical singularities. §3 and §4 describe important classes of canonical singularities which can be tackled by means of the toric geometry of Mumford and Kempf; this is a key technique in algebraic geometry which extends the range of computability in a spectacular way, and I refer to Danilov[15] for an extremely attractive treatment.

§4 contains in passing an implicit list of singularities which can reasonably be called "simple elliptic" 3-fold singularities.

§5 contains a formula for the plurigenera of 3-folds of f.g. general type, independent of the preceding partial classification of canonical singularities. §6 contains remarks and further problems arising out of the preceding sections.

In acknowledgement, I must plead guilty to shameless exploitation of my research student Nick Shepherd-Barron; several of the key ideas in this paper originated with him, and in particular the beautiful connection between rational Gorenstein 3-fold singularities and elliptic Gorenstein surface singularities (Theorem 2.6) was prompted by his determined and original attempts to prove Theorem 2.2.

All varieties and maps are defined over the complex ground field $k = \mathbf{C}$. A linguistic novelty introduced here is a *free* linear system to mean a linear system free from fixed components and base points.

Contents

§0. Du Val singularities

DEFINITION (0.1): A (non-singular, projective) variety V of dimension n is of *f.g. general type* if the canonical ring $R(V) = \bigoplus_{m \geq 0} H^0(V, \omega_V^{\otimes m})$ is finitely generated over k, and of the maximum transcendence degree $n+1$. In this case $X = \operatorname{Proj} R(V)$ is birationally equivalent to V, and is called the *canonical model* of V.

In this section X will be called a canonical variety, or $P \in X$ a canonical singularity, if X is the canonical model of some V (compare Definition 1.1 and Proposition 1.2, (II)). The question studied here is this: what do canonical 3-folds look like? Global properties will be considered in §5, but most of the paper will be concerned with the local question, or the study of canonical 3-fold singularities.

For surfaces it is well known that any canonical singularity is analytically isomorphic to a Du Val singularity, that is to the hypersurface singularity in $\mathbf{A}^3$ defined by one of the following polynomial equations:

Table (0.2)

A_n: $x^2 + y^2 + z^{n+1}$, for $n \geq 1$;
D_n: $x^2 + y^2 z + z^{n-1}$, for $n \geq 4$;
E_6: $x^2 + y^3 + z^4$
E_7: $x^2 + y^3 + yz^3$
E_8: $x^2 + y^3 + z^5$.

THEOREM (0.3): *The following are 4 characterisations of Du Val singularities:*

(I) **Adjunction-theoretic.** *$P \in X$ is a normal Gorenstein point such that if $s \in \omega_X$ is a local generator, then for any resolution $f: Y \to X$, $s \in \omega_Y$, that is, s remains regular on $f^{-1}P$ when considered as a differential on Y (in other words, $P \in X$ is a rational Gorenstein point).*

(II) **Inductive.** *$P \in X$ is normal, and for any chain $X_n \to \cdots \to X_0 = X$ of length $n \geq 0$, in which $s_i: X_i \to X_{i-1}$ is the blow-up of the maximal ideal m_{P_i} for some closed point $P_i \in X_{i-1}$, X_n has at most isolated double points*; (in particular, $P \in X$ is a hypersurface double point, and the tangent cone is a plane conic, which may be irreducible, a line pair, or a double line).

This is in fact a powerful algorithmic method for determining whether a given singularity is a Du Val point.

(III) **Quotient singularities.** *$P \in X$ is isomorphic to a quotient singularity $\mathbf{A}^2/G$, where $G \subset SL(2, k)$ is a finite group.*

(IV) **Quasi-homogeneous hypersurfaces.** *$P \in X$ is isomorphic to an isolated hypersurface singularity $0 \in X \subset \mathbf{A}^3$ defined by $f(x) = 0$, with f quasi-homogeneous with respect to some weighting α, with*

$$\alpha\left(\frac{x_1 x_2 x_3}{f}\right) = \sum \alpha(x_i) - \alpha(f) > 0.$$

(For example, $x^2 + y^3 + z^5$ is quasi-homogeneous of weight 1, where $\alpha(xyz) = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} > 1$.)

The four sections 1–4 of this paper correspond to the four parts of this theorem, and are concerned with trying to generalise the statements to higher dimension; the fact that all four characterisations lead to the same surface singularities is however a miracle which we cannot hope to see continue in the higher-dimensional case. The four characterisations in the theorem go back in the main to Du Val [1], although the first invariably appears in Artin's dual cohomological form $R^1 f_* \mathcal{O}_Y = 0$ (see [2] and [3]); see also Durfee [5].

The following is a list of the difficulties of classifying canonical singularities; these arise as much from known properties of canonical 3-folds as from technicalities of proof.

REMARK (0.4):

(i) It is quite inadequate to restrict attention to hypersurface singularities; indeed, the quotient singularities $X = \mathbf{A}^3/G$, with $G \subset SL(3, k)$ give examples of Gorenstein canonical points having embedding dimension $\dim T_{X,P} = \dim m_P/m_P^2$ arbitrarily large. In fact "typical" examples of 3-fold rational Gorenstein points are given by the affine cone over a del Pezzo surface.

(ii) The Weil divisor class K_X need not be a Cartier divisor; formally we know that rK_X must be an ample Cartier divisor for some $r \geq 1$. I define the smallest such r to be the *index* of X; there are canonical quotient singularities, first discovered by Shepherd-Barron, having arbitrary index (§3).

There is a cyclic covering trick (Corollary 1.9) which reduces canonical singularities of local index $r > 1$ to the $r = 1$ case. It is therefore sufficient for some purposes to concentrate on the case that ω_X is locally free.

(iii) Canonical 3-folds have in general 1-dimensional singular loci; in fact it seems to me perverse to distinguish isolated singularities, since even in very simple cases non-isolated singularities will unavoidably appear in the course of resolving isolated singularities.

(iv) As a technical difficulty, there is no very simple reason why canonical singularities should be Cohen–Macaulay in higher dimensions[1]; the 3-fold case has recently been settled by Shepherd–Barron [36].

(v) Even for hypersurface singularities the simple and elegant inductive criterion (II) above cannot extend as such to the higher dimensional case. For example consider the two hypersurface singularities

$$k = 2\colon x^2 + y^4 + z^4 + t^4 = 0;$$

and $$k = 1\colon x^2 + y^3 + z^6 + t^6 = 0.$$

These are weighted cones over del Pezzo surfaces of degree k, and are easily seen to be canonical (Proposition 4.5). However, on blowing up the maximal ideal of the origin in the first we get a non-normal variety; and in the second we get a normal 3-fold having a curve of singularities whose surface sections are simple elliptic singularities. Thus the blow-up need not be canonical, so that an inductive criterion analogous to (II) must be more complicated.

§1. Definition of canonical singularities

The appendix to this section deals with Weil divisors and divisorial sheaves on a a normal variety X, and introduces the divisorial sheaves $\omega_X^{[r]} = \mathcal{O}_X(rK_X)$ of regular r-differentials on X.

I have written this paper consistently using the language of the sheaves $\omega_X^{[r]}$ rather than the equivalent language of Weil divisors K_X; the reader who wishes to translate some of the definitions or arguments back into the language of Weil divisors will benefit immensely from the exercise.

Definition (1.1): A quasi-projective variety X is said to have *canonical singularities* if it is normal, and if the following 2 conditions hold:

(i) for some integer $r \geqq 1$ $\omega_X^{[r]}$ is locally free;

(ii) for some resolution $f\colon Y \to X$, and r as in (i), $f_*\omega_Y^{\otimes r} = \omega_X^{[r]}$.

Observe that these conditions are local on X; if they hold in a

[1] This has been proved in all dimensions by O. Gabber and R. Elkik.

neighbourhood of $P \in X$, I will say that X is canonical at P, or that $P \in X$ is a *canonical singularity*; the smallest r for which (i) holds at P is called the *index* of $P \in X$.

If furthermore X is projective and

(iii) $\omega_X^{[r]}$ is ample,

then X is said to be a *canonical variety*.

Proposition (1.2):

(I) *$P \in X$ is a canonical singularity if and only if for some integer $r \geqq 1$ $\omega_X^{[r]}$ is generated by a section $s \in \omega_X^{[r]}$, such that $s \in \omega_Y^{\otimes r}$ for a resolution $f\colon Y \to X$; that is, the r-differential s, when considered as a rational r-differential on Y, remains regular on a neighbourhood of $f^{-1}P$.*

(II) *X is a canonical variety if and only if it is the canonical model of a variety of f.g. general type.*

Remark (1.3): Assuming (i), (ii) is equivalent to

(ii') for every proper birational morphism $f\colon Y \to X$, and every $s \geqq 1$, $f_*\omega_Y^{[s]} = \omega_X^{[s]}$.

In particular, canonical singularities satisfy "Kempf's condition" $f_*\omega_Y = \omega_X$ for a resolution $f\colon Y \to X$, so that according to Kempf's duality argument ([6], p. 44), canonical singularities are rational if and only if they are Cohen–Macaulay. However, not all rational singularities are canonical, since (ii') is stronger than Kempf's condition.

Question (1.4) Does (ii') imply (i)?[1]

To emphasize that the condition in (I) of Proposition 1.2 is readily calculable once we know a resolution, let me give two examples, which give a foretaste of results of §3 and §4.

Example (1.5) (i) Let $X \subset \mathbf{A}^{n+1}$ be a hypersurface with an ordinary k-fold point at the origin 0; *then $0 \in X$ is canonical if and only if $k \leqq n$.*

(ii) Let $X = \mathbf{A}^n/\mu_k$, where μ_k is the cyclic group of k-th roots of 1, acting by $\epsilon\colon (x_1, \ldots, x_n) \mapsto (\epsilon x_1, \ldots, \epsilon x_n)$; X is isomorphic to the affine cone over the k-fold Veronese embedding of $\mathbf{P}^{n-1}$; *then X is canonical if and only if $k \leqq n$, and its index is the denominator of n/k.*

Computation. (i) Let X be given by $f = f(x_0, \ldots, x_n) = 0$, so that ω_X is generated by

$$s = \operatorname{Res}_X\left(\frac{dx_0 \wedge \cdots \wedge dx_n}{f}\right) = \frac{dx_1 \wedge \cdots \wedge dx_n}{(\partial f/\partial x_0)}.$$

[1] This has been answered negatively by Pinkham; the counter-example is one of a type of singularities studied by Laufer [35].

A typical piece of the blow-up of $\mathbf{A}^{n+1}$ has coordinates $y_0, \dots, y_n$ with $x_0 = y_0$, $x_i = y_0 y_i$, and in this piece the non-singular proper transform X' is given by $f' = \frac{1}{y_0^k} f(y_0, y_0 y_i) = 0$. In terms of the generator

$t = \operatorname{Res}\left(\frac{\mathrm{d}y_0 \wedge \cdots \wedge \mathrm{d}y_n}{f'}\right) \in \omega_{X'}$ we have $s = y_0^{n-k} t$, Q.E.D.

(ii) Among the coordinate functions on X I pick out the invariant monomials $u_1 = x_1^k$, $u_i = x_1^{k-1} x_i$; let $\frac{n}{k} = \frac{b}{a}$ with a and b coprime positive integers. Write

$$s = (\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n)^a = (\text{const.}) \cdot \frac{(\mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_n)^a}{u_1^{b(k-1)}};$$

this is a rational differential on X having no zeroes or poles, and is thus a generator of $\omega_X^{[a]}$.

This cone also becomes non-singular after a single blow-up; coordinates on a typical piece of X' are $v_1, \dots, v_n$, with $u_1 = v_1$, $u_i = v_1 v_i$. Since $\mathrm{d}u_1 \wedge \cdots \wedge \mathrm{d}u_n = v_1^{n-1}(\mathrm{d}v_1 \wedge \cdots \wedge \mathrm{d}v_n)$ we have

$$s = (\text{const.}) \cdot v_1^{\alpha} \cdot (\mathrm{d}v_1 \wedge \cdots \wedge \mathrm{d}v_n)^a,$$

where $\alpha = a(n-1) - b(k-1) = b - a$. Thus s remains regular on X' if and only if $b \geqq a$, that is $k \leqq n$. Q.E.D.

The proofs of Remark 1.3, of (I) and of the "only if" part of (II) in Proposition 1.2 are purely formal, and are left to the reader as interesting exercises.

Proof of "if" part of (II): Let V be a variety of f.g. general type. There exists an r such that the graded ring

$$R(V)^{(r)} = \bigoplus_{n \geq 0} H^0(V, \omega_V^{\otimes nr})$$

is generated by its elements of the least degree r. Blowing up the base locus of $|rK_V|$ on V, I may assume that

$$|rK_V| = |D| + F,$$

with F fixed and $|D|$ free.

Let $\varphi_D = \varphi_{rK_V}\colon V \to X$ be the morphism defined by $|D|$; I will show that $\varphi_D(F)$ has no components of codimension 1, so that for some open set $X^0 \subset X$ having complement of codimension $\geqq 2$, $\varphi_D\colon \varphi_D^{-1}(X^0) \overset{\sim}{\to} X^0$, providing an isomorphism of $\mathcal{O}_X(1)$ with $\omega_X^{[r]}$ over X^0. Thus $\omega_X^{[r]} = \mathcal{O}_X(1) = \varphi_{D*}\omega_V^{\otimes r}$, so that X is canonical. There only remains to prove the following assertion.

Lemma (1.6): *For every component Γ of F, $\varphi_D(\Gamma)$ has codimension $\geqq 2$.*

Proof: Write $n = \dim V$. By hypothesis, for every $m > 0$,

$$H^0(\mathcal{O}_V(mD + \Gamma)) \longrightarrow H^0(\mathcal{O}_\Gamma(mD + \Gamma))$$

is the zero map. An easy argument using the Leray spectral sequence for φ_D shows that $h^1(\mathcal{O}_V(mD))$, and with it $h^0(\mathcal{O}_\Gamma(mD + \Gamma))$ is bounded by $(\text{const.})m^{n-2}$; thus the Iitaka dimension $\kappa(\Gamma, D_\Gamma) \leqq n-2$, Q.E.D.

