7. V. A. Iskovskikh, *Fano 3-folds*. I and II, I, Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977), 516–562; English transl. in Math USSR Izv. **11** (1977); II, Izv. Akad. Nauk SSSR Ser. Mat. **42** (1978), 469–506; English transl. in Math USSR Izv. **11** (1977).

8. V. Kanev, *Intermediate Jacobians of threefolds with a pencil of del Pezzo surfaces and generalised Prym varieties*, C. R. Acad. Bulgare Sci. **36** (1983), 1015–1019.

9. Y. Kawamata, *Pluricanonical systems on minimal algebraic varieties*, Invent. Math. **79** (1985), 567–588.

10. Y. Miyaoka, *The Chern classes and Kodaira dimension of a minimal variety*, Algebraic Geometry, Sendai; to appear in Advanced Stud. Pure Math, Kinokuniya and North-Holland, 1987.

10′. ——, *On the Kodaira dimension of minimal threefolds*, Math. Ann. (1987) (to appear).

11. Y. Miyaoka and S. Mori, *A numerical criterion for uniruledness*, Ann. of Math. **124** (1986), 65–69.

12. S. Mori and S. Mukai, *On Fano 3-folds with $B_2 \geq 2$*, Algebraic Varieties and Analytic Varieties, S. Iitaka, editor, Advanced Stud. Pure Math. **1** (1983), 101–129.

13. S. Mori, *Classification in dimension ≥ 3*, this volume.

14. D. Mumford, *Enriques' classification of surfaces in characteristic p*. I, Global Analysis, papers in honour of K. Kodaira, Univ. of Tokyo Press, Tokyo, and Princeton Univ. Press, Princeton, N.J., 1969, pp. 325–339.

15. M. Reid, *Decomposition of toric morphisms*, Arithmetic and Geometry, papers dedicated to I. R. Shafarevich, Birkhäuser, Boston, Mass., 1983, pp. 395–418.

16. ——, *Minimal models of canonical 3-folds*, Algebraic Varieties and Analytic Varieties, S. Iitaka, editor, Advanced Stud. Pure Math. **1** (1983), 131–180.

17. ——, *Young person's guide to canonical singularities*, this volume.

18. V. G. Sarkisov, *Birational automorphisms of conic bundles*, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), 918–945; English transl. in Math. USSR Izv. **17** (1981).

19. ——, *On the structure of conic bundles*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), 371–408; English transl. in Math. USSR Izv. **20** (1983).

20. I. R. Shafarevich, *On certain tendencies in the development of mathematics*, English transl. in Math. Intelligencer **3** (1980/81), no. 4, 182–184.

21. V. V. Shokurov, *The non-vanishing theorem*, Izv. Akad. Nauk SSSR Ser. Matem. **49** (1985), 635–651; English transl. in Math. USSR Izv. **27** (1986).

22. P. M. H. Wilson, *Towards birational classification of algebraic varieties*, Bull. London Math. Soc. **19** (1987), 1–48.

23. R. Hartshorne, *Ample subvarieties of algebraic varieties*, Lecture Notes in Math., vol. 156, Springer-Verlag, 1970.

24. Y. Kawamata, *Crepant blowings-up of three-dimensional canonical singularities and applications to degenerations of surfaces*, Ann. of Math. (1987) (to appear).

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Young Person's Guide to Canonical Singularities

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In memory of Oscar Zariski

This article aims to do three things: (I) to give a tutorial introduction to canonical varieties and singularities, with some of the motivating examples; (II) to provide a skeleton key to the results of my two papers on canonical singularities [**C3-f**], [**Pagoda**], and those of [**Morrison-Stevens**] and [**Mori**, Terminal singularities]; and (III) to explain the recent "exact plurigenus formula." The expository intention is reflected in explanations of some well-known standard technical points (well-known to experts but maybe not to the algebraic geometer in the street), and also worked examples, exercises, and deliberate mistakes to entertain the reader; I apologise if any secrets of the priesthood are divulged despite my best efforts.

After §4, most of the material is new: §§5, 6 and 7 contain the material of [**Morrison-Stevens**] and [**Mori**] in substantially laundered form, and the results on equivariant RR and the plurigenus formula of Chapter III appear here for the first time. The juxtaposition of these two topics reveals quite amazing relations between the cyclotomic sums appearing traditionally in connection with equivariant RR and toric geometry; of course, these are linked in a primary way by the fact that quotient singularities make contributions (for example, to $H^0(\mathbf{P}, \mathcal{O}(k))$ for a weighted projective space $\mathbf{P}$) which can be computed either by equivariant RR or as the number of lattice points of a polyhedron. However, it was something of a shock to discover how intricately cyclotomy relates to the combinatorics of the Newton polyhedron at the heart of the classification of terminal singularities.

My contribution to the subject has mainly been concerned with the study of 3-fold singularities. It should be noted that most of the recent work on varieties of dimension ≥ 4 (in particular the two circles of ideas, cone, contraction, non-vanishing theorems of Kawamata and Shokurov, and positivity of $f_*\omega, C^+_{n,m}$ of Fujita, Viehweg, Kawamata and Kollár) uses only the definitions of canonical and terminal singularities (and their log generalisations), together with general properties such as rationality and behaviour in codimension 2, but does not

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in any essential way use specific results concerning the singularities. Indeed, counterexamples (see (3.13)) suggest that it is unlikely that we can expect any worthwhile classification of these singularities in dimension ≥ 4.

The bulk of this paper was written during a six-week visit to the NSF-sponsored special year in singularity theory and algebraic geometry at the University of North Carolina, Chapel Hill; I thank Jonathan Wahl, Jim Damon, and the other visitors for providing a stimulating environment. I am indebted to Y. Kawamata, S. Mori and D. Zagier for helpful conversations, and to A. R. Fletcher, who has repeatedly corrected false versions of the formulas of Chapter III; S. Mori has saved me from serious error in two places.

Contents

Chapter I
Overview of the Subject

1. Definitions and easy examples. Varieties are always assumed to be normal and quasiprojective, and defined over an algebraically closed field k of characteristic zero; my favourite is $k = \mathbf{C}$.

(1.1) DEFINITION. A variety X has *canonical singularities* if it satisfies the following two conditions:

(i) for some integer $r \geq 1$, the Weil divisor rK_X is Cartier;

(ii) if $f\colon Y \to X$ is a resolution of X and $\{E_i\}$ the family of all exceptional prime divisors of f, then

$$rK_Y = f^*(rK_X) + \sum a_i E_i, \quad \text{with } a_i \geq 0.$$

If $a_i > 0$ for every exceptional divisor E_i, then X has ***terminal singularities.***

This section is devoted to explaining and motivating by examples the terms in the definition (see (1.8–9) for easy examples); the definition is ultimately justified by the fact (see (2.5)) that the canonical model of a variety of general type has canonical singularities.

Although the definition of canonical singularities is abstract, there is a kind of classification in the 3-fold case which will reduce most problems to hypersurface singularities, cyclic quotient singularities, and cyclic quotients of isolated hypersurface singularities. (It will be shown in (3.14), see also **[Pagoda]**, that every terminal singularity is of this kind.) The reduction steps consist of various cyclic covers, partial resolutions, and partial smoothings, and each step involves of course some definite understanding of the "general" singularity.

Here is some more terminology: the smallest r for which rK_X is Cartier in a neighbourhood of $P \in X$ is called the *index* of the singularity P; the **Q**-divisor $\Delta = (1/r)\sum a_i E_i$ which satisfies the formal equality

$$K_Y = f^*K_X + \Delta$$

is called the *discrepancy* of f. (To remember which way round this equality goes, think of the adjunction formula for the blow-up $\sigma\colon Y \to X$ of a smooth point $P \in X$ of a surface: $\sigma^{-1}P = l$ is a (-1)-curve, and $K_Y = \sigma^*K_X + l$.)

(1.2) *The surface case.* Everyone knows that the ordinary double point of a surface (the singularity $X\colon (xz = y^2) \subset \mathbf{A}^3$) has a resolution $f\colon Y \to X$ for which the exceptional curve E is a (-2)-curve, that is, $E \cong \mathbf{P}^1$, $E^2 = -2$. It's easy to see by the adjunction formula (or by a direct calculation with differentials, as in (1.9) below) that $K_Y = f^*K_X$, so that this is a canonical singularity. In fact it can be proved that the surface canonical singularities are exactly nonsingular points, together with the *Du Val surface singularities*, the hypersurface singularities given by one of the equations

$$\begin{aligned}
&A_n\colon x^2 + y^2 + z^{n+1} \quad (\text{for } n \geq 1),\\
&D_n\colon x^2 + y^2 z + z^{n-1} \quad (\text{for } n \geq 4),\\
&E_6\colon x^2 + y^3 + z^4,\\
&E_7\colon x^2 + y^3 + yz^3,\\
&E_8\colon x^2 + y^3 + z^5.
\end{aligned}$$

(One derivation of this list is sketched in (4.9), (3).) Among the many extraordinary properties enjoyed by these singularities is the fact that each of them has a resolution $f\colon Y \to X$ such that the exceptional locus of f is a bunch of (-2)-curves (forming a configuration given by the corresponding Dynkin diagram), and such that $K_Y = f^*K_X$.

It is important to realise that this harmless-looking observation is central to the theory of minimal models of surfaces and canonical models of surfaces of general type. The point is this: if X is a canonical surface (a surface with at worst Du Val singularities and ample K_X), then a minimal resolution $f\colon Y \to X$ is a nonsingular surface Y with K_Y nef: conversely, if Y is a nonsingular surface with K_Y nef and big (a minimal nonsingular model of a surface of general type), then the curves E with $K_Y E = 0$ form bunches of (-2)-curves, and can be contracted to Du Val singularities. Thus for surfaces of general type, it's a matter of personal preference whether you take the canonical model X, or a nonsingular minimal model Y with K_Y nef. In fact the influence of the Du Val singularities extends throughout the classification of surfaces. Now let's see how this fails in higher dimensions.

(1.3) EXAMPLE: *The Veronese cone.* The simplest example of a singularity of index > 1 is the cone over the Veronese surface; this has a resolution Y with exceptional locus E satisfying $E \cong \mathbf{P}^2$ and $\mathcal{O}_E(-E) \cong \mathcal{O}(2)$. By the adjunction formula, $\mathcal{O}_E(K_Y + E) = K_{\mathbf{P}^2} = \mathcal{O}(-3)$, so that purely formally we should have

$$K_Y = f^* K_X + \tfrac{1}{2} E.$$

We'll see presently that this is meaningful in terms of differentials. The first context in which I met a variety with these singularities was the Kummer variety of an Abelian 3-fold A; dividing out A by the involution (-1), the resulting variety $X = A/(-1)$ has 64 Veronese cone singularities (at the 64 fixed points of (-1)), and $2K_X \sim 0$. You can simply blow up these singularities to get a smooth variety Y if you wish, but then $2K_Y \sim \sum_{64} E_i$, so you've lost the good numerical properties of K_X. This is one reason why the Kummer surface does not generalise to higher dimensions (at least, not in a very simple way).

Two examples where this singularity appears on canonical models of 3-folds are given in (2.8–9); compare [**Ueno**].

(1.4) *Canonical differentials.* If V is a smooth variety, $\omega_V = \mathcal{O}_V(K_V) = \Omega^n_V$ is the invertible sheaf generated by $dx_1 \wedge \cdots \wedge dx_n$, where $x_1, \ldots, x_n$ are local coordinates. Sections of ω_V are *canonical differentials*, and sections of $\omega_V^{\otimes m}$ are *m-canonical differentials*. Canonical differentials are important for the following reasons:

(1) *Intrinsic nature.* The sheaves ω_V and $\mathcal{O}_V(rK_V)$ are part of the bundled hardware which comes free when you buy V. This is particularly important in classification theory; for example, if ω_V is ample, then there is an intrinsic way of embedding V into projective space.

(2) *Duality.* ω_V is the dualising sheaf which makes Serre duality work; that is, there is a perfect pairing $H^i(V, \mathcal{F}) \times \mathrm{Ext}^{n-i}_V(\mathcal{F}, \omega_V) \to k$.

(3) *Vanishing.* Kodaira vanishing says that $H^i(V, \mathcal{L} \otimes \omega_V) = 0$ for an ample sheaf $\mathcal{L}$ and $i > 0$.

(4) *Birational nature.* If $V \dashrightarrow W$ is a birational map between nonsingular projective varieties, then it is easy to see (for example, [**Shafarevich**, p. 167]) that regular differentials on V and W coincide, so for example

$$H^0(V, \mathcal{O}_V(rK_V)) = H^0(W, \mathcal{O}_W(rK_W)).$$

(5) *Adjunction formulas.* If two varieties X and Y are closely related, then you expect to be able to compute K_X in terms of K_Y and vice-versa; a formula of this kind is called an *adjunction formula*. In practice this means that K_X is readily computable. The following are some of the many examples of adjunction formulas.

(a) If Y is a smooth variety and $X \subset Y$ is a hypersurface, then $K_X = (K_Y + X)|_X$.

(b) The Riemann-Hurwitz formula $K_X = f^* K_Y + R_f$ for a generically finite (separable) morphism $f\colon X \to Y$ between nonsingular varieties: there is a canonical map $J\colon f^*(\Omega^n_Y) \hookrightarrow \Omega^n_X$, and the *ramification divisor* can be defined by $R_f = \mathrm{div}(J)$; this is of course just an intrinsic way of saying the determinant of the Jacobian matrix

$$J = \det \frac{\partial(x_1, \ldots, x_n)}{\partial(y_1, \ldots, y_n)}.$$

(c) If $\sigma\colon Y \to X$ is the blow-up of a nonsingular point $P \in X$ of an n-fold, and $E = \sigma^{-1} P$, then $K_Y = \sigma^* K_X + (n-1)E$.

(d) If $p\colon F \to X$ is the $\mathbf{P}^{r-1}$-bundle $F = \mathbf{P}_X(\mathcal{E})$ associated to a rank r vector bundle $\mathcal{E}$ over X, and $\mathcal{O}_F(1)$ is the tautological line bundle (that is, $\mathcal{O}_F(1)$ is $\mathcal{O}(1)$ on each fibre, and $p_* \mathcal{O}_F(1) = \mathcal{E}$), then $K_F = p^*(K_X + \det \mathcal{E}) \otimes \mathcal{O}_F(-r)$.

(e) On a deeper level, Kodaira's canonical bundle formula for an elliptic surface (see, for example, [**Barth-Peters-Van de Ven**, p. 161]) should be viewed as an adjunction formula.

(1.5) *Definition of ω_X and $\mathcal{O}_X(mK_X)$ for singular X.* Assume X is normal. Then $\Omega^n_{k(X)}$, the space of rational canonical differentials of X (more precisely, I should write $\Omega^n_{k(X)/k}$), is a 1-dimensional vector space over $k(X)$, with basis $df_1 \wedge \cdots \wedge df_n$ for any $f_1, \ldots, f_n \in k(X)$ forming a separable transcendence basis of $k(X)$ over k. Write X^0 for the nonsingular locus of X. Then for $P \in X^0$ I can choose local coordinates $x_1, \ldots, x_n$ at P, and write any $s \in \Omega^n_{k(X)}$ as

$$s = f \cdot dx_1 \wedge \cdots \wedge dx_n \quad \text{with } f \in k(X).$$

Then s is *regular* at $P \in X^0$ if f is a regular function at P. Now by definition, s is regular at $P \in X$ if there is a neighbourhood $P \in U \subset X$ such that s is regular at every $x \in U \cap X^0$. This defines a sheaf ω_X, with

$$\Gamma(U, \omega_X) = \{ s \in \Omega^n_{k(X)} \mid s \text{ is regular on } U \cap X^0 \}.$$

That is, I don't attempt to define directly a regular differential at a singular point P, but just take rational differentials which are regular on the smooth points of a neighbourhood of P. There are several traditional alternative ways of defining the same sheaf.

(a) $\omega_X = j_*(\Omega^n_X)$, where $j\colon X^0 \hookrightarrow X$ is the inclusion of the smooth locus of X;

(b) ω_X is the *double dual* of Ω^n_X.

(1.6) *Explanation.* Although the Kähler differentials Ω^1 and Ω^n have good universal properties, they are often not right for (birational) geometrical purposes; the construction (a) of ω_X in terms of rational differentials which are regular in codimension 1 is one obvious geometrical alternative, due to Zariski. To explain (b), taking the dual kills any torsion which might be present in Ω^n_X, and then taking the double dual saturates Ω_X, in the sense that any rational sections of Ω^n_X which belong to Ω^n_X in codimension 1 actually belong to ω_X; or "kills the cotorsion" in the jargon of the trade. The universal constructions of tensor product and f^* of a sheaf are often not right for geometrical purposes for similar reasons. A simple example: if $f: Y \to X$ is the blow-up of a nonsingular point of a surface, many writers who should know better write $f^* m_P$ for the ideal $m_P \cdot \mathcal{O}_Y$; in fact there are at least 3 different pull-backs of m_P, namely, the sheaf-theoretic $f^{-1}m_P$, the ringed-space construction $f^*m_P = f^{-1}m_P \otimes \mathcal{O}_Y$ (which has torsion, as you should check for yourself), and $m_P \cdot \mathcal{O}_Y = \operatorname{Im}\{f^*m_P \to \mathcal{O}_Y\}$. In more complicated situations you might also contemplate saturating $m_P \cdot \mathcal{O}_Y$, etc.

(1.7) The sheaves

$$\mathcal{O}_X(mK_X) = \{s \in (\Omega^n_{k(X)})^{\otimes m} \mid s \text{ is regular on } X^0\} = j_*((\Omega^n_{X^0})^{\otimes m})$$

are defined in a similar way. Here the canonical divisor K_X is the Weil divisor (more precisely, divisor class) corresponding to ω_X; this means the following: take any nonzero rational differential $s \in \Omega^n_{k(X)}$, and let $K_X = \operatorname{div}(s)$ be the divisor of zeros and poles of s. The statement that rK_X is Cartier at $P \in X$ (that is, locally principal) is equivalent to saying that $\mathcal{O}_X(rK_X)$ is invertible.

(1.8) EXAMPLES. (1) Suppose that $P \in X: (f = 0) \subset \mathbf{A}^{n+1}$ is a (normal) hypersurface singularity. Consider the expression

$$s = \frac{dx_1 \wedge \cdots \wedge dx_n}{\partial f/\partial x_0} \in \Omega^n_{k(X)},$$

where $x_0, \ldots, x_n$ are local coordinates on $\mathbf{A}^{n+1}$ in a neighbourhood of P. At any point $Q \in X$ where $(\partial f/\partial x_0)(Q) \neq 0$, X is a manifold with local coordinates $x_1, \ldots, x_n$, and $s = (\text{unit}) \cdot (dx_1 \wedge \cdots \wedge dx_n)$ is a basis on Ω^n_X. Now because of the identifications involved in the definition of Ω^1_X and in taking the wedge product, it happens that under permutation of $x_0, \ldots, x_n$, the element $s \in \Omega^n_{k(X)}$ is invariant up to ± 1. Thus x_0 does not play any particular role, and *s is a basis of Ω^n_X at any nonsingular point of X*. This means that s is regular at P. In fact s is a basis of ω_X, that is, $\omega_X = \mathcal{O}_X \cdot s$: for given any $t \in \omega_X$, I can write $t = f \cdot s$ with $f \in k(X)$; but then f must be regular at every $Q \in X^0$, and so by normality of X, f is regular on X.

(2) Let X be the quotient $X = \mathbf{A}^2/\mu_3$ of $\mathbf{A}^2$ by the cyclic group μ_3 of cube roots of 1 acting by

$$\varepsilon: (x, y) \to (\varepsilon x, \varepsilon y) \quad \text{for all } \varepsilon \in \mu_3.$$

The ring of invariants of the action is

$$k[x^3, x^2y, xy^2, y^3] \cong k[u_0, u_1, u_2, u_3]/(u_0u_2 - u_1^2, u_1u_3 - u_2^2, u_0u_3 - u_1u_2),$$

and the quotient X is Spec of this ring, which as you can see is the affine cone over the twisted cubic.

I now write down a basis $s \in \mathcal{O}_X(3K_X)$ as a rational 3-canonical differential on X. The idea is that upstairs on $\mathbf{A}^2$, $dx \wedge dy$ is a basis of Ω^2, but under the group action, $\varepsilon: (dx \wedge dy) \to \varepsilon^2(dx \wedge dy)$; so $(dx \wedge dy)^{\otimes 3}$ is invariant under the group action, and should come from something on X. Now if I set

$$s = \frac{(du_0 \wedge du_1)^{\otimes 3}}{u_0^4} \in (\Omega^2_{k(X)})^{\otimes 3},$$

then differentiating $u_0 = x^3$, $u_1 = x^2y$ shows that

$$\pi^* s = (\text{unit}) \cdot (dx \wedge dy)^{\otimes 3},$$

(where $\pi: \mathbf{A}^2 \to X$ is the quotient map); since π is etale outside the origin, it is clear that s is a basis of $(\Omega^2_X)^{\otimes 3}$ everywhere on $X \setminus P$. Alternatively, note that from the equations defining X, (u_0, u_1) are local coordinates wherever $u_0 \neq 0$, and that (by direct calculation)

$$s = \frac{(du_0 \wedge du_1)^3}{u_0^4} = \frac{(du_2 \wedge du_3)^3}{u_3^4},$$

which works wherever $u_3 \neq 0$. Thus $s \in \mathcal{O}_X(3K_X)$ is a basis, and $3K_X$ is Cartier.

These two examples illustrate condition (1.1), (i). The next section tries to explain condition (ii).

(1.9) *Regularity of differentials on a resolution.* As discussed above, condition (1.1), (i) means that the sheaf $\mathcal{O}_X(rK_X)$ is invertible. Suppose that s is a local basis of $\mathcal{O}_X(rK_X)$ at a singular point $P \in X$, and that $f: Y \to X$ is a resolution. Then $s \in (\Omega^n_{k(X)})^{\otimes r}$, and since $k(Y) = k(X)$, I can consider s as a rational differential on Y, and ask again whether it is regular; of course, where f is an isomorphism there is no problem, but s can perfectly well have poles along exceptional divisors of f. So condition (1.1), (ii) is the condition that s remains regular on a resolution Y.

EXAMPLES. (1) In the notation of (1.8), suppose in addition that $P \in X \subset \mathbf{A}^{n+1}$ is an ordinary point of multiplicity k, so that the projectivised tangent cone is a nonsingular hypersurface $E \subset \mathbf{P}^n$ of degree k. Then $P \in X$ is terminal if $k < n$, canonical if $k = n$, and not canonical if $k > n$.

You can see this by an explicit calculation: let $\sigma: Y \to X$ be the blow-up of P; then Y is nonsingular and the exceptional locus $E = \sigma^{-1}P$ is the hypersurface $E \subset \mathbf{P}^n$. I'm interested in the zeros or poles along E of the rational canonical differential

$$s = \frac{dx_1 \wedge \cdots \wedge dx_n}{\partial f/\partial x_0} \in \Omega^n_{k(X)}.$$

To calculate this, write down one affine piece of the blow-up of $\mathbf{A}^{n+1}$, which is the map $\sigma: \mathbf{A}^{n+1} \to \mathbf{A}^{n+1}$ given by

$$x_n = y_n, \qquad x_i = y_i y_n \quad \text{for } i = 0, 1, \ldots, n-1,$$

where $y_0, \ldots, y_n$ are coordinates in a copy of $\mathbf{A}^{n+1}$. Then

$$\sigma^* f = f(y_0y_n, \ldots, y_n) = y_n^k \cdot g(y_0, \ldots, y_n),$$

where g is the equation of the affine piece of Y in $\mathbf{A}^{n+1}$. Now since

$$g = f(y_0y_n, \ldots, y_n) \cdot y_n^{-k},$$

it follows that

$$\partial g/\partial y_0 = y_n^{-k+1} \cdot (\partial f/\partial x_0);$$

hence at a point $Q \in E$ where $(\partial g/\partial y_0)(Q) \neq 0$,

$$s = \frac{dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n}{\partial f/\partial x_0} = y_n^{n-1} \cdot \frac{dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n}{\partial f/\partial x_0}$$
$$= y_n^{n-1} \cdot \frac{dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n}{y_n^{k-1} \cdot \partial g/\partial y_0} = y_n^{n-k} \cdot t,$$

where

$$t = \frac{dy_1 \wedge dy_2 \wedge \cdots \wedge dy_n}{\partial g/\partial y_0}$$

is a basis of Ω_Y^n near Q.

So the rational differential s has

a zero of order $n-k$ along E if $k < n$,

a pole of order $k-n$ along E if $k > n$,

and no zero or pole if $n = k$. More succinctly, the computation can be expressed as follows:

$$K_X = (K_{\mathbf{A}} + X)|_X \quad \text{and} \quad K_Y = (K_B + Y)|_Y$$

where $\sigma\colon B \to \mathbf{A}$ is the blow-up of P. However,

$$K_B = \sigma^* K_{\mathbf{A}} + nE \quad \text{and} \quad Y = \sigma^* X - kE,$$

where $E \subset B$ is the exceptional divisor. Adding these up gives

$$K_Y = \sigma^* K_X + (n-k)E.$$

(2) Use the notation of (1.8), (2). The quotient singularity X is resolved by a single blow-up $\sigma\colon Y \to X$, so that one affine piece of the resolution is a copy of $\mathbf{A}^2$ with coordinates (z,t), mapping to X by

$$(z,t) \mapsto (z, zt, zt^2, zt^3).$$

The exceptional curve $E = \sigma^{-1}P$ of the blow-up has $E \cong \mathbf{P}^1$, $\mathcal{O}_E(-E) \cong \mathcal{O}(3)$, and is given in the affine piece $\mathbf{A}^2$ by $z = 0$.

The rational 3-canonical differential $s = (du_0 \wedge du_1)^3/u_0^4 \in \mathcal{O}_X(3K_X)$ is a basis. Now think of s as a rational differential on the resolution $\sigma\colon \mathbf{A}^2 \to X$, by just writing $u_0 = z$, $u_1 = zt$. Then

$$s = \frac{(dz \wedge z\,dt)^3}{z^4} = \frac{(dz \wedge dt)^3}{z},$$

where $dz \wedge dt$ is a basis of the regular canonical differential on $\mathbf{A}^2$; so s has a pole along the exceptional curve E.

(1.10) *Exercise.* A similar calculation that the 3-fold quotient singularity $X = \mathbf{A}^3/\mu_3$ where

$$\mu_3 \ni \varepsilon\colon (x,y,z) \mapsto (\varepsilon x, \varepsilon y, \varepsilon^2 z)$$

is canonical of index 3. To see this, note that $X = \operatorname{Spec} A$, where

$$A = k[x^3, x^2y, xy^2, y^3, xz, yz, z^3] = k[u_0, u_1, u_2, u_3, v_0, v_1, w]/I,$$

and the ideal I of relations between the 7 generators is generated by 10 relations of monomial type; you will enjoy checking that the projectivised tangent cone to $P \in X$ consists of $E_1 \cup E_2$, where E_1 is a plane and E_2 is a quartic scroll, and the blow-up of $P \in X$ is nonsingular. Next,

$$s = \frac{(du_0 \wedge du_1 \wedge dv_0)^{\otimes 3}}{u_0^5}$$

satisfies

$$\pi^* s = (dx \wedge dy \wedge dz)^3,$$

and so is a basis of $\mathcal{O}_X(3K_X)$. It has zeros of order 1 and 2 along the two exceptional components of the resolution.

(1.11) *Historical note.* This example was first discovered in this context (as a counterexample to my primitive idea that the index is always ≤ 2) by N. Shepherd-Barron, although it had been previously hinted at in a letter of K. Ueno. The Veronese cone singularity of (1.3) and (2.8–9) was also the main ingredient in Ueno's paper of the same period [**Ueno**]. Note that the Veronese cone of (1.3) and the quotient singularity of (1.10) are the first of a well-understood series of terminal quotient singularities (see (5.2)); the nice resolution constructed in Exercise 1.10 is generalised in (5.7).

2. Brief introduction to global canonical 3-folds. This section is logically independent of the rest of the paper, giving some of the examples and historical motivation underlying [**C3-f**].

(2.1) *Is the canonical ring of a variety finitely generated?* Zariski's work [**Zariski**] implies that the canonical ring of a surface of general type is a f.g. k-algebra; geometric constructions of surfaces of general type since Enriques have often been closely related to a description of the canonical ring (see, for example, [**Catanese**, §1.3]). Experience has shown that many of the basic assertions in a traditional treatment of the classification of surfaces fail in higher dimensions; perhaps finite generation of the canonical ring generalises? At present this is not completely settled for 3-folds, although it looks good. Be that as it may, early examples of canonical models of 3-folds of general type for which the canonical ring is f.g. (see (2.8–11)) displayed interesting new features compared with surfaces of general type, and studying canonical 3-folds has led to some understanding of what seem to be typical features of higher-dimensional birational geometry.

