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A pencil of four-nodal plane sextics

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Introduction

Wiman, in 1895, found ((5), p. 208) an equation for a 4-nodal plane sextic W that admits a group S of 120 Cremona self-transformations; of these, 24 are projectivities, the other 96 quadratic transformations. S is isomorphic to the symmetric group of degree 5 and Wiman emphasizes that S does permute among themselves 5 pencils (4 pencils of lines and 1 of conics) and 5 nets (4 nets of conics and 1 of lines). But he gives no geometrical properties of W. The omission should be repaired because, as will be explained below, W can be uniquely determined by elementary geometrical conditions. Furthermore: W is only one, though admittedly the most interesting, of a whole pencil P of 4-nodal sextics; every member of P is invariant under S^+, the icosahedral subgroup of index 2 in S, while the transformations in the coset $S\backslash S^+$ transpose the members of P in pairs save for two that they leave fixed, W being one of these. When the triangle of reference is the diagonal point triangle of the quadrangle of its nodes the form of W is (7·2) below. Wiman referred his curve to a different triangle.

After the equianharmonic properties necessitated by invariance have been stressed in §§ 1–2 there follows an account of the geometry of P (§§ 3–6). Equations for the sextics are then found (§§ 7–8) and some quadratic transformations in S given (§§ 9–10).

In the two concluding sections (§§ 11–12) the plane maps a del Pezzo quintic surface. Indeed the main object of the paper is to promulgate what appears to be as yet unknown and unsuspected: *there is, on a del Pezzo quintic surface, a uniquely special single canonical curve* W *of genus* 6. It seems that W has no free moduli, being subject to 15 restrictions on its degree of freedom. Its discovery invites an exploration of the geometry of its $6(6^2-1) = 210$ Weierstrassian points at which the osculating [4] has (at least) six-point intersection, and of its $2^5(2^6-1) = 2016$ contact primes, whose ten intersections with W consist of five contacts.

The geometrical constraints imposed by invariance

1. A quadrangle in a plane π consists of four points A, B, C, D, no three of them collinear; it has three diagonal points

X, common to BC and AD,
Y, common to CA and BD,
Z, common to AB and CD.

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It determines five pencils:

α: lines through A. β: lines through B.
γ: lines through C. δ: lines through D.
ϵ: conics through A, B, C, D.

There is, by a fundamental theorem of projective geometry, a unique projectivity imposing any one of the 4! permutations on A, B, C, D. If a curve in π has a node at D and is invariant under the projectivity of period 3 that leaves D fixed and permutes A, B, C cyclically, its nodal tangents can only be the two lines d, d' completing equianharmonic pencils with DA, DB, DC; they are the Hessian pair of these three joins. With a 4-nodal curve the like circumstances occur at A, B, C with nodal tangents a, a'; b, b'; c, c'. Notice that a, a' and d, d' meet BC in the same pair of points, the Hessian duad of the triad B, C, X; the like concurrencies occur on the join of any two of A, B, C, D. If, then, a sextic has nodes at all of A, B, C, D and is invariant under the 24 projectivities that permute its nodes its nodal tangents are all determined as a, a'; b, b'; c, c'; d, d'.

The projectivities that impose the double transpositions

$$(BC)(AD), \quad (CA)(BD), \quad (AB)(CD)$$

are the harmonic inversions in the vertices and opposite sides of XYZ; when, as below, XYZ is triangle of reference for homogeneous coordinates x, y, z the equation of an invariant sextic will include only even powers of x, y, z. Wiman took his triangle of reference to be ABC; the pairs of tangents at these three nodes can thus be read off from his equation and the equianharmonic property confirmed.

2. The restrictions of invariance serve also to identify the intersections, other than the nodes themselves, with the six joins. Let J be the involutory standard quadratic transformation ((4), p. 47) whose homaloids are the conics through A, B, C and which has D for a fixed point; its other fixed points D', D'', D''' form with D a quadrangle whose diagonal points are A, B, C. The transform of any conic c through A, B, C, D is a line g through D; as any second intersection of g and c would be fixed under J, g is the tangent of c at D. Now take c to be Ω, that conic of ϵ which touches d. By the cross-ratio property of conics, and the fact that the condition to be equianharmonic involves four objects symmetrically, Ω also touches a, b, c – the notation being fixed accordingly. So *all* intersections of Ω and the sextic are absorbed, three at each of A, B, C, D; and therefore *all* intersections of d and the sextic are absorbed at D itself and ((4), pp. 48–9) at the intersections of d with BC, CA, AB all of whose points congregate, under J, at A, B, C, respectively. Analogous statements apply to all eight nodal tangents, so that the two intersections with the join of two nodes are, apart from the nodes themselves, the Hessian duad of the two nodes and the diagonal point. For three of the joins this can be checked instantly from Wiman's equation.