The remainder of this section is devoted to some easy but important formal consequences of the definition of canonical singularities.

Proposition (1.7): *Suppose that $\varphi\colon Y \to X$ is a proper morphism, with X and Y normal varieties, and suppose that φ is etale in codimension 1 on Y. Then*

(I) *if X has canonical singularities, so does Y.*

Suppose furthermore that ω_X is locally free; then

(II) *if Y has canonical singularities, so does X.*

Proof: Form a commutative diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{\varphi}} & \tilde{X} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\varphi} & X, \end{array} \qquad (*)$$

with f and g resolutions. Then if $s \in \omega_X^{[r]}$ is a generator, so is $\varphi^* s \in \omega_Y^{[r]}$. For (I) note that if $f^* s$ is regular on $\tilde{X}$ then $g^*(\varphi^* s) = \tilde{\varphi}^* f^* s$ is regular on $\tilde{Y}$.

(II) follows by computing $v_\gamma(s)$ for v_γ a valuation of $k(X)$ in terms of a valuation v_δ of $k(Y)$ lying over v_γ, and having ramification index e:

$$v_\gamma(s) = \frac{1}{e}(v_\delta(\varphi^* s) - r(e-1)); \qquad (**)$$

thus if $r = 1$ and Y is canonical, then $v_\delta(\varphi^* s) \geqq 0$, so that the integer $v_\gamma(s) \geqq 0$.

Remark (1.8): In fact X is canonical if and only if, for every r such that $\omega_X^{[r]}$ is locally free, generated by t, say, $v_\delta(t) \geqq r(e-1)$ for every valuation v_δ of $k(Y)$, where e is the ramification index of v_δ in the field extension $k(Y)/k(X)$. This criterion is of course useless in practice, since one has no hope of finding every relevant valuation v_δ without first resolving X.

A consequence of (II) is that all Gorenstein quotient singularities are canonical. From (I) we get a cyclic covering trick[1], which reduces the study of canonical singularities with $r > 1$ to the $r = 1$ case.

Corollary (1.9): *Let $P \in X$ be a canonical point of index r; then there exists a finite cover $Y \to X$ which is Galois with group $\mathbf{Z}/r$, and which is etale in*

[1] Due independently to J. Wahl.

codimension 1, such that Y is canonical with ω_Y locally free; the construction is defined locally and uniquely determined up to local analytic isomorphism.

PROOF: By means of a local generator, identify $\omega_X^{[r]}$ with $\mathcal{O}_X$, and define in the obvious way an algebra structure on $\mathcal{A} = \mathcal{O}_X \oplus \omega_X^{[1]} \oplus \cdots \oplus \omega_X^{[r-1]}$; the finite $\mathbf{Z}/r$-cover $Y = \operatorname{Spec}_X \mathcal{A} \to X$ is unramified in codimension 1 because $\omega_X^{[a]} \otimes \omega_X^{[b]} \to \omega_X^{[a+b]}$ is an isomorphism in codimension 1; Y is therefore non-singular in codimension 1, and is normal because each of the sheaves $\omega_X^{[a]}$ is divisorial, so that $\mathcal{A}$ is saturated in the sense of (iv) of Proposition 2 of the Appendix. Q.E.D.

Example (1.10). The Galois tower

$$\begin{array}{ccccc} & & \mathbf{A}^3/\boldsymbol{\mu}_2 & & \\ & \nearrow & & \searrow & \\ \mathbf{A}^3 & & & & X = \mathbf{A}^3/\boldsymbol{\mu}_6 \\ & \searrow & & \nearrow & \\ & & \mathbf{A}^3/\boldsymbol{\mu}_3 & & \end{array}$$

(where the actions of the $\boldsymbol{\mu}_k$ are as in Example 1.5, (ii)) shows that a group action which splits as a direct sum, and such that each factor has canonical quotient, need not have canonical quotient. Furthermore, since both X and $\mathbf{A}^3/\boldsymbol{\mu}_2$ have the same index 2, the hypothesis made before (II) in Proposition 1.6 cannot be replaced by "index X = index Y".

Problem (1.11). §2 will give some ideas towards the classification of canonical 3-fold singularities of index 1; together with Corollary 1.9 this puts an upper bound on the problem of classification of singularities of any index r, which can be obtained as quotients of index 1 singularities by a cyclic group $\boldsymbol{\mu}_r$ whose representation on ω_Y is faithful. However, I have no very precise idea as to when such a quotient $Y/\boldsymbol{\mu}_r$ will be canonical, apart from Remark 1.8 and the useful examples in §3.

Canonical varieties satisfy a trivial but important compatibility with taking hyperplane sections. The reader will excuse the following rather obscene digression.

LEMMA (1.12) (Seidenberg): *Let $X \subset \mathbf{P}^N$ be a quasi-projective scheme over any field. If X satisfies Serre's condition S_r then so does its general hyperplane section; if X is a normal variety then so is its general hyperplane section.*

PROOF: Let

$$\begin{array}{ccc} & Z \subset X \times \check{\mathbf{P}}^N & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ X & & \check{\mathbf{P}}^N \end{array} \qquad (***)$$

be the incidence relation $Z = \{(x, h) \mid x \in h\}$. $p_1 : Z \to X$ is a $\mathbf{P}^{N-1}$-fibre bundle, so that X satisfies S_r implies Z satisfies S_r. If $\eta \in \check{\mathbf{P}}^N$ is the generic point then the generic fibre Z_η of p_2 is also S_r, because the local rings of Z_η are particular local rings of Z; I conclude by E.G.A. IV$_3$, Proposition (9.9.2), (viii). For the final part one uses Serre's criterion for normality and the trivial Bertini theorem.

THEOREM (1.13): *If X has canonical singularities then so has its general hyperplane section.*

PROOF: Return to the incidence diagram (***) above. Since p_1 is a $\mathbf{P}^{N-1}$-fibre bundle, Z is canonical; the general hyperplane section Y of X is the general fibre of p_2: $Z \to \check{\mathbf{P}}^N$. Now let f: $\tilde{Z} \to Z$ be any resolution; the general fibre $\tilde{Y}$ of $g = p_2 \circ f$ is a resolution of Y (by Bertini's theorem), and Y is canonical by standard adjunction considerations.

COROLLARY (1.14): *Let X be a 3-fold with canonical singularities. Then with the exception of at most a finite number of "dissident" points $P \in X$, every point has an analytic neighbourhood which is (non-singular or) isomorphic to a Du Val surface singularity $\times \mathbf{A}^1$.*

The point is just that the Du Val singularities have no moduli.

The results of §5 on global properties of canonical 3-folds are consequences of this result, and do not depend on the attempts to classify canonical 3-fold singularities in §§2–4.

Appendix to §1; Weil divisors, divisorial sheaves and $\omega_X^{[r]}$

This section is intended to be complementary to Hartshorne's book Algebraic Geometry, II.6 and III.7.

Let X be a quasi-projective variety defined over a field k, and let $k(X)$ be its function field. Until further notice X is assumed normal. A *prime divisor* of X is an irreducible subvariety of codimension 1.

THEOREM (1): (i) *For every prime divisor Γ the local ring $\mathcal{O}_{X,\Gamma}$ is a discrete valuation ring, with valuation v_Γ: $k(X) \to \mathbf{Z} \cup \{-\infty\}$;*

(ii) *$\mathcal{O}_X = \cap\, \mathcal{O}_{X,\Gamma}$, in the following 2 senses:*

(a) *for all $P \in X$, $\mathcal{O}_{X,P} = \bigcap_{P \in \Gamma} \mathcal{O}_{X,\Gamma}$;*

(b) *for all open $U \subset X$, $\Gamma(U, \mathcal{O}_X) = \bigcap_{\Gamma \in U} \mathcal{O}_{X,\Gamma}$.*

PROOF: [8], p. 124.

Sections of $\mathcal{O}_X$ are thus rational functions $f \in k(X)$ which are regular along each prime divisor; this is an algebraic form of Hartog's lemma.

Let $\mathcal{L}$ be a coherent sheaf of $\mathcal{O}_X$-modules which is torsion-free and of rank 1. The generic stalk $\mathcal{L} \otimes k(X)$ is a 1-dimensional vector space over

$k(X)$, so that choosing a basis identifies $\mathscr{L}$ with a subsheaf of the constant sheaf $k(X)$.

Proposition (2): *Equivalent conditions*:

(i) $\mathscr{L} = \mathscr{L}^{**}$, *where for a sheaf of* $\mathcal{O}_X$*-modules* $\mathscr{F}$, $\mathscr{F}^*$ *denotes the dual* $\mathscr{F}^* = \mathcal{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathcal{O}_X)$;

(ii) $\mathscr{L} = \bigcap \mathscr{L}_\Gamma$ *in the sense of (ii) above*;

(iii) $\mathrm{Ass}(k(X)/\mathscr{L}) = \{\text{Prime divisors of } X\}$;

(iv) *for every inclusion* $\mathscr{L} \subset \mathscr{M}$ *where* $\mathscr{M}$ *is a torsion-free* $\mathcal{O}_X$*-module and* $\mathrm{Supp}(\mathscr{M}/\mathscr{L})$ *has codimension* $\geqq 2$, $\mathscr{L} = \mathscr{M}$;

(v) *if* $X^0 \subset X$ *is a non-singular open subvariety such that* $X \backslash X^0$ *has codimension* $\geqq 2$, *then* $\mathscr{L}_{|X^0}$ *is invertible, and* $\mathscr{L} = j_*(\mathscr{L}_{|X^0})$, *where j denotes the inclusion.*

Proof: [7], §4, no. 2, Theorem 2.

A sheaf satisfying these conditions is called *divisorial*; a *Weil divisor* is defined as a formal sum $D = \sum n_\Gamma \Gamma$, with Γ prime divisors, $n_\Gamma \in \mathbf{Z}$, and almost all $n_\Gamma = 0$. If D is a Weil divisor, the subsheaf $\mathcal{O}_X(D) \subset k(X)$ is defined by

$$\Gamma(U, \mathcal{O}_X(D)) = \{f \in k(X) \mid v_\Gamma(f) \geqq -n_\Gamma \text{ for all } \Gamma \in U\}.$$

Exactly as for Cartier divisors one has:

Theorem (3): *The correspondence* $D \mapsto \mathcal{O}_X(D)$ *defines a bijection*

$$\begin{Bmatrix}\text{Weil divisors}\\ \text{on } X\end{Bmatrix} \overset{\text{bij}}{\longleftrightarrow} \begin{Bmatrix}\text{divisorial sub-}\\ \text{sheaves } \mathscr{L} \subset k(X)\end{Bmatrix} / \Gamma(X, \mathcal{O}_X)^*.$$

Proof: [7], §1, no. 3, Theorem 2.

Lemma (4): *Let* $P \in X$, *and let D be a Weil divisor on X. Then equivalent conditions*:

(i) $\mathcal{O}_X(D)$ *is invertible at P*;

(ii) *there exists an* $f \in k(X)$ *such that* $v_\Gamma(f) = -n_\Gamma$ *for every prime divisor* Γ *with* $P \in \Gamma$;

(iii) *there exists a neighbourhood U of P, and a section* $s \in \Gamma(U, \mathcal{O}_X(D))$ *such that s generates* $\mathcal{O}_X(D)$ *over an open subset* $V \subset U$ *with codimension* $(U \backslash V) \geqq 2$.

Proof: Trivial.

Such a divisor D is called *principal* at P, or a *Cartier divisor* at P. Locally one can always choose a Cartier divisor $E \geqq D$, so that $\mathcal{O}_X(D)$ has an expression $\mathcal{O}_X(D) = \mathscr{I}_{E-D} \cdot \mathcal{O}_X(E)$ as a product of a divisorial ideal sheaf $\mathscr{I}_{E-D}$ with an invertible sheaf $\mathcal{O}_X(E)$; this holds globally if, as here, X is assumed to be quasi-projective.

Remark (5): It may well happen that $\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2) \neq \mathcal{O}_X(D_1 + D_2)$; the left-hand side may have torsion, and it may not map onto the right-hand side either: let X be the quadric cone $X = (x^2 - yz = 0) \subset \mathbf{A}^3$, and let D be the line $(x = y = 0)$; then $\mathcal{O}_X(-D) = \mathscr{I}_D$, and $\mathcal{O}_X(-D) \otimes \mathcal{O}_X(-D) \twoheadrightarrow \mathscr{I}_D^2$, which is clearly not the same as the principal ideal $y \cdot \mathcal{O}_X = \mathscr{I}_{2D}$.

It will however always be true that

$$(\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2))^{**} = \mathcal{O}_X(D_1 + D_2),$$

and this process of taking the product, and then the double dual, is similar to the procedure of taking the "symbolic power" of a prime ideal in the theory of primary decomposition.

In the remainder of this appendix I will show how to define a divisorial sheaf $\omega_X = \mathcal{O}_X(K_X)$, and set $\omega_X^{[r]} = \mathcal{O}_X(rK_X) = (\omega_X^{\otimes r})^{**}$.

Suppose now that $X \subset \mathbf{P}^N$ is an irreducible n-dimensional variety, *not necessarily normal.* Set

$$\omega_X = \mathcal{E}xt^{N-n}_{\mathcal{O}_{\mathbf{P}^N}}(\mathcal{O}_X, \omega_{\mathbf{P}^N}),$$

where $\omega_{\mathbf{P}^N} = \Omega^N_{\mathbf{P}^N} \cong \mathcal{O}_{\mathbf{P}^N}(-N-1)$; compare [9], p. 1.

Now let $X^0 \subset X$ denote the non-singular locus.

Proposition (6): $\omega_X | X^0 = \Omega^n_X | X^0$.

Proof: [9], p. 14 (the two sides can be calculated by means of an identical adjunction procedure).