(2.2) *Hilbert's 14th problem.* A standard method of constructing a graded ring: start from a nonsingular projective variety V and a divisor D on V, and set

$$R(V,D) = \bigoplus_{n \geq 0} H^0(\mathcal{O}_V(nD)).$$

Hilbert asked [**Mumford**, §3] if rings of this form are f.g. in general; this is false: the first counterexamples were given by Nagata and Zariski in the 1950s. However, Zariski gave the following sufficient condition for $R(V,D)$ to be f.g. (Graded rings appearing here are assumed to have $R_0 = k$, and f.g. means finitely generated as k-algebra.)

(2.3) THEOREM. *Suppose that the linear system $|mD|$ is free for some $m > 0$; then $R(V, D)$ is f.g.*

MODERN SKETCH PROOF. Suppose first that $m = 1$, so that $|D|$ itself is free; then $|D|$ defines a morphism $\varphi = \varphi_D : V \to \varphi(V) = Y \subset \mathbf{P}$ to a projective space $\mathbf{P}$ such that $\mathcal{O}_V(D) = \varphi^*\mathcal{O}_{\mathbf{P}}(1)$. If I set $\mathcal{A} = \varphi_*\mathcal{O}_V$ it follows that for any n,

$$\varphi_*\mathcal{O}_V(nD) = \mathcal{A} \otimes \mathcal{O}(n) = \mathcal{A}(n),$$

so that

$$H^0(V, \mathcal{O}_V(nD)) = H^0(\mathbf{P}, \varphi_*\mathcal{O}_V(nD)) = H^0(\mathbf{P}, \mathcal{A}(n)).$$

However, since $\mathcal{A}$ is a coherent sheaf of $\mathcal{O}_{\mathbf{P}}$-algebras, it follows easily from Serre's theorems that the ring $\bigoplus_{n\geq 0} H^0(\mathbf{P}, \mathcal{A}(n))$ is finite as a module over the homogeneous coordinate ring of $\mathbf{P}$. This gives the result in the case $m = 1$.

The more general case is similar, using the morphism $\varphi = \varphi_{mD}$ and considering the coherent sheaf of $\mathcal{O}_{\mathbf{P}}$-algebras $\mathcal{A} = \varphi_*\mathcal{O}_V$ together with the sheaves of $\mathcal{A}$-modules $\mathcal{M}_i = \varphi_*\mathcal{O}_V(iD)$ for $i = 1, \dots, m-1$. Q.E.D.

(2.4) *Projective normalisation.* Suppose V and D are as in (2.3). Set

$$X = \operatorname{Proj} R(V, D),$$

and consider the morphism $\varphi_{nD} : V \to \varphi(V) = Y \subset \mathbf{P}$ for any n such that $|nD|$ is free. Then as is clear from the proof just given, X coincides with $\operatorname{Spec}_Y \mathcal{A}$, which in Zariski's language is the normalisation of Y in the function field of V; write

$$V \xrightarrow{\varphi} X \xrightarrow{f} Y \subset \mathbf{P}$$

for the factorisation of φ_{nD} (the Stein factorisation). Then φ is a contraction morphism corresponding to D: it is the unique morphism such that $\varphi_*\mathcal{O}_V = \mathcal{O}_X$ and for every curve $C \subset V$,

$$\varphi(C) = \text{pt.} \iff CD = 0.$$

Furthermore, since $f : X \to Y$ is finite, $f^*\mathcal{O}_Y(1)$ is ample on X, and some multiple is very ample. This gives:

COROLLARY. $X = \operatorname{Proj} R(V, D) \cong \varphi_{nD}(V)$ *for every sufficiently large and divisible* n.

Note that if S is a minimal surface of general type then $|mK_S|$ is free for any $m \geq 4$ (in fact for $m \geq 2$ if $K_S^2 \geq 5$, see **[Catanese]**), so that it follows that the canonical ring of S is finitely generated.

(2.5) *Canonical models.*

DEFINITION. A *canonical variety* is a projective variety X with at worst canonical singularities such that the **Q**-Cartier divisor K_X is ample. If V is a variety of general type and X is a canonical variety birational to V, then X is the *canonical model* of V.

THEOREM **[C3-f**, (1.2), (II)**]**. *Let V be a smooth projective variety of general type. Then V has a canonical model X if and only if the canonical ring $R = R(V, K_V)$ is f.g., and then $X = \operatorname{Proj} R(V, K_V)$.*

PROOF. For a graded ring $R = \bigoplus_{k\geq 0} R_k$ and $m > 0$, the truncated ring $R^{(m)}$ is defined by $R^{(m)} = \bigoplus_{k\geq 0} R_{km}$; by the Veronese embedding, $\operatorname{Proj} R^{(m)} = \operatorname{Proj} R$.

First, let X be a canonical variety, and $m > 0$ an integer such that $\mathcal{O}(mK_X)$ is an ample Cartier divisor; then of course $R(X, \mathcal{O}_X(mK_X))$ is f.g. and $X = \operatorname{Proj} R(X, \mathcal{O}_X(mK_X))$. However, from the definition of canonical singularities and the birational invariance of $H^0(mK_V)$ it follows that

$$H^0(V, kK_V) = H^0(\mathcal{O}_X(kK_X))$$

for any nonsingular projective variety V birationally equivalent to X and any $k > 0$. Therefore $R(V, K_V)$ is finitely generated, and

$$X = \operatorname{Proj} R(X, \mathcal{O}_X(mK_X)) = \operatorname{Proj} R(V, mK_V) = \operatorname{Proj} R(V, K_V).$$

(2.6) I now prove the converse; suppose that $R(V, K_V)$ is f.g. and set $X = \operatorname{Proj} R(V, K_V)$.

It is well known that if R is a f.g. graded ring, there exists $m > 0$ such that $R^{(m)}$ is generated by elements of the smallest degree m (see, for example, EGA II, (2.1.6), (v)); fix such an m. In other words, for each $k \geq 1$, $H^0(\mathcal{O}_V(kmK_V))$ is spanned as a vector space by k-fold products of elements of $H^0(\mathcal{O}_V(mK_V))$.

Let $V' \to V$ be a resolution of the base locus of $|mK_V|$; in view of the birational invariance of $H^0(mK_V)$, I can replace V by V', so assume that

$$|mK_V| = |M| + F,$$

where $|M|$ is a free linear system and F the fixed part. Because of what I just said about $H^0(kmK_V)$, I also have

$$|kmK_V| = |kM| + kF \quad \text{for } k \geq 1.$$

By (2.4) applied to M, the map $\varphi = \varphi_M : V \to \varphi_M(V) \cong X \subset \mathbf{P}$ is birational, and X is normal. By construction $\mathcal{O}_V(M) \cong \varphi^*\mathcal{O}_X(1)$.

CLAIM. *Every irreducible component Γ of F is contracted by φ to a locus $\varphi(\Gamma)$ of dimension $\leq n-2$.*

The claim implies that X is a canonical variety: indeed, if I set

$$V^0 = V - \{\text{exceptional divisors of } \varphi\}$$

and write X^0 for the open subset of $\varphi(V^0)$ where φ^{-1} is regular, then X^0 is the complement of a subset of codimension ≥ 2 in X; and $\varphi : V^0 \xrightarrow{\sim} X^0$ is an isomorphism inducing

$$\mathcal{O}_X(1)|_{X^0} \cong \mathcal{O}_V(M)|_{V^0} \cong \mathcal{O}_V(mK_V)|_{V^0} \cong \mathcal{O}_X(mK_X)|_{X^0}.$$

So mK_X is a Cartier divisor, giving (1.1), (i); and $mK_V = \varphi^*mK_X + F$ gives (1.1), (ii).

(2.7) PROOF OF CLAIM. This is an easy result in the style of [**Zariski**]: notice first that if $\dim \varphi_M(\Gamma) = n-1$ then $h^0(\mathcal{O}_\Gamma(kM+A)) \sim (\text{const.}) \cdot k^{n-1}$ as $k \to \infty$ for any divisor A on Γ. Now Γ is fixed in $|kM+\Gamma|$ for every k, and hence the restriction map

$$r_k \colon H^0(\mathcal{O}_V(kM+\Gamma)) \to H^0(\mathcal{O}_\Gamma(kM+\Gamma))$$

is zero. Also, one gets a bound of the form

$$h^1(\mathcal{O}_V(kM)) < (\text{const.}) \cdot k^{n-2}.$$

In fact, since $\varphi_*\mathcal{O}_V(kM) = \mathcal{O}_X(k)$ is ample, $H^i(\varphi_*\mathcal{O}_V(kM)) = 0$ for $k \gg 0$, and the Leray spectral sequence gives

$$H^1(\mathcal{O}_V(kM)) = H^0(R^1\varphi_*\mathcal{O}_V(kM)) = H^0(R^1\varphi_*\mathcal{O}_V \otimes \mathcal{O}_X(k)),$$

so that h^0 grows like k^d where $d = \dim \operatorname{Supp} R^1\varphi_*\mathcal{O}_V \le n-2$.

This contradiction proves the claim. Q.E.D.

(2.8) EXAMPLES. The following is an example of a canonical model of a 3-fold of general type: take the weighted projective space $\mathbf{P}(1,1,2,2,7)$ with weighted homogeneous coordinates x_1, x_2, y_1, y_2, w, and the hypersurface $X = X_{14} \subset \mathbf{P}(1,1,2,2,7)$ given by $w^2 = f_{14}(x_1,x_2,y_1,y_2)$. To explain this variety in terms of ordinary projective spaces, consider the generically 2-to-1 morphism $\pi \colon X \to \mathbf{P}(1,1,2,2)$ given by omitting w. Then $\mathbf{P}(1,1,2,2)$ is isomorphic in an obvious way to the quadric of rank 3, $Q \subset \mathbf{P}^4$, and π is the double covering branched in the intersection $Q \cap F_7$ of Q with a general septic, and along the vertex of Q. It is easy to see that X has 7 Veronese cone points at the intersection of the vertex with F.

This example was psychologically important, because using the easy formalism of weighted projective spaces (see [**Dolgachev**]),

$$K_X = \mathcal{O}(14-7-2-2-1-1) = \mathcal{O}(1)$$

is an ample $\mathbf{Q}$-Cartier divisor satisfying $K_X^3 = 14/(2\cdot 2\cdot 7) = 1/2$; this number controls the growth of the plurigenera of X (that is, of a nonsingular model of X), so that

$$P_n(X) = h^0(X, \mathcal{O}(n)) \sim \left(\frac{1}{3!}\right)\cdot\left(\frac{1}{2}\right)\cdot n^3 \quad \text{as } n \to \infty.$$

However, if X had a nonsingular model Y with K_Y nef, RR would give $P_n \sim (1/3!)\cdot K_Y^3 \cdot n^3$, with $K_Y^3 \in 2\mathbf{Z}$; so the plurigenera of this X grow a lot slower than those of any 3-fold of general type having a nonsingular model with K_Y nef. Note also that φ_{nK} cannot be birational for $n \le 6$, so that this kind of example is analogous to Enriques' famous example $X_{10} \subset \mathbf{P}(1,1,2,5)$ of a surface of general type for which φ_{4K} is not birational (compare [**Catanese**]).

(2.9) Now consider the weighted complete intersection

$$X = X_{6,6,6} \subset \mathbf{P}(2^4, 3^3).$$

This is a rare case when the theoretical idea of embedding a weighted projective space $\mathbf{P}$ by means of some $\mathcal{O}_\mathbf{P}(n)$ actually helps to understand it. The embedding

of $\mathbf{P}(2^4,3^3)$ by means of $\mathcal{O}(6)$ looks as follows: take a copy of $\mathbf{P}^3$ in its Veronese embedding by $\mathcal{O}(3)$, and a copy of $\mathbf{P}^2$ in its Veronese embedding by $\mathcal{O}(2)$, then take the linear join:

$$\mathbf{P} = \mathbf{P}(2^4,3^3) = v_3(\mathbf{P}^3) * v_2(\mathbf{P}^2) \subset \mathbf{P}^{25}.$$

My 3-fold X is the intersection of $\mathbf{P}$ with 3 sufficiently general hyperplanes of $\mathbf{P}^{25}$. It's not hard to see that X has 27 Veronese cone singularities at its intersection with the 3-dimensional stratum $v_3(\mathbf{P}^3)$, and has $K_X = \mathcal{O}(1)$ and $K_X^3 = 1/2$; moreover, since there are no homogeneous polynomials of degree 1, $H^0(\mathbf{P}, \mathcal{O}(1)) = 0$, so it follows that $p_g(X) = 0$. Thus in contrast to the surface case, it's quite easy to write down 3-folds of general type with $p_g = 0$, even with the canonical ring a complete intersection.

(2.10) Since the 3-fold X of (2.9) has $p_g = 0$, it is of some interest to have an interpretation of the intermediate Jacobian JX in terms of families of 1-cycles; I am grateful to D. Ortland for permission to include a description of his beautiful solution of this. The idea is to look at the net (2-dimensional linear system) $F_\lambda = \sum \lambda_i F_i$ with $\lambda = (\lambda_1, \dots, \lambda_3) \in \Lambda = \mathbf{P}^2$ of weighted hypersurfaces of degree 6 through X, and to note that in coordinates $y_1, \dots, y_4, z_1, \dots, z_3$ of $\mathbf{P}$, each F_λ is of the form

$$F_\lambda = c_\lambda(y_1, \dots, y_4) + q_\lambda(z_1, \dots, z_3)$$

with c_λ cubic and q_λ quadratic. This can be viewed as a net of cubic surfaces in $\mathbf{P}^3$ and a net of plane conics parametrised by the same base space $\Lambda = \mathbf{P}^2$.

Now the conic $Q_\lambda \colon (q_\lambda = 0) \subset \mathbf{P}^2$ breaks up as a line pair when λ belongs to a discriminant curve $E \subset \Lambda$ (for general X this is a nonsingular cubic curve). Also, for general $\lambda \in \Lambda$, the cubic surface $S_\lambda \colon (c_\lambda = 0) \subset \mathbf{P}^3$ is nonsingular and contains 27 lines. Hence the set

$$B = \{\text{pairs } (l,m) \text{ of lines} \mid \exists \lambda \in \Lambda \text{ with } l \subset Q_\lambda \text{ and } m \subset S_\lambda\}$$

is a generically 54-to-1 cover $B \to E$; each pair $b = (l,m) \in B$ corresponds to a weighted linear subspace $\Pi_b = \mathbf{P}(2,2,3,3) \subset \mathbf{P}$ entirely contained in one of the hypersurfaces F_λ. It is easy to see that $C_b = \Pi_b \cap X$ is a curve of genus 2.

A general result of Lefschetz theory says that the Hodge structure on $H^3(X, \mathbf{Q})$ is irreducible for sufficiently general X, hence also the intermediate Jacobian JX, so that the family $\{C_b\}_{b\in B}$ induces an Abel-Jacobi (or cylinder) map $JB \to JX$ which must be either zero or surjective. Finally, Ortland uses methods of Clemens to interpret the derivative of $JB \to JX$ and to prove that it is nonzero.

Speculation. Note that the key to success in Ortland's example is to find some special representation of one of the defining equations: if the line $(l,m) \in B$ is given by $y_1 = y_2 = z_1 = 0$ then the corresponding F_λ is of the form

$$F_\lambda = y_1 q_1(y_1, \dots, y_4) + y_2 q_2(y_1, \dots, y_4) + z_1 z_2$$

(which looks almost like a quadric of rank 6); the 5-fold hypersurface $(F_\lambda = 0)$ has nontrivial 3-cycles, from which X inherits nontrivial curves.

Now there are plenty of other 3-folds of general type with $p_g = 0$ for which the defining equations do not seem to admit such nice representations. (Can you see what to do with the general quasihomogeneous polynomial $f_{18}(x, y_1, y_2, z, t)$, with $\deg x = 2$, $\deg y_i = 3$, $\deg z = 4$, $\deg t = 5$?) Here the generalised Hodge conjecture predicts a family of curves having a nontrivial Abel-Jacobi map, so that this is (to say the least) a substantial case where the Hodge conjecture has yet to be verified.

There is an analogy between these deep questions and the Bloch-Mumford conjecture on the Chow group of 0-cycles on a surface S with $p_g = 0$ (see, for example, [**Inose-Mizukami**]): in this case the traditional conjecture could also be destroyed by proving that there are no nontrivial curves on the 3-fold $S \times \mathbf{P}^1$.

(2.11) *Canonical hypersurfaces.* There are several methods of searching for canonical 3-folds which are weighted complete intersections; for example, this can be done by guessing the invariants going into the plurigenus formula of §10 (that is, an integer χ, a rational number K^3, and a basket of terminal cyclic quotient singularities), then computing the plurigenera, and determining whether or not there exists a complete intersection ring with this as its Hilbert function. This can all be done by computer, and systematic searches have been carried out by A. R. Fletcher, giving rise to many interesting families of varieties.

The following list of canonical weighted hypersurfaces was generated by a much cruder computer program. It is a complete list of $X = X_d \subset \mathbf{P}(a_1, \dots, a_5)$ in a "well-formed" weighted projective space (that is, no 4 of the a_i have a common factor, see [**Dolgachev**, (1.3)]) such that

(i) X has terminal quotient singularities (of the type described in (5.2));
(ii) $K_X = \mathcal{O}_X(1)$;
(iii) $d \le 100$.

(Probably there are no others for any d, but the list was obtained by starting an infinite search and switching off the computer after it stopped printing out data.)

Canonical 3-fold hypersurfaces

$X_6 \subset \mathbf{P}^4$	$p_g = 5,\ K^3 = 6$
$X_7 \subset \mathbf{P}(1^4, 2)$	$p_g = 4,\ K^3 = 7/2$
$X_8 \subset \mathbf{P}(1^3, 2^2)$	$p_g = 3,\ K^3 = 2$
$X_9 \subset \mathbf{P}(1^3, 2, 3)$	$p_g = 3,\ K^3 = 3/2$
$X_{10} \subset \mathbf{P}(1^2, 2^2, 3)$	$p_g = 2,\ K^3 = 5/6$
$X_{12} \subset \mathbf{P}(1, 2^2, 3^2)$	$p_g = 1,\ K^3 = 1/3$
$X_{12} \subset \mathbf{P}(1^2, 2, 3, 4)$	$p_g = 2,\ K^3 = 1/2$
$X_{10} \subset \mathbf{P}(1^4, 5)$	$p_g = 4,\ K^3 = 2$
$X_{15} \subset \mathbf{P}(1, 2, 3^2, 5)$	$p_g = 1,\ K^3 = 1/6$
$X_{16} \subset \mathbf{P}(1, 2, 3, 4, 5)$	$p_g = 1,\ K^3 = 2/15$
$X_{18} \subset \mathbf{P}(2, 3^2, 4, 5)$	$p_g = 0,\ P_2 \neq 0,\ K^3 = 1/20$
$X_{20} \subset \mathbf{P}(2, 3, 4, 5^2)$	$p_g = 0,\ P_2 \neq 0,\ K^3 = 1/30$
$X_{12} \subset \mathbf{P}(1^3, 2, 6)$	$p_g = 3,\ K^3 = 1$
$X_{14} \subset \mathbf{P}(1^2, 2^2, 7)$	$p_g = 2,\ K^3 = 1/2$
$X_{21} \subset \mathbf{P}(1, 3, 4, 5, 7)$	$p_g = 1,\ K^3 = 1/20$
$X_{16} \subset \mathbf{P}(1^2, 2, 3, 8)$	$p_g = 2,\ K^3 = 1/3$
$X_{28} \subset \mathbf{P}(3, 4, 5, 7, 8)$	$p_g = P_2 = 0,\ P_3 \neq 0,\ K^3 = 1/120$
$X_{18} \subset \mathbf{P}(1, 2^2, 3, 9)$	$p_g = 1,\ K^3 = 1/6$
$X_{22} \subset \mathbf{P}(1, 2, 3, 4, 11)$	$p_g = 1,\ K^3 = 1/12$
$X_{28} \subset \mathbf{P}(1, 3, 4, 5, 14)$	$p_g = 1,\ K^3 = 1/30$
$X_{30} \subset \mathbf{P}(2, 3, 4, 5, 15)$	$p_g = 0,\ P_2 \neq 0,\ K^3 = 1/60$
$X_{40} \subset \mathbf{P}(3, 4, 5, 7, 20)$	$p_g = P_2 = 0,\ P_3 \neq 0,\ K^3 = 1/210$
$X_{46} \subset \mathbf{P}(4, 5, 6, 7, 23)$	$p_g = P_2 = P_3 = 0,\ P_4 \neq 0,\ K^3 = 1/420$

Note that K^3 can get fairly small, although it is now known to be bounded below for canonical 3-folds with $\chi(\mathcal{O}_X) \le 1$; see [**Fletcher**] where it is proved (following ideas of J. Kollár) that $P_{12} \neq 0$, $P_{24} \ge 2$, and hence by results of Kollár, $\varphi_{mK}\colon X \to \mathbf{P}^N$ is birational for $m \ge 269$, and so in particular, $K_X^3 \ge (1/269)^3$.

(2.12) *Exercise.* Find the singularities of some of these canonical hypersurfaces; write (x, y, z, t, u) for homogeneous coordinates on the 4-dimensional weighted projective space. Consulting [**Dolgachev**] for information on weighted projective spaces, you can prove, for example, that

(i) $X_{15} \subset \mathbf{P}(1, 2, 3^2, 5)$ has $\chi(\mathcal{O}_X) = 0$, $K^3 = 1/6$, and singularities:

1 point of type $\frac{1}{2}(1,1,1)$ at $(0,1,0,0,0)$
and 5 points of type $\frac{1}{3}(2,1,1)$ along the (z,t)-axis.

(ii) $X_{18} \subset \mathbf{P}(2, 3^2, 4, 5)$ has $\chi(\mathcal{O}_X) = 1$, $K^3 = 1/20$, and singularities:

4 points of type $\frac{1}{2}(1,1,1)$ along the (x,t)-axis;
6 points of type $\frac{1}{3}(2,1,1)$ along the (y,z)-axis;
1 point of type $\frac{1}{4}(3,1,1)$ at $(0,0,0,1,0)$;
and 1 point of type $\frac{1}{5}(3,2,1)$ at $(0,0,0,0,1)$.

(The notation for the quotient singularities is explained in (4.2).)

3. The main reduction steps.

(3.0) *Overview.* This section gives a brief run-down of the general theory of canonical singularities under the following 6 headings:

(A) Canonical $\Rightarrow$ Du Val singularities in codimension 2.
(B) Reduction to index 1 by cyclic covers.
(C) Index 1 canonical $\Rightarrow$ rational $\Rightarrow$ Cohen-Macaulay.

(D) The general section through a rational Gorenstein singularity is a rational or elliptic singularity.

(E) Reduction to cDV singularities by crepant blow-ups.

(F) Further reduction to isolated cDV singularities.

The final 7th topic

(G) Mori's detailed study of terminal singularities

will be the subject of Chapter II, §§6–7. (The material of (A)–(D) is valid in all dimensions, but (E)–(G) is restricted to $\dim X = 3$; see (3.13).)

(3.1) The overview (3.0) has introduced two new definitions:

DEFINITION. A *cDV singularity* is a 3-fold hypersurface singularity

$$P \in X\colon (F = 0) \subset \mathbf{A}^4$$

given by an equation of the form

$$F(x, y, z, t) = f(x, y, z) + tg(x, y, z, t).$$

where f is the equation of a Du Val singularity (as in (1.2)), and g is an arbitrary polynomial. So a cDV point is just a 3-fold singularity which has a Du Val surface singularity as a hyperplane section; on the other hand, a cDV point can be viewed as a 1-parameter deformation of a Du Val singularity.

DEFINITION. A birational morphism $f\colon Y \to X$ between normal varieties is *crepant* if $K_Y = f^*K_X$. If rK_X is a Cartier divisor, this means that a local basis element $s \in \mathcal{O}_X(rK_X)$ at $P \in X$ remains a local basis around $f^{-1}P$; (here and elsewhere there is an abuse of notation in writing f^*K_X, since K_X is not Cartier: by definition this means $(1/r)f^*(rK_X) \in \operatorname{Pic} Y \otimes \mathbf{Q}$). Note that the definitions of canonical and terminal singularities differ only in that canonical singularities are allowed to have exceptional divisors E_i appearing with multiplicity 0 in the discrepancy. A crepant partial resolution of a variety X with canonical singularities is one which pulls out only such exceptional divisors; the key example is a blow-up of a Du Val surface singularity.

(3.2) The goal of steps (A)–(F) is the following theorem, the main result of [**Pagoda**]:

THEOREM. (a) *Any terminal 3-fold point $P \in X$ is of the form Y/μ_r, where $Q \in Y$ is an isolated cDV singularity (or nonsingular), and μ_r acts on Y freely outside Q and such that on a generator $s \in \omega_Y$,*

$$\mu_r \ni \varepsilon\colon s \mapsto \varepsilon s.$$

(b) *If X is a 3-fold with canonical singularities then there exists a crepant partial resolution $f\colon Y \to X$ where Y has only terminal singularities.*

(3.3) The result (a) is a partial classification of terminal singularities. (Most of §§5–7 will be devoted to the further classification of the singularities of (a).) On the other hand (b) represents a certain reduction of all canonical singularities to terminal singularities. Compare the situation with the surface case: for surfaces, the canonical points are just the Du Val singularities; the terminal singularities

are just nonsingular points. As mentioned in (1.2), the key fact is that there is a resolution $f\colon Y \to X$ such that $K_Y = f^*K_X$. (Note however that for 3-folds, the partial resolution given by (b) is not unique, so that if you've ever heard of the "absolutely minimal models" of surface theory you should do your best to forget about them in higher dimension.)

Finally, one of the key consequences of (G) (see (6.4), (A)) will be that if $P \in X$ is a terminal singularity, then it has a *Q-smoothing*, that is, a deformation X_t such that all the singular points $P_i \in X_t$ of a neighbouring fibre are terminal cyclic quotient singularities $\mathbf{A}^3/\mu_r$, where the action is

$$\mu_r \ni \varepsilon\colon (x, y, z) \mapsto (\varepsilon^a x, \varepsilon^{-a} y, \varepsilon z).$$

This reduces certain problems on 3-folds with canonical singularities to this special class of quotient singularities; in particular, this is the key to the plurigenus formula of §10 below. One can think of this as saying vaguely that a representative sample of canonical 3-folds has only this type of cyclic quotient singularities, in the same way that "most" canonical surfaces are nonsingular (so that the minimal model does not contain (-2)-curves); beware that there is definitely no theorem to this effect, even for surfaces.

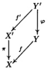
I now run through the points (A)–(F) in more detail; (G) will be the main subject of §§5–7.

(3.4) (A) *Canonical $\Rightarrow$ Du Val singularities in codimension* 2. Canonical singularities are not necessarily isolated, but if X has canonical singularities then X is analytically of the form

$$X \cong (\text{Du Val singularity}) \times \mathbf{A}^{n-2}$$

in a neighbourhood of a general point of any codimension 2 stratum. This is what you would expect, and is easy to prove; see [**C3-f**, (1.14)].

(3.5) (B) *Reduction to index* 1 *by cyclic covers.* There are several points to make here, since cyclic covers are used in various ways. Firstly, if $P \in X$ is a canonical singularity then there is a standard local μ_r-cover $\pi\colon X' \to X$ with $K_{X'}$ a Cartier divisor and $K_{X'} = \pi^*K_X$. This construction is discussed in detail in (3.6) below. Next, in various contexts there are diagrams of the form



where f and f' are partial resolutions. In this set-up, I have

$$K_{X'} = \pi^*K_X \quad \text{and} \quad K_{Y'} = \varphi^*K_Y + R_\varphi,$$

where R_φ is the ramification divisor of φ. Taking f and f' to be resolutions in this diagram, it is easy to see that if $P \in X$ is canonical, then $Q \in X'$ is canonical of index 1 (where $Q = \pi^{-1}P$).

A different way of using the same kind of diagram is to take f' to be some construction made in an intrinsic way from an index 1 point, for example, a crepant blow-up. Then I can construct $f: Y \to X$ by taking $Y = Y'/\mu_r$ to be the quotient of Y' by the group action, and use the same kind of discrepancy calculation to show that f is also crepant. In short, this kind of argument allows me to reduce the study to the index 1 case; for details, see **[Pagoda**, §2].