The pencil of 4-nodal sextics

3. It might at first sight appear that there are now sufficient restrictions on W to determine it: one has to impose linear conditions on an arbitrary ternary sextic in number

(*a*) 5 at each of A, B, C, D in order to have a node with prescribed tangents;

(*b*) 2 in order to pass through prescribed points on each of BC, CA, AB, AD, BD, CD. These make a total of 32, but a ternary sextic can only satisfy 27 linearly independent linear conditions so that the 32 cannot be independent. Indeed *they are insufficient* because there are two sextic curves satisfying all of them, namely

$$\Sigma \equiv \Omega a'b'c'd' \quad \text{and} \quad \Sigma' \equiv \Omega' abcd.$$

There is, therefore, at least a pencil P of sextics satisfying all 32 conditions; and there cannot be more than a pencil because they all have the equivalent of 36 common points – 6 at each of 4 nodes, 2 on each of 6 joins. A third member of P is Π, the product of the 6 joins: every line through A, not only a and a', has 3-point intersection with Π at A. The place of W in P can, as will be seen in § 5, be identified.

4. It was noted in § 1 that a, a' meet BC in the same pair of points as do d, d'. But the intersection of a and d is the pole of AD with respect to Ω and so is on YZ because XYZ is self-polar for Ω. Thus it cannot be on BC because the intersection of YZ and BC is the harmonic conjugate of X with respect to B and C and so does not belong to the Hessian duad of B, C, X. Thus a, d' meet on BC, as do a', d; corresponding concurrencies occur on the other joins.

5. Σ and Σ' are, it can now be shown, interchanged by any transposition, and so by any odd permutation, in S. They must then both be invariant under S^+.

It is enough to show this for, say, $(\alpha\beta)$ and $(\delta\epsilon)$. Under the projectivity $(\alpha\beta)$ $A \leftrightarrow B$ while C, D are both fixed, it is the harmonic inversion with axis CD and centre the intersection of AB and XY. Thus its effect is

$$a \leftrightarrow b', \quad b \leftrightarrow a', \quad c \leftrightarrow c', \quad d \leftrightarrow d', \quad \Omega \leftrightarrow \Omega'$$

and so

$$\Sigma \leftrightarrow \Sigma'.$$

Under the quadratic transformation $(\delta\epsilon)$

$$\Omega \leftrightarrow d, \quad \Omega' \leftrightarrow d'$$

as remarked above. The lines of α are ((4), p. 48) permuted among themselves; here, as $(\delta\epsilon)$ has period 2, ADD' and $AD''D'''$ are both fixed while the other lines of α are paired by $(\delta\epsilon)$ as harmonic conjugates with respect to these; one pair is AB, AC. As cross-ratio is unaltered by this pairing of lines of α, and as a, a' are not fixed, they are transposed. The circumstances are similar in β and γ:

$$a \leftrightarrow a', \quad b \leftrightarrow b', \quad c \leftrightarrow c'.$$

Thus

$$\Omega a'b'c'd' \leftrightarrow dabc\Omega',$$

$$\Sigma \leftrightarrow \Sigma'$$

Σ and Σ' are transposed by every odd permutation of $\alpha, \beta, \gamma, \delta, \epsilon$ and so are both invariant under every operation of S^+.

Now Π is invariant not only under S^+ but under the whole of S. Under $(\alpha\beta)$ its components

$$BC,\quad CA,\quad AB,\quad AD,\quad BD,\quad CD$$

become

$$AC,\quad CB,\quad BA,\quad BD,\quad AD,\quad CD.$$

Under $(\delta\epsilon)$ AD, BD, CD are all fixed while the set of three lines BC, CA, AB is transformed into itself: any scrutiny here of the subtleties of a standard quadratic transformation can be avoided by a reference to page 49 of (4). Thus three distinct members of P are all invariant under S^+, and therefore every member of P is invariant too – on a projective line the only projectivity with three distinct fixed points is the identity.