Theorem (7): ω_X *is a torsion-free sheaf of rank* 1, *satisfying the saturation condition (iv) of Proposition* 2; *in particular, if X is normal,* ω_X *is a divisorial sheaf.*

Proof (compare [9], p. 8): (a) ω_X has rank 1 at the generic point, according to Proposition 6.

(b) ω_X is torsion-free; for if $\mathscr{F} \subset \omega_X$ is a torsion part, $\dim(\mathrm{Supp}\, \mathscr{F}) \leqq n-1$, so that $H^n(\mathscr{F}) = 0$, and hence dually $\mathrm{Hom}(\mathscr{F}, \omega_X) = 0$, so that $\mathscr{F} = 0$.

(c) Let $\omega_X \subset \mathscr{F}$, with $\dim(\mathrm{Supp}\, \mathscr{F}/\omega_X) \leqq n-2$; then $H^{n-1}(\mathscr{F}/\omega_X) = 0$, and hence $H^n(\omega_X) = H^n(\mathscr{F}) \cong k$. By duality I obtain a non-zero element of $\mathrm{Hom}(\mathscr{F}, \omega_X)$, which provides a splitting of the exact sequence

$$0 \longrightarrow \omega_X \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}/\omega_X \longrightarrow 0,$$

and since $\mathscr{F}$ was supposed torsion free, $\mathscr{F}/\omega_X = 0$. Q.E.D.

Corollary (8): *Suppose that X is normal; then*

$$\omega_X = (\Omega^n_X)^{**} = j_*(\omega_{X^0}) = \Omega_X = \mathcal{O}_X(K_X),$$

where (i) $j\colon X^0 \hookrightarrow X$ *is the inclusion of the smooth locus, or more generally of any smooth part of X having complement of codimension* $\geqq 2$;

(ii) *Ω_X is the sheaf of Zariski differentials regular in codimension* 1 (see [10], Proposition 8.7):

$$\Gamma(U, \Omega_X) = \{s \in \Omega^n_{k(X)} \mid s \in \Omega^n_{X,\Gamma} \text{ for all prime divisors } \Gamma \subset U\};$$

and (iii) *K_X is the Weil divisor class corresponding to the sheaf ω_X under Theorem* 3 *above, or the class constructed below.*

For each prime divisor Γ of X the stalk $\Omega^n_{X,\Gamma}$ is the $\mathcal{O}_{X,\Gamma}$-module generated by $s_\Gamma = dx_1 \wedge \cdots \wedge dx_n$, where x_1 is any local parameter of $\mathcal{O}_{X,\Gamma}$, and $x_2, \ldots, x_n \in \mathcal{O}_{X,\Gamma}$ are elements whose residues in $k(\Gamma)$ form a separating transcendence basis. Thus for any rational differential $s \in \Omega^n_{k(X)}$ there is a unique integer $v_\Gamma(s)$ such that

$$s = (\text{unit}) \cdot x_1^{v_\Gamma(s)} \cdot s_\Gamma,$$

and the *divisor* of s is the finite sum

$$(s) = \sum v_\Gamma(s)\Gamma;$$

then $K_X \sim (s)$.

As a coda to this appendix I include two remarks on the non-normal case.

Firstly, the structure sheaf $\mathcal{O}_X$ has a natural saturation in the sense of condition (iv), Proposition 2, consisting of rational functions $f \in k(X)$ which belong to $\mathcal{O}_X$ in codimension 1; it is natural to call this the S_2-isation, $S_2(\mathcal{O}_X)$,

$$\mathcal{O}_X \subset S_2(\mathcal{O}_X) \subset \bar{\mathcal{O}}_X,$$

since $S_2(X) = \operatorname{Spec}(S_2(\mathcal{O}_X)) \xrightarrow{\pi} X$ is the unique finite morphism which is an isomorphism in codimension 1, and such that $S_2(X)$ satisfies Serre's condition S_2. A consequence of Theorem 7 is that ω_X is an $S_2(\mathcal{O}_X)$-module and coincides with $\pi_*\omega_{S_2(X)}$. Thus in discussing ω_X there is little loss of generality in assuming that X satisfies S_2.

Secondly, there is a sense in which the computation of ω_C for a curve C in terms of Rosenlicht differentials (see [11], p. 76) determines ω_X on any quasi-projective variety. To be precise, X has a linear section which is a reduced curve C (Lemma 1.12), and such that at each point $P \in C$ the equations $x_1, \ldots, x_{n-1}$ of the linear section $C \subset X$ form a regular sequence. In a neighbourhood of $P \in X$ one can then construct an isomorphism

$$\omega_X \otimes \mathcal{O}_C \xrightarrow{\sim} \omega_C, \qquad (*)$$

denoted $s \longmapsto \operatorname{Res}_C\left(\dfrac{s}{\prod x_i}\right)$,

and constructed as follows: let $X = X_n \supset X_{n-1} \supset \cdots \supset X_1 = C$ be the chain

of local divisors defined in turn by $x_1, \ldots, x_{n-1}$; each X_{i-1} is a Cartier divisor in X_i, locally defined by $x_{n-i} = 0$. Then the construction of [9], p. 7 provides a "residue" isomorphism

$$\omega_{X_i}(X_{i-1}) \otimes \mathcal{O}_{X_{i-1}} \xrightarrow{\sim} \omega_{X_{i-1}},$$

which can be fitted together into an isomorphism

$$\operatorname{Res}_C\colon \omega_X(D) \otimes \mathcal{O}_C \xrightarrow{\sim} \omega_C,$$

where D is the Cartier divisor defined by $x_1 \cdot x_2 \cdots x_{n-1}$. (*) is then obtained by composing this with $\omega_X \to \omega_X(D)$, given by $s \mapsto \dfrac{s}{\prod x_i}$. It is then easily seen that ω_X is uniquely determined as the sheaf of rational differentials on X whose residue (in the sense of (*)) belongs to ω_C for every sufficiently general section C.

§2. Inductive treatment of 3-fold rational Gorenstein points

DEFINITION (2.1): A point $P \in X$ of a 3-fold is called a *compound Du Val point* if for some section H through P, $P \in H$ is a Du Val singularity. Equivalently, $P \in X$ is *cDV* if it is locally analytically isomorphic to the hypersurface singularity given by

$$f + tg = 0,$$

where $f \in k[x, y, z]$ is one of the polynomials listed in Table 0.2, and $g \in k[x, y, z, t]$ is arbitrary.

A cDV point point may be isolated or otherwise. It will be shown below that it must be canonical.

As pointed out in Remark 0.4 (v), the blow-up of a canonical 3-fold point need not be normal, and if it is normal need not be canonical. However, if $f\colon Y \to X$ is any proper birational morphism with Y normal and X canonical of index 1, and if $\omega_Y = f^*\omega_X$ then Y is also canonical (this is an easy consequence of Proposition 1.2 (I)). For general $f\colon Y \to X$ $f^*\omega_X = \omega_Y(-\Delta)$ (or $K_Y = f^*K_X + \Delta$), with $\Delta = \Delta(f) \geq 0$ a Weil divisor, the *discrepancy* of f; a prime divisor of $f^{-1}p$ which occurs in Δ with strictly positive coefficient is called *discrepant*, and one not occuring in Δ is called *crepant*.

THEOREM (2.2): *If $P \in X$ is a canonical point of index* 1 *which is not cDV then there exists a proper birational morphism $f\colon Y \to X$ with*

(i) *f is crepant, that is $f^*\omega_X = \omega_Y$, and*

(ii) *$f^{-1}P$ contains at least one prime divisor of Y.*

This theorem will be proved here using the fact [36] that $P \in X$ is

Cohen–Macaulay; on the other hand, Shepherd–Barron (who gave a proof under extra conditions) points out that the result implies that $P \in X$ is Cohen–Macaulay, using the Grauert and Riemenschneider vanishing theorem [27].

The fact that the inductive process must terminate follows from this easy result:

LEMMA (2.3): *Let $P \in X$ be a canonical point of index 1; as $f: Y \to X$ runs through all proper birational morphisms to X, the number of crepant prime divisors of Y is bounded.*

PROOF: Let $\pi: \tilde{X} \to X$ be some resolution; then by Hironaka's resolution theorem every $Y \to X$ can be housed

$$\begin{array}{ccccc} & & \tilde{\tilde{X}} & & \\ & \swarrow & & \searrow & \\ \tilde{X} & & & & Y \\ & \searrow & & \swarrow & \\ & & X & & \end{array}$$

under a blow-up $\tilde{\tilde{X}}$ of $\tilde{X}$. Every crepant prime divisor of Y must then lie under a crepant prime divisor of $\tilde{X}$. But the exceptional divisors of the blow-ups in $\tilde{\tilde{X}} \to \tilde{X}$ are certainly discrepant, so that the crepant prime divisors of Y can be mapped injectively to those of the fixed $\tilde{X}$. Q.E.D.

Recall that a variety X is called *Gorenstein* if it is Cohen–Macaulay and the sheaf ω_X is locally free. I will assume from now on (see [36]) that my index 1 point $P \in X$ is Cohen–Macaulay; this assumption will always be used in the form that a hyperplane section H through P having an isolated singularity is normal. To say that $P \in X$ is Cohen–Macaulay and canonical of index 1 is equivalent to saying that $P \in X$ is rational Gorenstein.

DEFINITION (2.4): A Gorenstein point $P \in X$ of an n-dimensional variety X is *rational* (respectively *elliptic*) if for a resolution $f: Y \to X$ we have $f_*\omega_Y = \omega_X$ (respectively $f_*\omega_Y = m_P \cdot \omega_X$, where m_P is the ideal of P).

(This is equivalent via duality to the cohomological assertion $R^{n-1}f_*\mathcal{O}_Y = 0$ (respectively, is a 1-dimensional k-vector space at P)).

It is convenient to make intrinsic (and generalise slightly) the notion of "a general hyperplane section through P":

DEFINITION (2.5): Let $(\mathcal{O}_{X,P}, m_P)$ be the local ring of a point $P \in X$ of a k-scheme, and let $V \subset m_P$ be a finite-dimensional k-vector space which maps onto m_P/m_P^2 (equivalently, by Nakayama's lemma, V generates the $\mathcal{O}_{X,P}$-ideal m_P); by a *general hyperplane section through P* is meant the subscheme $H \subset X_0$ defined in a suitable open neighbourhood X_0 of P by the ideal $\mathcal{O}_X \cdot v$, where $v \in V$ is a sufficiently general element (that is, v is a k-point of a certain dense Zariski open $U \subset V$).

THEOREM (2.6): (I) *If $P \in X$ is a rational Gorenstein point (with $n = \dim X \geqq 2$) then for a general hyperplane section H through P, $P \in H$ is elliptic or rational Gorenstein*;

(II) *if there exists a hyperplane section H through P such that $P \in H$ is rational Gorenstein then $P \in X$ is rational Gorenstein; in particular cDV points are canonical.*

PROOF: The fact that the Cohen–Macaulay condition passes to and from a hyperplane section is obvious; the fact that ω_X is locally free if and only if ω_H is locally free follows from the residue isomorphism

$$\omega_X(H) \otimes \mathcal{O}_H \xrightarrow{\sim} \omega_H,$$

so that if ω_X is generated by s at P, ω_H is generated by $\mathrm{Res}_H\left(\frac{s}{h}\right)$, where $h \in \mathcal{O}_{X,P}$ is the local equation of H.

For (I), let $\sigma: X_1 \to X$ be the blow-up of $P \in X$, and let $g: Y \to X_1$ be any resolution; by construction of the blow-up $m_P \cdot \mathcal{O}_{X_1}$ is an invertible sheaf of ideals and the same continues to hold for Y, so that $m_P \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E)$. Under these conditions the Cartier divisor E on Y is called the *strong geometric fundamental cycle* of the resolution $f = g \circ \sigma: Y \to X$.

As the hyperplane section H through P runs through any linear system whose local equations generate m_P, $f^*H = L + E$, where L runs through a linear system on Y which is free near $f^{-1}P$. Thus by Bertini's theorem a general L is a resolution of the corresponding $P \in H$:

$$\begin{array}{ccc} f^*H = L + E & \subset & Y \\ \downarrow & & \downarrow f \\ H & \subset & X. \end{array}$$

Since X is canonical, the generator $s \in \omega_X$ remains regular on Y; $\frac{s}{h}$ generates the sheaf $\omega_X(H)$. At any point of Y, h factorises as $h = \ell \cdot e$, where ℓ is a local equation for L, and e one for E. Thus if $a \in m_{X,P}$, $\frac{as}{h} = \frac{as}{\ell e}$, and since a vanishes along E, $\frac{as}{h}$ is a regular section of $\omega_Y(L)$. It follows that for any element $\bar{a} \in m_{H,P}$, the product $\bar{a} \cdot \mathrm{Res}_H\left(\frac{s}{h}\right)$ of $\bar{a}$ with a generator of ω_H remains regular on L, $\bar{a} \cdot \mathrm{Res}_H\left(\frac{s}{h}\right) = \mathrm{Res}_L\left(\frac{as}{h}\right) \in \omega_L$. Thus $m_P \cdot \omega_H \subset f_*\omega_L \subset \omega_H$. Q.E.D.

(II) follows from one of the main results of Elkik [14], Theorem 4, p. 146, once I observe that X is a flat deformation of the variety $H \times \mathbb{A}^1$, which has rational singularities.

LEMMA (2.7): *Let X be an affine variety, and H a hyperplane section of X; then there exists a flat family $\mathfrak{X}\to\mathbb{A}^1$ having fibres $X_t\cong X$ if $t\neq 0$, and $X_0\cong H\times\mathbb{A}^1$.*

PROOF: If $X\subset\mathbb{A}^N$ is given by the ideal $I=I(X)\subset k[T_1,\dots,T_N]$, with H the hyperplane $T_N=0$, let $\varphi\colon k[T_1,\dots,T_N]\to k[S_1,\dots,S_{N+1}]$ be given by $T_i\mapsto S_i$ for $i\leqq N-1$, $T_N\mapsto S_NS_{N+1}$; it is then easy to check that the ideal $J\subset k[S_1,\dots,S_{N+1}]$ generated by $\varphi(I)$ defines a variety $\mathfrak{X}\subset\mathbb{A}^{N+1}$, with a morphism to $\mathbb{A}^1$ given by S_{N+1}, having the required property. Q.E.D.