(3.6) *Cyclic covering trick.* Suppose that $P \in X$ is a point of a normal variety, and D is a Weil divisor on X which is **Q**-Cartier; this means that D is a Weil divisor of X, and rD is a Cartier divisor near P for some $r \in \mathbf{Z}$, $r > 0$. Suppose that r is the smallest such r (the *index* of D). Let $s \in \mathcal{O}_X(-rD)$ be a local basis near P; by taking a smaller neighbourhood of P, I will assume that s is a basis of $\mathcal{O}_X(-rD)$ over the whole of X, and view s as giving an isomorphism $s: \mathcal{O}_X(rD) \xrightarrow{\sim} \mathcal{O}_X$.

PROPOSITION. *There exists a cover $\pi: Y \to X$ which is Galois with group μ_r, and such that the sheaves $\mathcal{O}_X(iD)$ are the eigensheaves of the group action on $\pi_*\mathcal{O}_Y$, that is,*

$$(*) \qquad \mathcal{O}_X(iD) = \{f \in \pi_*\mathcal{O}_Y \mid \varepsilon(f) = \varepsilon^i \cdot f \text{ for all } \varepsilon \in \mu_r\}.$$

*Also, Y is normal, π is etale over the locus X_0 where D is Cartier, and $\pi^{-1}P = Q$ is a single point. The **Q**-divisor $\pi^*D = E$ is a Cartier divisor on Y.*

(3.7) This is an important reduction of the problem, since working directly with the singular sheaves $\mathcal{O}_X(iD)$ is likely to be difficult. The proof can be understood as follows: over X_0, the invertible sheaf $\mathcal{O}_X(-D)$ corresponds to a line bundle $L_0 \to X_0$. A local generator $z \in \mathcal{O}_X(-D)$ is a coordinate on the fibres of L_0; now consider the locus $Y_0\colon (z^r = s) \subset L_0$. Since s is a nowhere vanishing section of $\mathcal{O}_X(-rD)$, the projection map $\pi_0: Y_0 \to X_0$ is etale. The idea of the proof is to extend this over the whole of X.

PROOF. Using the given section s, construct the sheaf of $\mathcal{O}_X$-algebras

$$\mathcal{A} = \mathcal{O}_X \oplus \mathcal{O}_X(D) \oplus \cdots \oplus \mathcal{O}_X((r-1)D),$$

with multiplication defined by

$$\mathcal{O}_X(aD) \otimes \mathcal{O}_X(bD) \to \mathcal{O}_X((a+b)D) \quad \text{if } a+b < r,$$

or

$$\mathcal{O}_X(aD) \otimes \mathcal{O}_X(bD) \to \mathcal{O}_X((a+b)D) \xrightarrow{s} \mathcal{O}_X((a+b-r)D) \quad \text{if } a+b \geq r.$$

There is a natural action of μ_r on $\mathcal{A}$ given by multiplication by ε^i in the summand $\mathcal{O}_X(iD)$. Then $\pi: Y = \operatorname{Spec}_X \mathcal{A} \to X$ is a cyclic Galois cover which is etale over X; the fact that Y is normal follows by the Serre criterion (see (3.18)). $\pi^{-1}P = Q$ is a single point, since otherwise the subgroup of μ_r stabilizing a point $Q \in \pi^{-1}P$ would be a proper subgroup, and then it is easy to get a contradiction to $r = \operatorname{index}(D)$.

To see the last sentence, suppose without loss of generality that D is an effective divisor. Then the inclusion maps $\mathcal{O}_X((i-1)D) \hookrightarrow \mathcal{O}_X(iD)$, together

with the isomorphism $s: \mathcal{O}_X(rD) \xrightarrow{\sim} \mathcal{O}_X$ defines an $\mathcal{A}$-linear map $g: \mathcal{A} \to \mathcal{A}$ such that g^r is the local equation of $\pi^*(rD)$. Thinking of g as a section of $\mathcal{O}_Y$, clearly $\operatorname{div}(g) = E = \pi^*D$.

(3.8) (C) *Canonical singularities are rational.* It is known quite generally that in characteristic zero, canonical singularities are rational and therefore Cohen-Macaulay; since a quotient of a rational singularity is again rational (this is easy for a quotient $Y = X/G$ by a finite group G, essentially because if

$$\begin{array}{ccc} Y' & \xrightarrow{\pi} & X' \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

where the vertical arrows are resolutions, then $\mathcal{O}_{X'}$ is a direct summand of $\pi_*\mathcal{O}_{Y'}$; see, for example, **[Pinkham**, p. 150] for details), it is in any case enough to prove this for index 1 singularities:

THEOREM **[Elkik; Flenner**, (1.3)]. *Let $P \in X$ be a canonical singularity of index 1 (that is, K_X is Cartier at P, and $f_*\omega_Y = \omega_X$ for a resolution $f: Y \to X$). Then for a resolution $f: Y \to X$,*

$$R^i f_*\mathcal{O}_Y = 0 \quad \textit{for all } i > 0.$$

There are two ingredients in any proof of this: (1) vanishing and (2) duality. Let me run through Shepherd-Barron's proof in the 3-fold case, where these appear in a transparent way; although this is now a standard result, it still seems rather miraculous to me.

PROOF. Let $P \in X$ be a canonical index 1 point, and $f: Y \to X$ a resolution which is the minimal resolution along the Du Val locus. Grothendieck duality for the morphism f gives at once that

$$R^2 f_*\mathcal{O}_Y \stackrel{d}{=} \omega_X / f_*\omega_Y,$$

which is zero by assumption, so that I must prove that $R^1 f_*\mathcal{O}_Y = 0$.

Now $K_Y = f^*K_X + \Delta$, where Δ is an effective divisor with $f(\Delta) = P$. Vanishing gives

$$R^i f_*\omega_Y = 0 \quad \text{for all } i > 0.$$

However, above a neighbourhood of P, $\omega_Y = \mathcal{O}_Y(\Delta)$, so that also $R^i f_*\mathcal{O}_Y(\Delta) = 0$. Bearing this in mind, consider the cohomology long exact sequence of

$$0 \to \mathcal{O}_Y \to \mathcal{O}_Y(\Delta) \to \mathcal{O}_\Delta(\Delta) \to 0.$$

Since X is normal, $f_*\mathcal{O}_Y = f_*\mathcal{O}_Y(\Delta) = \mathcal{O}_X$, and the long exact sequence becomes

$$H^i(\Delta, \mathcal{O}_\Delta(\Delta)) = R^{i+1} f_*\mathcal{O}_Y \quad \text{for } i = 0, 1, 2.$$

Since the fibres of f have dimension ≤ 2, it follows that

$$H^2(\Delta, \mathcal{O}_\Delta(\Delta)) = R^3 f_*\mathcal{O}_Y = 0.$$

On the other hand, Δ is a Gorenstein scheme with dualising sheaf

$$\mathcal{O}_\Delta(K_Y + \Delta) = \mathcal{O}_\Delta(2\Delta),$$

and therefore Serre duality on Δ gives

$$H^0(\Delta, \mathcal{O}_\Delta(\Delta)) \overset{d}{=} H^2(\Delta, \mathcal{O}_\Delta(\Delta)) = 0.$$

This proves that $R^1 f_*\mathcal{O}_Y = 0$, so that $R^i f_*\mathcal{O}_Y = 0$ for $i > 0$, and $P \in X$ is a rational singularity. Q.E.D.

(3.9) Rational singularities are known to be Cohen-Macaulay; the Appendix to §3 contains all you need to know about this notion, including a direct and self-contained proof of the case of the result required here: the general hypersurface section through a rational 3-fold singularity $P \in X$ is again normal (see (3.19)).

(3.10) (D) *The general section.*

THEOREM. *Let $P \in X$ be a rational Gorenstein singularity (of an n-fold X, with $n \geq 3$). Then the general hyperplane section $P \in S \subset X$ through P is a rational or elliptic Gorenstein singularity.*

Here elliptic Gorenstein means that for a resolution $f\colon T \to S$,

$$\varphi_*\omega_T = m_p \cdot \omega_S$$

(or equivalently $R^{n-1}f_*\mathcal{O}_T$ is 1-dimensional).

PROOF. Suppose that S runs through any linear system of sections $P \in S \subset X$ whose equations generate the maximal ideal m_P of $\mathcal{O}_{X,P}$. Then as noted in (3.19), a general element S of this linear system is normal.

Let $f\colon Y \to X$ be any resolution of X which dominates the blow-up of the maximal ideal m_P; by definition of the blow-up, the scheme-theoretic fibre over P is an effective divisor E such that $m_P{\cdot}\mathcal{O}_Y = \mathcal{O}_Y(-E)$. Hence $f^*S = T+E$, where T runs through a free linear system on Y. By Bertini's theorem, $\varphi = f|_T\colon T \to S$ is a resolution of S. Now I use the adjunction formula to compare K_T and φ^*K_S.

In the diagram

$$\begin{array}{ccc} Y & \supset & T+E \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \varphi} \\ X & \supset & S, \end{array}$$

I have

$$K_Y = f^*K_X + \Delta \quad \text{with } \Delta \geq 0$$

and

$$T = f^*S - E,$$

so that

$$K_Y + T = f^*(K_X + S) + \Delta - E$$

and

$$K_T = (K_Y + T)_T = \varphi^*K_S + (\Delta - E)_S.$$

This just means that any $s \in \omega_S$ has at worst $(E - \Delta)_T$ as pole on T. On the other hand, since the maximal ideal $m_{S,P} \subset \mathcal{O}_{S,P}$ is the restriction to S of the maximal ideal $m_{X,P} \subset \mathcal{O}_X$ (this is where the argument uses Cohen-Macaulay in an essential way), it follows that every element of $m_{S,P}$ vanishes along $E \cap T$. Hence every element of $m_{S,P} \cdot \omega_S$ is regular on T, that is,

$$\varphi_*\omega_T \supset m_P \cdot \omega_S. \quad \text{Q.E.D.}$$

(3.11) (D) says that if $\dim X = 3$ and $P \in X$ is a canonical index 1 singularity, then *either* it is cDV, *or* a general hyperplane section through P is an elliptic Gorenstein surface singularity. From now on I will use special results on the classification of elliptic Gorenstein surface singularities, so that the remainder of the discussion is restricted to $\dim X = 3$.

(3.12) (E) *Reduction to cDV points by crepant blow-ups.* If $P \in X$ is a canonical index 1 point which is not cDV then there exists a blow-up $\sigma\colon X' \to X$ such that $K_{X'} = \sigma^*K_X$. This follows by using analogous properties of elliptic Gorenstein singularities; see [**C3-f**, (2.11–12)] for details.

COROLLARY. (i) *A rational Gorenstein 3-fold singularity $P \in X$ is terminal if and only if it is cDV.*

(ii) *Let $P \in X$ be a canonical index 1 point; then there exists a partial resolution $f\colon Y \to X$ which is crepant (that is, $K_Y = f^*K_X$) and such that Y has cDV singularities.*

EXAMPLES. (i) Suppose that $P \in X$ is the hypersurface singularity X: $(f = 0)$ where $f = x^3 + y^3 + z^3 + t^n$ with $n \geq 3$; then the blow-up $\sigma\colon X' \to X$ of m_P is the variety X': $(f' = 0)$, where $f' = x^3 + y^3 + z^3 + t^{n-3}$. Essentially the same calculation as in (1.9), (1) shows that $K_{X'} = \sigma^*K_X$.

(ii) A hypersurface double point is rather special, and in this case the required blow-up σ is not just the blow-up of m_P. For example, let $P \in X$: $(f = 0)$ where $f = x^2 + y^3 + z^6 + t^n$ with $n \geq 6$; then the required blow-up is given in one affine piece by setting

$$x = x_1 t^3, \qquad y = y_1 t^2, \qquad z = z_1 t.$$

The blown-up variety is then X': $(f' = 0)$, where $f' = x_1^2 + y_1^3 + z_1^6 + t^{n-6}$. The proof that $K_{X'} = \sigma^*K_X$ in this case is similar to the calculation of (1.9), (1), and you can try it as an amusing exercise.

(3.13) It is important to understand that Corollary 3.12, (i) is proved via the classification of elliptic Gorenstein surface singularities. The statement that a terminal index 1 singularity has a rational hypersurface section is false for 4-fold singularities, as shown by the following example (one of a large class related to weighted $K3$ hypersurfaces):

$$0 \in X\colon (x^3 + y_1^4 + y_2^4 + z_1^6 + z_2^6 = 0) \subset \mathbf{A}^5;$$

here X is a terminal 4-fold singularity. This follows by the argument of Theorem 4.6 (see also [**C3-f**, (4.3)]), essentially because

$$\tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{6} + \tfrac{1}{6} = 1 + \tfrac{2}{12} > 1 + \tfrac{1}{12}.$$

However, any hypersurface section $0 \in H \subset X$ is an irrational singularity: for example the hyperplane section $(z_2 = 0)$ is a weighted cone over a $K3$ surface.

This is one aspect of the fact that there are very many more terminal singularities in dimension ≥ 4 than in dimension 3, and it seems unrealistic to expect any useful classification.

(3.14) (F) *Further reduction to isolated cDV singularities.* The procedure of (3.12) reduces 3-fold canonical singularities of index 1 to cDV points, but the singularities are not necessarily isolated. Let X be a 3-fold with at worst cDV singularities, and suppose that Γ is an irreducible curve component of $\operatorname{Sing} X$; then as mentioned in (3.4), X is analytically isomorphic to $\Gamma \times$ (Du Val singularity) in a neighbourhood of a general point of Γ (and possibly worse at a finite set of dissident points). Above this neighbourhood, the blow-up of X along Γ is just $\Gamma\times$ the blow-up of a Du Val singularity, which is crepant. The idea is to extend this crepant blow-up along the whole of Γ, so that the nonisolated singular locus of X can be reduced by a crepant morphism $f\colon Y \to X$.

The key to understanding this situation is the Brieskorn-Tyurina theory of simultaneous resolution of families of Du Val singularities. If $P \in X$ is a cDV point and $t \in m_P$ is such that the section $X_0\colon (t = 0) \subset X$ is a Du Val singularity, then the map $t\colon X \to \mathbf{A}^1 = T$ represents X as a deformation of X_0 over a 1-dimensional parameter space T. Now it is well known [**Brieskorn**] that after making a base change

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ T' & \longrightarrow & T \end{array}$$

by a cyclic branched cover $T' \to T$ of the base, the family admits a simultaneous resolution, that is, there is a morphism $f\colon Y \to X'$ which fibre-by-fibre is the minimal resolution of the fibres of $X \to T$. Since we know everything about Du Val singularities and their minimal resolution, there is a lot of information available on the resolution $Y \to X'$. On the other hand $X' \to X$ is just a cyclic cover, so that the relation between singularities of X' and those of X can be studied by the methods mentioned in (3.5).

Putting together these ideas shows firstly that isolated cDV points are exactly the terminal singularities of index 1, and that nonisolated cDV singularities can be blown-up along the Du Val locus to give crepant partial resolutions; this leads to a proof of Theorem 3.2 (see [**Pagoda**] for details).

Appendix to §3. Cohen-Macaulay and all that.

(3.15) The following two properties are the main things you need to know about Cohen-Macaulay (CM):

(i) *Invariance under passing to a hyperplane section.* CM is a property of the local ring $\mathcal{O}_{X,P}$ of a point $P \in X$ of a scheme X (think of P as the generic point of an irreducible subvariety of X); if P has codimension 0 then X is CM at P by definition. Otherwise $P \in X$ is CM if and only if there exists an element $h \in m_{X,P}$ of the maximal ideal which is a non-zerodivisor of $\mathcal{O}_{X,P}$ and such that $P \in Y$ is CM, where $Y \subset X$ is the subscheme defined locally by the principal ideal sheaf $\mathcal{I}_Y = h\mathcal{O}_X$, that is, $Y\colon (h = 0) \subset X$.

(ii) *The Serre criterion.* An isolated surface singularity is CM if and only if it is normal.

So by definition, a 3-fold X which is nonsingular in codimension 1 is CM if and only if there is a normal surface section through every point $P \in X$. Despite the geometric significance of this notion, young persons seem to find it hard to grasp, and I give a brief treatment.

(3.16) *The definition of depth and Serre's condition S_k.* Given a point P of a scheme X, there is a well-defined integer

$$d = \operatorname{depth}_P \mathcal{O}_X$$

with the property that there exist chains of subschemes of length d

$$(*) \qquad P \in X_d \subset X_{d-1} \subset \cdots \subset X_1 \subset X_0 = X,$$

where for each i, $X_{i+1} \subset X_i$ is the subscheme defined locally by the principal ideal sheaf $h_i\mathcal{O}_{X_i}$ with h_i a non-zerodivisor of $\mathcal{O}_{X_i,P}$, and no such chains of length $d+1$. In this situation,

either $\dim X_d = 0$, and $P \in X$ is Cohen-Macaulay,
or $\dim X_d \geq 1$, and every element of $m_{X_d,P}$ is a zero-divisor of $\mathcal{O}_{X_d,P}$;

by elementary results in primary decomposition, the second possibility happens if and only if there exists $0 \neq f \in \mathcal{O}_{X_d,P}$ such that $m_{X_d,P} \cdot f = 0$, that is, f is a section of $\mathcal{O}_{X_d}$ whose support is just $\{P\}$.

EXAMPLE. Let $X \subset \mathbf{A}^2$ be the subscheme given by $(x^2 = xy = 0)$; then $0 \neq x \in \Gamma(\mathcal{O}_X)$ is killed by the maximal ideal $m = (x, y)$; hence m does not contain any non-zerodivisor, and $\operatorname{depth}_0 \mathcal{O}_X = 0$.

This is of course just a geometrical translation of the algebraic definition of depth in terms of regular sequences. There is only one thing to be checked: that the property is independent of the choice of the chain $(*)$, or equivalently, that the statement of (3.15), (i) is independent of the choice of h; see for example [**Matsumura**, (16.3)].

DEFINITION. A scheme X *satisfies condition S_k* if for every point $P \in X$,

$$\operatorname{depth}_P X \geq \min(k, \operatorname{codim} P).$$

It follows at once from the above discussion that X fails to satisfy S_1 if and only if there exists a section $f \in \Gamma(U, \mathcal{O}_X)$ of $\mathcal{O}_X$ (over an open $U \subset X$) whose support has codimension ≥ 1 in X; since f is necessarily nilpotent, this can't happen for an integral scheme: a variety automatically satisfies S_1.

(3.17) Now I discuss the S_2 condition. Let X be an integral scheme and $k(X)$ its function field.

LEMMA. *Let $Q \in X$ be a point of an integral scheme; then*

$$\operatorname{depth}_Q \mathcal{O}_X = 1 \iff \exists f \in k(X) \text{ s.t. } f \notin \mathcal{O}_{X,Q} \text{ but } m_{X,Q} \cdot f \subset \mathcal{O}_{X,Q}$$

PROOF. ($\Leftarrow$) For any $0 \neq x$, $y \in m_{X,Q}$, both $xf, yf \in \mathcal{O}_{X,Q}$, but $xf \notin (x)$ (otherwise $f \in \mathcal{O}_{X,Q}$). Then y is a zero-divisor in $\mathcal{O}_{X,Q}/(x)$, since $y(xf) = x(yf)$; this proves that $\operatorname{depth}_Q \mathcal{O}_X \leq 1$.

($\Rightarrow$) Let $0 \neq x \in m_{X,Q}$; x is automatically a non-zerodivisor. Then by the assumption $\operatorname{depth}_Q \mathcal{O}_X = 1$, there exists $0 \neq \overline{g} \in \mathcal{O}_{X,Q}/(x)$ which is killed by $m_{X,Q}$. Let $g \in \mathcal{O}_{X,Q}$ be any lift of $\overline{g}$; then $m_{X,Q} \cdot g \subset (x)$, so that $f = g/x \in k(X)$ satisfies $f \notin \mathcal{O}_{X,Q}$ but $m_{X,Q} \cdot f \subset \mathcal{O}_{X,Q}$. Q.E.D.

EXAMPLE (Macaulay). Let $S = \operatorname{Spec} k[x^4, x^3y, xy^3, y^4]$; this is the affine cone over an embedding $C \subset \mathbf{P}^3$ of $\mathbf{P}^1$ as a quartic in $\mathbf{P}^3$, which is not linearly normal. Then S has depth 1 only, since $x^2y^2 \notin \mathcal{O}_S$.

(3.18) *The Serre criterion: normality and* S_2. It is well known that a rational function on a normal variety X with no poles along divisors is regular on X. (In commutative algebra, the assertion is that a normal Noetherian domain is an intersection of DVR's, see [**Matsumura**, (11.5)].)

Now let X be an affine integral scheme; say that a rational function $f \in k(X)$ is *quasiregular* if $f \in \mathcal{O}_{X,P}$ for every codimension 1 point $P \in X$.

THEOREM. *An integral scheme X satisfies S_2 if and only if*

$$\textit{quasiregular} \Rightarrow \textit{regular};$$

in other words, for an open set $U \subset X$,

$$\Gamma(U, \mathcal{O}_X) = \{f \in k(X) | f \in \mathcal{O}_{X,P} \ \forall \text{ codim. 1 points } P \in U\}.$$

In particular,

$$\textit{normal} \iff \textit{regular (nonsingular) in codimension 1 and } S_2.$$

PROOF. Given a quasiregular element $f \in k(X)$, the set $\Sigma = \Sigma(f)$ of points $P \in X$ such that $f \notin \mathcal{O}_{X,P}$ is closed, and $\operatorname{codim} \Sigma \geq 2$. By the Nullstellensatz, if Q is a generic point of a component of Σ then $(m_Q)^k \cdot f \subset \mathcal{O}_{X,Q}$, so that a suitable multiple g of f satisfies $g \notin \mathcal{O}_{X,Q}$ but $m_Q \cdot g \subset \mathcal{O}_{X,Q}$, and by the lemma, $\operatorname{depth}_Q \mathcal{O}_X = 1$. This proves the first part: if X is S_2 then $\Sigma = \emptyset$, and conversely.

If X is regular in codimension 1 and S_2 then the local ring $\mathcal{O}_{X,P}$ at each prime divisor is normal, and by what I've just proved, $\Gamma(U, \mathcal{O}_X)$ is an intersection of these, hence again normal. Conversely, if X is normal then so is the local ring $\mathcal{O}_{X,P}$ at each prime divisor, and hence $\mathcal{O}_{X,P}$ is a DVR (see [**Matsumura**, (11.2)]); this gives regular in codimension 1, and by the fact that a Noetherian normal domain is an intersection of DVRs, quasiregular implies regular, which gives S_2. Q.E.D.

Notice that the theorem must be stated in terms of scheme-theoretic points of X; for example, if S is as in Example 3.17 and $X = \mathbf{A}^1 \times S$ then the closed points of X have depth ≥ 2.

(3.19) *Rational $\Rightarrow$ Cohen-Macaulay.* The general proof that rational singularities are Cohen-Macaulay in characteristic 0 seems to involve vanishing and two applications of duality; the 3-fold case can be done much more simply.

THEOREM. *Let $P \in X$ be a normal 3-fold singularity and $P \in S \subset X$ a general hypersurface section. Then $R^1 f_* \mathcal{O}_Y = 0$ for a resolution $f\colon Y \to X$ implies that P is a normal point of S, and hence $P \in X$ is CM.*

PROOF. Write $m_P = m_{X,P} \subset \mathcal{O}_{X,P}$ for the maximal ideal, and $|m_P|$ for the linear system of all hypersurface sections through P. An element $S \in |m_P|$ is just a surface section $P \in S \subset X$. Then since $|m_P|$ is very ample outside P, a suitably general $S \in |m_P|$ is nonsingular outside P (by the trivial Bertini theorem). So the question is to prove that S is normal.

Let $f\colon Y \to X$ be a resolution which dominates the blow-up of m_P; then $f^*S = T + E$, where E is the scheme-theoretic fibre, and T is a surface moving in a free linear system. By Bertini (using characteristic 0), T is a resolution of S, so that $\mathcal{O}_S \subset f_*\mathcal{O}_T$ and S is normal if and only if $f_*\mathcal{O}_T = \mathcal{O}_S$. Since $\mathcal{O}_S$ is generated as a vector space by k and m_P, this is equivalent to saying that $f_*\mathcal{O}_T(-E) = m_P$. But this follows from the cohomology long exact sequence of

$$0 \to \mathcal{O}_Y(-f^*S) \to \mathcal{O}_Y(-E) \to \mathcal{O}_T(-E) \to 0.$$

Indeed, $R^1 f_* \mathcal{O}_Y(-f^*S) = (R^1 f_* \mathcal{O}_Y) \otimes \mathcal{O}_X(-S) = 0$ (by $R^1 f_* \mathcal{O}_Y = 0$); hence

$$f_*\mathcal{O}_Y(-E) \to f_*\mathcal{O}_T(-E)$$

is surjective. However, $f_*\mathcal{O}_Y(-E) = m_{X,P}$, and $m_{X,P}$ maps to $m_{S,P} \subset \mathcal{O}_S$. This proves S is normal.

Chapter II
Classification of 3-Fold Terminal Singularities

4. Toric methods for hyperquotient singularities.

(4.1) *Hyperquotient singularities in general.* This section is pure toric geometry. I put together the notation in force throughout §§4–7.

Pedantry. Recall that μ_r denotes the cyclic group of rth roots of unity in k; the choice of a primitive rth rooth of unity ε defines an isomorphism $\mathbf{Z}/r \cong \mu_r$, but I want to avoid making this choice. The point is that the action of μ_r on any k-vector space V will break it up into 1-dimensional irreducible eigenspaces, where the action is given by $\mu_r \ni \varepsilon\colon v \mapsto \varepsilon^a v$ for some $a \in \mathbf{Z}/r$; in the notation ε^a, think of the element $a \in \mathbf{Z}/r$ as a character of μ_r (the endomorphism $\mu_r \to \mu_r$ given by $\varepsilon \mapsto \varepsilon^a$). The advantage of distinguishing elements $\varepsilon \in \mu_r$ from characters $a \in \mathbf{Z}/r$ is analogous to that of distinguishing between a vector space and its dual.

Suppose that

$$Q \in Y : (f = 0) \subset \mathbf{A}^{n+1}$$

is a hypersurface singularity with an action of μ_r, and $P \in X = Y/\mu_r$ is the quotient; I'm interested in saying when the singularity $P \in X$ is canonical (or terminal, log canonical, etc.) in terms of the action of μ_r and the Newton polyhedron of f. I always assume that the group action is free in codimension 1 on Y (so "no quasireflections"). The two cases

$$r = 1 \quad \text{and} \quad Y\colon (x_0 = 0) = \mathbf{A}^n \subset \mathbf{A}^{n+1}$$

are not excluded, so that this class includes both cyclic quotient singularities and hypersurface singularities. Points $P \in X$ of this kind are hereby christened *hyperquotient singularities.*

(4.2) *Type of a singularity.* Any cyclic quotient singularity is of the form $X = \mathbf{A}^n/\mu_r$; the action μ_r on $\mathbf{A}^n$ can be diagonalised, and is then given by

$$\mu_r \ni \varepsilon\colon (x_1, \dots, x_n) \mapsto (\varepsilon^{a_1} x_1, \dots, \varepsilon^{a_n} x_n)$$

for certain $a_1, \dots, a_n \in \mathbf{Z}/r$. The singularity is determined by a knowledge of r and $a_1, \dots, a_n$, and I define $\frac{1}{r}(a_1, \dots, a_n)$ to be the *type* of X; there is a reason for the fractional notation in toric geometry, although you can think of it as purely symbolic.