Every operation of the coset $S\backslash S^+$ transposes the members of P in pairs; this involution in P has two fixed members, invariant under the whole of S; one of these is Π. The other, since Σ and Σ' are a pair of the involution, *is harmonic to* Π *in* P *with respect to* Σ *and* Σ'. So W is identified.

It is notable that W is uniquely determined when its four nodes are assigned; it is an essential constituent of the geometry of the quadrangle. It has, like the quadrangle which determines it, freedom 8. But a plane sextic with four nodes has, in general, freedom $27-4 = 23$, so that W is specialized 15 times. As a non-hyperelliptic curve of genus 6 has $3.6-3 = 15$ moduli it would seem that W has no free modulus. But other 4-nodal members of P will have a single modulus which varies with the curve in P.

An equation for W is given in (7·2). W is indeed the sum of the squares of ten cubics, but the details of this relation must, with its geometrical significance, be left to some possible future communication.

The pair of rational curves

6. The harmonic inversion h with axis YZ and centre X imposes the double transposition $(\alpha\delta)(\beta\gamma)$, an even permutation: h belongs to S^+ and leaves every curve C in P invariant. Should an intersection of C with YZ be non-singular the tangent passes through X but it is, *a priori*, possible for C to have a node on YZ, the nodal tangents being transposed by h.

The six intersections of C and YZ comprise three pairs of the involution I whose foci are Y and Z. The parameters of any three pairs in I are zeros of a binary cubic; each member of P yields such a cubic, the different cubics belonging to a pencil. But, in a pencil of binary cubics, four members have repeated factors; each corresponding sextic in P has coincident intersections with YZ at both members of a pair in I. This has already been observed to happen with Σ and Σ', the intersections concerned being $b'c'$ and $a'd'$ for Σ, bc and ad for Σ'. There remain two further members R, R' of P to be accounted for. Note, in passing, that Π does not qualify, for while two of its intersections with YZ are at Y and two at Z these coincidences are of members of a single pair, not of members of different pairs.

The indications are, therefore, that P includes sextics R, R' each having two nodes on YZ; these, owing to invariance under S^+, will be accompanied on the same sextic by two nodes on ZX and two on XY; R and R' will be *rational*, having the maximum of ten nodes.

The discussion in § 12 below will show that all members of P save Π, Σ, Σ', R, R' are non-singular. R and R' are transposed by the operations of $S \backslash S^+$.

Equations for the plane sextics

7. When XYZ is triangle of reference, and D the unit point, for homogeneous coordinates the nodes are

$$A(1,-1,-1); \quad B(-1,1,-1); \quad C(-1,-1,1); \quad D(1,1,1) \tag{7·1}$$

and

$$\Pi \equiv (y^2-z^2)(z^2-x^2)(x^2-y^2).$$

Also, if ω is a complex cube root of 1,

$$d \equiv x+\omega y+\omega^2 z, \quad d' \equiv x+\omega^2 y+\omega z;$$

d and d' having to be fixed lines when x, y, z are cyclically permuted:

$$\Omega \equiv x^2+\omega y^2+\omega^2 z^2, \quad \Omega' \equiv x^2+\omega^2 y^2+\omega z^2,$$

$$\begin{aligned}\Sigma \equiv \Omega a'b'c'd' &\equiv \Omega(x-\omega^2 y-\omega z)(-x+\omega^2 y-\omega z)(-x-\omega^2 y+\omega z)(x+\omega^2 y+\omega z)\\ &\equiv (x^2+\omega y^2+\omega^2 z^2)(x^4+\omega^2 y^4+\omega z^4-2y^2z^2-2\omega^2 z^2x^2-2\omega x^2y^2)\\ &\equiv x^6+y^6+z^6-6x^2y^2z^2-\omega(y^4z^2+z^4x^2+x^4y^2)-\omega^2(y^2z^4+z^2x^4+x^2y^4);\end{aligned}$$

$$\Sigma' \equiv \Omega' abcd \equiv x^6+y^6+z^6-6x^2y^2z^2-\omega^2(y^4z^2+z^4x^2+x^4y^2)-\omega(y^2z^4+z^2x^4+x^2y^4).$$

Thus

$$\Sigma-\Sigma' \equiv (\omega-\omega^2)\,\Pi,$$

so that one must take $\Sigma+\Sigma'$ for W, producing the symmetric form

$$W \equiv x^6+y^6+z^6+(x^2+y^2+z^2)(x^4+y^4+z^4)-12x^2y^2z^2. \tag{7·2}$$

Since W is symmetric in x, y, z and consists entirely of even powers it is invariant under all permutations and all changes of sign. Indeed these $2^3 \times 3! = 48$ operations impose the 24 permutations on A, B, C, D as is clear from (7·1), a point not changing position when all three of its coordinates are multiplied by -1. W is of course also invariant under 96 quadratic transformations; some of these appear in (10·1) below.