It seems worthwhile to illustrate Theorem 2.6 with a summary of the low-dimensional cases.

Table (2.8):

dim.	rational Gor.	elliptic Gor.
1	non-sing. point	node or cusp
2	Du Val point	Laufer–Reid
3	this paper	??
4	???	

Theorem 2.6 is extremely strong, due to the fact that elliptic Gorenstein surface singularities form an extremely well-defined and tightly controlled class of singularities; see [13], where they are called "minimally elliptic", or my unpublished manuscript [12]. The following is a summary of some results of [12] and [13]; (see especially [13], Theorem 3.13, p. 1270 and Theorem 3.15, p. 1275).

PROPOSITION (2.9): *One can attach a natural number $k=-Z^2$, $k\geqq 1$ to each elliptic Gorenstein surface point $P\in S$, in such a way that*

(i) if $k\geqq 2$ then $k=\mathrm{mult}_P S$;

(ii) if $k\geqq 3$ then $k=$ minimal embedding dimension $=\dim m_P/m_P^2$; if $k\geqq 3$ then the blow-up $S_1\to S$ of (the reduced point) P in S is a normal surface having only Du Val singularities.

If $k=2$ then $P\in S$ is isomorphic to a hypersurface given by $x^2+f(y,z)=0$, with f a sum of monomials y^az^b of degree $a+b\geqq 4$; if α is the weighting $\alpha(x)=2$, $\alpha(y)=\alpha(z)=1$ then the α-blow-up (see §4) $S_1\to S$ is a normal surface having only Du Val points.

If $k=1$ then $P\in S$ is given by $x^2+y^3+f(y,z)$, where f is a sum of monomials yz^a with $a\geqq 4$ and z^a with $a\geqq 6$; if α is the weighting $\alpha(x)=3$, $\alpha(y)=2$, $\alpha(z)=1$ then the α-blow-up $S_1\to S$ is a normal surface having at most 1 Du Val point.

The given blow-up $\sigma\colon S_1\to S$ has the following effect on the canonical sheaf: $\omega_{S_1}=\sigma^*\omega_S(-Z_1)$, where Z_1 is the geometric fundamental cycle for σ; that is, Z_1 is a Cartier divisor, and for $k\geqq 2$ $m_P\cdot\mathcal{O}_{S_1}=\mathcal{O}_{S_1}(-Z_1)$, so that Z_1 is the strong geometric fundamental cycle. If $k=1$ then there is a point $Q\in Z_1$, non-singular on Z_1 and on S_1 such that $m_P\cdot\mathcal{O}_{S_1}=m_Q^2\cap\mathcal{I}_{Z_1}=m_Q\cdot\mathcal{O}_{S_1}(-Z_1)$.

The assertions about the weighted blow-up of the $k=2$ or $k=1$ points are not in [12] or [13]; morally they should be proved by relating the weighting α to the higher adjunction ideals $\mathcal{I}_n\subset\mathcal{O}_S$ (that is, the ideals $\mathcal{I}_n$ such that $f_*\omega_Y^{\otimes n}=\mathcal{I}_n\cdot\omega_S^{\otimes n}$, where $f\colon Y\to S$ is a resolution), and proving general results about the "relative canonical model" $\mathrm{Proj}_S(\bigoplus\mathcal{I}_n)$. However, as a practical alternative they can be proved case-by-case by performing the α-blow-up on each of the $k=2$ or $k=1$ points, listed in [12] or [13], p. 1290; this amounts to making a projective transformation, one affine piece of which is given by setting

$$k=2\colon\quad x=z^2x_1,\ y=zy_1,\ z=z,$$
$$k=1\colon\quad x=z^3x_1,\ y=z^2y_1,\ z=z,$$

and deleting the unwanted factor z^4 or z^6 from the resulting equation. For the $k=2$ points this can also be described as the ordinary blow-up followed by normalisation.

The assertions about the canonical sheaf and the fundamental cycles follow easily from similar results for the minimal resolution (see [12], p.3.4, and compare [13], Lemma 3.12, p. 1268).

COROLLARY (2.10): *To a rational Gorenstein 3-fold point $P\in X$ one can attach a natural number $k\geqq 0$ such that*

$$k=0\Leftrightarrow P\in X \text{ is a cDV point}\Leftrightarrow\begin{cases}\text{the general section } H \text{ through}\\ P \text{ has a Du Val point } P\in H;\end{cases}$$

$k\geqq 1$: the general section H through P has an elliptic Gorenstein point $P\in H$ with invariant k. In particular,

(i) if $k\geqq 2$ then $k=\mathrm{mult}_P X$;

(ii) if $k\geqq 3$ then $k+1=$ minimal embedding dimension $=\dim m_P/m_P^2$.

If $k=2$ then $P\in X$ is isomorphic to a hypersurface given by $x^2+f(y,z,t)=0$, with f a sum of monomials of degree $\geqq 4$; if $k=1$ then $P\in X$ is given by $x^2+y^3+f(y,z,t)=0$, where $f=yf_1(z,t)+f_2(z,t)$ and f_1 (respectively f_2) is a sum of monomials z^at^b of degree $a+b\geqq 4$ (respectively $\geqq 6$).

The next result is a precise form of Theorem 2.2.

THEOREM (2.11): *Let $P\in X$ be a rational Gorenstein point with invariant $k\geqq 1$, and let $\sigma\colon X_1\to X$ be defined as follows: if $k\geqq 3$, $\sigma\colon X_1\to X$ is the blow-up of (the reduced point) P. If $k=2$ or 1, choose coordinates so that $P\in X$ is the hypersurface point in $\mathbb{A}^4$ given by an equation as in the last sentence of Corollary 2.10; let α be the weighting*

$$k=2\colon\quad \alpha(x)=2,\ \alpha(y)=\alpha(z)=\alpha(t)=1$$
$$\text{or}\quad k=1\colon\quad \alpha(x)=3,\ \alpha(y)=2,\ \alpha(z)=\alpha(t)=1,$$

and let $\sigma\colon X_1\to X$ be the α-blow-up (see §4).

Then X_1 is normal and Cohen–Macaulay, and $\sigma^*\omega_X=\omega_{X_1}$, so that X_1 is again rational Gorenstein.

PROOF: The blow-up $X_1 \to X$ has a geometric fundamental cycle E_1 which is a Cartier divisor; in case $k \geqq 2$, E_1 is a strong fundamental cycle, because σ dominates the blow-up of m_P. If $k = 1$ the reader can check by writing down the equations of the α-blow-up that E_1 is still a Cartier divisor, although now only a weak geometric fundamental cycle (that is, $\mathcal{O}_{X_1}(-E_1) = (m_P \cdot \mathcal{O}_{X_1})^{**}$).

If $H \subset X$ is a sufficiently general section through P then $\sigma^*H = H_1 + E_1$, where H_1 is a Cartier divisor, and the restriction $\sigma: H_1 \to H$ is the standard blow-up of the elliptic Gorenstein point $P \in H$ referred to in Proposition 2.9. Because H_1 is relatively very ample and is itself normal (by Proposition 2.9) it follows that X_1 is normal except possibly at a finite number of points. I now want to prove that X_1 is Cohen–Macaulay at each point of E_1; this is obvious for the $k = 2$ or 1 points, since X_1 remains a hypersurface. For $k \geqq 3$, $P \in X$ is a Gorenstein point having embedding dimension $k+1$ and multiplicity k; it follows from the main result of Sally [20] that the tangent cone E_1 is Cohen–Macaulay, and hence so is X_1.

The assertion $\sigma^*\omega_X = \omega_{X_1}$ is now a simple consequence of the last paragraph of Proposition 2.9 and the technique of proof used in (I) of Theorem 2.6. The equation $h \in m_P$ of the general section H through P splits locally on X_1 as $h = h_1 \cdot e$, where h_1 defines H_1, and e defines E_1. Now $H_1 \to H$ is the standard blow-up, and the restriction to H_1 of the cycle E_1 is the fundamental cycle referred to in Proposition 2.9. Now let $s \in \omega_X$ be a local generator near P; the generator $\mathrm{Res}_H \left(\frac{s}{h}\right) \in \omega_H$, when considered as a rational differential on H_1, generates $\omega_{H_1}(Z_1)$, according to Proposition 2.9. Thus by the adjunction formula $\frac{s}{h}$ must generate $\omega_{X_1}(Z_1 + H_1)$ in a neighbourhood of H_1; but $Z_1 + H_1 = \sigma^*H$ is the divisor defined on X_1 by h, so that s must generate ω_{X_1} in a neighbourhood of H_1. Since H_1 meets every component of $\sigma^{-1}P$, s can have no zeroes on X_1, proving the theorem.

COROLLARY (2.12): *If X is a 3-fold with rational Gorenstein singularities then there exists a partial resolution $f: Y \to X$ which is proper and birational, such that*

(i) *f is crepant, $f^*\omega_X = \omega_Y$;*
(ii) *Y has only cDV singularities.*

I do not wish at present to go into the various interesting questions concerned with resolving *cDV* points; for many purposes it seems natural to leave them alone! However, merely the existence of a crepant $Y \to X$ with Y having only hypersurface singularities implies that the local invariant $(-c_2 \cdot \Delta)$ defined in §5 is zero for $P \in X$ rational Gorenstein (see Corollary 5.6).

PROPOSITION (2.13): *Let $P \in X$ be a rational Gorenstein point with invariant $k \geqq 1$; let $T = T_{X,P}$ be the projectivised tangent cone if $k \geqq 3$, or the α-tangent cone if $k = 2$ or 1. Then T is a (generalised)* del Pezzo surface, *in the sense that it satisfies the following host of conditions:*

(i) *T is a 2-dimensional Gorenstein scheme;*
(ii) *the dual invertible sheaf to ω_T is ample, $\omega_T^{\otimes -1} = \mathcal{O}_T(1)$;*
(iii) *$h^1(\mathcal{O}_T(m)) = 0$ for all m, and*

$$h^0(\mathcal{O}_T(m)) = \begin{cases} 0 \text{ if } m < 0 \\ 1 + k\binom{m}{2} \text{ for } m \geqq 0; \end{cases}$$

(iv) *form the graded ring $R = R(T, \omega_T^{\otimes -1}) = \bigoplus_{m \geq 0} H^0(\mathcal{O}_T(m))$;*

then if $k \geqq 3$, R is generated by its elements of degree 1. If $k = 2$ (respectively $k = 1$) then

$$R = k[x, y, z, t]/f$$

where x, y, z, t and f have the weights 2, 1, 1, 1 *and* 4 (*respectively* 3, 2, 1, 1 *and* 6).

(v) *the reduced irreducible components of T are projectively normal surfaces of degree $a - 1$ or a in $\mathbf{P}^a$, and in particular are either rational or elliptic ruled surfaces.*

Sketch proof. The affine tangent cone remains Gorenstein according to Sally [20]; then T is Gorenstein with $\omega_T = \mathcal{O}_T(m)$ for some m, as follows from the main theorem of Goto and Watanabe [22]. The fact that $m = -1$ then follows from the adjunction formula: $T \subset X_1$, with $\mathcal{O}_X(T) \otimes \mathcal{O}_T = \mathcal{O}_T(-1)$, and $\omega_{X_1} = \sigma^*\omega_X$, so that $\omega_{X_1} \otimes \mathcal{O}_T = \mathcal{O}_T$.

The remaining assertions depend on similar assertions for the tangent cone to an elliptic Gorenstein surface singularity, which follow by considering the minimal resolution, as in [12] or [13]; in particular it is easily seen that every component of the projectivised tangent cone to an elliptic Gorenstein point is a normal rational or elliptic curve, or a nodal or cuspidal rational curve embedded normally. Q.E.D.

COROLLARY (2.14): *Let $P \in X$ be a rational Gorenstein point; then there exists a resolution $f: Y \to X$ such that $f^{-1}P$ is a union of rational and ruled surfaces.*

For the partial resolution $f: Y \to X$ of Corollary 2.12, $f^{-1}P$ is a union of rational and elliptic ruled surfaces by (v) of Proposition 2.13; but Sing Y may contain curves of positive genus above P.

§3. Toric and quotient singularities

In this section I review some notions of toric geometry, and give criteria for toric varieties to be canonical; for more details of the definitions and properties of differentials on toric varieties see [15]. Toric methods have appeared implicitly in the last section in the form of weighted blow-ups, and they will play a crucial part in §4; a more immediate aim is the proof of the following result, which was suggested by some examples of Shepherd-Barron, who also proved the theorem in a particular case.

THEOREM (3.1): *Let $G \subset GL(n, k)$ be a finite group acting linearly on $\mathbf{A}^n$. Suppose that G has no quasi-reflections, so that the map $\mathbf{A}^n \to \mathbf{A}^n/G = X$ is etale in codimension 1. Then X is canonical if and only if for every element $g \in G$ of order r, and ϵ any primitive rth root of 1, the diagonal form of the action of g is*

$$g: x_i \longrightarrow \epsilon^{a_i} x_i, \text{ such that } 0 \leqq a_i < r,$$

with $\Sigma\, a_i \geqq r$.

REMARK (3.2): X is Gorenstein if and only if $\Sigma\, a_i \equiv 0 \mod r$, in which case it is already canonical by Proposition 1.7, (II). (This is a theorem of Watanabe and Khinich, [31] and [32]).