Now return to the set-up of (4.1); in suitable local analytic coordinates, the group action on Y extends to an action on $\mathbf{A}^{n+1}$ (it acts on the tangent space $T_{Y,Q}$), and I can assume that there, μ_r acts diagonally by

$$\mu_r \ni \varepsilon\colon (x_0, \dots, x_n) \mapsto (\varepsilon^{a_0} x_0, \dots, \varepsilon^{a_n} x_n).$$

Since Y is fixed by the action of μ_r, it follows that f is an eigenfunction, so that

$$\mu_r \ni \varepsilon\colon f \mapsto \varepsilon^e f;$$

the symbol $\frac{1}{r}(a_0, \dots, a_n; e)$ is the *type* of the hyperquotient singularity $P \in X$. It will be useful to note that the action of μ_r on the standard generator

$$s = \operatorname{Res}_Y \frac{dx_0 \wedge \dots \wedge dx_n}{f} = \frac{dx_1 \wedge \dots \wedge dx_n}{\partial f/\partial x_0} \in \omega_Y$$

(see (1.8)) is given by

$$\mu_r \ni \varepsilon\colon s \mapsto \varepsilon^c s, \quad \text{with } c = a_0 + \dots + a_n - e.$$

The assumption that the group acts freely in codimension 1 on Y implies that for any divisor $d|r$,

$$\#\{i | d \text{ divides } a_i\} \le n - 1.$$

(4.3) Write $\overline{M} \cong \mathbf{Z}^{n+1}$ for the lattice of monomials on $\mathbf{A}^{n+1}$, and $\overline{N}$ for the dual lattice; then define N to be the overlattice of $\overline{N}$ given by

$$N = \overline{N} + \mathbf{Z} \cdot \tfrac{1}{r}(a_0, \dots, a_n).$$

Thus

$$\alpha \in N \iff \alpha \equiv \tfrac{1}{r}(ja_0, \dots, ja_n) \mod \mathbf{Z}^{n+1} \text{ for some } j = 0, \dots, r-1.$$

Let $M \subset \overline{M}$ be the dual sublattice. (Each of these lattices is $\cong \mathbf{Z}^{n+1}$, so it is important to give each its own name; think of M as monomials, and N as

weightings or valuations of monomials.) The point of this construction is just that μ_r acts on a monomial $x^m = x_0^{m_0} \cdots x_n^{m_n}$ by

$$\mu_r \ni \varepsilon\colon x^m \mapsto \varepsilon^{\alpha(m)} x^m, \quad \text{with } \alpha(m) = \textstyle\sum a_i m_i,$$

so that

$$x^m \text{ is invariant under } \mu_r \iff \alpha(m) \equiv 0 \bmod r \iff m \in M.$$

Write σ for the positive quadrant in $N_{\mathbf{R}}$ and $\sigma^\vee$ for the dual quadrant in $M_{\mathbf{R}}$. Then as usual in toric geometry,

$$\mathbf{A}^{n+1} = \operatorname{Spec} k[x_0, \dots, x_n] = \operatorname{Spec} k[\overline{M} \cap \sigma^\vee],$$

and the quotient, corresponding to polynomials invariant under the action of μ_r, is

$$A = \mathbf{A}^{n+1}/\mu_r = \operatorname{Spec} k[M \cap \sigma^\vee].$$

Notice that $P \in X \subset A$, so that the quotient singularity still lives naturally in an ambient space, but is not necessarily a Cartier divisor there: the ideal

$$I_X \subset k[M \cap \sigma^\vee]$$

is the intersection of the ring of invariants $k[M \cap \sigma^\vee]$ with the principal ideal (f) of $k[x_0, \dots, x_n]$, and is generated by some set $\{x^m \cdot f\}$ of invariant products of f with suitable monomials.

It's nevertheless useful to think of X as being $X\colon (f = 0) \subset A$, but beware that this can lead to error. A typical paradox of this kind is the fact that the hypersurface $X\colon (f = 0) \subset \mathbf{P}$ in a weighted projective space $\mathbf{P}$ (defined globally by a weighted homogeneous polynomial) is not necessarily defined locally by one equation.

(4.4) There is of course no purely toric method of getting a resolution of a general singularity $P \in X$ of this type (it includes all hypersurfaces). However, given a resolution $f\colon B \to A$ of the ambient space A, the proper transform $X' \subset B$ of X can be thought of as lying between X and its resolution, and the conditions

$$f_* \mathcal{O}_{X'}(rK_{X'}) = \mathcal{O}_X(rK_X)$$

for various toric resolutions $B \to A$ provide necessary conditions for $P \in X$ to be canonical; taking B related to the Newton polyhedron of X is most likely to produce useful information. If X is nondegenerate with respect to its Newton polyhedron (by definition this means that a suitable toric resolution $B \to A$ of the ambient space leads to a nonsingular X') then these conditions will also be sufficient.

(4.5) Let $\alpha = (b_0, \dots, b_n) \in N \cap \sigma$ be a vector; this means that α is a weighting $\alpha(x_i) = b_i \in \mathbf{Q}$ on monomials such that

(i) $\alpha \in N$, that is, $\alpha \equiv \frac{1}{r}(ja_0, \dots, ja_n) \bmod \mathbf{Z}^{n+1}$ for some $j = 0, \dots, r-1$; and (ii) $\alpha \in \sigma$, that is, $b_i \ge 0$ for each i.

I can extend this weighting to $k[x_0,\dots,x_n]$ in the obvious way: say that a monomial $x^m = x_0^{m_0}\cdots x_n^{m_n}$ *appears in* f (or just write $x^m \in f$) if its coefficient in f is nonzero; then define

$$\alpha(f) = \min\{\alpha(x^m) \mid x^m \in f\}.$$

In these terms, the Newton polyhedron of f is the lattice polyhedron of $M_{\mathbf{R}}$ defined by

$$\text{Newton}(f) = \{u \in M_{\mathbf{R}} \mid \alpha(u) \geq \alpha(f) \text{ for every } \alpha \in N \cap \sigma\};$$

note that the "inside" of Newton(f) is the part above the polygon, except in some contexts when it is the part below.

(4.6) THEOREM. *A necessary condition for $P \in X$ to be terminal (or canonical) is that*

$$(*) \qquad \alpha(x_0\cdots x_n) > \alpha(f) + 1$$

(respectively $\geq$) for every primitive vector $\alpha \in N\cap\sigma$. (For a quotient singularity, that is, when Y is nonsingular, the condition is $\alpha(x_1\cdots x_n) > 1$, formally the case $f = x_0$ of $(*)$; some of the arguments below may need minor modification to deal with this case.)

REMARK. If $r = 1$, then $(*)$ is the condition that the point $(1,\dots,1) \in M$ is in the interior of Newton(f). For $r > 1$, the analogous statement involves a slightly nonobvious notion of interior of a lattice polyhedron due to J. Fine which is important in other "canonical" contexts, and I discuss this in the appendix to §4.

(4.7) *The unit cube $\square$ and the weightings α_k.* I run through notation and ideas which will appear throughout the rest of this chapter. Write $\square$ for the unit cube of $N_{\mathbf{R}}$; this is the unit cell of the sublattice $\mathbf{Z}^{n+1} = \overline{N} \subset N$. Both the unit point $(1,\dots,1) \in N$ and the symmetry $i\colon \alpha \mapsto \alpha' = (1,\dots,1) - \alpha$ will appear in what follows, and obviously $\square = \sigma \cap i(\sigma)$.

$(*)$ applies to any weighting of $N\cap\sigma$, but in practice the most important ones to consider are the points of $\overline{N}$ (which correspond to Newton(f) independently of the $\boldsymbol{\mu}_r$-action), and those of $N \cap \square$.

In the case of a cyclic group action, $N = \mathbf{Z}^{n+1} + \mathbf{Z}\cdot\frac{1}{r}(a_0,\dots,a_n)$, and it is easy to check that $N \cap \square$ consists of (the vertices of $\square$ together with) the $r-1$ weightings

$$\alpha_k = \tfrac{1}{r}(\overline{a_0k},\dots,\overline{a_nk})$$

for $k = 1,\dots,r-1$ (where $\overline{}$ denotes smallest residue mod r). I will usually be interested in cases where the fixed locuses of elements of $\boldsymbol{\mu}_r$ have small dimension, so that most of the a_i are coprime to r.

(4.8) PROOF OF (4.6). This is a tutorial on standard toric stuff. Roughly speaking, the residue of $s = ((dx_0\wedge\cdots\wedge dx_n)/f)^{\otimes r}$ on X bases $\mathcal{O}_X(rK_X)$ and $(*)$ is the condition that s is regular and vanishes along an exceptional prime divisor corresponding to α. It's easy to apply the methods without understanding the proof, and some elementary applications are given in (4.9); the reader not in need of remedial instruction on these topics should GOTO (4.10).

For each primitive vector $\alpha \in N\cap\sigma$, there is toric blow-up $\varphi\colon B \to A$ of the ambient space $A = \mathbf{A}^{n+1}/\boldsymbol{\mu}_r$ with a single exceptional divisorial stratum $\Gamma \subset B$, such that the valuation of a monomial x^m (for $m \in M$) along Γ is given by

$$(**) \qquad v_\Gamma(x^m) = \alpha(m).$$

φ is given in toric geometry by the barycentric subdivision of the cone σ at α. A neighbourhood of the generic point of Γ in B is given as follows: the lattice $\alpha^\perp \subset M$ is isomorphic to $\mathbf{Z}^n$, and there is a complementary vector m_0 such that $\alpha(m_0) = 1$. The semigroup $M \cap (\alpha \geq 0)$ is then of the form $\alpha^\perp \oplus \mathbf{N}\cdot M_0$, and Spec of this is $\mathbf{G}_m^n \times \mathbf{A}^1$, with a single remaining toric stratum $\Gamma \cong \mathbf{G}_m^n$ given by $(z = 0)$, where $z = x^{m_0}$. Then for any $m \in M$, $x^m = x^{m'}\cdot z^{\alpha(m)}$ with $\alpha(m') = 0$, which proves $(**)$.

Write $X' \subset B$ for the proper transform of $X \subset A$ under the blow-up $\varphi\colon B \to A$ of the ambient space, and $\psi\colon Y \to X'$ for a resolution of X'; as usual, the pull-back of $\mathbf{Q}$-divisors is denoted by φ^*. The following proposition obviously implies (4.6).

PROPOSITION. *In the above notation,*
(I) $X' = \varphi^*X - \alpha(f)\Gamma$ *and* $K_B = \varphi^*K_A + (\alpha(x_0\cdots x_n) - 1)\Gamma$.
(II) *Suppose that* $a = \alpha(x_0\cdots x_n) - \alpha(f) - 1 \leq 0$, *so that*

$$K_B + X' = \varphi^*(K_A + X) + a\Gamma \qquad \textit{with } a \leq 0;$$

then the resolution $h = \varphi\circ\psi\colon Y \to X$ *satisfies*

$$K_Y = h^*K_X - Z,$$

where every exceptional component of h lying over a component of $X' \cap \Gamma \subset B$ appears in Z with coefficient ≥ 0. In particular, X is not terminal.

PROOF OF (I). If follows from the definition of $\alpha(f)$ that $g = f^r/z^{r\alpha(f)}$ is a regular function in a neighbourhood of Γ not vanishing along Γ. Then rX' is the divisor given by $(g = 0)$ in a neighbourhood of $X' \cap \Gamma$; it follows that $g = x^m f'^r$ where $m \in \alpha^\perp$ and X' is given by $(f' = 0)$. This proves the first equality in (I).

Now to compare differentials on A and B. The following manoeuvre is the standard treatment of the canonical class in toric geometry: let $m_0,\dots,m_n \in M$ be a $\mathbf{Q}$-basis of M, and write $z_i = x^{m_i}$ for the corresponding monomials; then write down the rational canonical differential

$$t = \frac{dz_0}{z_0}\wedge\cdots\wedge\frac{dz_n}{z_n}.$$

The point is that t is independent of the particular choice of $\mathbf{Q}$-basis of M (up to a scalar factor): to see this, consider the Jacobian of a monomial coordinate change.

Clearly t is a basis of $\mathcal{O}(K_{\mathbf{T}})$ over the big torus $\mathbf{T} \subset A$, with logarithmic poles along all codimension 1 strata of A. For any toric variety A, write D_A for the reduced divisor of A made up of all the codimension 1 strata of A. Then $t \in \mathcal{O}_A(K_A + D_A)$ is a basis element, essentially independent of the choice of basis $m_0, \dots, m_n$. In particular, the toric blow-up $\varphi: B \to A$ satisfies

$$K_B + D_B = \varphi^*(K_A + D_A).$$

(This proves that, quite generally, a toric variety A marked with its divisor D_A has log canonical singularities.) To prove the second formula in (I), note that $\varphi^* D_A$ is a **Q**-Cartier divisor which coincides with D_B outside the exceptional locus Γ of φ; on the other hand, by construction, Γ has multiplicity 1 in D_B, and it clearly has multiplicity $\alpha(x_0 \cdots x_n)$ in $\varphi^* D_A$ (since rD_A is a Cartier divisor with defining equation $(x_0 \cdots x_n)^r$).

PROOF OF (II). The statement in (II) looks obvious enough: the adjunction formula should give

$$K_{X'} \stackrel{?}{=} (K_B + X')|_{X'} = (\varphi^*(K_A + X) + a\Gamma)|_{X'} = \varphi^* K_X + a(\Gamma \cap X'),$$

so that if $a \le 0$, the components of $\Gamma \cap X'$ make a negative contribution to the canonical class. The simple case is when the generic points of $\Gamma \cap X'$ are all contained in the nonsingular locus of B, and X' is nonsingular there; then there is no problem about using the adjunction formula, and $K_{X'} = \varphi^* K_X + a\Gamma$, where $a = \alpha(x_0 \cdots x_n) - \alpha(f) - 1$.

Unfortunately this argument doesn't work in general: the plausible-looking adjunction formula for **Q**-divisors (indicated by $\stackrel{?}{=}$) is false whenever B is singular along a divisor of X'. (Consider, for example, a generator of the ordinary quadratic cone in $\mathbf{P}^3$.) The point is to see that singularities of B along a divisor of X' make a negative contribution to the canonical class.

By easy results on surface singularities, B can be resolved by a morphism $f: C \to B$ such that, restricting to a neighbourhood of the general point of any codimension 2 singular stratum, $K_C = f^* K_b - Z_C$ with $Z_C \ge 0$; if Y is the proper transform of X' in C then $Y = f^* X' - D$ with $D > 0$, and then, writing $g: Y \to B$ for the composite $Y \to X' \hookrightarrow B$, I get

$$\begin{aligned} K_Y &= (K_C + Y)|_Y = g^*(K_B + X') - (Z_C + D)|_Y \\ &= h^* K_X + a g^* \Gamma - (Z_C + D)|_Y. \end{aligned}$$

Thus if $a \le 0$, any exceptional divisor lying over a component of $X' \cap \Gamma$ appears in the formula for the canonical class with coefficient ≤ 0, so that X is not terminal. (If Y is not normal, then passing to its normalisation again makes a negative contribution to the canonical class.) This proves (II). Q.E.D.

REMARK. A component of $X' \cap \Gamma$ can easily be contained in the boundary of Γ, that is, in a codimension 2 toric stratum of B; this will happen, for example, if there is only a single monomial $x^m \in f$ for which the minimal value $\alpha(x^m) = \alpha(f)$ is achieved, since then $g = f^r / z^{r\alpha(f)}$ restricted to Γ is a monomial.

(4.9) *Elementary applications.* (1) Applying (*) to the simplest possible weighting $(1, \dots, 1)$ shows at once that a hypersurface singularity

$$P \in X: (f = 0) \subset \mathbf{A}^{n+1}$$

is terminal (or canonical) only if

$$\operatorname{mult}_P f < n \quad (\text{respectively } \le n);$$

for an ordinary multiple point, the condition is also sufficient (compare (1.9), (1)).

(2) Let $X = \mathbf{A}^2/\mu_r$ be a cyclic quotient singularity of type $\frac{1}{r}(1, a)$, for some a coprime to r; applying (*) to the weightings

$$\alpha_i = \tfrac{1}{r}(i, \overline{ai})$$

for $i = 1, \dots, r-1$ (where $\overline{}$ denotes minimal residue mod r), it is easy to see that X has a canonical singularity if and only if $a = -1$. Every quotient singularity (cyclic or otherwise) is log terminal.

The next example is important, and you won't get much further in this paper without understanding it thoroughly.

(3) Condition (*) gives at once Du Val's analysis of canonical points of an embedded surface $P \in X: (f = 0) \subset \mathbf{A}^3$: first, the quadratic part f_2 of f is nonzero, by (*) applied to $(1, 1, 1)$, so that by a choice of coordinates I can arrange that

$$f = f_2 + \cdots, \quad \text{where } f_2 = x^2 + y^2 + z^2 \text{ or } xy \text{ or } x^2,$$

and $\cdots$ denotes terms of degree ≥ 3. If $f_2 = xy$ then some power of z appears in f, and this gives the A_n points $f = xy + z^{n+1}$ (in suitable coordinates).

If $f_2 = x^2$ then (*) applied to $(2, 1, 1)$ implies that

$$f = x^2 + g(y, z)$$

where g has a nonzero cubic part, and in suitable coordinates,

$$f = x^2 + g_3 + \cdots \quad \text{with } g_3 = y^3 + z^3 \text{ or } y^2 z \text{ or } y^3.$$

Now $x^2 + y^2 z$ gives the D_n points $x^2 + y^2 z + z^{n-1}$.

If $f = x^2 + y^3$, then (*) applied to $(3, 2, 1)$ shows that one of the three monomials z^4, yz^3, or z^5 appears in f, which gives E_6, E_7, and E_8.

(4.10) It turns out to be pleasurable and important to consider the curious-looking question of which hyperquotient surface singularities are canonical. This means the following: take $Q \in Y: (f = 0) \subset \mathbf{A}^3$ and an action of μ_r on Y which is free outside Q, and consider the quotient $P \in X$. If I ask for X to be canonical, it is a Du Val singularity, so the question is equivalent to asking for all the ways in which one Du Val singularity can be a cyclic unramified cover of another. This can of course be done in a number of different ways, but I particularly recommend it as an exercise in toric technique.

Exercise. In the above situation, suppose that Y is singular and $r > 1$. Prove that one of the following six cases occurs (in suitable coordinates).

	r	Type	f	Description
(1)	any	$\frac{1}{r}(1,-1,0;0)$	$xy + z^n$	$A_{n-1} \xrightarrow{\text{r-to-1}} A_{rn-1}$
(2)	4	$\frac{1}{4}(1,3,2;2)$	$x^2 + y^2 + z^{2n-1}$	$A_{2n} \xrightarrow{\text{4-to-1}} D_{2n+1}$
(3)	2	$\frac{1}{2}(0,1,1;0)$	$x^2 + y^2 + z^{2n}$	$A_{2n-1} \xrightarrow{\text{2-to-1}} D_{n+2}$
(4)	3	$\frac{1}{3}(0,1,2;0)$	$x^2 + y^3 + z^3$	$D_4 \xrightarrow{\text{3-to-1}} E_6$
(5)	2	$\frac{1}{2}(1,1,0;0)$	$x^2 + y^2 z + z^n$	$D_{n+1} \xrightarrow{\text{2-to-1}} D_{2n}$
(6)	2	$\frac{1}{2}(1,0,1;0)$	$x^2 + y^3 + z^4$	$E_6 \xrightarrow{\text{2-to-1}} E_7$

The question may look artificial, since surface quotient singularities of this type have been well understood for more than a century, and the coverings can be classified in many other ways (and were of course, by Felix Klein and subsequently by Patrick Du Val and Coxeter). However, both the list and its derivation by toric methods seem inextricably linked to questions on terminal 3-fold singularities, and in particular to Mori's Theorem 6.1. [*Hint*: if you have trouble doing the exercise, glance at the proof of Mori's theorem in §§6–7, starting at (6.7).]

(4.11) *Canonical quotient singularities.* In the case of cyclic quotient singularities $\mathbf{A}^n/\mu_r$ of type $\frac{1}{r}(a_1, \dots, a_n)$, condition (*) of (4.6) can be rewritten in terms of the $r-1$ weightings

$$\alpha_k = \tfrac{1}{r}(\overline{ka_1}, \dots, \overline{ka_n})$$

for $k = 1, \dots, r-1$ (here $^-$ denotes smallest residue mod r).

THEOREM. *A quotient singularity $X = \mathbf{A}^n/\mu_r$ of type $\frac{1}{r}(a_1, \dots, a_n)$ is terminal (or canonical) if and only if*

$$\alpha_k(x_1 \cdots x_n) = \frac{1}{r}\sum_k \overline{ka_k} > 1 \quad \textit{for } k = 1, \dots, r-1$$

(*respectively* $\geq$).

This holds because (*) of (4.6) is trivially satisfied for any lattice point α not contained in the unit cube $\square$. This theorem implies at once the Reid–Shepherd-Barron–Tai criterion [**C3-f**, (3.1); **Tai**, (3.2)]: a quotient singularity $\mathbf{A}^n/G$ by an arbitrary group G acting without quasireflections is canonical if and only if every element $g \in G$ of order r when written diagonally

$$g\colon (x_1, \dots, x_n) \mapsto (\varepsilon^{a_1}x_1, \dots, \varepsilon^{a_n}x_n)$$

in terms of any primitive $\varepsilon \in \mu_r$ satisfies $\sum a_i \geq r$.

The combinatorics of the condition in (4.11) and some of its consequences will be studied in §5, see also [**Morrison**].

Appendix to §4. The Fine interior, plurigenera and canonical models. The ideas of this appendix are due to J. Fine (around 1980).

(4.12) *The Fine interior.* If $r = 1$, then (*) of (4.6) is the condition that the point $(1, \dots, 1) \in M_{\mathbf{R}}$ is in the interior of Newton(f). For $r > 1$, the statement involves the following notion of interior of a lattice polyhedron Δ (a *lattice polyhedron* is a convex polyhedron with vertices in M): define the *Fine interior* of Δ to be the closed (!) polyhedron Fine(Δ) given by

$$\mathrm{Fine}(\Delta) = \{u \in M_{\mathbf{R}} \mid \alpha(u) \geq \alpha(\Delta) + 1 \text{ for every } \alpha \in N \text{ with } \alpha(\Delta) \geq 0\}.$$

To describe this in words, take each supporting hyperplane of Δ and push it into Δ until it hits another point of M (see Figure 1); now take the intersection of all of these half-spaces. It is definitely not sufficient to take just the walls of Δ; for example if $M = \{m_1, m_2 \mid m_1 + m_2 \equiv 0 \bmod 3\}$ and $\Delta = \sigma$ is the first quadrant (giving the quotient singularity X of type $\frac{1}{3}(1,1)$), the monomial xy (which gives the canonical class of X) is in the interior of σ, but not in Fine(σ) (see Figure 2); this is equivalent to the computation of (1.9), (2).

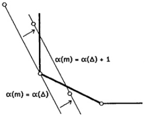


FIGURE 1

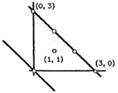


FIGURE 2

Fine(Δ) is rather tricky to calculate, because the definition involves in principle all the supporting hyperplanes of Δ; the correspondence with the canonical class in toric geometry shows how to reduce this to the finitely many vectors α involved in a toric resolution of the variety X_Δ. This proves that Fine(Δ) is a finite rational polyhedron. Note that it is not in general a lattice polyhedron.

(4.13) *Plurigenera, canonical models.* The point of the construction is that, by [**Khovanskiĭ**], lattice points in the interior of Newton(f) correspond in various set-ups to the geometric genus of the variety or singularity given by $(f = 0)$; Fine observed that the plurigenera correspond in a similar way to the lattice points of multiples of Fine(f). Clearly (4.6) just says that $P \in X$ is canonical only if (sometimes also if) $(1, \dots, 1) \in$ Fine(f).

Quite generally, these ideas determine the plurigenera (and canonical models) of nondegenerate toric hypersurfaces in terms of the Fine interior.

EXAMPLE. Consider the hypersurface singularity $X\colon (f = 0) \subset \mathbf{A}^3$, where

$$f = x^2 + y^3 + z^k \quad \text{for } 6 \leq k \leq 11;$$

then Fine(f) is the polyhedron in $M_{\mathbf{R}} = \mathbf{R}^3$ given by

$$(*) \qquad l \geq 1, \quad m \geq 1, \quad n \geq 1, \quad 3l + 2m + n \geq 6 + 1 = 7.$$

Of course, $\alpha = (3k, 2k, 6)$ is also a supporting hyperplane of Newton(f), but the condition

$$3kl + 2km + 6n \geq 6k + 1$$

is already implied by ($*$) (because

$$15(l-1) + 10(m-1) + (11-k)(n-1) + (k-5)(3l + 2m + n - 7) \geq 0).$$

This means that the pluricanonical invariants of the singularity only take account of the weighting $(\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$.

In this case if $\varphi\colon Y \to X$ is a resolution of X then the pluri-adjunction ideal $\mathcal{J}_a$, defined by $\varphi_*\mathcal{O}_Y(aK_Y) = \mathcal{J}_a \cdot \mathcal{O}_X(aK_X)$, is the ideal generated by all monomials $x^l y^m z^n$ with $3l + 2m + n \geq a$. It is easy to calculate from this the (genuine) plurigenera

$$P_n(X) := \dim \mathcal{O}_X/\mathcal{J}_a = 1 + \binom{a}{2}.$$

(4.14) REMARK. There are (at least) two other notions of plurigenera in the literature on singularities.

(i) The *log plurigenera* correspond to taking the resolution Y marked with its exceptional divisor E (assumed to be a reduced normal crossing divisor) and working with $\varphi_*\mathcal{O}_Y(a(K_Y + E)) \subset \mathcal{O}_X(aK_X)$.

(ii) The L^2 *plurigenera* correspond to differentials on the nonsingular locus $X^0 \subset X$ which are square-integrable; it is known that this is the same as considering $\varphi_*\mathcal{O}_Y(aK_Y + (a-1)E)$.

These invariants are determined by the usual interior of Newton(f) in a much simpler way than the genuine plurigenera.

(4.15) EXAMPLE. Let $M \subset \mathbf{Z}^4$ be the 3-dimensional affine lattice defined by

$$M = \{(m_1, m_2, m_3, m_4) \in \mathbf{Z}^4 \mid \sum m_i = 5 \text{ and } \sum i m_i \equiv 0 \bmod 5\},$$

and let $\Delta \subset M$ be the simplex spanned by the 4 points $(0, \dots, 5, \dots, 0)$, that is, the monomials $x_1^5, x_2^5, x_3^5, x_4^5$. A general polynomial f_Δ supported on Δ is the equation of a nonsingular quintic $Y \subset \mathbf{P}^3$ invariant under the μ_5-action

$$\mu_5 \ni \varepsilon\colon x_i \mapsto \varepsilon^i x_i \quad \text{for } (x_1, x_2, x_3, x_4) \in \mathbf{P}^3.$$

The toric space $\mathbf{P}_\Delta$ constructed from Δ is in this case $\mathbf{P}^3/\mu_5$, and the hypersurface $X_\Delta\colon (f_\Delta = 0) \subset \mathbf{P}_\Delta$ is the Godeaux surface $X_\Delta = Y/\mu_5$. It is a nice exercise to prove that

$$\text{Fine}(\Delta) = (1,1,1,1) + \tfrac{1}{5}\Delta.$$

Problem (I. Dolgachev). It is an interesting problem to look for other examples of this phenomenon: a lattice M ($\cong \mathbf{Z}^3$), and a lattice polyhedron Δ having no interior points (so that the toric hypersurface $X_\Delta \subset \mathbf{P}_\Delta$ has $p_g(X_\Delta) = 0$), but with Fine(Δ) $\neq 0$ (so that $\kappa(X_\Delta) \geq 0$), or better still with Fine(Δ) of positive 3-dimensional volume (so that $\kappa(X_\Delta) = 2$). Note that J. Fine observed in 1981 that the traditional model of an Enriques surface as a space sextic passing doubly through the edges of the coordinate tetrahedron is a construction of this form (here Δ is a cube, and M the lattice generated by the vertices of Δ and the midpoints of the six faces).

5. The terminal lemma.

(5.1) *The terminal quotient singularities* $\frac{1}{r}(a, -a, 1)$. Condition ($**$) of (4.11) has a nice geometric interpretation: it says that all the lattice points of N contained in the cube $\square$ actually live in the middle strip (see Figure 3 for a picture). In fact ($**$) says that every point of $N \cap \square$ lies on or above the hyperplane $\sum y_i = 1$, and since $n \mapsto (1, \dots, 1) - n$ is a symmetry of N (as an affine lattice), they clearly also lie below the hyperplane $\sum y_i = n - 1$ (Figure 3).

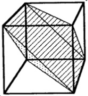


FIGURE 3

(5.2) The condition for terminal singularities is of course that the points of $N \cap \square$ live strictly in the middle strip of the cube. In the 3-dimensional case, this situation is completely understood by the following theorem of G. K. White, D. Morrison, G. Stevens, V. Danilov and M. Frumkin.

THEOREM. *A 3-fold cyclic quotient singularity* $X = \mathbf{A}^3/\mu_r$ *is terminal if and only if (up to permutations of* (x, y, z) *and symmetries of* μ_r*) it is of type* $\frac{1}{r}(a, -a, 1)$ *with a coprime to r.*

(5.3) An obvious, but nevertheless key step in the proof of (5.2) is to replace the inequality ($**$) by $r - 1$ equalities:

LEMMA. *Let X be of type* $\frac{1}{r}(a, b, c)$*, and write* $d = a + b + c$. *Then X is terminal if and only if*

$$\overline{ak} + \overline{bk} + \overline{ck} = \overline{dk} + r \quad \text{for } k = 1, \dots, r-1$$

(*where* $\bar{\ }$ *denotes smallest residue* mod r).