8. The sextics R, R' are among the curves $W+\lambda\Pi = 0$. These meet $x = 0$ where

$$y^6+z^6+(y^2+z^2)(y^4+z^4)-\lambda y^2z^2(y^2-z^2) = 0;$$

this yields, on writing η for y^2 and ζ for z^2, the binary cubic

$$f \equiv 2\eta^3+(1-\lambda)\eta^2\zeta+(1+\lambda)\eta\zeta^2+2\zeta^3$$

whose Hessian is

$$(5+8\lambda-\lambda^2)\eta^2+(\lambda^2+35)\eta\zeta+(5-8\lambda-\lambda^2)\zeta^2.$$

But if f has a repeated factor its Hessian is a square, so that

$$(\lambda^2+35)^2 = 4\{(5-\lambda^2)^2-64\lambda^2\},$$
$$\lambda^4-122\lambda^2-375 = 0,$$

making λ^2 either -3 or 125. The former alternative, $\lambda = \pm(\omega-\omega^2)$, leads back to Σ and Σ' which, as noted earlier, each have two nodes on $x = 0$:

$$W+(\omega-\omega^2)\,\Pi \equiv 2\Sigma, \quad W-(\omega-\omega^2)\,\Pi \equiv 2\Sigma'.$$

When
$$\lambda = \omega-\omega^2,$$
$$f = 2(\eta^3-\omega\eta^2\zeta-\omega^2\eta\zeta^2+\zeta^3) = 2(\eta-\omega\zeta)^2(\eta+\omega\zeta)$$

and the (composite) curve has nodes on $x = 0$ where $y^2 = \omega z^2$, i.e. at $(0, \omega^2, 1)$ and $(0, -\omega^2, 1)$. These points are $b'c'$ and $a'd'$, nodes of Σ.

If
$$\lambda = 5\sqrt{5}$$
then
$$\begin{aligned} f &\equiv 2\eta^3+(1-5\sqrt{5})\,\eta^2\zeta+(1+5\sqrt{5})\,\eta\zeta^2+2\zeta^3 \\ &= \left(\eta-\frac{3+\sqrt{5}}{2}\zeta\right)^2(2\eta+(7-3\sqrt{5})\,\zeta) \\ &= 2(\eta-\tau^2\zeta)^2(\eta+\tau^{-4}\zeta), \end{aligned}$$

where $\tau = \frac{1}{2}(1+\sqrt{5})$, a root of $\theta^2 = \theta+1$, the other root being $-\tau^{-1}$. So one sextic R in P has nodes on $x = 0$ where $y^2 = \tau^2z^2$, i.e. at $(0, \tau, 1)$ and $(0, -\tau, 1)$. R therefore has nodes at A, B, C, D and at $(0, \tau, 1)$, $(\tau, 1, 0)$, $(1, 0, \tau)$, $(0, -\tau, 1)$, $(-\tau, 1, 0)$, $(1, 0, -\tau)$. The other rational curve R' is obtained from R by the replacement $\tau \leftrightarrow -\tau^{-1}$ so that its nodes, other than A, B, C, D, are $(0, -1, \tau)$, $(-1, \tau, 0)$, $(\tau, 0, -1)$, $(0, 1, \tau)$, $(1, \tau, 0)$, $(\tau, 0, 1)$. The replacement is, of course, simply changing the sign of $\sqrt{5}$.

As R and R', in common with all other members of P, are invariant under S^+ they admit self-transformations of period 5. Such a transformation will leave one node of the hexad fixed and permute the others in a single cycle. The six cyclic subgroups C_5 of S^+ are thus associated one with each node (other than A, B, C, D) of R; this node is twinned with one of R' fixed for the same C_5. Indeed the six pairs of twinned nodes undergo, under S^+, analogous permutations to those undergone by the six diagonals of a regular icosahedron under the icosahedral group of its rotations.