By Remark 1.7, the condition for X to be canonical can be expressed in terms of the ramification of valuations v_δ in the field extension $k(\mathbf{A}^n)/k(X)$; standard ramification theory (see for example [7], p. 284, together with the fact that the ramification group must be cyclic in characteristic 0) then reduces the condition to the cyclic subgroup $R_{v_\delta} \subset G$. Thus for the proof of Theorem 3.1, which I defer to the end of this section (Theorem 3.9), I can assume that G is an Abelian group acting diagonally.

Let $\bar{M} = \mathbf{Z}^n$, and for $m = (m_1, \dots, m_n) \in \bar{M}$ write x^m for the monomial $x^m = \Pi x_i^{m_i} \in k[\mathbf{A}^n]$. The action of a diagonal group G on $\mathbf{A}^n$ is given by a homomorphism

$$\alpha: G \longrightarrow \mathrm{Hom}_{Ab}(\bar{M}, \mathbf{G}_m),$$

so that $g \in G$ acts as $x^m \mapsto \alpha_g(m) \cdot x^m$. The invariant monomials are x^m, with $m \in M$, where $M \subset \bar{M}$ is the sublattice of finite index

$$M = \bigcap_g \mathrm{Ker}\, \alpha_g \subset \bar{M}.$$

Let $\sigma \subset M_{\mathbf{R}}$ be the first quadrant $\sigma = \{m \mid m_i \geqq 0 \text{ for each } i\}$. Then $\mathbf{A}^n = \mathrm{Spec}\, k[\sigma \cap \bar{M}]$, and $X = \mathrm{Spec}\, k[\sigma \cap M]$, with $\mathbf{A}^n \to X$ corresponding to the inclusion of the exponent semigroups $\sigma \cap M \subset \sigma \cap \bar{M}$.

Quotients $\mathbf{A}^n/G$ by an Abelian group acting without quasi-reflections correspond precisely to simplicial toric varieties: if $\sigma \subset M_{\mathbf{R}}$ is a rational simplicial cone, then there exists a unique overlattice $\bar{M} \supset M$ such that $\sigma = \langle e_1, \dots, e_n \rangle$, with $\{e_i\}$ a basis of $\bar{M}$, and such that the following condition holds:

$$\text{for every } i, \ \bar{M} = M + \sum_{j \neq i} \mathbf{Z} \cdot e_j \qquad (*)$$

this condition corresponds to the fact that G has no quasi-reflections.

Now let M be any lattice of rank n, and let $\sigma \subset M_{\mathbf{R}}$ be a cone spanning $M_{\mathbf{R}}$; set $X = \mathrm{Spec}\ k[\sigma \cap M]$. X contains a big torus, $\mathbf{T} \subset X$, with $\mathbf{T} = \mathrm{Spec}\ k[M] \cong \mathbf{G}_m^n$. For every wall $\tau \subset \sigma$, $\sigma - \tau$ is the half-space of $M_{\mathbf{R}}$ containing σ and bounded by τ, and the corresponding variety $X^\tau = \mathrm{Spec}\, k[(\sigma - \tau) \cap M]$ is isomorphic to $\mathbf{A}^1 \times \mathbf{G}_m^{n-1}$, with $\mathbf{T} \subset X^\tau \subset X$. The complement $X \backslash \mathbf{T}$ is a union of prime divisors Γ_τ, and the generic point of Γ_τ corresponds to the last remaining coordinate hyperplane in $X^\tau \cong \mathbf{A}^1 \times \mathbf{G}_m^{n-1}$. Thus to check the regularity of a differential on X it is sufficient to know it on $\mathbf{T}$, and to check at each prime divisor $\Gamma_\tau \subset X^\tau$.

For $\{m_1, \dots, m_n\} \subset M$ a linearly independent set, the rational differential

$$s = \frac{\pm 1}{[M : \bigoplus \mathbf{Z} \cdot m_i]} \cdot \frac{dx^{m_1}}{x^{m_1}} \wedge \cdots \wedge \frac{dx^{m_n}}{x^{m_n}}$$

does not depend on the choice of $\{m_1, \dots, m_n\}$, and is a generator of the $\mathcal{O}_{\mathbf{T}}$-module $\omega_{\mathbf{T}}$.

LEMMA (3.3): *$x^m s^r \in \Gamma(X, \omega_X^{[r]})$ if and only if for every $\tau \in \mathrm{Walls}(\sigma)$ we have*

$$m \in r \cdot \mathrm{Int}((\sigma - \tau) \cap M).$$

PROOF: If I set $M_2 = \mathrm{Span}(\tau) \cap M$, and let $m_1 \in (\sigma - \tau) \cap M$ be a complementary element, then the semigroup $(\sigma - \tau) \cap M$ decomposes as $\langle m_1 \rangle \times M_2$. The discrete valuation ring $\mathcal{O}_{X, \Gamma_\tau}$ then has x^{m_1} as a local parameter, and x^m is a unit for $m \in M_2$. Thus taking a basis $m_2, \dots, m_n$ of M_2, I can write

$$s = \frac{dx^{m_1}}{x^{m_1}} \wedge \cdots \wedge \frac{dx^{m_n}}{x^{m_n}};$$

thus $x^m s^r$ is regular along Γ_τ if and only if $m - rm_1 \in (\sigma - \tau)$. Q.E.D.

Write

$$r\text{-}\mathrm{Int}(\sigma \cap M) = \bigcap_{\tau \in \mathrm{Walls}(\sigma)} r \cdot \mathrm{Int}((\sigma - \tau) \cap M);$$

note that for $r = 1$, $1\text{-}\mathrm{Int}(\sigma \cap M) = \mathrm{Int}(\sigma) \cap M$. Let $\mathfrak{A}^{[r]} \subset k[\sigma \cap M]$ be the ideal generated by x^m, $m \in r\text{-}\mathrm{Int}(\sigma \cap M)$. Then the map

$$\mathfrak{A}^{[r]} \longrightarrow \Gamma(X, \omega_X^{[r]})$$

given by $x^m \longrightarrow x^m s^r$

is an isomorphism. Compare Danilov [15], §4.

COROLLARY (3.4): *$\omega_X^{[r]}$ is locally free if and only if the semigroup ideal $r\text{-}\mathrm{Int}(\sigma \cap M) \subset \sigma \cap M$ is principal.*

Let $\sigma' = \langle f_1, \dots, f_n \rangle$ be any basic cone with $\sigma' \supset \sigma$; then $X' = \operatorname{Spec} k[\sigma' \cap M] \cong \mathbf{A}^n$, and has a birational morphism $X' \to X$ defined by the inclusion $\sigma \cap M \subset \sigma' \cap M$. A resolution $f: Y \to X$ can be made by glueing together such affine constructions as σ' runs through a fan of cones (see [15], II, §8). Since for a basic cone $\sigma' = \langle f_1, \dots, f_n \rangle$, $r\text{-Int}(\sigma' \cap M)$ is the principal ideal generated by $r(f_1 + \cdots + f_n)$, I get the following result.

Corollary (3.5): *X satisfies the condition (ii') of Remark 1.3 if and only if, for every $r \geqq 1$, and for every basic cone $\sigma' = \langle f_1, \dots, f_n \rangle$ with $\sigma' \supset \sigma$ we have*

$$r\text{-Int}(\sigma \cap M) \subset r\text{-Int}(\sigma' \cap M) = r(f_1 + \cdots + f_n) + \sigma' \cap M.$$

Since for $r = 1$ this amounts to $\operatorname{Int}(\sigma \cap M) \subset \operatorname{Int}(\sigma' \cap M)$, which is trivially satisfied, the next result follows.

Corollary (3.6): *If X is toric then for every proper birational morphism $f: Y \to X$, $f_* \omega_Y = \omega_X$. In particular if X is Gorenstein then it is canonical.*

This also follows from the fact that toric varieties are Cohen–Macaulay ([15], §3) and rational ([15], §8) by using Kempf's duality argument ([6], p. 50).

Now assume that σ is simplicial[1], and let $M \subset \bar{M} = \mathbf{Z}^n$ be the overlattice in which σ becomes basic, $\sigma = \langle e_1, \dots, e_n \rangle$, with $\{e_i\}$ a basis of $\bar{M}$ and condition (*) satisfied.

Lemma (3.7): *$\omega_X^{[r]}$ is locally free if and only if $r(e_1 + \cdots + e_n) \in M$.*

Proof: σ has walls τ_i given by $m_i = 0$. Furthermore, according to (*) each $\operatorname{Int}((\sigma - \tau_i) \cap M)$ contains an element $e_i + \sum_{j \neq i} a_j e_j$; by adding an element of $\tau_i \cap M$ I can even assume that each $a_j \geqq N$ for any chosen $N \in \mathbf{Z}$, so that for each i, $r\text{-Int}(\sigma \cap M)$ contains a vector $re_i + \sum_{j \neq i} b_j e_j$. If $m = \sum m_i e_i$ is a generator for the ideal $r\text{-Int}(\sigma \cap M)$ it follows that each $m_i \leqq r$; the inequality the other way is trivial, so that the only possible single generator of $r\text{-Int}(\sigma \cap M)$ is $r(e_1 + \cdots + e_n)$. Q.E.D.

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[1] The general case is covered by the following result, kindly communicated by Danilov.
Theorem: Let M be a lattice, $\sigma \subset M_{\mathbf{R}}$ a cone, and set $X = \operatorname{Spec} k[\sigma \cap M]$. Let N be the dual lattice, and $\check{\sigma} \subset N_{\mathbf{R}}$ the dual cone; write $1\text{-}Sk\,\check{\sigma}$ for the set of primitive $\alpha \in \check{\sigma} \cap N$ such that $\tau_\alpha = (\alpha = 0)$ is a wall of σ. Then

(I) $\omega_X^{[r]}$ is locally free for some r if and only if there exists $f \in M_{\mathbf{Q}}$ such that, considering f as a linear map $f: N \to \mathbf{Q}$,

$$f_{|1\text{-}Sk\,\check{\sigma}} = 1;$$

(II) X is canonical if and only if furthermore

$$f_{|\check{\sigma} \cap N \setminus \{0\}} \geqq 1.$$

The proof is similar to that of 3.7–3.9.

Corollary (3.8): *Suppose that $\omega_X^{[r]}$ is locally free; then X is canonical if and only if, for every basic cone $\sigma' = \langle f_1, \dots, f_n \rangle \supset \sigma$, with $\{f_i\}$ a basis of M, we have*

$$(e_1 + \cdots + e_n) - (f_1 + \cdots + f_n) \in \sigma'.$$

Proof: The condition in Corollary 3.5 can be rewritten

$$r(e_1 + \cdots + e_n) + \sigma \cap M \subset r(f_1 + \cdots + f_n) + \sigma' \cap M. \qquad \text{Q.E.D.}$$

Now let $N = M^*$ be the lattice dual to M; N consists of linear forms $\alpha(m) = \sum q_i m_i$, with $q_i \in \mathbf{Q}$ and $\alpha(m) \in \mathbf{Z}$ for every $m \in M$. Dual to σ we have the positive quadrant $\check{\sigma} = \{\alpha \mid q_i \geqq 0 \text{ for each } i\}$.

The following criterion is equivalent to Theorem 3.1 for Abelian G.

Theorem (3.9): *X is canonical if and only if for every non-zero $\alpha \in \check{\sigma} \cap N$ we have $\alpha(e_1 + \cdots + e_n) = \sum q_i \geqq 1$.*

Of course this condition need only be tested on primitive α in the unit cube.

Proof: Given any primitive vector $\alpha \in \check{\sigma} \cap N$, I can extend it to a basis $\alpha = f_1^*, \dots, f_n^*$ of N lying in $\check{\sigma}$. The dual basis $f_1, \dots, f_n$ spans a basic cone $\sigma' \supset \sigma$, and every such basic cone σ' arises in this way. But $\sum e_i - \sum f_i \in \sigma'$ is the assertion that for each i we have $f_i^*(\sum e_i - \sum f_i) \geqq 0$; that is, $f_i^*(e_1 + \cdots + e_n) \geqq 1$. Q.E.D.

Example (3.10): The "Shepherd–Barron node"[1] $X_r = \mathbf{A}^3/\boldsymbol{\mu}_r$, where $\rho \in \boldsymbol{\mu}_r$ acts by

$$(x, y, z) \longmapsto (\rho x, \rho y, \rho^{r-1} z),$$

is a canonical singularity of index r; for $\rho = \epsilon^k$ acts with eigenvalues ϵ^k, ϵ^k, ϵ^{r-k}, and $k + k + (r - k) \geqq r$.

These singularities actually occur as the only singularities of a general weighted hypersurface $X_{dr(r-1)} \subset \mathbf{P}(1, 1, r-1, r, r)$; (see [17] for the techniques needed to justify this assertion). This is a 3-fold with canonical singularities, and $\omega_X = \mathcal{O}_X(k)$, with $k = dr^2 - dr - 3r - 1$; if $k \geqq 1$, X is a canonical 3-fold; we have

$$K_X^3 = dr(r-1)k^3/r^2(r-1) = dk^3/r,$$

and since $k \equiv -1 \mod r$, the invariant K_X^3 defined in §5 is a rational number which can have arbitrary denominator.

Problem (3.11): Give necessary and sufficient combinatorial conditions on a sequence of integers $(b_1, \dots, b_k; a_1, \dots, a_{k+n+1})$ for the general weighted

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[1] Considered independently by J. Wahl.

complete intersection

$$X_{b_1,\dots,b_k} \subset \mathbf{P}(a_1, \dots, a_{k+n+1})$$

to have canonical singularities. This condition might resemble (***) in Theorem 4.5.

§4. Hypersurfaces and quasi-homogeneity

Let $X \subset \mathbf{A}^n$ be a hypersurface, $P \in X$, and let $x_1, \dots, x_n$ be analytic coordinates on $\mathbf{A}^n$ around P; near P, $X \subset \mathbf{A}^n$ is given by an equation $g = g(x)$.