PROOF. The points $P_k = \frac{1}{r}(\overline{ak}, \overline{bk}, \overline{ck})$ are just the points of $N \cap \square$. The two sides are congruent mod r, and equality holds if and only if the left-hand side is in the interval $(r, 2r)$.

(5.4) The lemma reduces Theorem 5.2 to the case $n = 3$, $m = 1$ of the following more general result.

THEOREM (TERMINAL LEMMA). *Let n and m be integers with $n \equiv m$ mod 2, and suppose that $\frac{1}{r}(a_1, \dots, a_n; b_1, \dots, b_m)$ is an $(n+m)$-tuple of rational numbers with denominator r.*

(A) *Suppose each a_i and b_j is coprime to r; then the following two conditions are equivalent:*

(i) $$\sum_{i=1}^{n} \overline{a_i k} = \sum_{j=1}^{m} \overline{b_j k} + \frac{n-m}{2} \cdot r \quad \text{for } k = 1, \dots, r-1.$$

(ii) *The $n+m$ elements $\{a_i, -b_j\}$ can be split up into $(n+m)/2$ disjoint pairs of the form $(a_i, a_{i'})$ or $(b_j, b_{j'})$ or $(a_i, -b_j)$ which add to 0 mod r. (That is, each a_i is either paired with another $a_{i'}$ such that $a_{i'} \equiv -a_i$ mod r, or with one of the b_j such that $b_j \equiv a_i$ mod r, and similarly for the b_j's.)*

(B) *More generally (without the coprime condition), (i) is equivalent to (ii) plus the following condition:*

(iii) *For every divisor q of r,*

$$\#\{\text{pairs}(a_i, a_{i'}) \mid q = hcf(a_i, r)\} = \#\{\text{pairs}(b_j, b_{j'}) \mid q = hcf(b_j, r)\}.$$

Note that (ii) ⇒ (i) is trivial, since

$$\overline{ak} + \overline{(r-a)k} = \begin{cases} r & \text{if } \overline{ak} \neq 0, \\ 0 & \text{if } \overline{ak} = 0; \end{cases}$$

the implication (i) ⇒ (iii) in (B) is similar and I leave it as an easy exercise.

(5.5) REMARK. For (5.2) I only need the case $n = 3$, $m = 1$ with a_i and b_j coprime; the case $n = 4$, $m = 2$ will be used in §6 in connection with terminal hyperquotient singularities in the form of Corollary 5.6. The tuple might more generally correspond to an action of $\boldsymbol{\mu}_r$ on a complete intersection singularity $Q \in Y \subset \mathbf{A}^n$, for example with $\frac{1}{r}(a_1, \dots, a_n)$ specifying the type of the action on the coordinates, $\frac{1}{r}(b_1, \dots, b_{m-1})$ that on the defining equations, and $\frac{1}{r}(b_m)$ corresponding to a choice of generator of the class group of the singularity $P \in X = Y/\boldsymbol{\mu}_r$ (a "polarisation" of the singularity).

(5.6) COROLLARY. *Let $\frac{1}{r}(a_1, \dots, a_4; e, 1)$ be a 6-tuple of rational numbers with denominator r such that*

$$q = \text{hcf}(e, r) = \text{hcf}(a_4, r) \quad \text{and} \quad a_1, a_2, a_3 \text{ are coprime to } r;$$

assume that

$$\sum_{i=1}^{4} \overline{a_i k} = \overline{ek} + k + r \quad \text{for } k = 1, \dots, r-1.$$

Then $a_4 \equiv e$ mod r, and the remaining 4 elements can be paired together as $a_1 \equiv 1$, $a_2 + a_3 \equiv 0$ mod r (or permutations).

The proof of Theorem 5.4, taken mainly from **[Morrison-Stevens]**, is something of a digression from the main theme, and is left as an appendix so that the reader can skip over it.

(5.7) *Economic resolutions.* An important consequence of Theorem 5.2 is that the $r-1$ points of $N \cap \square$ are $Q_i = \frac{1}{r}(\overline{ai}, \overline{(r-a)i}, i)$ for $i = 1, \dots, r-1$, or equivalently,

$$P_j = \tfrac{1}{r}(j, r-j, \overline{bj}) \quad \text{for } j = 1, \dots, r-1,$$

where $ab \equiv 1$ mod r; hence they all lie on the affine plane $x + y = 1$ of $\mathbf{R}^3$. It follows from this that there is a class of "economic" toric resolutions of X, due to Danilov and R. Barlow (see **[Danilov**, §4]): these correspond to subdivisions of the cone σ obtained from the picture of Figure 4 (the shaded polygonal area is to be subdivided into basic triangles).

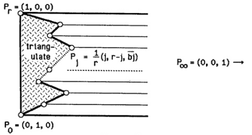


FIGURE 4

This set-up has the following nice properties.

(1) Each of the r triples (P_j, P_{j+1}, P_∞) is a basis of N, so that the subdivision leads to a toric resolution $f\colon Y \to X$ of $P \in X$.

(2) Each new vertex P_j of the subdivision lies in the interior of $\square$, so that $f^{-1}P = \bigcup E_j$, where E_j has fractional discrepancy a_j with $0 < a_j < 1$; these are the so-called "essential" or "semicrepant" exceptional components, which necessarily occur in any resolution of $P \in X$. In fact there is one exceptional component with each discrepancy $1/r, 2/r, \dots, (r-1)/r$, generalising the resolution in Exercise 1.10.

(3) The shaded polygon is planar, so that triangulating it leads only to curves l with $K_Y l = 0$; in fact these curves are always $(-1,-1)$-curves, that is, $l \cong \mathbf{P}^1$ with $N_Y|_l \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$, and there are $(r-1)$ of them. Although the triangulation, and hence the toric resolution $Y \to X$, is not unique, you can get from one to another by a composite of flops (symmetric flips) in these curves.

Various properties of the singularities $\frac{1}{r}(a,-a,1)$ can be read off quite conveniently from the explicit resolution given here; for example the plurigenus contributions $l(n)$ of [**C3-f**, §5] (discussed in §10 below) could in principle be calculated from it, and this is in fact a reasonable approach to calculating the main term

$$c_2 \cdot \Delta = \frac{r^2-1}{12r}$$

(compare (10.3) and [**Kawamata**, (2.2)]).

Exercise. If $\overline{b(j+1)} = \overline{bj} + b$ (one of the two possible cases), then

(i) the affine piece $U_j \cong \mathbf{A}^3$ of Y corresponding to $\langle P_j, P_{j+1}, P_\infty\rangle$ has coordinates

$$u = x^{-b}(xy)^{[bj/r]}, \quad v = x^r(xy)^{-j}, \quad w = x^{-r}(xy)^{j+1},$$

and in terms of these, the invariant monomials xy, z^r, etc. are given by

$$xy = vw, \quad z^r = u^r v^b (vw)^{\overline{bj}}, \quad x^r = v(vw)^j, \quad y^r = v^{-1}(vw)^{r-j}, \text{ etc.}$$

(ii) $\langle P_k, P_j, P_{j+1}\rangle$ is a basic cone contained in the shaded area of Figure 4 if and only if $[bk/r] = [bj/r] + 1$ (see Figure 5); the corresponding affine piece $V_{j,k}$ of Y then has coordinates

$$u' = u^{-1}, \quad v' = u^{k-j}x^r(xy)^{-j}, \quad w' = u^{-k+j+1}x^{-r}(xy)^{j+1},$$

and in terms of these,

$$xy = u'v'w', \quad z^r = u'^{\overline{bk}} v'^b (v'w')^{\overline{bj}}.$$

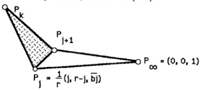


FIGURE 5

Appendix to §5. Cyclotomy and the proof of (5.4).

(5.8) Write s_a for the r-tuple

$$s_a = (\overline{ak})_{k=0,\dots,r-1} \in \mathbf{Q}^r = V;$$

then (i) of (5.4) is a certain relation between s_a for different values of a, and the implication (i) $\Rightarrow$ (ii) is deduced by proving that the s_a are as linearly independent as you could reasonably expect. As remarked above,

$$(s_a)_k + (s_{r-a})_k = \overline{ak} + \overline{(r-a)k} = \begin{cases} r & \text{if } r \nmid ak, \\ 0 & \text{if } r \mid ak; \end{cases}$$

because of this, for each $a = 0, \dots, r-1$, define a new r-tuple $S_a \in V$ by

$$(S_a)_k = \begin{cases} \overline{ak} - r/2 & \text{if } r \nmid ak, \\ 0 & \text{if } r \mid ak. \end{cases}$$

Then the above relation takes the form

$$(*) \qquad S_a + S_{r-a} = 0.$$

In fact only the subspace $V^- = \{(v_k) \mid v_k + v_{r-k} = 0\ \forall k\} \subset V$ will play any part in the following argument; this is a $\mathbf{Q}$-vector space of dimension $[(r-1)/2]$, and in due course of time I will prove that

$$\{S_a \mid \text{ for } a = 1, \dots, [(r-1)/2]\}$$

is a basis. This easily implies (5.4).

For the purpose of proving (5.4), (A) restrict attention to a coprime to r (the more general case will be dealt with later). If all the a_i and b_j are coprime to r, then condition (i) of (5.4) is of the form

$$\sum S_{a_i} = \sum S_{b_j},$$

and the following result clearly gives the implication (i) $\Rightarrow$ (ii).

(5.9) PROPOSITION. *Consider only a coprime to r; then the relations $(*)$ are the only linear dependence relations holding between the S_a. In other words, the $\varphi(r)/2$ elements S_a with $a = 1, \dots, [(r-1)/2]$ coprime to r are linearly independent.*

Discussion. The proof which follows, due to [**Morrison-Stevens**], is somewhat abstract. To be absolutely concrete, write out the multiplication table of the ring $\mathbf{Z}/r$ in terms of smallest residues mod r, and let

$$M_{a,k} = \{\overline{ak} - r/2 \text{ or } 0\}$$

be the $r \times r$ matrix obtained by subtracting $r/2$ from all nonzero entries; the result I am after is equivalent to saying that the top left quarter determinant of M formed by taking rows and columns $1, 2, \dots, [(r-1)/2]$ is nonzero; for example, if $r = 11$ then

$$\begin{vmatrix} -9/2 & -7/2 & -5/2 & -3/2 & -1/2 \\ -7/2 & -3/2 & 1/2 & 5/2 & 9/2 \\ -5/2 & 1/2 & 7/2 & -9/2 & -3/2 \\ -3/2 & 5/2 & -9/2 & -1/2 & 7/2 \\ -1/2 & 9/2 & -3/2 & 7/2 & -5/2 \end{vmatrix} = -\frac{1}{2} \cdot (11)^4;$$

or if $r = 14$ then

$$\begin{vmatrix} -6 & -5 & -4 & -3 & -2 & -1 \\ -5 & -3 & -1 & 1 & 3 & 5 \\ -4 & -1 & 2 & 5 & -6 & -3 \\ -3 & 1 & 5 & -5 & -1 & 3 \\ -2 & 3 & -6 & -1 & 4 & -5 \\ -1 & 5 & -3 & 3 & -5 & 1 \end{vmatrix} = -7 \cdot (14)^4.$$

These determinants are quite fun to evaluate by row and column operations (for example, on a home computer using commercial spreadsheet software); if you difference successive rows then you see that the essential information contained in the matrix is whether

$$\overline{(a+1)k} = \overline{ak} + k \quad \text{or} \quad \overline{ak} - (r-k)$$

and whether or not $\overline{ak} = 0$ (that is, whether a "carry" occurs) for each a, k.

The nonvanishing of these determinants is equivalent to Dirichlet's theorem $L(1, \chi) \neq 0$ for odd characters χ; due to my ignorance of number theory, I can't help wondering if there isn't an elementary proof of this fact. The actual value of the determinant is presumably a power of r times a factor of number-theoretical interest, compare [**Lang**, p. 92].

(5.10) PROOF OF (5.9). I will find $\varphi(r)/2$ linearly independent vectors $w_\chi \in V \otimes \mathbf{C}$ which are linear combinations of the S_a for $a \in (\mathbf{Z}/r)^*$; this obviously proves the proposition. The vectors w_χ are constructed as eigenvectors of the natural action of $(\mathbf{Z}/r)^*$ on V with distinct characters χ^{-1} as eigenvalues, so that by a standard argument of linear algebra, to prove they are linearly independent it will be enough to prove that each $w_\chi \neq 0$.

$a \in (\mathbf{Z}/r)^*$ acts on V by $(v_k)_k \mapsto (v_{ak})_k$; note that $a(S_1) = S_a$. Given any character $\chi\colon (\mathbf{Z}/r)^* \to \mathbf{C}^*$ and any $v \in V$, the linear combination

$$w_\chi(v) = \sum_{a \in (\mathbf{Z}/r)^*} \chi(a) \cdot a(v) \in V \otimes \mathbf{C}$$

is of course an eigenvector of the action of $(\mathbf{Z}/r)^*$ on V, with eigenvalue χ^{-1}. I apply this with $v = S_1$ and χ an odd character (that is, $\chi(-1) = -1$), getting eigenvectors

$$w_\chi = w_\chi(S_1) = \sum_{a \in (\mathbf{Z}/r)^*} \chi(a) \cdot S_a;$$

there are just $\varphi(r)$ characters of $(\mathbf{Z}/r)^*$ of which $\varphi(r)/2$ are odd, so this does what I want, except that I must still prove that $w_\chi \neq 0$.

(5.11) *Proof that* $w_\chi \neq 0$. Suppose first for simplicity that χ is a primitive character mod r (that is, it does not factor through the quotient map $(\mathbf{Z}/r)^* \to (\mathbf{Z}/r')^*$ for any divisor $r'|r$), and consider the first coordinate of $w_\chi \in V$:

$$(w_\chi)_1 = \sum_{a \in (\mathbf{Z}/r)^*} \chi(a) \cdot (S_a)_1 = \sum \chi(a) \cdot \left(a - \frac{r}{2}\right) = \sum \chi(a) \cdot a$$

(the last equality uses $\sum \chi(a) = 0$). The fact that this number is nonzero now follows from results of analytic number theory:

(a) The sum for $(w_\chi)_1$ is by definition r times the generalised Bernoulli number $B_{1,\chi}$, where

$$B_{1,\chi} = \frac{1}{r} \sum_{a \in (\mathbf{Z}/r)^*} \chi(a) \cdot \left(a - \frac{r}{2}\right) = \frac{1}{r} \sum_{a \in (\mathbf{Z}/r)^*} \chi(a) \cdot a;$$

see [**Washington**, p. 30].

(b) A contour integration shows that the Dirichlet L-function $L(s, \chi)$ for χ satisfies

$$-L(0, \chi) = B_{1,\chi}$$

(see [**Washington**, p. 31]).

(c) The functional equation relating $L(s, \chi)$ and $L(1-s, \chi^{-1})$ shows that $L(0, \chi)$ is a nonzero multiple of $L(1, \chi^{-1})$ (see [**Washington**, p. 29]).

(d) Finally, the statement that $L(1, \chi^{-1}) \neq 0$ is a famous theorem of Dirichlet (see [**Washington**, p. 33] or any book on analytic number theory).

(5.12) A suitable modification of the argument will in fact go through for any odd character χ: if $r = qf$ is a factorisation of r, and χ is induced by a primitive character $\chi\colon (\mathbf{Z}/f)^* \to \mathbf{C}^*$ mod f (that is, χ has *conductor* f) then I prove that the qth coordinate $(w_\chi)_q \neq 0$. (In fact it is true that $(w_\chi)_1 \neq 0$, as will be proved in Proposition 5.17; but the calculation is quite a lot more involved.) First, for each $a \in (\mathbf{Z}/r)^*$,

$$(S_a)_q = \overline{aq} - \frac{r}{2}$$

so that $(S_a)_q$ also only depends on a mod f. Then

$$(w_\chi)_q = \sum_{a \in (\mathbf{Z}/r)^*} \chi(a') \cdot q \left(a' - \frac{f}{2}\right),$$

where a' is the smallest residue of a mod f; now since $(\mathbf{Z}/r)^* \to (\mathbf{Z}/f)^*$ is a surjective group homomorphism, to every $a' \in (\mathbf{Z}/f)^*$ there correspond $\varphi(r)/\varphi(f)$ values of a, so that

$$(w_\chi)_q = q \cdot \frac{\varphi(r)}{\varphi(f)} \sum_{a' \in (\mathbf{Z}/f)^*} \chi(a') \cdot \left(a' - \frac{f}{2}\right) = r \cdot \frac{\varphi(r)}{\varphi(f)} \cdot B_{1,\chi}.$$

Apart from the initial factor, this is the same sum as before, now for the primitive character χ mod f, so I conclude as before.

This completes the proof of Proposition 5.9, and with it (5.4), (A).

(5.13) I now go on to prove that the only linear relations between all the S_a are given by $(*)$ of (5.8), that is, that the $[(r-1)/2]$ elements

$$\{S_a \mid \text{for } a = 1, \dots, [(r-1)/2]\}$$

are linearly independent. The aim is as before: to show that the vectors $w_\chi(S_a)$ provide the right number of linear combinations of the S_a which are linearly independent. This time however I take $w_\chi(S_a)$ for different divisors $a|q$, where $q = r/f$. (Note that $w_\chi(S_a)$ is a linear combination of the vectors $S_{a'}$ for $a' \in [0, \dots, r-1]$ such that $\operatorname{hcf}(a', r) = a$; the divisor $a|r$ is just a natural choice of representative of this set.)

MAIN CLAIM. Let $r = fq$ be a factorisation, and $\chi\colon (\mathbf{Z}/r)^* \to \mathbf{C}^*$ an odd character which is induced from a primitive character $\chi\colon (\mathbf{Z}/f)^* \to \mathbf{C}^*$. Then the vectors $w_\chi(S_a)$ with $a|q$ are linearly independent.

The claim implies Theorem 5.4. Since there cannot be a nontrivial linear dependence relation between eigenvectors belonging to different characters, the claim gives that the set

$$\{w_\chi(S_a) \mid r = as \text{ and } \chi \text{ is an odd character } \bmod s\}$$

is linearly independent; by analogy with the formula $r = \sum_{s|r} \varphi(s)$, if I write $\varphi^-(n)$ for the number of odd characters of $(\mathbf{Z}/n)^*$ then clearly

$$\left[\frac{r-1}{2}\right] = \dim V^- = \sum_{s|r} \varphi^-(s),$$

so that I have just the right number $[(r-1)/2]$ of vectors to base V^-; since they are linear combinations of the S_a for $a = 1, 2, \dots, [(r-1)/2]$, it follows that these also base V^-. Given condition (iii), the relation (i) is (as before) just of the form

$$\sum S_{a_i} = \sum S_{b_j},$$

so that this implies (5.4), (B).

(5.14) For the proof of Claim 5.13, note first the following easy facts:

LEMMA. (i) $(w_\chi(S_a))_c$ *depends only on the product* $ac \bmod r$;
(ii) *if* $ac = q$ *then*

$$w_\chi(S_a)_c = w_\chi(S_1)_q = r \cdot \varphi(r)/\varphi(f) \cdot B_{1,\chi} \neq 0;$$

(iii) *if* $a'|r$ *and* $\operatorname{hcf}(a'c, r) \nmid q$ *then*

$$w_\chi(S_{a'})_c = 0.$$

PROOF. If you write out the definition of $(w_\chi(S_a))_c$ then (i) is trivial, and (ii) follows by the argument of (5.12).

(iii) Write $q' = \operatorname{hcf}(a'c, r)$ and $f' = r/q'$. Arguing as in (5.12),

$$w_\chi(S_{a'})_c = \sum_{b \in (\mathbf{Z}/f)^*} \chi(b) \cdot f(b), \qquad f(b) = \begin{cases} \overline{ba'c} - \frac{r}{2} \\ 0 \quad \text{if } r|ba'c, \end{cases}$$

where the term $f(b)$ multiplying $\chi(b)$ depends only on $b \bmod f'$; so it's invariant under $b \mapsto bk$ for $k \in \operatorname{Ker}\{(\mathbf{Z}/r)^* \to (\mathbf{Z}/f')^*\}$. By definition of f, this is a subgroup on which χ is nontrivial, and hence the sum is zero. Q.E.D.

(5.15) It's important to understand the distinction made in (iii); if p is a number with some factor in common with f then of course $\operatorname{hcf}(pq, r)$ is a strict multiple of q; however, if p is coprime to f then $\operatorname{hcf}(pq, r) = \operatorname{hcf}(q, r)$, and it follows that

$$pq \equiv \beta q \qquad \bmod r,$$

with $\beta \in (\mathbf{Z}/r)^*$; in fact β is uniquely determined $\bmod f$.

If I could replace the unpleasant second hypothesis in (iii) by the nice condition $a'c \nmid q$, then it is easy to order the divisors a and c of q in such a way that the matrix

$$\{w_\chi(S_a)_c\} \quad \text{as } a \text{ and } c \text{ run through the divisors of } q$$

is upper-triangular with nonzero diagonal entries. (Just write $a_i c_i = q$, and order the a_i such that $i < j \Rightarrow a_j \nmid a_i$.) This already proves (5.13) in the special case that every prime divisor of q also divides f. I now endeavour to modify the $w_\chi(S_a)$ to get an upper-triangular matrix in the general case. There's essentially only one way to proceed ("Möbius inversion").

(5.16) Write $\mathcal{P}$ for the set of prime divisors of q which are coprime to f; for each $p \in \mathcal{P}$ let p^α be the highest power of p dividing r, and choose $\beta_p \in (\mathbf{Z}/r)^*$ such that $\beta_p \equiv p \bmod r/p^\alpha$. (This is of course possible, and I could even require $\beta_p \equiv 1 \bmod p^\alpha$, since the numbers $p + i \cdot r/p^\alpha$ for $i = 0, \dots, p^\alpha - 1$ take every value $\bmod p^\alpha$.) Then

$$pa \equiv \beta_p a \bmod r \quad \text{for every } a \text{ with } p^\alpha|a.$$

To simplify notation, if $d = \prod_{i=1}^m p_i$ is a product of distinct primes $p_i \in \mathcal{P}$, I write $\beta_d = \prod \beta_{p_i}$. Now define vectors $v_\chi(a) \in V \otimes \mathbf{C}$ for each a by the following formula:

$$\begin{aligned} v_\chi(a) &= w_\chi(S_a) - \sum_{\substack{p \in \mathcal{P} \\ p|a}} w_\chi(S_{\beta_p \cdot a/p}) + \sum_{\substack{p,p' \in \mathcal{P} \\ pp'|a}} w_\chi(S_{\beta_p \beta_{p'} \cdot a/pp'}) - \cdots \\ &= \sum_{d|a} \mu(d) w_\chi(S_{\beta_d \cdot a/d}), \end{aligned} \tag{1}$$

where the sum runs over products of distinct primes in $\mathcal{P}$ dividing a, and $\mu(d)$ is the Möbius function defined on square-free integers by

$$\mu(d) = (-1)^m \quad \text{where } m = \#\{\text{distinct prime factors of } d\}.$$

This means of course that if a is not divisible by any $p \in \mathcal{P}$ then $v_\chi(a) = w_\chi(S_a)$; and for $p \in \mathcal{P}$,

$$v_\chi(p^t a) = w_\chi(S_{p^t a}) - w_\chi(S_{\beta_p p^{t-1} a}).$$

In this case, Lemma 5.14, (i) together with the definition of β_p gives

$$(v_\chi(p^t a))_c = 0 \quad \text{for all } c \text{ with } p^{\alpha - t + 1}|c. \tag{2}$$

(5.17) PROPOSITION. *Let* a, c, c' *be divisors of* q.

(i) *If* $ac' \nmid q$ *then* $(v_\chi(a))_{c'} = 0$.

(ii) *Suppose that* $ac = q$, *and let* $d|a$ *be a product of distinct primes in* $\mathcal{P}$; *for each divisor* $d'|d$, *write* $d = d'd''$. *Then (sorry folks!)*

$$\begin{aligned} (w_\chi(S_{\beta_d \cdot a/d}))_c &= \left\{ \sum_{d'|d} \mu(d') \chi^{-1}(d'') \right\} \cdot r \cdot \frac{\varphi(r)}{\varphi(df)} \cdot B_{1,\chi} \\ &= \prod_{p|d} \left(\frac{\chi^{-1}(p) - 1}{p - 1} \right) \cdot r \cdot \frac{\varphi(r)}{\varphi(f)} \cdot B_{1,\chi}; \end{aligned} \tag{3}$$

and

$$(v_\chi(a))_c = \prod_{p|a} \left(\frac{p - \chi^{-1}(p)}{p - 1} \right) \cdot r \cdot \frac{\varphi(r)}{\varphi(f)} \cdot B_{1,\chi} \neq 0. \tag{4}$$

Notice that (i) and (ii) together prove Claim 5.13, and Theorem 5.4, since the matrix $\{(v_\chi(a))_c\}_{a,c}$ is upper-triangular as discussed in (5.15).

PROOF. (i) is easy: if $ac' \nmid q$ then there is some prime p appearing in ac' with higher power than in q. If p divides f then $\mathrm{hcf}(ac', r) \nmid q$, so that Lemma 5.14, (iii) gives $(w_\chi(S_a))_{c'} = 0$, and the same after dividing primes in $\mathcal{P}$ out of a, which gives the result at once. Otherwise $p \in \mathcal{P}$, and (i) comes from grouping the sum for $(v_\chi(a))_{c'}$ into pairs of terms

$$\mu(d)(w_\chi(S_{\beta_d\cdot a/d}))_{c'} + \mu(pd)(w_\chi(S_{\beta_p\beta_d\cdot a/pd}))_{c'}, \tag{5}$$

where $p \nmid d$, so that $\mu(pd) = -\mu(d)$. Under the assumption that $p^{\alpha+1}|ac'$ it follows from the definition of β_p that $\beta_p\cdot ac'/p \equiv ac' \bmod r$, so that by Lemma 5.14, (i), this pair of terms cancels out.

(5.18) The proof of (ii) breaks up into several steps. By definition

$$\begin{aligned} w_\chi(S_{\beta_d\cdot a/d})_c &= \sum_{b\in(\mathbf{Z}/r)^*} \chi(b)\cdot\overline{(b\beta_d\cdot \tfrac{q}{d})} \\ &= \frac{\varphi(r)}{\varphi(df)} \sum_{b'\in(\mathbf{Z}/f)^*} \chi(b') \sum_{\substack{b\mapsto b'\\ b\in(\mathbf{Z}/df)^*}} \overline{(b\beta_d\cdot \tfrac{q}{d})}. \end{aligned} \tag{6}$$

Step 1. Consider first the range of summation of the internal sum. This can be replaced by a sum over the additive group $(\mathbf{Z}/df)$ using a Möbius inversion: for each divisor $d'|d$, write $d = d'd''$, and let

$$T(d') = \sum_{\substack{b\mapsto b'\\ b\in(d'\mathbf{Z}/df\mathbf{Z})}} (\text{summand}); \tag{7}$$

then clearly, since $(\mathbf{Z}/df)\smallsetminus(\mathbf{Z}/df)^*$ is the union of $(d'\mathbf{Z}/df\mathbf{Z})$ for different divisors $d' > 1$, I have

$$\sum_{\substack{b\mapsto b'\\ b\in(\mathbf{Z}/df)^*}} (\text{summand}) = \sum_{d'|d} \mu(d')T(d'). \tag{8}$$

Step 2. I now make the range of summation in $T(d')$ more explicit. First, since d and f are coprime, for any b', exactly one of the integers $b' + if$ for $i = 0, \dots, d-1$ is divisible by d, so that there exists i_0 and x with $b' + i_0 f = dx$. Since the sum $T(d')$ only involves b which are divisible by d', and the summand only depends on $b \bmod df$, I can take the range of summation to be

$$b = b' + i_0 f + jd'f \quad \text{where } j = 0, \dots, d'' - 1.$$

Step 3. For each divisor d', I claim that

$$T(d') = \sum_{\substack{b\mapsto b'\\ b\in(d'\mathbf{Z}/df\mathbf{Z})}} \overline{\left(b\beta_d\cdot\frac{q}{d}\right)} = \sum_{i=0}^{d''-1} \overline{b'q + i\cdot\frac{r}{d''}} = q\cdot\overline{\overline{b'd''}} + \binom{d''}{2}\cdot\frac{r}{d''}, \tag{9}$$

where $\overline{\overline{}}$ denotes smallest residue $\bmod f$.

PROOF. By definition,

$$T(d') = \sum_{\substack{b\mapsto b'\\ b\in(d'\mathbf{Z}/df\mathbf{Z})}} \overline{\left(b\beta_d\cdot\frac{q}{d}\right)}. \tag{10}$$

Using Step 2, this becomes

$$\sum_{j=0}^{d''-1} \overline{(b' + i_0 f + jd'f)\beta_d\cdot\frac{q}{d}}; \tag{11}$$

now

$$(b' + i_0 f)\beta_d\cdot\frac{q}{d} = dx\beta_d\cdot\frac{q}{d} = x\beta_d q \equiv xdq = (b' + i_0 f)q \equiv b'q \quad \bmod r, \tag{12}$$

where the middle congruence uses the defining property of β_d. Gathering together the terms $d'fq/d$ into r/d'', this gives

$$\sum \overline{b'q + \beta_d j\cdot\frac{r}{d''}}, \tag{13}$$

and since $\beta_d \in (\mathbf{Z}/r)^*$, multiplication by β_d just permutes the range of summation. This proves the first part of (9).

To get the second equality, note that the numbers

$$\overline{(b'd'' + if)\cdot\frac{q}{d''}} \quad \text{as } i = 0, \dots, d'' - 1$$

take on a smallest value

$$\frac{q}{d''}\cdot\overline{\overline{b'd''}} \quad \text{when } 0 \le b'd'' + i_1 f < f,$$

and if I then change the range of summation to $i = i_1 + j$ with $j = 0, \dots, d'' - 1$, the summand simplifies, and the r.h.s. of (9) emerges after a brief struggle.

Step 4. Writing (8) for the internal sum in (6) gives

$$w_\chi(s_{\beta_d\cdot a/d})_c = \frac{\varphi(r)}{\varphi(df)} \sum_{b'\in(\mathbf{Z}/f)^*} \chi(b') \sum_{d'|d} \mu(d')T(d'), \tag{14}$$

which by (9) is equal to

$$\frac{\varphi(r)}{\varphi(df)} \sum_{b'\in(\mathbf{Z}/f)^*} \chi(b') \sum_{d'|d} \mu(d')\cdot\left\{q\cdot\overline{\overline{b'd''}} + \binom{d''}{2}\cdot\frac{r}{d''}\right\}. \tag{15}$$

Now it is clear that for fixed d', summing the two terms in curly brackets against $\chi(b')$ leads respectively to $\chi^{-1}(d'')\cdot r\cdot B_{1,\chi}$ and 0. The first equality of (3) then comes out at once.

The second equality is easy: $\varphi(df) = \varphi(d)\cdot\prod(p-1)$ gives the denominator, and expanding $\prod(\chi^{-1}(p) - 1)$ the numerator.

Step 5. This is also easy: each factor on the right-hand side of (4) is

$$\frac{p - \chi^{-1}(p)}{p - 1} = 1 - \frac{\chi^{-1}(p) - 1}{p - 1};$$

the product of these taken over the primes $p \in \mathcal{P}$ with $p|a$ is obviously of the form

$$\sum_{d|a} \mu(d) \cdot \prod_{p|d} \left(\frac{\chi^{-1}(p) - 1}{p - 1} \right).$$

Now by (3), after multiplication by $r \cdot \varphi(r)/\varphi(f) \cdot B_{1,\chi}$, each summand becomes

$$(w_\chi(S_{\beta_d \cdot a/d}))_c,$$

and comparing with the definition of $v_\chi(a)$ in (5.16), the sum is just $(v_\chi(a))_c$.

Amen.

(5.19) *Interpretation.* Suppose I take it into my head to write down the sum

$$B(\varepsilon) = B_r(\varepsilon) = \frac{1}{2} + \frac{1}{r} \sum_{k=1}^{r-1} k\varepsilon^k$$

for any $\varepsilon \in \boldsymbol{\mu}_r - 1$, $B(1) = 0$. Then for $a \in \mathbf{Z}$ coprime to r,

$$B(\varepsilon^a) = \frac{1}{2} + \frac{1}{r} \sum k\varepsilon^{ak} = \frac{1}{2} + \frac{1}{r} \sum_{k=1}^{r-1} \overline{a'k} \cdot \varepsilon^k,$$

where $aa' \equiv 1 \bmod r$. So the sum pulled out of a hat organizes the apparently random combinatorial data of the periodic values of $\overline{a'k}$ into a single element $B(\varepsilon) \in K = \mathbf{Q}(\boldsymbol{\mu}_r)$ of the cyclotomic field.

If I multiply the kth equality in (i) of (5.4) by ε^k for some primitive ε and sum over k, then (provided the a_i and b_j are coprime to r), I get

$$\sum_{i=1}^{n} B(\varepsilon^{a'_i}) = \sum_{j=1}^{m} B(\varepsilon^{b'_j}).$$

This reduces the proof of (i) $\Rightarrow$ (ii) in (5.4) to linear dependence relations between the $B(\varepsilon) \in K$ for primitive elements $\varepsilon \in \boldsymbol{\mu}_r$. Since

$$B_r(\varepsilon) = -\frac{1}{2} \left\{ \frac{1+\varepsilon}{1-\varepsilon} \right\} = \frac{i}{2} \cot \frac{\pi k}{r}$$

if $\varepsilon = \exp(2\pi k i/r)$, the problem is equivalent to proving the following result:

PROPOSITION. *The $\varphi(r)/2$ numbers*

$$\cot \frac{\pi k}{r} \quad \textit{for } k = 1, \dots, \left[\frac{r}{2}\right] \textit{ coprime to } r$$

are linearly independent over $\mathbf{Q}$.

REMARK. The preprint version of this paper contained a false proof of this proposition.

PROOF. The vector space $\mathbf{Q}^r = V$ of (5.8) can also be thought of as $\mathbf{Q}[X]/(X^r - 1)$; now corresponding to the factorisation $X^r - 1 = \prod \Phi_d(X)$ of $X^r - 1$ into the product of cyclotomic polynomials Φ_d for divisors $d|r$, the $\mathbf{Q}$-algebra $\mathbf{Q}[X]/(X^r - 1)$ splits in a canonical way as the product of cyclotomic fields $\mathbf{Q}(\varepsilon_d)$ of degree d (where ε_d denotes a primitive dth root of 1).

Now by construction, if a is coprime to r, the projection to $\mathbf{Q}(\varepsilon_r)$ of $S_a \in V$ is

$$\sum_{k=1}^{r-1} \left(\overline{ak} - \frac{r}{2} \right) \cdot \varepsilon^k = \frac{r}{2} + \sum k \cdot \varepsilon^{a'k} = r \cdot B(\varepsilon^{a'});$$

on the other hand if $\mathrm{hcf}(a, r) > 1$, it is not hard to check that S_a projects to zero in $\mathbf{Q}(\varepsilon_r)$. It follows from this that the S_a with a coprime to r map to linearly independent elements of $\mathbf{Q}(\varepsilon_r)$. Q.E.D.

6. Terminal 3-fold singularities according to Mori.

(6.1) The results of §§6–7 are taken essentially from **[Mori]**. Part (I) of the main theorem gives necessary conditions on the possible equations and group actions for terminal hyperquotient (= *quick*) singularities, whereas (II) is a first attempt at giving sufficient conditions.

THEOREM (S. MORI). (I) *Let $P \in X = (Q \in Y)/\boldsymbol{\mu}_r$ be a terminal hyperquotient singularity, where $r > 1$ and $Q \in Y$ is singular. Then $P \in X$ belongs to one of the following 6 families:*

	r	Type	f	Conditions
(1)	any	$\frac{1}{r}(a, -a, 1, 0; 0)$	$xy + g(z^r, t)$	$g \in m^2$, a, r coprime
(2)	4	$\frac{1}{4}(1, 1, 3, 2; 2)$	$xy + z^2 + g(t)$	$g \in m^3$
			or $x^2 + z^2 + g(y, t)$	$g \in m^3$
(3)	2	$\frac{1}{2}(0, 1, 1, 1; 0)$	$x^2 + y^2 + g(z, t)$	$g \in m^4$
(4)	3	$\frac{1}{3}(0, 2, 1, 1; 0)$	$x^2 + y^3 + z^3 + t^3$	
			or $x^2 + y^3 + z^2t + yg(z, t) + h(z, t)$	$g \in m^4$, $h \in m^6$
			or $x^2 + y^3 + z^3 + yg(z, t) + h(z, t)$	$g \in m^4$, $h \in m^6$
(5)	2	$\frac{1}{2}(1, 0, 1, 1; 0)$	$x^2 + y^3 + yzt + g(z, t)$	$g \in m^4$
			or $x^2 + yzt + y^n + g(z, t)$	$g \in m^4$, $n \geq 4$
			or $x^2 + yz^2 + y^n + g(z, t)$	$g \in m^4$, $n \geq 3$
(6)	2	$\frac{1}{2}(1, 0, 1, 1; 0)$	$x^2 + y^3 + yg(z, t) + h(z, t)$	$g, h \in m^4$, $h_4 \neq 0$.

(II) *The general element of each of the families* (1)–(6) *is terminal.*

(6.2) REMARKS. (i) Some of the families can be tidied up into discrete normal forms using standard methods of singularity theory: for example, the second alternative case of (4) can be reduced to one of

$$\begin{aligned} &x^2 + y^3 + z^2t + yt^a \text{ with } a \equiv 1 \bmod 3,\ a \geq 4 \\ \text{or}\quad &x^2 + y^3 + z^2t + t^b \text{ with } b \equiv 0 \bmod 3,\ b \geq 6 \\ \text{or}\quad &x^2 + y^3 + z^2t + \alpha y t^{2c} + \beta t^{3c} \text{ with } c \equiv 2 \bmod 3,\ c \geq 2,\ 4\alpha^3 + 27\beta^2 \neq 0. \end{aligned}$$

More information is given in [**Mori**, (12.1), (23.1), and (25.1)].

(ii) In fact, for each of (1)–(6), every isolated singularity is terminal, as has recently been proved by [**Kollár and Shepherd-Barron**, §6] (compare (6.5), (2)). For the most important family (1), this can be proved as follows: any singularity in (1) is a Cartier divisor $P \in X$: $(xy + g(z^r, t)) \subset \mathbf{A}^1 \times A$, where

$\mathbf{A}^1$ corresponds to the invariant coordinate t, and A is the terminal quotient singularity $\frac{1}{r}(a,-a,1)$. Now let $B \to A$ be Danilov's resolution as in (5.7), and

$$\varphi\colon \mathbf{A}^1 \times B \to \mathbf{A}^1 \times A.$$

Clearly $X' = \varphi^{-1}(X) \subset \mathbf{A}^1 \times B$ is irreducible, and it can be seen that X' is normal; $\varphi\colon X' \to X$ is totally discrepant, so that to prove that $P \in X$ is terminal it is sufficient to prove that X' has only canonical hypersurface singularities. This can be done (with quite a lot of pain) by a direct calculation using the explicit coordinates for B described in Exercise 5.7.

(6.3) Part (I) of Theorem 6.1 is proved in §7 after some preliminary work at the end of §6. The main point will be to determine what the μ_r-action looks like by making knowledgeable use of the terminal lemma (5.4). I should emphasise that this is merely a technical reworking of Mori's original argument, and that each step is derived more or less directly from **[Mori]**.

(6.4) *Q-smoothing and the general elephant.* (A) If $P \in X$ is a terminal singularity it belongs to one of the families of (6.1) and is the quotient of an isolated cDV singularity $Q \in Y\colon (f = 0) \subset \mathbf{A}^4$. It is then possible to write down deformations of X by just varying the equation f in its eigenspace; the singularity of X just varies inside its toric ambient space $\mathbf{A}^4/\mu_r$.

Now in every case of (6.1), one can write down a 1-parameter deformation $\{Y_\lambda\}$ of Y compatible with the action of μ_r such that Y_λ is nonsingular for $\lambda \neq 0$ and meets the fixed locuses of μ_r transversally; this is possible because in each case at least one of the coordinates x_i has the same eigenvalue as f. Define a deformation of X by $X_\lambda = Y_\lambda/\mu_r$. I call this situation a *Q-smoothing* of $P \in X$: it is a deformation $\{X_\lambda\}$ of $P \in X$ such that the general fibre has as its only singularities a number of terminal quotient singularities $\frac{1}{r}(a,-a,1)$.

Notice that since $\{X_\lambda\}$ is constructed as a quotient of a flat deformation $\{Y_\lambda\}$ of Y, the eigensheaves $\mathcal{L}_i$ of the action of μ_r on $\mathcal{O}_Y$ are sheaves over X_λ which vary in a flat family with λ; this will be important for the proof of Theorem 10.2.

For example, in (1) the fixed locus of the group action is the t-axis l; suppose that t^n is the smallest power of t appearing in f. Then $f|_l$ has 0 as a root with multiplicity n; taking $f_\lambda = f + \lambda t$, this root splits up into n simple roots. The picture for X is given in Figure 6.

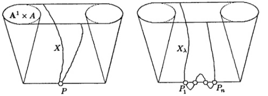


FIGURE 6

(B) If $P \in X$ is a Cohen-Macaulay point of a 3-fold then it follows by standard formalism that the general elephant $S \in |-K_X|$ has a normal Gorenstein singularity at P. The following is an interesting and important question: when does S have a Du Val singularity at P? There is some hope that under this condition, the 3-fold singularity can be treated as a kind of generalised cDV point, so that, for example, problems related to the class group and small partial resolutions of X can be dealt with by some (quite considerable) generalisation of the Brieskorn techniques for cDV points. For 3-fold terminal singularities $P \in X$, the lists of Theorem 6.1 allow me to write down an explicit $S \in |-K_X|$ with a Du Val singularity by writing down a hyperplane section of $Q \in Y$ which belongs to the eigenvalue corresponding to K_X. This shows clearly the close relationship between the list of Theorem 6.1 and that of (4.10).

	Type	Q-smoothing	General elephant	Section
(1)	$\frac{1}{r}(a,-a,1,0;0)$	$f+\lambda t$	z	$A_{n-1} \xrightarrow{r-1} A_{rn-1}$
(2)	$\frac{1}{4}(1,1,3,2;2)$	$f+\lambda t$	$x-y$	$A_{2n} \xrightarrow{4-1} D_{2n+1}$
(3)	$\frac{1}{2}(0,1,1,1;0)$	$f+\lambda x$	$\lambda z+\mu t$	$A_{2n-1} \xrightarrow{2-1} D_{n+2}$
(4)	$\frac{1}{3}(0,2,1,1;0)$	$f+\lambda x$	$\lambda z+\mu t$	$D_4 \xrightarrow{3-1} E_6$
(5)	$\frac{1}{2}(1,0,1,1;0)$	$f+\lambda y$	$\lambda z+\mu t$	$D_{n+1} \xrightarrow{2-1} D_{2n}$
(6)	$\frac{1}{2}(1,0,1,1;0)$	$f+\lambda y$	$\lambda z+\mu t$	$E_6 \xrightarrow{2-1} E_7$

(6.5) *Remaining problems.* As discussed in (6.2), the converse statement Theorem 6.1, (II) is in a rather unsatisfactory state. The lists of Theorem 6.1 should perhaps be regarded as just the start of the study of these singularities.

Problems. (1) Is there a direct proof that terminal $\Rightarrow$ the general elephant $S \in |-K_X|$ is a Du Val singularity?

(2) Is there a proof of the converse statement (II) based on the fact that the general elephant is Du Val?

(3) How should one resolve these singularities? In particular, is there an analogue of the economic resolution (5.7)? It would be useful to have a partial resolution $f\colon Y \to X$ such that all exceptional primes have discrepancy < 1 and Y has only isolated cDV points.

For example, the equation in family (1) is of the form $xy + g(z^r, t)$; in singularity theory it is traditional to ignore quadratic factors ("Morse lemma"), so that one might look for a resolution of the 3-fold singularity in terms of that of the curve singularity $(g(Z,t) = 0)$. See also **[Kollár and Shepherd-Barron, §6]**.

(6.6) The proof of (6.1) will break up into proofs in 3 disjoint cases, which are carried over to §7; I start by setting up the general framework.

Notation. Introduce the following terminology: first, $(x,y,z,t) = (x_1,\dots,x_4)$ are local analytic coordinates on $\mathbf{A}^4$, and the group action is given by

$$\mu_r \ni \varepsilon\colon (x,y,z,t;f) \mapsto (\varepsilon^a x, \varepsilon^b y, \varepsilon^c z, \varepsilon^d t; \varepsilon^e f),$$

where $(a,b,c,d) = (a_1, \dots, a_4)$. Note that all the monomials in f belong to the same eigenvalue of the action, so that for example if $xy \in f$ then $a+b=e$ and $\alpha(f) \equiv \alpha(xy) \bmod \mathbf{Z}$ for all $\alpha \in N \cap \sigma$. The group action on the generator

$$s = \operatorname{Res}_Y \frac{dx \wedge dy \wedge dz \wedge dt}{y} \in \omega_Y$$

is given by

$$\mu_r \ni \varepsilon\colon s \mapsto \varepsilon^{a+b+c+d-e} \cdot s,$$

and since the quotient X has index r, $a+b+c+d-e$ is coprime to r; it is therefore reasonable to fix this to be 1, which corresponds to the fact that the local class group of $P \in X$ has a canonical generator K_X.

Rules of the game.

Rule I: For every primitive $\alpha \in N \cap \sigma$,

$$\alpha(f) + 1 < \alpha(xyzt).$$

Rule II: (i) If $\operatorname{hcf}(a_i, r) \neq 1$ then a_i divides e;
(ii) $\operatorname{hcf}(a_i, a_j, r) = 1$;
(iii) $a+b+c+d-e=1$.

Rule III: An analytic change of coordinates compatible with the group action can be used to put f in normal form with respect to its leading terms. That is, I can assume that:

(i) $$f = q(x_1, \dots, x_k) + f'(x_{k+1}, \dots, x_4)$$

with q a nondegenerate quadratic form in $x_1, \dots, x_k$; moreover
(ii) if the 3-jet of f is $x^2 + y^2 z$ then

$$f = x^2 + y^2 z + yg(t) + h(z,t),$$

or if the 3-jet is $x^2 + y^3$ then

$$f = x^2 + y^3 + yg(z,t) + h(z,t).$$

Rule IV: Only one entry per household; employees of Kelloggs' or of their subsidiaries are not eligible for entry; the referees' decision is final.

Here Rule I is the condition of Theorem 4.6, and Rule II, (i) is a consequence of the fact that μ_r acts freely on Y outside the origin: if the action of some element of μ_r fixes the x_i-axis pointwise then some power of x_i must appear in f, hence $a_i | e$; Rule II, (ii) is similar, and (iii) has already been discussed. Rule III is a standard manipulation in singularity theory. The normal forms can be got by explicit ad hoc coordinate changes, and it is easy to see that these can be done in an equivariant way. For example, if $x^2 \in f$ then the terms in f divisible by x^2 can be gathered together to give

$$f = x^2(1+\eta) + xg(y,z,t) + h(y,z,t)$$

with $\eta \in m$. Then $\xi = \sqrt{1+\eta}$ is an analytic function invariant under the group action, and

$$f = x'^2 + h'(y,z,t),$$

where $x' = \xi x + (1/2)g/\xi$.

(6.7) *First division into cases.* [*Hint*: if you have trouble following the proof of Theorem 6.1, redo Exercise 4.10.]

I know from general theory that $Q \in Y$ is a cDV point, which implies f contains certain quadratic, cubic terms, etc.; as in (4.9), (3), these restrictions can of course be deduced from Rule I applied to suitable weightings $\alpha \in \mathbf{Z}^4$, but I will not spend time on this, and just assert what I need about f without proof. Write $f = f_2 + \cdots$, or $f_2 + f_3 + \cdots$, where f_2 is the quadratic part, f_3 the cubic part, and $\cdots$ indicates terms of higher degree. I write m for the maximal ideal of $k[x,y,z,t]$, so that for example $g(z,t) \in m^2$ means that $g \in (z,t)^2 \subset k[z,t]$. The following proposition deals with the quadratic part of f.

PROPOSITION. *Exactly one of the following 5 cases holds in suitable eigencoordinates. To be more precise, I can make a μ_r-equivariant analytic change of coordinates (including possibly permuting the coordinates), to achieve one of the following, where the coordinate functions x, y, z and t are eigenfunctions of the μ_r-action; the 5 cases are disjoint.*

cA case: $f = xy + g(z,t)$ *with* $g \in m^2$;
odd case: $f = x^2 + y^2 + g(z,t)$ *with* $g \in m^3$, *and* $a \neq b$;
cD_4 case: $f = x^2 + g(y,z,t)$ *with* $g \in m^3$ *and* g_3 *a reduced cubic;*
cD_n case: $f = x^2 + y^2 z + g(z,t)$ *with* $g \in m^4$;
cE case: $f = x^2 + y^3 + yg(z,t) + h(z,t)$ *with* $g \in m^3$ *and* $h \in m^4$.

Note that by the proposition, the proof of Theorem 6.1 breaks up into the 3 disjoint proofs of the implications

$$cA \Rightarrow (1) \text{ or } (2);$$
$$\text{odd case} \Rightarrow (2) \text{ or } (3);$$
$$cD_4, cD_n \text{ or } cE \Rightarrow (4), (5) \text{ or } (6).$$

(6.8) PROOF. If $\operatorname{rank} f_2 \leq 1$ then there's essentially nothing to prove, since $f_2 = x^2$ with x an eigenfunction; completing the square (by Rule III) gives

$$f = x^2 + g(y,z,t) \quad \text{with } g \in m^3 \text{ and } g_3 \neq 0.$$

Now if g_3 is a reduced cubic I'm in the cD_4 case. Failing that, since the tangent cone to $(g=0)$ is invariant under the group action, either $g_3 = y^2 z$ with y and z eigenfunctions, or $g_3 = y^3$ with y an eigenfunction.

It's convenient to deal with the case $\operatorname{rank} f_2 \geq 2$ by means of a general result.

(6.9) LEMMA. *Let V be a vector space (over an algebraically closed field k of characteristic $\neq 2$) with a linear action of μ_r, and let $q\colon V \to k$ be a nonzero quadratic form which is an eigenform. Then there exist integers $k \geq 0$ and $l \leq 2$ with $2k + l = \operatorname{rank} q$ and a basis of V consisting of eigenvectors*

$u_1, \ldots, u_k, v_1, \ldots, v_k, w_1, \ldots, w_{n-2k}$ *(when $n = \dim V$) such that*

(i) *q is given in the dual coordinates (x_i, y_i, z_i) by*

$$(*) \qquad q = \sum_{i=1}^{k} x_i y_i + \sum_{j=1}^{l} z_j^2;$$

and (ii) *if $l = 2$ then w_1 and w_2 have different eigenvalues.*

(6.10) This proves (6.7), since if $\operatorname{rank} f_2 \geq 3$ then by (6.9) I can make a coordinate change to get $f_2 = xy + g_2(z,t)$, and by Rule III this reduces to the cA case; if $\operatorname{rank} f_2 = 2$, a coordinate change will give either $f_2 = xy$, or $x^2 + y^2$ with different eigenvalues for x and y, giving the cA or odd cases.

PROOF OF (6.9). (GOTO §7 if you already know this.) This can be done as an exercise in undergraduate algebra. The main point is that if $q = \sum \lambda_i x_i^2$ is diagonal in eigencoordinates $(x_1, \ldots, x_n)$, and no two x_i's with $\lambda_i \neq 0$ have the same eigenvalue then $\operatorname{rank} q \leq 2$; this is trivial, since the two square roots of the eigenvalue of q are the only two possibilities for the eigenvalues of x_i. On the other hand, if it is not diagonal then you can pick a term say $x_1 x_2 \in q$, and make coordinate transformations arguing on the other terms in which x_1 and x_2 appear.

(6.11) Now here's the real proof of (6.9). By choosing a μ_r-invariant complement of the kernel of q, I can assume that q is of maximal rank n. Now the existence of the normal form $(*)$ is formally equivalent to the existence of an isotropic subspace $E \subset V$ of dimension $k = (n-l)/2$ invariant under the action of μ_r. Indeed, let E be an invariant k-dimensional subspace with $q|_E \equiv 0$; then $E \subset E^\perp$, where $E^\perp$ is the orthogonal of E with respect to the associated bilinear form. Choose $F \subset V$ to be an invariant complement of $E^\perp$ in V, necessarily k-dimensional. By construction the quadratic form induces an isomorphism from E to the dual of F, and using this it is not hard to adjust F so that it is also isotropic. The orthogonal complement of $E \oplus F$ is then l-dimensional with $l \leq 2$, and it is easy to complete the proof.

The form q defines a nonsingular quadric $Q \subset \mathbf{P}^{n-1}$; an isotropic linear subspace $E \subset V$ of dimension k corresponds exactly to a $(k-1)$-plane in Q (that is, a linear subspace $\mathbf{P}^{k-1} \subset Q \subset \mathbf{P}^{n-1}$), and the problem is to find an invariant one. This is very well known material (see, for example [Hodge and Pedoe, vol. II, pp. 230–237]). Since $k = (\operatorname{rank} q - l)/2$, there are lots of $(k-1)$-planes on Q; these are maximal linear subspaces of Q if $l = 0$ or 1, or one less than maximal if $l = 2$. The space parametrising them is a nonsingular variety $G_{k-1}(Q)$, and is irreducible if $l = 1$ or 2, or has two components if $l = 0$; each component is rational. Now μ_r acts on $Q \subset \mathbf{P}(V)$, and hence on $G_{k-1}(Q)$; if $n = \operatorname{rank} q$ is even and for $l = 0$ the group action interchanges the two components of $G_{k-1}(Q)$, then there can't be any fixed point and so there's no normal form $(*)$ with $l = 0$. But in case $l = 1$ or 2, or in case $l = 0$ if the group action takes each component to itself, the action must have at least one fixed point (since the component of $G_{k-1}(Q)$ is irreducible and rational). This corresponds to an invariant isotropic subspace of V of the required dimension. Q.E.D.

7. Case-by-case proof of Theorem (6.1).

(7.1) *Plan of proof.* In (6.7) I divided the proof into the 3 implications

$$\begin{aligned} cA&: \quad cA \Rightarrow (1) \text{ or } (2);\\ odd&: \quad \text{odd case} \Rightarrow (2) \text{ or } (3);\\ cD-E&: \quad cD_4, cD_n, \text{ or } cE \Rightarrow (4), (5) \text{ or } (6). \end{aligned}$$

Although the proofs in each case are logically disjoint, they all follow the same 4-step pattern; the first 3 steps involve only the quadratic terms xy or x^2+y^2 or x^2 of f.

Step 1. Reduce to the terminal lemma. This part is analogous to Lemma 5.3 in the classification of terminal quotient singularities; it goes from the inequalities $\alpha(f) < \alpha(xyzt) - 1$ of Rule I, applied to the weightings

$$\alpha_k = \tfrac{1}{r}(\overline{ak}, \overline{bk}, \overline{ck}, \overline{dk}) \quad \text{for } k = 1, \ldots, r-1$$

to a set of $r-1$ equalities; see (7.2).

Step 2. Coprime problem. This step discusses the possible common factors of a, b, c, d and e with r to verify the assumption of Theorem 5.4, (A) or of Corollary 5.6.

Step 3. Using the terminal lemma, write down in a mechanical way a list containing all possibilities for the type.

Step 4. Final method. The situation will be that the terms xy or x^2+y^2 or x^2 of f_2 will satisfy the conditions of Rule I for one-half of all the weightings α_k: the remaining half of the weightings will then impose further monomials on f, and the condition that $f \in m^2$ will then exclude many of the cases written down in Step 3. See (7.8) for a more precise statement in the cA case.

(7.2) *How to reduce to the terminal lemma.*

PROPOSITION. (a) *Suppose that $xy \in f$. Then for any $\alpha \in N \cap \square$,*

either $\alpha(f) = \alpha(xy) < 1$ *and* $\alpha(zt) > 1$;
or $\alpha(f) = \alpha(xy) - 1$ *and* $\alpha(zt) < 1$.

(b) *Suppose that $x^2 \in f$. Then for any $\alpha \in N \cap \square$,*

either $\alpha(f) = 2\alpha(x) < 1$ *and* $\alpha(yzt) > 1 + \alpha(x)$;
or $\alpha(f) = 2\alpha(x) - 1$ *and* $\alpha(yzt) < 1 + \alpha(x)$.