I will use the following notations: $M = \mathbf{Z}^n$, with $\{e_i\}$ the natural basis; $m \in M$ corresponds to, and is sometimes identified with, the monomial x^m, with $x^{e_i} = x_i$. The first quadrant is $\sigma \subset M_{\mathbf{R}}$, N is the dual lattice to M, $\check{\sigma} \subset N_{\mathbf{R}}$ is the dual first quadrant. For $m = \sum m_i e_i \in M$ and $\alpha = (q_1, \dots, q_n) \in N_{\mathbf{Q}}$, $\alpha(m) = \sum m_i q_i$. I will abuse the notation by writing $\alpha(x^m) = \alpha(m)$ for a monomial x^m, and extend the definition of α to the whole of $k[x_1, \dots, x_n]$ by setting, for $g = \sum a_m x^m$,

$$\alpha(g) = \inf\{\alpha(m) \mid a_m \neq 0\}.$$

For example, if $\alpha = e_i^*$ then $\alpha(g) = 1$ if and only if g vanishes along the coordinate hyperplane $x_i = 0$ to multiplicity 1.

THEOREM (4.1): *The following is a necessary condition for X to have canonical singularities:*

$$(*)\quad \begin{cases} \text{for all } P \in X, \\ \text{for all analytic coordinates } x_1, \dots, x_n \text{ around } P, \\ \text{for all } \alpha \in \check{\sigma} \cap N_{\mathbf{Q}}, \text{ with } \alpha \neq q e_i^*, \\ \alpha\left(\dfrac{x_1 \ \dots \ x_n}{g}\right) = \sum q_i - \alpha(g) > 0. \end{cases}$$

It is often useful to make an obvious normalisation, and to assume that $\sum q_i = 1$; I will occasionally assume without warning that $q_1 \geqq \dots \geqq q_n$.

CONJECTURE (4.2): *The condition (*) in Theorem 4.1 is also sufficient, that is*

$$\left.\begin{array}{l} P \in X \text{ is} \\ \text{non-rational} \end{array}\right\} \Leftrightarrow \begin{cases} \text{there exist analytic coordinates,} \\ \text{there exists } \alpha \in \check{\sigma} \cap N_{\mathbf{Q}},\ \alpha \neq e_i^*, \text{ with } \alpha(e_1 + \dots + e_n) = 1, \\ \text{such that } \alpha(g) \geqq 1. \end{cases}$$

The condition certainly implies that X is "naïvely canonical" in the sense that X has multiplicity $\operatorname{mult}_Y X < r$ along every subvariety $Y \subset X$ of dimension $\dim Y = n - r$ (for any $r \geqq 2$); for at a general point of Y, Y can

be given by $x_1 = \dots = x_r = 0$, and setting $\alpha = \frac{1}{r}(e_1^* + \dots + e_r^*)$, the condition $\alpha(g) < 1$ holds if and only if $\operatorname{mult}_Y X < r$.

The next result is a feeble approximation[1] to Conjecture 4.2.

PROPOSITION (4.3): *The hypersurface $X \subset \mathbf{A}^n$ defined by $g = \sum_{i=1}^{n} x_i^{a_i} = 0$ has a canonical singularity at 0 if and only if $\sum \frac{1}{a_i} > 1$.*

The proofs of Theorem 4.1 and Proposition 4.3 are both based on the notion of weighted blow-up, which is a particular case of the toric morphism defined by a subdivision of a fan (see [15], §5). Let $\alpha \in \check{\sigma} \cap N_{\mathbf{Q}}$, and let d be the least denominator of α, so that $d\alpha \in N$. For each $i = 1, \dots, n$ (later for clarity I will take $i = 1$) $\sigma_i \subset M_{\mathbf{R}}$ denotes the cone

$$\sigma_i = \{m \mid \alpha(m) \geqq 0, \text{ and } m_j \geqq 0 \text{ for each } j \neq i\}.$$

If $\alpha = e_i^*$ then $\sigma_i = \sigma$, so that the construction will be trivial. Set $Z_i = \operatorname{Spec} k[\sigma_i \cap M]$, and let $\varphi_i \colon Z_i \to \mathbf{A}^n$ be the birational map corresponding to $\sigma \cap M \subset \sigma_i \cap M$; for $i \leqq 1 \leqq n$, the φ_i glue together into a projective morphism $\varphi \colon Z \to \mathbf{A}^n$, the α-blow-up of $\mathbf{A}^n$.

It is easy to give a toric description of the weighted projective space $\mathbf{P}(\alpha)$, and to check that Z is none other than the normalised graph of the rational map $\mathbf{A}^n \dashrightarrow \mathbf{P}(\alpha)$ which makes $\mathbf{A}^n \backslash \{0\}$ into a $\mathbf{G}_m$-bundle.

Write $E \subset Z$ for the exceptional locus $E = \varphi^{-1} 0$ ($\varphi \colon E \to \mathbf{P}(\alpha)$ is a finite cover); in Z_1, $E_1 = E \cap Z_1$ is the stratum of Z_1 corresponding to the wall $\tau_1^{(0)} \subset \sigma_1$ given by $\alpha = 0$ (see [15], §2.5). If $m_0 \in M$ is such that $d\alpha(m_0) = 1$ then x^{m_0} is a local parameter of the discrete valuation ring $\mathcal{O}_{Z,E}$, and $v_E(x_i) = d\alpha(e_i)$. Note that the reduced E is not necessarily a Cartier divisor, although in Z_1, x_1 is a local equation for $d\alpha(x_1) \cdot E_1$.

Z contains 2 kinds of coordinate hyperplanes: the proper transforms $(x_i = 0) \subset \mathbf{A}^n$, which is $Z \backslash Z_i$, and E itself; the intersection $\bigcap_i Z_i = Z_{(0)}$ is a neighbourhood of the generic point of E, and any monomial x^m with $\alpha(m) = 0$ becomes a unit when restricted to $Z_{(0)}$.

Now let $d\alpha(g) = c$, and suppose that X is irreducible and not contained in any coordinate hyperplane of $\mathbf{A}^n$; if I write $g = (x^{m_0})^c g'$ then g' is a unit in $\mathcal{O}_{Z,E}$, and defines the proper transform X' of X in $Z_{(0)}$:

$$\varphi^* X = X' + cE.$$

X' will in general not be a Cartier divisor on the whole of Z.

PROOF OF THEOREM 4.1: I will assume that $\alpha(e_1 + \dots + e_n) = 1$ but

[1] The method of proof given here also proves the conjecture if g is non-degenerate with respect to its Newton polygon, in the sense of Koushnirenko[38]; compare[37], Theorem 2.3.1.

$\alpha(g) \geqq 1$, that is $c \geqq d$, and deduce that X is not canonical. Let s be the usual basis of ω_T (as in §3), so that ω_{A^n} is based by $x_1 \dots x_n \cdot s$ and $t = g^{-1} \cdot x_1 \dots x_n \cdot s$ generates $\omega_{A^n}(X)$. If I show that $\varphi^* t$ has E as a pole on Z when considered as a rational section of $\omega_Z(X')$, then X is not canonical; indeed, t will be of the form $t = u \cdot v^{-1}$, where $u \in \omega_Z(X')$ is a basis and $v \in \mathcal{O}_Z$ has a zero along E. The Poincaré residue of t is then a product of $\mathrm{Res}_{X'} u$, which bases $\omega_{X'}$ by the adjunction formula ([9], p. 7), and the restriction of v^{-1}, where $v \in \mathcal{O}_{X'}$ is a non-unit; note that this argument does not assume that X' is normal.

ω_Z is generated near E by $x^{m_0} \cdot s$, so that $x_1 \dots x_n \cdot s$, considered as a differential on Z, has divisor of zeroes eE, where $e = d\alpha(x_1 \dots x_n) - 1 = d - 1$; hence it generates $\omega_Z(-eE)$, and $\varphi^* t$ generates $\omega_Z(\varphi^* X - eE) = \omega_Z(X' + (c - e)E)$. Under the hypothesis $c \geqq d$,

$$c - e = c - (d - 1) > 0,$$

showing that $\varphi^* t$ has a pole. Q.E.D.

Proof of Proposition 4.3: Let $\alpha = \sum \frac{1}{a_i} e_i^*$, and carry out the α-blow-up as above. It will follow from Lemma 4.4 below that X' has the following two virtues:

(i) X' is normal;

(ii) there exists a resolution $f\colon Y \to X'$ with $f_* \omega_Y = \omega_{X'}$.

In view of (i) and the computation in the proof of Theorem 4.1, the generator of ω_X, which is the Poincaré residue of t, lifts to a regular differential on X'. Combining this with (ii), X is canonical. It remains to prove the following result.

Lemma (4.4): *X' is toroidal.*

Proof: For each $j \neq 1$ (for clarity I will later take $j = 2$), write

$$a_1 = b_j \gamma_j \text{ and } a_j = c_j \gamma_j,$$

with b_j and c_j coprime integers.

Write $z_j = x_1^{-b_j} x_j^{c_j}$, which is a coordinate function on Z_1. On Z_1, X' is given by

$$x_1^{-a_1} \cdot g = g' = 1 + \sum_{j \neq 1} z_j^{\gamma_j}.$$

It follows that at each point of X' one of the z_i is non-zero, say $z_2 \neq 0$.

Let $Z_{1:2} = \{P \in Z_1 \mid z_2 \neq 0\}$, and $X_2' = X' \cap Z_{1:2}$ be a typical piece of X'; then there is a decomposition $M = M_1 \times M_2$, with M_1 the 1-dimensional lattice based by $(-b_2, c_2, 0, \dots, 0)$, and M_2 a complementary lattice, which induces a decomposition

$$Z_{1:2} \cong \mathbf{G}_m \times Y,$$

with Y the toric variety corresponding to $\sigma_1 \cap M_2$; z_2 is the coordinate in $\mathbf{G}_m$. Now $X_2' \subset Z_{1:2}$ is given by the equation

$$z_2^{\gamma_2} = 1 - \sum_{j \geqq 3} z_j^{\gamma_j},$$

and since on this piece z_2 never vanishes it follows that the restriction of the second projection $Z_{1:2} \to Y$ is an etale morphism $X_2 \to Y$. Q.E.D.

There is no doubt that Conjecture 4.2 is true, at least in the case $n = 4$. Here are two (related) ideas for its proof. Firstly one can make a list (in hierarchical order) of all g which satisfy (*), and show that this list satisfies the inductive property analogous to Theorem 2.6. Indeed, it is easily seen that for any g satisfying (*) and not defining a cDV point, one of the α-blow-ups used in the proof of Theorem 2.6 is appropriate ($\alpha = (1, 1, 1, 1)$ or $(2, 1, 1, 1)$ or $(3, 2, 1, 1)$), and leads to a variety X' which again has only hypersurface singularities; to prove the conjecture one has to show that for any g in our list the singularities of X' are either cDV points or points occurring earlier in the list. Although this is a perfectly feasible program, I have only scratched the surface; apart from the fact that the effort involved in making the list seems to be about 10 times that required for the analogous lists of elliptic surface singularities (see the tables in [12] and [13]), a more serious difficulty is that there do not seem to be any checks to eliminate errors—in the surface case the equations of the singularities and the shape of the resolution (a configuration of curves on a non-singular surface) both fit into nicely controllable hierarchical patterns.

The second possible proof of Conjecture 4.2 is to try to prove directly that in making the appropriate α-blow-up $X' \to X$, condition (*) for X implies (*) for X'; for some fixed set of coordinates on X, and the coordinates on X' resulting from the toric description of the α-blow-up, this is trivial. What is therefore required is some theoretical understanding of which analytic changes of coordinates on X' are relevant to (*), and which of these come from changes of coordinates on X.

I conclude this section with a discussion of the combinatoric condition in (*). This condition is formally similar to the numerical condition for stability of a hypersurface $X \subset \mathbf{P}^n$ of given degree under the action of $PGL(n)$ (see [23], pp. 48 and 80, and also [24]). As in that theory, it should be possible, for combinatorial reasons, to write down a *finite* set $\{\alpha_i\}_{i \in I}$ of $\alpha_i \in \check{\sigma} \cap N_{\mathbf{Q}}$ with $\alpha_i(e_1 + \cdots + e_n) = 1$ which have the same effect as *all* α, that is:

$$(**) \quad \begin{cases} \textit{for all } \alpha \in \check{\sigma} \cap N_{\mathbf{Q}} \textit{ with } \alpha(e_1 + \cdots + e_n) = 1 \\ \textit{there exists an } i \in I \textit{ such that} \\ \alpha(m) \geqq 1 \Rightarrow \alpha_i(m) \geqq 1 \textit{ for all } m \in \sigma \cap M. \end{cases}$$

For $n = 2$ and 3 this blessèd purpose is accomplished by the sets

$$A_2 = \{\tfrac{1}{2}(1, 1)\}.$$

and $$A_3 = \{\tfrac{1}{3}(1, 1, 1), \tfrac{1}{4}(2, 1, 1), \tfrac{1}{6}(3, 2, 1)\} \cup \{\tfrac{1}{2}(1, 1, 0)\};$$

this explains the significance of these weightings for surface rational and elliptic singularities; the first component of A_3 is easily characterised as the set of solutions of $\sum_{i=1}^{3} \frac{1}{a_i} = 1$.

THEOREM (4.5): *For $n = 4$, the following set A_4 satisfies (**):*

$$A_4 = A_4' \cup \{(q_1, q_2, q_3, 0) \mid (q_1, q_2, q_3) \in A_3\},$$

where $A_4' = \{\alpha \in \check{\sigma} \cap N_{\mathbf{Q}} \mid \alpha(e_1 + \cdots + e_4) = 1, \text{ and (***) holds}\}$,

where (***) is the condition

$$(***) \quad \begin{cases} \text{for each } i \text{ there is a monomial } x_i^{a_i} \cdot r_i \text{ with } \alpha(x_i^{a_i} \cdot r_i) = 1, \\ \text{with } a_i \in \mathbf{Z},\ a_i \geqq 2, \text{ and with } r_i \text{ a monomial of degree} \leqq 1; \\ \text{and for each } j \text{ not all of these are divisible by } x_j. \end{cases}$$

In terms of the geometry of the lattice, (***) means that the tetrahedron $(\alpha = 1) \cap \sigma \subset M_{\mathbf{R}}$ has "near-vertices" $x_i^{a_i}$ or $x_i^{a_i}x_j$.