In (a) *and* (b), *the alternative cases are interchanged by the symmetry*

$$\alpha \mapsto \alpha' = (1, \ldots, 1) - \alpha.$$

(c) *If either xy or $x^2 \in f$, then*

$$\overline{ak} + \overline{bk} + \overline{ck} + \overline{dk} = \overline{ek} + k + r$$

for each $k = 1, \ldots, r-1$.

(7.3) PROOF OF (a). Assume that $xy \in f$; it follows that $a + b \equiv e \bmod r$, and therefore

$$\alpha(f) \equiv \alpha(xy) \quad \text{for all } \alpha \in N \cap \sigma;$$

also from Rule II, (iii), $c+d \equiv 1 \bmod r$. From this it is clear that

$$\alpha_k(zt) > 1 \iff \alpha_{r-k}(zt) < 1.$$

In view of $\alpha(xy) \equiv \alpha(f) \bmod \mathbf{Z}$ and $0 < \alpha(f) \le \alpha(xy) < 2$, the two cases $\alpha(f) = \alpha(xy)$ or $\alpha(xy) - 1$ seem pretty well inevitable. Also Rule I gives

$$\alpha(f) = \alpha(xy) \Rightarrow \alpha(zt) > 1.$$

So I only have to show that

$$\begin{array}{ll}\text{neither} & \alpha(f) = \alpha(xy) \ge 1 \text{ and } \alpha(zt) > 1;\\ \text{nor} & \alpha(f) = \alpha(xy) - 1 \text{ and } \alpha(zt) \ge 1\end{array}$$

can happen for any α. Write $\alpha' = (1, \dots, 1) - \alpha$. Then clearly in either of the two cases

$$\alpha(xy) \ge 1 \Rightarrow \alpha'(f) \le \alpha'(xy) \le 1,$$

and therefore $\alpha'(f) = \alpha'(xy)$, whereas $\alpha(zt) \ge 1$ implies $\alpha'(zt) \le 1$. This contradicts Rule I.

(7.4) PROOF OF (b). (This is word-for-word the same proof as for (a).) Assume that $x^2 \in f$. Note that since $x^2 \in f$ it follows that $2a \equiv e \bmod r$, and therefore

$$\alpha(f) \equiv 2\alpha(x) \quad \text{for all } \alpha \in N \cap \sigma;$$

also from Rule II, (iii), $b+c+d \equiv 1+a \bmod r$. From this it is an easy computation to see that

$$\alpha_k(yzt) > 1 + \alpha(x) \iff \alpha_{r-k}(yzt) < 1 + \alpha(x).$$

Now in view of $2\alpha(x) \equiv \alpha(f) \bmod \mathbf{Z}$ and $0 < \alpha(f) \le 2\alpha(x) < 2$, one of the two cases $\alpha(f) = 2\alpha(x)$ or $2\alpha(x) - 1$ must hold. Also Rule I gives

$$\alpha(f) = 2\alpha(x) \Rightarrow \alpha(yzt) > 1 + \alpha(x).$$

So I only have to show that

$$\begin{array}{ll}\text{neither} & \alpha(f) = 2\alpha(x) \ge 1 \text{ and } \alpha(yzt) > 1 + \alpha(x);\\ \text{nor} & \alpha(f) = 2\alpha(x) - 1 \text{ and } \alpha(yzt) \ge 1 + \alpha(x)\end{array}$$

can happen for any α. Write $\alpha' = (1, \dots, 1) - \alpha$. Then clearly in either case

$$2\alpha(x) \ge 1 \Rightarrow \alpha'(f) \le 2\alpha'(x) \le 1.$$

and therefore $\alpha'(f) = 2\alpha'(x)$, whereas

$$\alpha(yzt) \ge 1 + \alpha(x) \text{ implies } \alpha'(yzt) \le 1 + \alpha(x).$$

This contradicts Rule I.

(7.5) PROOF OF (c). This is easy: if $xy \in f$ then in the two cases of (a),

$$\overline{ak} + \overline{bk} = \overline{ek} \quad \text{and} \quad \overline{ck} + \overline{dk} = k + r$$

or

$$\overline{ak} + \overline{bk} = \overline{ek} + r \quad \text{and} \quad \overline{ck} + \overline{dk} = k.$$

Similarly, if $x^2 \in f$ then in the cases of (b),

$$2\overline{ak} = \overline{ek} \quad \text{and} \quad \overline{bk} + \overline{ck} + \overline{dk} = \overline{ak} + k + r$$

or

$$2\overline{ak} = \overline{ek} + r \quad \text{and} \quad \overline{bk} + \overline{ck} + \overline{dk} = \overline{ak} + k. \quad \text{Q.E.D.}$$

(7.6) *The cA case.* The time has now come to divide up into the cases of (6.7). The case cA: $f = xy + g(z,t)$ will occupy me from now until (7.8).

cA. Step 2 (*coprimeness*). Lemma 7.2 gives me the conditions of the terminal lemma (5.4), (A) for the $(4+2)$-tuple $\frac{1}{r}(a,b,c,d;e,1)$, except that some of the numbers may not be coprime to r. However, this does not make too much trouble, thanks to the following argument.

Define $q = \mathrm{hcf}(e,r)$; the cases $q = 1$ and $q = r$ are not excluded in what follows.

LEMMA. *After interchanging z and t if necessary, the following hold:*
(1) $q = \mathrm{hcf}(d,r)$,
(2) *a, b and c are coprime to r,*
(3) $g = g(z^r, t)$.

PROOF. Since $xy \in f$, I have $e \equiv a + b \bmod r$, and it's easy to prove that a and b are coprime to r using Rule II, (ii). By Rule II, (i), any common factor of c or d and r divides q, so that I only need to prove that q divides either c or d. Note that if I set $k = r/q$ then $\overline{ek} = 0$, so I must be in the second case of the computation in (7.5), and hence

$$\overline{ck} + \overline{dk} = k \quad \text{and} \quad \overline{c(r-k)} + \overline{d(r-k)} = r - k;$$

therefore not both of

$$\overline{ck} + \overline{c(r-k)} \quad \text{and} \quad \overline{dk} + \overline{d(r-k)}$$

can be equal to r. This proves that q divides d, say.

(7.7) *cA. Step* 3. I list all the possibilities for the type; one of the following cases holds (after possibly interchanging x and y or z and t):

If $q > 1$,
(A) $a + b \equiv 0$, $c \equiv 1$, $d \equiv e \bmod r$, that is, $\frac{1}{r}(a, -a, 1, 0; 0)$;
(B) $a \equiv 1$, $b + c \equiv 0$, $d \equiv e \bmod r$, that is, $\frac{1}{r}(1, b, -b, b+1; b+1)$.

If $q = 1$,
(C) $\frac{1}{r}(a, 1, -a, a+1; a+1)$;
(D) $\frac{1}{r}(a, -a-1, -a, a+1; -1)$

with a and $a+1$ coprime to r.

PROOF. Recall that $a + b \equiv e$ and $a + b + c + d - e \equiv 1 \bmod r$.

The terminal lemma tells me that the 6 elements a, b, c, d, e and 1 must be paired off: if $q > 1$ then by Corollary 5.6 I must pair d and e, giving (A) and (B). If $q = 1$ then by Theorem 5.4 it's easy to see that (C) and (D) are the only two possibilities.

(7.8) cA. *Step* 4. Case (A) gives $\frac{1}{r}(a,-a,1,0;0)$, which is case (1) of (6.1). To complete the proof of (6.1) in case cA, I must prove that (C) and (D) are impossible, and that (B) is only possible if either

$$\tfrac{1}{r}(a,b,c,d;e)=\tfrac{1}{4}(1,1,3,2;2) \quad \text{and} \quad z^2\in f$$

or $\frac{1}{r}(a,b,c,d;e)$ also falls in case (A).

Method. The condition which remains to use is that, by (7.2), (a),

$$\alpha(f)=\alpha(xy)-1 \quad \text{for every } \alpha \text{ such that } \alpha(zt)\le 1,$$

so that g must have a monomial x^m with $\alpha(x^m)=\alpha(xy)-1$; on the other hand, $g=g(z^r,t)\in m^2$.

To kill (C), let $k(a+1)\equiv 1 \bmod r$ with $k<r$. Then $\alpha_k(zt)=k/r<1$, but $\alpha_k(xy)-1=1/r$, so that no monomial in m^2 stands a chance. To kill (D), choose $k=r-1$. Then $\alpha_k(zt)=(r-1)/r<1$, but again $\alpha_k(xy)-1=1/r$ so that again no monomial in m^2 can work.

Case (B) is $\frac{1}{r}(1,b,-b,b+1;b+1)$ with b coprime to r; if $b+1\equiv 0 \bmod r$, this simplifies to $\frac{1}{r}(1,-1,1,0;0)$ which is in (A). So assume I have

$$\tfrac{1}{r}(1,b,-b,b+1;b+1) \quad \text{with } b \text{ coprime to } r \text{ and } b+1<r.$$

Consider the weighting $\alpha=\alpha_{r-1}=\frac{1}{r}(r-1,r-b,b,r-b-1)$; then $\alpha(zt)=(r-1)/r<1$ and so by the method of this step $\alpha(g)\le(r-b-1)/r$. This means that there is some monomial $x^m\in(z,t)^2$ with

$$x^m\in g \quad \text{and} \quad \alpha(x^m)=\alpha(xy)-1=(r-b-1)/r.$$

Obviously no multiple of t will work, so that the monomial can only be z^n for some n with $nb=r-b-1$. Notice that since $n\ge 2$, it follows that $r\ge 3b+1\ge 2b+2$.

CLAIM. $r=4$, $b=1$, *and* $n=2$.

PROOF. I choose another weighting

$$\beta=\alpha_{r-2}=\tfrac{1}{r}(r-2,r-2b,2b,r-2b-2);$$

since $\beta(zt)=(r-2)/r<1$, the method gives

$$\beta(g)=\beta(xy)-1=(r-2b-2)/r.$$

However, the same monomial z^n can't possibly work, nor can any multiple of zt; hence some power t^m must appear in g with $m\ge 2$ and

$$m(r-2b-2)=r-2b-2.$$

This implies $r=2b+2$, and the claim follows at once.

This completes the proof of case $cA\Rightarrow$ (1) or (2) of (6.1).

(7.9) *The odd case*: $f=x^2+y^2+g(z,t)$ with $g\in m^3$ and $a\ne b$; this will take until (7.11). Note that $2a\equiv 2b\equiv e$ and $a\not\equiv b \bmod r$ implies that r is even, and $b\equiv a+r/2$.

Odd. Step 2 (*coprimeness*). Either (after interchanging x and y if necessary) $a\equiv e\equiv 0 \bmod r$, and b,c,d are coprime to r, or (after interchanging z and t if necessary) I have: $q=2=\operatorname{hcf}(d,r)$ and a,b,c are coprime to r.

PROOF. First of all I claim that if $a\not\equiv 0 \bmod r$ then $\operatorname{hcf}(a,r)=1$; for otherwise there exists some divisor k of r with $0<k<r$ such that $\overline{ak}=\overline{a(r-k)}=0$. Then by the computation in (7.5), I have

$$\overline{bk}+\overline{ck}+\overline{dk}=k+r$$

and

$$\overline{b(r-k)}+\overline{c(r-k)}+\overline{d(r-k)}=r-k;$$

(or the same with k and $(r-k)$ interchanged). Adding these together would show that not all of

$$\overline{bk}+\overline{b(r-k)}, \qquad \overline{ck}+\overline{c(r-k)} \quad \text{and} \quad \overline{dk}+\overline{d(r-k)}$$

can be r; so one of b,c,d has a common factor with a and r, contradicting Rule II, (ii).

Now since $e=2a$ and $\operatorname{hcf}(a,r)=1$ and r is even, it follows that $q=\operatorname{hcf}(e,r)=2$. Then setting $k=r/2$ I get $\overline{ak}=r/2$ and $\overline{ek}=0$, so that from (7.5),

$$\overline{bk}+\overline{ck}+\overline{dk}=r/2+r/2,$$

and exactly one of b,c,d is even.

(7.10) *Odd. Step* 3 (*listing possible types*). I claim that the type is

$$\begin{array}{ll}\text{either} & \tfrac{1}{2}(0,1,1,1;0)\\ \text{or} & \tfrac{1}{r}(1,\tfrac{r+2}{2},\tfrac{r-2}{2},2;2) \quad \text{for some } r \text{ with } 4|r.\end{array}$$

In fact $2a\equiv 2b\equiv e \bmod r$ so that r is even and $b\equiv a+r/2$. If $a=0$ then by Rule II, (ii), I must have $r=2$, giving the first conclusion. Otherwise by (7.9) and (5.6), I must pair d with e, and

$$\text{either } a\equiv 1,\ b+c\equiv 0 \quad \text{or} \quad a+b\equiv 0,\ c\equiv 1.$$

In the first case, $b=1+r/2=(r+2)/2$ is odd so $4|r$, as required. It's easy to see that the second possibility gives $a=r/4$, $b=3r/4$, and then using Rule II, (ii), necessarily $r=4$, which gives $\frac{1}{4}(1,3,1,2;1)$.

(7.11) *Odd. Step* 4 (*final method*). To prove that this case implies (3) of Theorem 1, I only have to show that the case

$$\tfrac{1}{r}\left(1,\tfrac{r+2}{2},\tfrac{r-2}{2},2;2\right) \quad \text{with } 4|r \text{ and } r>4$$

is impossible. By (7.2), (b), for every α with $\alpha(yzt)<1+\alpha(x)$, there exists a monomial of weight $2\alpha(x)-1$. So let

$$\alpha=\alpha_{r-2}=\tfrac{1}{r}(r-2,r-2,2,r-4);$$

then $\alpha(yzt)=(2r-4)/r<\alpha(x)+1=(r-2)/r+1$. Also

$$2\alpha(x)-1=(r-4)/r;$$

if $r > 4$ then it is easy to see that the only monomial in $(z,t)^2$ of weight $\le (r-4)/r$ and in the same eigenspace as f is z^2, which is excluded by the case assumption. This proves that I am in case $\frac{1}{4}(1,3,1,2;2)$ with $f = x^2 + y^2 + g(z,t)$, and permuting y and z gives (6.1), (2).

This completes the proof of odd case $\Rightarrow$ (2) or (3) of (6.1).

(7.12) Now consider the remaining cases $f = x^2 + g(y,z,t)$ with $g \in m^3$.

cD-E. Step 2 (coprimeness). Suppose that $x^2 \in f$; then

either $a \equiv e \equiv 0 \bmod r$ and b, c, d are coprime to r,
or r is odd and a, b, c, d, e are coprime to r,

or (after interchanging y, z, and t if necessary)

$$q = 2 = \mathrm{hcf}(d, r) \text{ and } a, b, c \text{ are coprime to } r.$$

PROOF. First of all, exactly as in (7.9), if $a \not\equiv 0 \bmod r$ then $\mathrm{hcf}(a,r) = 1$.

Now since $e = 2a$, it follows that $\mathrm{hcf}(e,r) = 1$ if r is odd, or 2 if r is even. Also, in the first case, a, b, c, and d are coprime to r, since any common factor with r would have to divide e by Rule II, (i). On the other hand, if r is even then setting $k = r/2$ gives $\overline{ak} = r/2$ and $\overline{ek} = 0$, so that from (7.5),

$$\overline{bk} + \overline{ck} + \overline{dk} = r/2 + r/2,$$

and exactly one of b, c and d is even. Q.E.D.

(7.13) *cD-E. Step 3 (listing possible types).* After possibly permuting y, z and t, the possible types are:

If $a \equiv e \equiv 0 \bmod r$, then

(a) $\frac{1}{r}(0, b, -b, 1; 0)$ with b coprime to r.

If a and r are coprime and $q = 2$, then

(b) $\frac{1}{r}(a, -a, 1, 2a; 2a)$ with r even and a coprime to r;

or (c) $\frac{1}{r}(1, b, -b, 2; 2)$ with r even and b coprime to r.

If $q = 1$, then

(d) $\frac{1}{r}((r-1)/2, -(r-1)/2, c, -c; -1)$ with r odd and c coprime to r;

or (e) $\frac{1}{r}(a, -a, 2a, 1; 2a)$ with r odd and a coprime to r;

or (f) $\frac{1}{r}(1, b, -b, 2; 2)$ with r odd and b coprime to r.

PROOF. As before, this follows easily from the terminal lemma (5.4), (A) and (5.6).

(7.14) *cD-E. Step 4 (Final method).* First of all, in cases (b) and (c), I claim that $r = 2$; in both cases this gives $\frac{1}{2}(1,1,1,0;0)$ which implies (5) or (6) of Theorem 6.1. Indeed, suppose that r is even and $r > 2$; so choose k such that $\overline{ka} = (r+2)/2 < r$. Then in case (b),

$$\alpha_k = \tfrac{1}{r}\left(\tfrac{r+2}{2}, \tfrac{r-2}{2}, k, 2\right)$$

satisfies

$$\alpha_k(yzt) = \tfrac{1}{r}\left(\tfrac{r+2}{2} + k\right) < \alpha_k(x) + 1 = \tfrac{1}{r}\left(\tfrac{r+2}{2} + r\right).$$

Therefore $\alpha_k(f) = 2\alpha_k(x) - 1 = 2/r$; but nothing in m^3 has weight $\le 2/r$, which gives a contradiction. A similar calculation also leads to a contradiction in case (c).

Next (d), (e), and (f) are impossible; indeed, choose k such that $\overline{ka} = (r+1)/2$, and consider

$$\alpha_k = \tfrac{1}{r}(\overline{ak}, \overline{bk}, \overline{ck}, \overline{dk}).$$

It is easy to check (separately in the 3 cases) that $\alpha_k(yzt) < \alpha_k(x) + 1$, so that by (7.2), $\alpha_k(f) = 2\alpha_k(x) - 1 = 1/r$. But no monomial in m^2 can have weight less that $1/r$.

(7.15) The only remaining case is (a) of (7.13), and for this I need to translate Rule I for f into a similar condition for $g(y,z,t)$. Note that $\alpha(g) \equiv 2\alpha(x) \bmod \mathbf{Z}$ for any $\alpha \in N \cap \sigma$.

LEMMA. *Assume case* (a). *Then*

(1) *For any* $\alpha \in N \cap \sigma$,

if $\alpha(g) - 2\alpha(x)$ *is even, then* $\alpha(g) < 2\alpha(yzt) - 2$;
if $\alpha(g) - 2\alpha(x)$ *is odd, then* $\alpha(g) < 2\alpha(yzt) - 1$.

(2) *The weightings*

$$\alpha_k = \tfrac{1}{r}(0, \overline{bk}, r - \overline{bk}, k) \quad \text{for } k = 1, \dots, r-1$$

satisfy $\alpha_k(g) = 1$.

PROOF OF (1). Define $\beta = \alpha + i \cdot (1,0,0,0)$, where $i = \frac{1}{2}(\alpha(g) - 2\alpha(x))$ or $\frac{1}{2}(\alpha(g) - 2\alpha(x) + 1)$ in the two cases. Then

$$\beta(f) = \min\{\alpha(g), 2\alpha(x) + 2i\} = \alpha(g).$$

Also

$$\beta(xyzt) = \alpha(x) + i + \alpha(yzt),$$

so that Rule I gives

$$\beta(f) < \beta(xyzt) - 1,$$

that is,

$$\alpha(g) < \alpha(x) + i + \alpha(yzt) - 1,$$

and writing out the definition of i gives the statement in the lemma.

PROOF OF (2). In fact $\alpha_k(g)$ even implies that

$$0 < \alpha_k(g) < 2\alpha_k(yzt) - 2 = 2k/r < 2,$$

which has no solutions, and $\alpha_k(g)$ odd implies

$$0 < \alpha_k(g) < 2\alpha_k(yzt) - 1 = (2k + r)/r < 3,$$

which has the single solution $\alpha_k(g) = 1$. Q.E.D.

(7.16) To prove that (a) $\Rightarrow$ (6.1), (4) I need to show that $r = 3$. So choose k such that $\overline{bk} = (r-1)/2$. By interchanging y and z if necessary, I can assume that $k \ge (r-1)/2$. Then consider

$$\alpha_k = \tfrac{1}{r}\left(0, \tfrac{r-1}{2}, \tfrac{r+1}{2}, k\right).$$

Since $\alpha_k(g) = 1$, there must be a monomial in y, z, t of weight 1, and of degree ≥ 3 (since $g \in m^3$). Note that $\alpha_k(yz) = 1$ and $\alpha_k(z^2) > 1$, so that no multiple of yz or z^2 can work. However, since $k \geq (r-1)/2$,

$$\alpha_k(y^3, y^2t, yt^2, t^3) \geq (3r-3)/2r = 1 + (r-3)/2r.$$

So the only way to get a monomial of weight 1 is if $r = 3$.

This shows that the type of X is $\frac{1}{3}(0,2,1,1)$; it is an exercise to see that f must contain both y^3 and a monomial of degree 3 in (z,t), giving (4).

Chapter III
Contributions of Q-divisors to RR

8. Quotient singularities and equivariant RR.

(8.1) This chapter introduces a number of formulas of the type

$$\chi(X, \mathcal{L}) = (\text{RR-type expression in } D) + \sum_Q c_Q(D);$$

here X is a normal variety, and $\mathcal{L} = \mathcal{O}_X(D)$, where D is a Weil divisor which is **Q**-Cartier, and Cartier outside a finite set of points; the terms $c_Q(D)$ are contributions due to the singularities of the sheaf $\mathcal{O}_X(D)$, and are local analytic invariants of the "polarised singularity" ($Q \in X$ and $\mathcal{O}_X(D)$). Notice that this is not immediately related to the "singular Riemann-Roch theorems" in the literature, which deal in the sheaf $\mathcal{L}$, singularities and all: the formula here deals only in the **Q**-divisor class of D in $\operatorname{Pic} X \otimes \mathbf{Q}$, so involves a substantial abuse of notation.

(8.2) The existence of such a formula is not in itself particularly exciting, but in several cases of interest the computation of $c_Q(D)$ can be reduced to the contributions of a "basket"

$$\{P_\alpha \in X_\alpha \text{ and } \mathcal{O}_{X_\alpha}(D_\alpha)\}$$

of cyclic quotient singularities, which can in turn be calculated by equivariant RR. I introduce the term "basket of singularities" to emphasise the fact that the singularities $P_\alpha \in X_\alpha$ are not points of X, but only "fictitious singularities": the singularities of X and D make contributions to RR equal to those which would occur if X had these singularities. For example, X and D might in good cases deform to a variety really having these singularities.

(8.3) The notation for cyclic quotient singularities is as in (4.1).

DEFINITION. The quotient $X = \mathbf{A}^n/\mu_r$, where μ_r acts on $\mathbf{A}^n$ by

$$\mu_r \ni \varepsilon \colon (x_1, \dots, x_n) \mapsto (\varepsilon^{a_1}x_1, \dots, \varepsilon^{a_n}x_n)$$

is a *cyclic quotient singularity of type* $\frac{1}{r}(a_1, \dots, a_n)$. Write $\pi \colon \mathbf{A}^n \to X$ for the quotient map; then the group μ_r acts on $\pi_*\mathcal{O}_{\mathbf{A}^n}$, and so decomposes it into r eigensheaves

$$\mathcal{L}_i = \{f \mid \varepsilon(f) = \varepsilon^i \cdot f \text{ for all } \varepsilon \in \mu_r\}$$

for $i = 0, \dots, r-1$. A singularity $P \in X$ with a Weil divisor D is a *cyclic quotient singularity of type* $_i(\frac{1}{r}(a_1, \dots, a_n))$ if $P \in X$ is (locally isomorphic to) a point of type $\frac{1}{r}(a_1, \dots, a_n)$, and $\mathcal{O}_X(D) \cong \mathcal{L}_i$.

REMARK. In writing the action $(x_1, \dots, x_n) \mapsto (\varepsilon^{a_1}x_1, \dots, \varepsilon^{a_n}x_n)$, I am thinking of the x_i as *coordinates* on $\mathbf{A}^n$, as usual in algebraic geometry. (This is the dual of what the topologist would write; since the tangent space is based by $\{\partial/\partial x_i\}$, the action of ε on the tangent space $T_{\mathbf{A}^n,0}$ has eigenvalues $(\varepsilon^{-a_1}, \dots, \varepsilon^{-a_n})$. I am repeating the bizarre pronouncement that the relation between points of $\mathbf{A}^n$ and coordinates is contravariant; sorry.)

(8.4) The following result is useful in reducing local problems concerning quotient singularities to the projective case.

PROPOSITION. *Given* $\frac{1}{r}(a_1, \dots, a_n)$ *(with a_i coprime to r), there exists a smooth projective n-fold Y with an action of μ_r having a number N of fixed points at which μ_r acts by*

$$\mu_r \ni \varepsilon \colon (x_1, \dots, x_n) \mapsto (\varepsilon^{a_1}x_1, \dots, \varepsilon^{a_n}x_n)$$

and freely outside these points.

PROOF. This is very easy. Suppose that $k >$ maximum number of the a_i which are congruent modulo any prime p dividing r (for example, $k > n$). Consider the action of μ_r on $\mathbf{P}^{n+k}$ given in homogeneous coordinates by

$$(\underbrace{1, \dots, 1}_{k+1 \text{ times}}, \varepsilon^{a_1}, \dots, \varepsilon^{a_n}).$$

The action of μ_r on $\mathbf{P}^{n+k}$ has a fixed locus $\mathbf{P}^k$ where the action in the normal direction is given by $(x_1, \dots, x_n) \mapsto (\varepsilon^{a_1}x_1, \dots, \varepsilon^{a_n}x_n)$, and other fixed locuses of smaller dimension. Let $X \subset \mathbf{P}^{n+k}/\mu_r$ be the intersection of k general very ample divisors, and Y its inverse image under $\mathbf{P}^{n+k} \to \mathbf{P}^{n+k}/\mu_r$; then it is easy to check that Y satisfies the conditions of the proposition. Q.E.D.

(8.5) THEOREM. *Under the conditions of* (8.4), *write* $\pi \colon Y \to X$ *for the quotient map, and let $\mathcal{L}_i$ be the ith eigensheaf of the action of μ_r on $\pi_*\mathcal{O}_Y$. Then for* $i = 0, 1, \dots, r-1$,

$$\chi(X, \mathcal{L}_i) = \frac{1}{r}\chi(\mathcal{O}_Y) + \frac{N}{r}\sigma_i\left(\tfrac{1}{r}(a_1, \dots, a_n)\right),$$

where

$$\sigma_i\left(\tfrac{1}{r}(a_1, \dots, a_n)\right) = \sum_{\varepsilon \neq 1} \frac{\varepsilon^i}{(1-\varepsilon^{a_1}) \cdots (1-\varepsilon^{a_n})},$$

the sum extending over all $\varepsilon \in \mu_r - \{1\}$.

PROOF. This follows easily from equivariant RR. Let Y be a nonsingular projective n-fold and g an automorphism of Y acting with only finitely many fixed points $\{Q\}$. The Lefschetz number $L(g \colon \mathcal{O}_Y)$ of g acting on $\mathcal{O}_Y$ is defined by

$$L(g \colon \mathcal{O}_Y) = \sum (-1)^j \operatorname{Tr}(g \colon H^j(\mathcal{O}_Y)). \tag{1}$$

Write $dg_Q\colon T_{Y,Q} \to T_{Y,Q}$ for the differential of the action at a fixed point Q. Then the Atiyah-Singer equivariant RR formula [**Atiyah-Segal**, p. 541] takes the simple form

$$L(g\colon \mathcal{O}_Y) = \sum_Q \frac{1}{\det(1 - dg_Q)}. \tag{2}$$

In the present case, this can be used as follows: since

$$\pi_*\mathcal{O}_Y = \bigoplus_{i=0}^{r-1} \mathcal{L}_i,$$

it follows that

$$H^j(Y, \mathcal{O}_Y) = \bigoplus_{i=0}^{r-1} H^j(X, \mathcal{L}_i);$$

moreover, any $\varepsilon \in \mu_r$ acts on $\mathcal{L}_i$ by multiplication by ε^i, so that,

$$\mathrm{Tr}(\varepsilon\colon H^j(Y, \mathcal{O}_Y)) = \sum_{i=0}^{r-1} h^j(X, \mathcal{L}_i) \cdot \varepsilon^i.$$

Therefore

$$L(\varepsilon\colon \mathcal{O}_Y) = \sum (-1)^j \, \mathrm{Tr}(\varepsilon\colon H^j(Y, \mathcal{O}_Y)) = \sum_{i=0}^{r-1} \chi(X, \mathcal{L}_i) \cdot \varepsilon^i. \tag{3}$$

By construction, ε acts on $T_{Y,Q}$ by $(\varepsilon^{-a_1}, \dots, \varepsilon^{-a_n})$ at each fixed point Q. Then applying (2) to $g = \varepsilon$ gives

$$\sum_{i=0}^{r-1} \chi(X, \mathcal{L}_i) \cdot \varepsilon^i = \frac{N}{(1 - \varepsilon^{-a_1}) \cdots (1 - \varepsilon^{-a_n})} \tag{4_ε}$$

for any $\varepsilon \in \mu_r - \{1\}$.