The idea of the proof is that if $\gamma \in \check{\sigma} \cap N_{\mathbf{Q}}$ also has $\gamma(e_1 + \cdots + e_n) = 1$, and if there are *no* monomials with $\alpha(x_i^{a_i} \cdot r) = 1$ and with $\gamma(r) < 1$, then I can replace α by $\alpha' = (1 - \lambda)\alpha + \lambda\gamma$ (with small λ), and have

$$\alpha(m) \geqq 1 \Rightarrow \alpha'(m) \geqq 1; \tag{0}$$

as λ increases we eventually acquire a new monomial $x_i^{a_i} \cdot r$ with $\alpha'(x_i^{a_i}r) = 1$, $\gamma(r) < 1$. Repeating this with a different γ satisfying $\gamma(x_i^{a_i}r) = 1$, we eventually get an α' still satisfying (0), but now forced to have an assortment of solutions of $\alpha'(x_i^{a_i} \cdot r) = 1$ for certain very restricted r, which forces α' to have a solution with $\deg(r) \leqq 1$.

A_4' is a finite set, since the a_i satisfy $a_i q_i \leqq 1$, and hence

$$1 = \sum q_i \leqq \sum \frac{1}{a_i},$$

and for each solution of this inequality (essentially a finite set) there are at most 4^4 $(= 256)$ hyperplanes in $M_{\mathbf{R}}$ spanned by some choice of the monomials $x_i^{a_i}$ or $x_i^{a_i}x_j$, one for each i; some of these hyperplanes will of course fail to pass through $(1, 1, 1, 1)$. A_4' seems to be quite large, with (apparently) 95 elements.

For $\alpha \in A_4'$ I can write $\alpha = \frac{1}{d}(b_1, b_2, b_3, b_4)$, with $b_i \in \mathbf{Z}$ and d the least common denominator of the $q_i = b_i/d$; the condition on α guarantees that a sufficiently general hypersurface

$$X_d \subset \mathbf{P}(b_1, b_2, b_3, b_4),\ \alpha \in A_4'$$

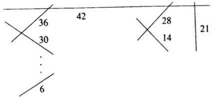
has singularities "not much worse" than those of $\mathbf{P}$ itself (see [17]), and ensures that X_d is a $K3$ surface with at worst Du Val points. A_4' should thus also provide a complete list of all weighted hypersurfaces which are $K3$ surfaces; these can be constructed from a non-singular $K3$ surface S, together with a rational divisor class $h \in \operatorname{Pic} S \otimes \mathbf{Q}$, by a

construction due to Demazure [25]. These projective surfaces have corresponding affine cones, the general weighted hypersurface of weight α, which correspond to "simple elliptic" 3-fold singularities.

Thus Conjecture 4.2 implies that one of the beautiful features of the hierarchy of surface singularities carries over in some form to higher dimensions: lurking on the fringe of the rational singularities there are simple elliptic ones.

Examples (4.6): $\frac{1}{42}(21, 14, 6, 1)$; $\frac{1}{5}(2, 1, 1, 1)$; $\frac{1}{42}(21, 14, 4, 3)$; $\frac{1}{14}(7, 4, 2, 1)$.

$X_{42} \subset \mathbf{P}(21, 14, 6, 1)$ has Du Val points A_6, A_2 and A_1 at the transverse intersection of X_{42} with the 1-dimensional singular strata of $\mathbf{P}$. On the $K3$ resolution, t^{42} defines the following divisor (all the components are rational non-singular with self-intersection -2):

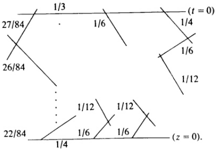


$X_5 \subset \mathbf{P}(2, 1, 1, 1)$ is a divisor in the cone on the Veronese $\mathbf{P}$, having a simple node at the vertex; this example occurs in Saint-Donat [26].

I have included the last two examples to show that things can get quite complicated: the hypersurface $X_{42} \subset \mathbf{P}(21, 14, 4, 3)$ given by $x^2 + y^3 + yz^7 + z^9t^2 + t^{14}$ has the following singularities:

$$\left.\begin{array}{ll} z = t = 0 & 1 \times A_6 \\ y = z = 0 & 2 \times A_2 \\ x = t = 0 & 1 \times A_1 \end{array}\right\} \text{transverse intersections of singularities of } \mathbf{P}$$

and an A_3 point at $(0, 0, 1, 0)$. The corresponding desingularisation and Demazure divisor is:



§5. The plurigenus formula

Let X be a canonical 3-fold of index r. The fact that $\omega_X^{[r]}$ is ample, together with the easy Corollary 1.14, allows us to construct a resolution $f: Y \to X$ satisfying the following two mild conditions.

DEFINITION (5.1): A resolution $f: Y \to X$ is *0-minimal* if $f^*\omega_X^{[r]} = \omega_Y^{\otimes r}(-\Delta_r)$, with $\Delta_r \geqq 0$ a divisor on Y such that $f(\Delta_r) \subset X$ is a finite set.

DEFINITION (5.2): A resolution $f: Y \to X$ is *elegant* if for $s = 1, \ldots, r-1$ (hence for all s) the subsheaf $f'\omega_X^{[s]} = \operatorname{Im} f^*\omega_X^{[s]} \subset \omega_Y^{\otimes s}$ is invertible.

REMARK (5.3): For a torsion-free sheaf $\mathcal{F}$ on X, $f'\mathcal{F} = f^*\mathcal{F}/\text{Torsion}$ is the sheaf denoted $\mathcal{F} \circ f$ by Grauert and Riemenschneider ([27], p. 267), who pointed out that in general torsion turns up in taking the sheaf-of-$\mathcal{O}$-module theoretical f^*, defined by setting the stalk $(f^*\mathcal{F})_P = \mathcal{F}_{fP} \otimes_{\mathcal{O}_{X,fP}} \mathcal{O}_{Y,P}$ for $P \in Y$. There is of course no problem in taking f^* if $\mathcal{F}$ is locally free.

For an elegant resolution, it follows that $f'\omega_X^{[n]} = \omega_Y^{\otimes n}(-\Delta_n)$ for each $n \geqq 0$, with $\Delta_n \geqq 0$ of the form

$$\Delta_n = m\Delta_r + \Delta_i,$$

where $n = mr + i$, $0 \leq i \leq r-1$.

Since $\omega_X^{[r]}$ is invertible and ample, $\omega_X^{[mr+i]} = \omega^{[i]} \otimes (\omega^{[r]})^{\otimes m}$ is generated by its global sections for all sufficiently large m; the same is therefore true of $f'\omega_X^{[mr+i]}$, so that elegance is equivalent to demanding that for all sufficiently large n

$$|nK_Y| = |D_n| + \Delta_n,$$

with Δ_n fixed and $|D_n|$ free.

PROPOSITION (5.4): *X has an elegant 0-minimal resolution $f: Y \to X$.*

PROOF: Both of these conditions are very easy to satisfy. Firstly, in order that Y be elegant it is necessary and sufficient that Y dominates each of the blow-ups of the divisorial sheaves $\omega_X^{[i]}$ for $i = 1, \ldots, r-1$. By the blow-up of a divisorial sheaf $\mathcal{L}$ is intended the following: express $\mathcal{L} \cong \mathcal{O}_X(D)$ in the form $\mathcal{O}_X(D) = \mathcal{I}_{E-D} \cdot \mathcal{O}_X(E)$, where $\mathcal{O}_X(E)$ is invertible, and $\mathcal{I}_{E-D}$ is a divisorial ideal sheaf (as in the Appendix to § 1); the blow-up of $\mathcal{L}$ is the blow-up of the sheaf of ideals $\mathcal{I}_{E-D}$, which is obviously independent of the choice of D and E. Since each of the $\omega_X^{[i]}$ is invertible outside finitely many dissident points of X, the condition that Y dominates the blow-up of each of them does not affect zero-minimality.

A 0-minimal resolution can be obtained by any sequence of steps $Y = X_n \to \cdots \to X_0 = X$ which leads to a non-singular Y, such that each

step $s_i: X_{i+1} \to X_i$ satisfies one of the following two conditions:

(i) s_i is an isomorphism above all but a finite number of points of X;

(ii) s_i is the blow-up with centre $C_i \subset \operatorname{Sing} X_i$ a reduced curve C_i which lies over a 1-dimensional component of the singular locus of X. In case (ii) s_i is necessarily a blow-up of a curve of singularities which is generically (Du Val point) $\times \mathbf{A}^1$; s_i will be crepant (Definition 2.1) outside a finite number of points of X_i. Q.E.D.

THEOREM (5.5): *The following formula for the plurigenera of Y holds for all $n \equiv 1 \bmod r$, $n \geqq 2$, and for all sufficiently large n:*

$$P_n(Y) = P_n(X) = \frac{K_X^3}{12} \cdot (2n-1)n(n-1) - (2n-1)\chi(\mathcal{O}_X) + \ell(n). \qquad (*)$$

Here $\ell(n) \in \mathbf{Q}$ is linear with periodic adjustments, and $K_X^3 \in \mathbf{Q}$ are invariants of X defined by their appearance in (*); K_X^3 is also determined by $r^3 \cdot K_X^3 = (rK_X)^3$, where the right-hand side makes sense because rK_X is a Cartier divisor.

It has already been pointed out in Example 3.10 that K_X^3 can have arbitrary denominator. Further information on the invariants appearing in (*), and a discussion of its significance, will be given after the proof.

PROOF: If $r \mid n$ then $f'\omega_X^{[n]} = f^*\omega_X^{[n]}$ is the inverse image of an ample sheaf under a birational morphism, and so is quasi-positive ([27], p. 265); if $n \gg 0$ then as already observed $f'\omega_X^{[n]}$ is generated by its global sections, and taking n bigger still it defines a birational map. Thus by the vanishing theorem of Grauert and Riemenschneider ([27], p. 273),

$$H^p(Y, f'\omega_X^{[n]} \otimes \omega_Y) = 0$$

for all $p > 0$, and for all n with $r \mid n$ or $n \gg 0$.

Thus $P_{n+1}(Y) = \chi(Y, f'\omega_X^{[n]} \otimes \omega_Y)$, which can be computed by the Hirzebruch–Riemann–Roch formula. Let $n = mr + i$, with $0 \leqq i \leqq r-1$. Thus

$$P_{n+1} = \chi = \kappa_3[\operatorname{ch}(D) \cdot \operatorname{Td}(X)]$$
$$= \kappa_3\left[\left(1 + D + \frac{1}{2}D^2 + \frac{1}{6}D^3\right) \cdot \left(1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2\right)\right],$$

where the Chern classes are those of Y, and

$$D = mf^*(rK_X) + (i+1)K_Y - \Delta_i. \qquad (1)$$

This will simplify to (*), using

$$c_1 = -K_Y, \qquad (2)$$

$$\frac{1}{24}c_1c_2 = \chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X), \qquad (3)$$

$$(f^*(rK_X)) \cdot \Delta_i = 0, \tag{4}$$

and

$$rK_Y = f^*(rK_X) + \Delta_r; \tag{5}$$

here (3) holds because X is Cohen–Macaulay [36], and so $R^i f_* \mathcal{O}_Y = 0$ for $i > 0$, and (4) because $f(\Delta_i) \subset X$ is a finite set for each i.

Thus

$$P_{n+1} = \frac{1}{6}D^3 - \frac{1}{4}K_Y D^2 + \frac{1}{12}c_1^2 D + \frac{1}{12}c_2 D + \frac{1}{24}c_1 c_2. \tag{6}$$

Using (5),

$$D = (n+1)K_Y - m\Delta_r - \Delta_i, \tag{7}$$

so that the last two terms in (6) are

$$\frac{1}{12}c_2 D + \frac{1}{24}c_1 c_2 = -\frac{1}{24}(2n+1)\chi(\mathcal{O}_X) - \frac{1}{12}[mc_2\Delta_r + c_2\Delta_i]. \tag{8}$$

The first three terms can be rewritten using (5) and (7) in terms of $f^*(rK_X)$ and the Δ:

$$\begin{aligned} &\frac{1}{12}D(2D - K_Y)(D - K_Y) \\ &= \frac{1}{12r^3}\{(n+1)f^*rK_X - r\Delta_i\}\{(2n+1)f^*rK_X - 2r\Delta_i - \Delta_r\} \\ &\times \{nf^*rK_X - r\Delta_i - \Delta_r\} \\ &= \frac{1}{12}(2n+1)(n+1)nK_X^3 - \frac{1}{12r^2}\Delta_i\{2r\Delta_i + \Delta_r\}\{r\Delta_i + \Delta_r\}; \end{aligned} \tag{9}$$

the final equality has involved (4).

(6), (8) and (9) imply (*), together with the following formula for $\ell(n)$:

$$\ell(n+1) = \frac{1}{12}m(-c_2\Delta_r) - \frac{1}{12}c_2\Delta_i - \frac{1}{12}\left(2\Delta_i^3 + \frac{3}{r}\Delta_i^2\Delta_r + \frac{1}{r^2}\Delta_i\Delta_r^2\right),$$

where $n = mr + i$, $0 \leqq i \leqq r-1$. In particular, if $r \mid n$,

$$\ell(n+1) = \frac{1}{12}\frac{n}{r}(-c_2\Delta_r),$$

so that $\frac{n}{r}(-c_2\Delta_r)$ is the linear part of ℓ.

The fact that P_n is an integer implies varies congruences modulo $12r$ on the invariants appearing in (*). In particular, the denominator of K_X^3 divides r.

The divisors Δ_i (for $i = 1, \ldots, r$) occur naturally as unions of connected components $\Delta_i(P)$ with $f(\Delta_i(P)) = P$, lying over finitely many points $P \in X$. A consequence of their appearance in the formula (*) for the birationally invariant plurigenera of Y is the following result.