I consider (4_ε) as giving $r - 1$ equations for the r unknowns $\chi(X, \mathcal{L}_i)$. The final equation is

$$\sum_{a=0}^{r-1} \chi(X, \mathcal{L}_i) = \chi(\mathcal{O}_Y). \tag{4_1}$$

I can now solve the r equations (4_ε) for $\chi(X, \mathcal{L}_i)$ by inverting a Vandermonde matrix: multiply (4_ε) through by ε^{-i} and sum over all $\varepsilon \in \mu_r$. This gives

$$\chi(\mathcal{L}_i) = \frac{1}{r}\chi(\mathcal{O}_Y) + \frac{N}{r}\sigma_i, \tag{5}$$

where for $i = 0, \dots, r - 1$,

$$\sigma_i\left(\tfrac{1}{r}(a_1, \dots, a_n)\right) = \sum_{\varepsilon \neq 1} \frac{\varepsilon^{-i}}{(1 - \varepsilon^{-a_1}) \cdots (1 - \varepsilon^{-a_n})}, \tag{6}$$

the sum extending over all $\varepsilon \in \mu_r - \{1\}$. Since ε^{-1} runs throught $\mu_r - \{1\}$ together with ε, I can ignore the minus signs throughout. Q.E.D.

(8.6) It is easy to eliminate $\chi(\mathcal{O}_Y)$ from this formula, to get

$$\chi(X, \mathcal{L}_i) = \chi(\mathcal{O}_X) + \left(\frac{N}{r}\right)\{\sigma_i - \sigma_0\},$$

a prototype of the kind of formula referred to in (8.1): $\mathcal{L}_i$ is a sheaf on X with isolated singularities, and since as a $\mathbf{Q}$-divisor, $L_i = 0 \in \operatorname{Div} X \otimes \mathbf{Q}$, I can think of $\chi(\mathcal{O}_X)$ as a RR-type expression in L, and the remainder of the formula as being a sum of N local contributions $(1/r)\{\sigma_i - \sigma_0\}$ coming from the singular points of $\mathcal{L}_i$.

COROLLARY. *Let X be an n-fold having a finite set of quotient singularities $\{Q\}$, and $\mathcal{L} = \mathcal{O}_X(D)$ a divisorial sheaf on X. Then there is a formula of type*

$$\chi(X, \mathcal{L}) = (\textit{RR-type expression in } D) + \sum_Q c_Q(D);$$

where (i) *the contributions $c_Q(X)$ are of the form*

$$c_Q(X) = \sigma_i\left(\tfrac{1}{r}(a_1, \dots, a_n)\right) - \sigma_0\left(\tfrac{1}{r}(a_1, \dots, a_n)\right)$$

if $\mathcal{L}$ is locally of type $_i(\frac{1}{r}(a_i, \dots, a_n))$ at Q; and (ii) *the RR-type expression in D is the usual $\mathrm{ch}(D) \cdot \mathrm{Td}_X$ interpreted formally in the following ad hoc way: for the terms in which D appears, just take Td_X on the nonsingular locus of X and intersect with D (this is right because some multiple of D is a Cartier divisor, so can be moved away from the singularities); for the remaining term, just substitute $\chi(\mathcal{O}_X)$.*

SKETCH OF PROOF. The fact that a formula of this kind exists can be readily seen by comparing X and $\mathcal{L}$ with a suitable resolution; the contribution which the argument gives is a sum of local analytic invariants at each of the singularities, so can be computed on any example where only this singularity appears, for example that of (8.4-5).

Appendix to §8. Computing the σ_i.

(8.7) The sums $\sigma_i(\frac{1}{r}(a_1, \dots, a_n))$ are defined by

$$\sigma_i\left(\tfrac{1}{r}(a_1, \dots, a_n)\right) = \sum_{\varepsilon \neq 1} \frac{\varepsilon^i}{(1 - \varepsilon^{a_1}) \cdots (1 - \varepsilon^{a_n})};$$

in general one cannot expect to get a closed formula for them, but I evaluate them here in two cases of interest. The σ_i are a kind of Dedekind sum, and a lot of information on them is contained in [**Hirzebruch-Zagier**], although I prefer to work from first principles.

The sums $\sigma_i(\frac{1}{r}(a_1, \dots, a_n))$ are determined recursively by the following two conditions:

$$\sum_{i=0}^{r-1} \sigma_i = 0; \tag{A}$$

and

$$(\sigma_{i+a_n} - \sigma_i)\left(\tfrac{1}{r}(a_1, \dots, a_n)\right) = -\sigma_i\left(\tfrac{1}{r}(a_1, \dots, a_{n-1})\right). \tag{B}$$

Here (A) follows from the fact that

$$\sum_{\varepsilon\in\mu_r}\varepsilon = 0,$$

and (B) is easy to check. Formula (B) is particularly useful if one or more of the a_j is ± 1.

Of course the recursion starts with

$$\text{(C)}\qquad \begin{cases} \sigma_0\left(\frac{1}{r}(\emptyset)\right) = r-1, & \\ \sigma_i\left(\frac{1}{r}(\emptyset)\right) = -1 & \text{if } i = 1,\dots,r-1. \end{cases}$$

Another useful fact is the relation

$$\text{(D)}\qquad \sigma_i\left(\tfrac{1}{r}(a_1,\dots,a_n)\right) = \sigma_{bi}\left(\tfrac{1}{r}(ba_1,\dots,ba_n)\right)$$

if b is coprime to r, which comes at once from the fact that $\varepsilon\mapsto\varepsilon^b$ is a bijection of μ_r.

(8.8) Using (A) and (B), it is easy to verify that

$$\sigma_i(\tfrac{1}{r}(-1)) = (r-1)/2 - i \quad \text{for } i = 0,\dots,r-1.$$

(8.9) To calculate $\sigma_i(\frac{1}{r}(-1,1))$, note that

$$\sigma_{i+1}-\sigma_i = -(r-1)/2 + i \quad \text{for } i = 0,\dots,r-1,$$

and therefore

$$\sigma_i = \sigma_0 + \sum_{j=0}^{i-1}\left\{-\frac{(r-1)}{2}+j\right\} = \sigma_0 - \frac{i(r-i)}{2}.$$

So from (A) I get

$$r\sigma_0 = \sum_{i=1}^{r-1}\frac{i(r-i)}{2} = \frac{r(r^2-1)}{12};$$

hence

$$\sigma_0 = (r^2-1)/12.$$

PROPOSITION. *For* $i = 0,\dots,r-1$,

$$\sigma_i\left(\tfrac{1}{r}(-1,1)\right) = \frac{(r^2-1)}{12} - \frac{i(r-i)}{2}.$$

Using (D), *it follows that for any* a *coprime to* r, *and* $j = 0,\dots,r-1$,

$$\sigma_j\left(\tfrac{1}{r}(a,-a)\right) = \frac{(r^2-1)}{12} - \frac{\overline{bj}(r-\overline{bj})}{2},$$

where $ba\equiv 1 \bmod r$ *and* $\overline{}$ *denotes smallest residue* mod r.

(8.10) To calculate $\sigma_i(\frac{1}{r}(a,-a,1))$ I proceed in the same way:

$$\sigma_{i+1}-\sigma_i = -\sigma_i\left(\tfrac{1}{r}(a,-a)\right) = \frac{(r^2-1)}{12} - \frac{\overline{bi}(r-\overline{bi})}{2},$$

where $ba\equiv 1 \bmod r$ and $\overline{}$ denotes smallest residue mod r. This gives

$$\sigma_i = \sigma_0 - \frac{i(r^2-1)}{12} + \sum_{j=0}^{i-1}\frac{\overline{bj}(r-\overline{bj})}{2}.$$

As before $0 = \sum_{i=0}^{r-1}\sigma_i$, and taking account of the number of times each of the summands $\overline{bj}(r-\overline{bj})/2$ appears in the sum over i, I get

$$0 = r\sigma_0 - r(r-1)\cdot\frac{(r^2-1)}{24} + \sum_{j=0}^{r-1}(r-j-1)\cdot\frac{\overline{bj}(r-\overline{bj})}{2}.$$

Now notice that the complicated sum here can be evaluated: since the factors $\overline{bj}(r-\overline{bj})/2$ in the jth and $(r-j)$th terms are the same, it simplifies to

$$\sum = \frac{(r-2)}{4}\sum_{j=0}^{r-1}\overline{bj}(r-\overline{bj}) = \frac{(r-2)}{4}\sum_{i=0}^{r-1}i(r-i) = r(r-2)\cdot\frac{(r^2-1)}{24};$$

this proves the following result.

PROPOSITION. $\sigma_0 = (r^2-1)/24$, *and for* $i = 0,\dots,r-1$,

$$\sigma_i - \sigma_0 = -i\cdot\frac{r^2-1}{12} + \sum_{j=0}^{i-1}\frac{\overline{bj}(r-\overline{bj})}{2}$$

(*the sum is by convention* 0 *if* $i = 0$ *or* 1).

The proposition was first proved by A. R. Fletcher, using results of **[Hirzebruch-Zagier]** on Dedekind sums.

(8.11) *Exercise.* Set $\sum a_i = k \bmod r$, so that the divisorial sheaf $\mathcal{O}_X(K_X)$ is of type ${}_k(\frac{1}{r}(a_1,\dots,a_n))$ near Q. Give two different proofs of the proposition

$$\sigma_i = (-1)^n\sigma_{i'} \quad \text{whenever } i+i'\equiv k \bmod r;$$

(one based on Serre duality and (8.5), one by induction on n using (A)–(C)).

9. Contributions from Du Val surface singularities. Let X be a projective surface with at worst Du Val singularities, and D a Weil divisor on X. Since D is $\mathbf{Q}$-Cartier, the intersection numbers $D^2, K_XD\in\mathbf{Q}$ are well defined (in fact $K_XD\in\mathbf{Z}$ since K_X is Cartier).

(9.1) THEOREM. (I) *There is a formula*

$$\chi(X,\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + (1/2)(D^2 - DK_X) + \sum_Q c_Q(D)$$

where $c_Q(D) = c_Q(\mathcal{O}_X(D))\in\mathbf{Q}$ *is a contribution due to the singularity of* $\mathcal{O}_X(D)$ *at* Q, *depending only on the local analytic type of* $Q\in X$ *and* D; *the sum takes place over the singularities of* D (*the points* $Q\in X$ *at which* D *is not Cartier*).

(II) *If* $P\in X$ *and* D *is a cyclic quotient singularity of type* ${}_i(\frac{1}{r}(1,-1))$ *then*

$$c_P(D) = -i(r-i)/2r.$$

(III) *For every Du Val singularity $Q \in X$ and Weil divisor D on X, there exists a basket of points of $\{P_\alpha \in X_\alpha$ and $D_\alpha\}$ of types ${}_{i_\alpha}(\frac{1}{r_\alpha}(1,-1))$ and with i_α coprime to r_α, such that*

$$c_Q(D) = \sum_\alpha c_{P_\alpha}(D_\alpha) = -\sum_\alpha \frac{i_\alpha(r_\alpha - i_\alpha)}{2r_\alpha}.$$

REMARKS. (i) (I) is rather trivial; it is proved by comparing X and D with a resolution as in (8.6). The proof gives an expression for $c_Q(D)$ which is not very useful for computational purposes.

(ii) In (II), i does not have to be coprime to r.

(iii) The point of (III) is that the contribution $c_Q(D)$ can be expressed as a sum of a basket of contributions of the type described in (II) with i and r coprime. To prove (III), I show that there is a deformation which replaces the local analytic singularity $Q \in X$ and D with a basket $\{(P_\alpha \in X_\alpha$ and $D_\alpha)\}$.

(9.2) PROOF OF (I). It is easy to see that there is a resolution $f: Y \to X$ and a Cartier divisor E on Y such that $f^*\mathcal{O}_X(D) \to \mathcal{O}_Y(E)$ is surjective; then also $f_*\mathcal{O}_Y(E) = \mathcal{O}_X(D)$ and $R^if_*\mathcal{O}_Y(E) = 0$, so that

$$\chi(Y, \mathcal{O}_Y(E)) = \chi(X, \mathcal{O}_X(D)).$$

(1) now follows by writing out RR for E on Y; in more detail, let $\{\Gamma_i\}$ be the exceptional curves of f. Then I can write

$$K_Y = f^*K_X + A, \quad \text{where } A = \sum a_i\Gamma_i,$$

and

$$E = f^*D + B, \quad \text{where } B = \sum b_i\Gamma_i,$$

with $a_i \in \mathbf{Z}$ and $b_i \in \mathbf{Q}$. Then (since $f^*D\Gamma_i = f^*K_X\Gamma_i = 0$ for all Γ_i),

$$\begin{aligned}\chi(X, \mathcal{O}_X(D)) &= \chi(Y, \mathcal{O}_Y(E)) = \chi(\mathcal{O}_Y) + (1/2)E(E - K_Y)\\ &= \chi(\mathcal{O}_X) + (1/2)(f^*D + B)(f^*(D - K_X) + B - A)\\ &= \chi(\mathcal{O}_X) + (1/2)(D^2 - DK_X) + \sum_Q c_Q(D),\end{aligned}$$

where $c_Q(D) = (1/2)(B_Q)(B_Q - A_Q)$.

(II) follows from (8.5–6), together with the result of (8.9),

$$\sigma_i - \sigma_0 = -i(r-i)/2.$$

(9.4) PROOF OF (III). This is a deformation argument; I show that given the singularity $Q \in X$ and D, there is a flat deformation $\{X_t$ and $D_t\}$ of X together with the Weil divisor class D such that X_t has only cyclic quotient singularities. The deformation family $\{X_t\}$ extends to a family of projective surfaces. Since $\chi(\mathcal{O}_X(D))$ and the invariants D^2 and DK_X are continuous in a flat family, the contribution $c_Q(D)$ is equal to the sum of the contributions from the cyclic quotient singularities of X_t.

Consider $Q \in X$ and D; let $\pi: Y \to X$ be the cyclic cover corresponding to D. Then $O \in Y$ is a Du Val surface singularity (if it's nonsingular, there's nothing

to prove), together with an action of $\boldsymbol{\mu}_r$ on $O \in Y$ satisfying the properties:

(i) the action is free outside O;

(ii) it acts trivially on a generator of ω_Y (corresponding to the fact that X is Gorenstein).

This is exactly the situation classified in Exercise 4.10. From the list given there, one sees that suitable Q-smoothings of X are as follows:

(1) $f + \lambda$; (-1) has two fixed points on the y-axis, given by $y^2 + \lambda = 0$.

(2) $f + \lambda z$; the subgroup $\boldsymbol{\mu}_2 \subset \boldsymbol{\mu}_4$ has $(2n+1)$ fixed points on the z-axis, given by $z^{2n+1} + \lambda z = 0$.

(3) $f + \lambda$; $\boldsymbol{\mu}_r$ has n fixed points on the z-axis given by $z^n + \lambda = 0$.

(4) $f + \lambda$; $\boldsymbol{\mu}_3$ has 2 fixed points on the x-axis given by $x^2 + \lambda = 0$.

(5) $f + \lambda$; (-1) has n fixed points on the z-axis given by $z^n + \lambda = 0$.

(6) $f + \lambda$; (-1) has 3 fixed points on the y-axis given by $y^3 + \lambda = 0$.

This completes the proof of Theorem 9.1.

10. The plurigenus formula. Let X be a projective 3-fold with canonical singularities, and D a Weil divisor on X such that $\mathcal{O}_X(D) \cong \mathcal{O}_X(iK_X)$ in a neighbourhood of every $P \in X$ for some i (possibly varying with P). Note that there are only finitely many points at which D is not Cartier.

(10.1) *Definition of $D \cdot c_2(X)$.* By definition,

$$D \cdot c_2(X) = (f^*D) \cdot c_2(Y),$$

where $f: Y \to X$ is a resolution (and as usual, f^*D refers to the pull-back of $\mathbf{Q}$-Cartier divisors). This does not depend on the resolution (if X has only 0-dimensional singular locus, $D \cdot c_2(X) = (1/r)E \cdot c_2(X)$, $E \sim rD$, where E is a divisor linearly equivalent to rD not passing through any singular points of X).

(10.2) THEOREM. (1) *There is a formula of the form*

$$\chi(X, \mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \tfrac{1}{12}D(D - K_X)(2D - K_X) + \tfrac{1}{12}D \cdot c_2(X) + \sum_Q c_Q(D),$$

where the summation takes place over the singularities of the sheaf $\mathcal{O}_X(D)$, and $c_Q(D) \in \mathbf{Q}$ is a contribution due to the singularity at Q, depending only on the local analytic type of $Q \in X$ and $\mathcal{O}_X(D)$.

(2) *If $P \in X$ is the terminal cyclic quotient singularity $X = \mathbf{A}^3/\boldsymbol{\mu}_r$ of type $\frac{1}{r}(a,-a,1)$ and $\mathcal{O}_X(D)$ is locally isomorphic to $\mathcal{O}_X(iK_X)$ (so that the pair (X and D) is of type ${}_i(\frac{1}{r}(a,-a,1))$ in the terminology of (8.3)) then*

$$c_P(D) = -i \cdot \frac{(r^2-1)}{12r} + \sum_{j=1}^{i-1} \frac{\overline{bj}(r - \overline{bj})}{2r},$$

where b satisfies $ab \equiv 1 \bmod r$, and $\overline{}$ denotes smallest residue mod r *(the sum $\sum_{j=1}^{i-1}$ is zero by convention if $i = 0$ or 1).*

(3) *For every 3-fold canonical singularity $Q \in X$ and Weil divisor D on X such that $\mathcal{O}_X(D) \cong \mathcal{O}_X(iK_X)$ for some i, there exists a basket of points*

$$\{P_\alpha \in X_\alpha \text{ and } D_\alpha\} \text{ of type } {}_{i_\alpha}\left(\tfrac{1}{r_\alpha}(a_\alpha, -a_\alpha, 1)\right)$$

such that

$$c_Q(D) = \sum c_{P_\alpha}(D_\alpha).$$

This means that the contribution from any singularity $Q \in X$ and D can be expressed as a sum of contributions from a basket of terminal cyclic quotient singularities.

PROOF. As before, (1) is proved by comparing X and D with a suitable resolution Y and E. Since X has only a finite set of dissident points, it is not hard to choose a resolution $f\colon Y \to X$ whose discrepancy Δ_f is concentrated above a finite set of X. For details of the argument, compare [**C3-f**, (5.5)].

(2) follows directly from (8.6) and (8.10).

There are two reductions in the proof of (3): the first step reduces to terminal singularities by a crepant partial resolution as described in (3.12) and (3.14), and the second to terminal quotient singularities by a flat deformation as described in (6.4), (A); the second of these is easy using (6.4), (A), and the reader should think through the details for himself.

CLAIM. *Let $P \in X$ be a canonical 3-fold singularity, and $g\colon X' \to X$ a crepant partial resolution such that X' has only terminal singularities. Then the contribution $c_P(iK_X)$ is equal to a sum of contributions $c_Q(iK_{X'})$ over the finite set of points $Q \in g^{-1}P$ at which $K_{X'}$ is not Cartier.*

This can be seen by looking more closely at the proof of (1): if I choose the resolution of X by first constructing $g\colon X' \to X$ and then resolving the singularities of X by $h\colon Y \to X'$, then obviously the discrepancy of $f\colon Y \to X$ equals that of $h\colon Y \to X'$.

Alternatively, argue as follows: there is no loss of generality in assuming that X is projective with $P \in X$ its only dissident singularity. By the fact that $X' \to X$ is crepant it follows that $g_*\mathcal{O}_{X'}(iK_{X'}) = \mathcal{O}_X(iK_X)$; also, by standard use of vanishing, $R^ig_*\mathcal{O}_{X'}(iK_{X'}) = 0$. The Leray spectral sequence then gives

$$\chi(\mathcal{O}_X(iK_X)) = \chi(\mathcal{O}_{X'}(iK_{X'}));$$

together with (1) this proves the claim.

(10.3) COROLLARY. (4) *Let X be a projective 3-fold with canonical singularities and*

$$\left\{P_\alpha \in X_\alpha \text{ and } K_{X_\alpha}, \text{ of type } {}_1(\tfrac{1}{r_\alpha}(a_\alpha, -a_\alpha, 1))\right\}$$

the basket for X and K_X in the sense of (3). Then $\chi(\mathcal{O}_X)$ and $K_X \cdot c_2(X)$ are related by

$$\chi(\mathcal{O}_X) = -\frac{1}{24}K_X \cdot c_2(X) + \sum_{P_\alpha} \frac{(r^2-1)}{24r},$$

or alternatively,

$$\frac{1}{12}K_X \cdot c_2 = -2\chi(\mathcal{O}_X) + \sum_{P_\alpha} \frac{(r^2-1)}{12r}.$$

(5) *Suppose that in addition H is a Cartier divisor on X; then*

$$\chi(\mathcal{O}_X(H + mK_X)) = (1-2m)\chi(\mathcal{O}_X)$$
$$+ \frac{1}{12}(H + mK_X)(H + (m-1)K_X)(H + (2m-1)K_X) + \frac{1}{12}H \cdot c_2(X)$$
$$+ \sum_Q \left\{ \frac{(r^2-1)}{12r}(m - \overline{m}) + \sum_{j=1}^{\overline{m}-1} \frac{\overline{bj}(r - \overline{bj})}{2r} \right\},$$

where inside the curly blankets, $\overline{}$ *denotes smallest residue of $m \bmod r$ and $ab \equiv 1 \bmod r$ (note that r varies with Q). In particular,*

$$\chi(\mathcal{O}_X(mK_X)) = (1-2m)\chi(\mathcal{O}_X) + \frac{1}{12}m(m-1)(2m-1)K^3$$
$$+ \sum_Q \left\{ \frac{(r^2-1)}{12r}(m - \overline{m}) + \sum_{j=0}^{\overline{m}-1} \frac{\overline{bj}(r - \overline{bj})}{2r} \right\},$$

where the sum takes place over the basket of singularities for X.

Historical remark. The correct version of the formula in (5) is due to Anthony Fletcher; his paper [**Fletcher**] gives several alternative versions of the formula which are convenient in applications. The contribution in (4) has been computed by several people, probably first by Rebecca Barlow (around 1980) using her version of Danilov's economic resolution (see (5.10)); essentially her computation is given in [**Kawamata**, (2.2)].

(10.4) *Exercise.* Check that the formula gives the right values for the first few plurigenera of the 3-folds of Exercise 2.12. For example, (i) has

$$P_2 = \chi(\mathcal{O}_X(2K_X)) = \tfrac{1}{2} \cdot \tfrac{1}{6} + \{\tfrac{1}{4}\} + 5 \times \{\tfrac{1}{3}\} = 2,$$
$$P_3 = \chi(\mathcal{O}_X(3K_X)) = \tfrac{5}{2} \cdot \tfrac{1}{6} + \{\tfrac{1}{4}\} + 5 \times \{\tfrac{2}{3}\} = 4.$$

REFERENCES

[**Atiyah-Segal**] M. F. Atiyah and G. Segal, *The index of elliptic operators.* II, Ann. of Math. (2) **87** (1968), 531–545.

[**Barth-Peters-Van de Ven**] W. Barth, C. Peters, and A. Van de Ven, *Compact complex surfaces*, Springer-Verlag, 1984.

[**Brieskorn**] E. Brieskorn, *Über die Auflösung gewisser Singularitäten von holomorphen Abbildungen*, Math. Ann. **166** (1966), 76–102; *Die Auflösung der rationalen Singularitäten von holomorphen Abbildungen*, Math. Ann. **178** (1968), 255–270.

[**Catanese**] F. Catanese, *Canonical rings of surfaces*, these Proceedings, Part 1, pp. 175–194.

[**Danilov**] V. I. Danilov, *The birational geometry of toric 3-folds*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (1982), 972–981; English transl. in Math USSR-Izv. **21** (1983).

[**Dolgachev**] I. Dolgachev, *Weighted projective spaces*, Group Actions and Vector Fields, Lecture Notes in Math., vol. 956, Springer-Verlag, 1982, pp. 34–71.

[**Elkik**] R. Elkik, *Rationalité des singularités canoniques*, Invent. Math. **64** (1981), 1–6.

[**Flenner**] H. Flenner, *Rational singularities*, Arch. Math. (Basel) **36** (1981), 35–44.

[**Fletcher**] A. R. Fletcher, *Contributions to Riemann-Roch on projective 3-folds with only canonical singularities and applications*, these Proceedings, Part 1, pp. 221–231.

[**Hirzebruch-Zagier**] F. Hirzebruch and D. Zagier, *The index theorem and elementary number theory*, Publish or Perish, 1974.

[Inose-Mizukami] H. Inose and M. Mizukami, *Rational equivalence of 0-cycles on surfaces of general type with $p_g = 0$*, Math. Ann. **244** (1979), 205–217.

[Kawamata] Y. Kawamata, *On the plurigenera of minimal algebraic 3-folds with $K \approx 0$*, Math. Ann. **275** (1986), 539–546.

[Khovanskiĭ] A. G. Khovanskiĭ, *Newton polyhedra and the genus of complete intersections*, Functional Anal. Appl. **12** (1978), 51–61.

[Kollár and Shepherd-Barron] J. Kollár and N. Shepherd-Barron, *Threefolds and deformations of surface singularities*, preprint.

[Lang] S. Lang, *Cyclotomic fields*, Springer-Verlag, 1978.

[Markushevich] D. G. Markushevich, *Canonical singularities of three-dimensional hypersurfaces*, Izv. Akad. Nauk SSSR Ser. Mat. **49** (1985), 334–368; English transl. in Math USSR-Izv. **26** (1986).

[Matsumura] H. Matsumura, *Commutative ring theory*, Cambridge Univ. Press, 1986.

[Mori] S. Mori, *On 3-dimensional terminal singularities*, Nagoya Math J. **98** (1985), 43–66.

[Morrison-Stevens] D. R. Morrison and G. Stevens, *Terminal quotient singularities in dimension 3 and 4*, Proc. Amer. Math Soc. **90** (1984), 15–20.

[Morrison] D. R. Morrison, *Canonical quotient singularities in dimension 3*, Proc. Amer. Math Soc. **93** (1985), 393–396.

[Mumford] D. Mumford, *Hilbert's fourteenth problem—the finite generation of subrings such as rings of invariants*, Math. Developments Arising from Hilbert Problems, Proc. Sympos. Pure Math., vol. 28, Amer. Math. Soc., Providence, R.I., 1976, pp. 431–444.

[Pinkham] H. Pinkham, *Singularités rationelles de surfaces*, Lecture Notes in Math., vol. 777, Springer-Verlag, 1980, pp. 147–168.

[C3-f] M. Reid, *Canonical 3-folds*, Journées de Géométrie Algébrique d'Angers, A. Beauville, editor, Sijthoff and Noordhoff, Alphen aan den Rijn, 1980, pp. 273–310.

[Pagoda] M. Reid, *Minimal models of canonical 3-folds*, Algebraic Varieties and Analytic Varieties, S. Iitaka and H. Morikawa, editors, Adv. Stud. Pure Math., vol. 1, Kinokuniya Book Store, Tokyo, and North Holland, Amsterdam, 1983, pp. 131–180.

[Shafarevich] I. R. Shafarevich, *Basic algebraic geometry*, Springer-Verlag, 1974.

[Tai] Y. S. Tai, *On the Kodaira dimension of the moduli space of Abelian varieties*, Invent. Math. **68** (1982), 425–439.

[Ueno] K. Ueno, *On the pluricanonical systems of algebraic manifolds*, Math. Ann. **216** (1975), 173–179.

[Washington] L. C. Washington, *Introduction to cyclotomic fields*, Springer-Verlag, 1982.

[Wilson] P. M. H. Wilson, *Towards birational classification of algebraic varieties*, Bull. London Math. Soc. **19** (1987), 1–47.

[Zariski] O. Zariski, *The theorem of Riemann-Roch for high multiples of an effective divisor on an algebraic surface*, Ann. of Math. (2) **76** (1962), 560–615.

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