COROLLARY (5.6): *The quantities* $-c_2\Delta_r(P)$ *and*

$$-\left(c_2\Delta_i + 2\Delta_i^3 + \frac{3}{r}\Delta_i^2\Delta_r + \frac{1}{r^2}\Delta_i\Delta_r^2\right)(P) \quad (\text{for } i = 1, \ldots, r-1)$$

are invariants of the canonical singularity $P \in X$, *independent of the resolution* $f: Y \to X$.

A similar argument based on calculating $H^0(\mathcal{O}_X(n_1 H) \otimes \omega_X^{[n_2]})$, where H is an ample divisor and $n_1 \gg n_2 \gg 0$, and using the birational invariance of logarithmic differentials, proves Corollary 5.6 for a local canonical singularity $P \in X$, without assuming that $P \in X$ is isomorphic to a point of a global canonical variety.

It is obvious that for a hypersurface rational point the single invariant $-c_2\Delta = 0$; and Corollary 2.12 implies that this continues to hold for all rational Gorenstein 3-fold points.

PROBLEM (5.7): (i) For a canonical point $P \in X$ of index r, relate the invariants of Corollary 5.6 to the following numerical functions of the $\mathcal{O}_{X,P}$-modules $\omega_X^{[i]}$:

(a) the Hilbert functions $H(n, i) = \dim_k \omega_X^{[i]}/m_P^n \omega_X^{[i]}$;

(b) the lengths $r(i,j)$ and $s(i,j)$ of the kernels and cokernels of $\omega_X^{[i]} \otimes \omega_X^{[j]} \to \omega_X^{[i+j]}$.

(ii) Calculate these invariants for the quotient singularities $\mathbf{A}^3/\boldsymbol{\mu}_r$; (it's quite likely that these are representative of all index r points).

(iii) Topological interpretation?

(iv) Is it true that $\ell(n) \geqq 0$?

§6. Open problems and concluding remarks

6.1. *Is the canonical ring finitely generated?*

Wilson has shown that on a non-singular 3-fold V with $\kappa(V) \neq -\infty$, K_V is ample if and only if $K_V C > 0$ for every curve $C \subset V$. On the other hand we have the adjunction formula

$$K_V C + \deg N_{V|C} = 2p_a(C) - 2, \tag{*}$$

where $N_{V|C}$ is the normal sheaf; by the Riemann–Roch theorem

$$h^0(N_{V|C}) - h^1(N_{V|C}) = -K_V C.$$

By deformation theory, if $K_V C < 0$, C should then move in a positive-dimensional family; C will thus lie in a surface F, which it would be highly desirable[1] to contract by a birational modification of V. The techniques for

[1] The possibility of carrying out this contraction has been proved by S. Mori in a precise form; unfortunately, this is as yet only the first step (and not the inductive step) in the direction of finite generation.

such modifications have been pioneered by Kulikov [28], and simplified by Persson and Pinkham [29].

6.2. Now suppose that $K_V C = 0$; the following remark is partly suggested by a conversation with Bombieri: if K_V is ample on $V \backslash C$, but $\mathcal{O}_C(K_V) \in \operatorname{Pic} C$ is not a torsion class, then $R(V)$ is *not* finitely generated. Compare Zariski [30], p. 562.

CONJECTURE: If K_V is ample on $V \backslash C$ and $K_V C = 0$ then $p_a C = 0$.

For S a surface of general type, $p_a C \geqq 1$ implies that $K_S C > 0$ (without using minimal models). Using the index theorem and minimal models, $K_S C \geqq \sqrt{C^2}$, so that the first term in (*) cannot be too small.

6.3. *The adjunction sequence.*

The following is a local version of the problem of finite generation. If $f: Y \to X$ is a resolution of a variety X (suppose *either* that X is normal, *or* that ω_X is invertible), define the *adjunction sequence* to be the sequence of subsheaves $f_* \omega_Y^{\otimes n} \subset \omega_X^{[n]}$; if ω_X is invertible, $f_* \omega_Y^{\otimes n} = \mathcal{J}_n \cdot \omega_X^{\otimes n}$, where $\mathcal{J}_n$ is the *n-th higher adjunction ideal.*

Problem. Is the $\mathcal{O}_X$-algebra $\bigoplus_{n \geqq 0} f_* \omega_Y^{\otimes n}$ finitely generated?

This is equivalent to knowing that the ring $R(Y, f^*\mathcal{O}_X(k) \otimes \omega_Y)$ is f.g. for $k \gg 0$. If this is true then

$$\operatorname{Proj} R(Y, f^*\mathcal{O}_X(k) \otimes \omega_Y) = \operatorname{Proj}_X \Big(\bigoplus_{n \geqq 0} f_* \omega_Y^{\otimes n}\Big) \to X$$

is called the *relative canonical model* of Y, or the *canonical blow-up* of X.

6.4. For simple types of hypersurface singularities one expects the sequence of ideals $\{\mathcal{J}_n\}$ to be defined by weighting conditions as in §4. The following conjecture would extend to 3-fold hypersurface singularities the most fundamental properties of elliptic surface singularities:

CONJECTURE: *Let* $0 \in X \subset \mathbf{A}^n$ *be an elliptic singularity* (*Definition* 2.4); *then there exist coordinates* x_i *on* $\mathbf{A}^4$, *and an* $\alpha \in A'_4$ (*Theorem* 4.5) *which is uniquely determined by any of the following statements*:

(i) $\alpha(g) = 1$; *where* g *is the defining equation of* X;

(ii) $\mathcal{J}_n = \left\{ f \in \mathcal{O}_X \,\middle|\, \alpha(f) \geqq \frac{n}{d} \right\}$, *where* d *is the least denominator of* α;

(iii) *the* α*-blow-up* $X_1 \to X$ *is a variety with canonical singularities along* $f^{-1}0$.

6.5. The varieties of f.g. general type for which the canonical model is Cohen–Macaulay have the following property: after making a cyclic cover of degree r ramified in a general element of $|mrK_V|$, $m \gg 0$, one can make a

birational modification W such that W has only cDV points and $|nK_W|$ is free for all $n \gg 0$; in particular, $K_W C \geqq 0$ for every curve $C \subset W$.

6.6. One does not expect to get a unique minimal non-singular model of a 3-fold; instead, one could ask for a class of "nice resolutions" of the canonical model X. One might hope to index nice resolutions by some kind of combinatorial data, and characterising canonical points as subvarieties (for example hypersurfaces) in toric varieties might be a first step in this direction. However, one should not merely restrict to complete intersections in toric varieties, since this would exclude many interesting varieties which are Weil divisors but not Cartier divisors—the weighted blow-up of a hypersurface is a case in point. The following is a rather vague hope.

CONJECTURE: *Every canonical singularity is isomorphic to a toric section* $P \in X \subset A$, *defined by an ideal* I_X *with* $\alpha(A) > \alpha(I_X)$ *for a class of weightings* α.

Here a *toric section* (quasi-complete intersection in a toric space) is an irreducible subvariety $X \subset A$ of codimension r, such that r equations $f_1, \dots, f_r$ define X outside the coordinate hyperplanes: $X \cap \mathbf{T} = (f_1 = \cdots = f_r = 0) \subset \mathbf{T}$. The notion of weighting awaits clarification.

6.7. The simplest kind of normal 3-fold singularity which is not Cohen–Macaulay would be a *fake cDV point* $P \in X$, that is a point $P \in X$ for which a general section H through P has a non-normal isolated singularity $P \in H$, whose normalisation $P \in S$ is a Du Val point; the existence of such points is related to the deformation theory of $P \in H$.

CONJECTURE: *Let* $P \in S$ *be a Du Val point, and let* $P \in H$ *be an isolated singularity with* $\mathcal{O}_{H,P} \subset \mathcal{O}_{S,P}$ *of finite codimension. Then any deformation of* H *arises from deforming the normalisation* S *or from moving* $\mathcal{O}_{H,P}$ *inside* $\mathcal{O}_{S,P}$ *as a subvector space of fixed codimension. In particular* H *is not a section of a normal 3-fold.*

For example, the simplest such $P \in H$, $H = \operatorname{Spec} k[x^2, x^3, y, xy]$ (obtained by pinching out the vector $(x^2 = y = 0) \subset \mathbf{A}^2$) is rigid.

There is now some evidence for this conjecture: by [36] a fake cDV singularity cannot be canonical; as kindly pointed out by Jonathan Wahl, Mumford's Theorem in IV of [39] shows that a fake cDV point cannot be isolated. However, the deformation theory is much harder to deal with.

6.8. One can continue Theorem 2.6 (I) to the assertion that if a Gorenstein point $P \in X$ satisfies $m_P^r \cdot \omega_X \subset f_* \omega_Y$, then the general section $P \in H$ will satisfy $m_P^{r+1} \cdot \omega_H \subset f_* \omega_L$. By an induction this can be chased down to the curve section: if $P \in X$ is a rational Gorenstein point of an n-fold then the general curve section through P, $P \in C$ say, satisfies

$m_P^{n-1} \subset \mathscr{C}$,

where $\mathscr{C} = [\mathcal{O}_{\bar{C}} : \mathcal{O}_C]$ is the conductor. In this context Theorem 2.11 is partly explained by Shepherd–Barron's remark that if $P \in C$ is a Gorenstein curve point such that $m_P^2 \subset \mathscr{C}$, then either $m_P^2 = \mathscr{C}$, or $P \in C$ is a very special curve such as the plane curve given by $(x^2 + y^n = 0,\ n \leqq 5)$.

6.9. The reader will have observed that despite my ideological commitment to replacing the cohomological arguments involving R^if_*'s by adjunction-theoretic argument, I have basely betrayed my principles in the proof of (II) of Theorem 2.6. I would like to know if a proof of this result could be given on the lines of the proof of (I).

6.10. The combinatorics involved in resolving a rational Gorenstein 3-fold point as in §2 deserves further study. It is easily seen that on blowing up a point with invariant $k \geqq 3$ the invariant of any resulting point is at most k. Thus the resolution of these singularities consist of trees of del Pezzo surfaces. It is not clear as yet if there are any restrictions of the branching of these trees, say arising from some kind of index theorem.

6.11. The existence of canonical 3-folds of arbitrary index means that there can be no bound n such that the canonical ring of every 3-fold of f.g. type is generated by elements of degree $\leqq n$; it is not clear whether K_X^3 can be arbitrarily small, although it becomes fairly small for some complete intersections (see Problem 3.11). Most probably there should exist some bound depending only on r such that for every canonical 3-fold of index r $|mrK_X|$ is very ample. See [18], [19] and [41].

6.12. This problem is suggested by a remark of Beauville's: if ω_X is ample on a non-singular 3-fold X then it is a consequence of Yau's inequalities [33] that

$$\chi(\mathcal{O}_X) = \tfrac{1}{24}c_1c_2 \leqq \tfrac{1}{64}c_1^3 < 0.$$

CONJECTURE: *Let X be a canonical 3-fold with Gorenstein singularities, and let $f\colon Y \to X$ be a 0-minimal resolution. Then $c_2(Y)$ is quasi-positive in the sense that for every prime divisor $\Gamma \subset Y$*

$$c_2(Y) \cdot \Gamma \geqq 0,$$

with equality if and only if $\dim f(\Gamma) \leqq 1$.

This might be a consequence of some kind of index theorem for 3-folds.

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BIRATIONAL GEOMETRY OF ALGEBRAIC THREEFOLDS

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In the present notes, by an algebraic variety V we mean that V is an irreducible complete algebraic variety defined over the complex number field **C**. A non-singular algebraic variety is called an algebraic manifold. A compact complex manifold M is called a *Moishezon manifold*, if we have tr. deg $\mathbf{C}(M) = \dim M$, where $\mathbf{C}(M)$ is the meromorphic function field of M. A Moishezon manifold has the structure of a proper smooth algebraic space over **C** and it is bimeromorphically equivalent to a projective manifold. Hence from the view point of the birational geometry, we need not distinguish Moishezon manifolds from algebraic manifolds.

Let D be a Cartier divisor (a line bundle) on a normal algebraic variety V. If the complete linear system $|mD|$ is not empty, we can define a rational mapping $\Phi_{mD}: V \to P^{\dim|mD|}$ associated with the complete linear system $|mD|$. The D-dimension $\kappa(D, V)$ is defined by

$$\kappa(D, V) = \begin{cases} -\infty, \text{ if } |mD| = \emptyset \text{ for all positive integers,} \\ \max_{m\geq 1} \dim \Phi_{mD}(V), \text{ otherwise.} \end{cases}$$

Let V be an algebraic manifold and K_V a canonical divisor (the canonical line bundle) of V. Then $\kappa(K_V, V)$ is called the *Kodaira dimension* of V and is written as $\kappa(V)$. The Kodaira dimension is a birational invariant. Hence for a singular algebraic variety V, the Kodaira dimension $\kappa(V)$ of V is defined as $\kappa(V) = \kappa(V^*)$ where V^* is a non-singular model of V.

By definition $\kappa(V) = -\infty$ if and only if $P_m(V) = h^0(V, mK_V) = 0$ for all $m \geq 1$ and $\kappa(V) = 0$ if and only if $P_m(V) \leq 1$ for all $m \geq 1$ and there exists a positive integer m_0 such that $P_{m_0}(V) = 1$. Moreover $\kappa(V) > 0$ if and only if there exists a positive integer m such that $P_m(V) \geq 2$.

Using the notion of Kodaira dimension, we can classify all n-dimensional algebraic varieties into $n+2$ classes. When $n = 2$, the detailed study of these 4 classes was done by Italian algebraic geometers more than sixty years ago. Recently we obtained several important structure theorems on algebraic threefolds. In the present notes we shall briefly review these results and discuss some important unsolved problems.

In §1 we shall discuss differences between algebraic surfaces and algebraic manifolds of dimension $n \geq 3$. The main results on the