Quadratic transformations

9. The transpositions of pairs of $\alpha, \beta, \gamma, \delta$ are produced by harmonic inversions; these replace x, y, z by the following sets of three:

$$\begin{array}{llll} (\beta\gamma)\; x, z, y, & (\gamma\alpha)\; z, y, x, & (\alpha\beta)\; y, x, z, & \\ (\alpha\delta)\; x, -z, -y, & (\beta\delta)\; -z, y, -x, & (\gamma\delta)\; -y, -x, z. & \qquad (9\cdot1) \end{array}$$

In order to have the whole group S one also uses $(\delta\epsilon)$, which changes lines through D into conics through A, B, C, and conversely. Since BC, CA, AB are

$$y+z = 0, \quad z+x = 0, \quad x+y = 0$$

it follows, by analogy with the standard quadratic transformation ((4), p. 47), that $(\delta\epsilon)$ will replace each of these lines by the product of the other two, and so x by a quadratic form proportional to

$$(x+y)(y+z)+(y+z)(z+x)-(z+x)(x+y) \equiv H-2x^2,$$

where H is the sum of the homogeneous products of degree 2:

$$H \equiv x^2+y^2+z^2+yz+zx+xy.$$

So, in $(\delta\epsilon)$, replace x, y, z by

$$H-2x^2, \quad H-2y^2, \quad H-2z^2. \tag{9·2}$$

When these quadratics are substituted for x, y, z in $H-2x^2$ the outcome is

$$8x(y+z)(z+x)(x+y)$$

in accordance with the period being 2 and the jacobian of the homaloidal net of conics being $(y+z)(z+x)(x+y)$. D is a fixed point; the others are

$$D'(-3,1,1), \quad D''(1,-3,1), \quad D'''(1,1,-3).$$

10. All operations in S are products of transpositions; examples, of periods 3, 4, 5, are, if operations on the right in products act first,

$$(\alpha\delta\epsilon) = (\alpha\delta)(\delta\epsilon), \quad (\alpha\beta\delta\epsilon) = (\beta\delta)(\alpha\delta)(\delta\epsilon), \quad (\alpha\beta\gamma\delta\epsilon) = (\gamma\delta)(\beta\delta)(\alpha\delta)(\delta\epsilon).$$

The quadratics that replace x, y, z in these transformations are found in succession by using (9·1). The replacements are as shown.

$$\begin{array}{cccc} & x & y & z \\ (\delta\epsilon) & H-2x^2 & H-2y^2 & H-2z^2 \\ (\alpha\delta\epsilon) & H-2x^2 & -H+2z^2 & -H+2y^2 \\ (\alpha\beta\delta\epsilon) & H-2y^2 & -H+2z^2 & -H+2x^2 \\ (\alpha\beta\gamma\delta\epsilon) & H-2z^2 & -H+2y^2 & -H+2x^2 \end{array} \tag{10·1}$$

Each of the six C_5's in S – indeed in S^+ – consists of powers of an operation obtained from $(\alpha\beta\gamma\delta\epsilon)$ by one of the six permutations of α, β, γ.

Any fixed point of $(\alpha\beta\gamma\delta\epsilon)$ satisfies

$$\frac{H-2z^2}{x} = \frac{H-2y^2}{-y} = \frac{H-2x^2}{-z} = \frac{2(x^2-z^2)}{x+z} \tag{10·2}$$

and, as the line $x+z=0$ is fundamental, joining two base points A, C of the homaloidal net of conics, one can suppose here that $x+z$ is not zero so that all ratios (10·2) are equal to $2(x-z)$. But they are also equal to

$$\frac{2(z^2+yz+zx+xy)}{-y-z} = -2(z+x),$$

the vanishing of $y+z$ being discounted for the same reason. So one is forced to the conclusion that, at any fixed point of $(\alpha\beta\gamma\delta\epsilon)$, $x = 0$. Now, taking $x = 0$, the equations (10·2) give

$$y^2+yz+z^2 = 2z^2, \quad \frac{-y^2+z^2+yz}{-y} = \frac{y^2+z^2+yz}{-z},$$

all of which hold when, and only when, $z^2 = yz+y^2$, so that there are two fixed points

$$(0, 1, \tau) \text{ and } (0, 1, -\tau^{-1}) \sim (0, \tau, -1),$$

twinned nodes of R' and R. Permuting the three coordinates gives the other five twinned pairs of nodes of R' and R; each pair is fixed for one of the six cyclic subgroups C_5 of S^+.

The pencil of canonical curves

11. The natural setting for the foregoing geometry is the del Pezzo quintic surface F in [5]; the prime sections of this rational surface are mapped in π by the cubics through A, B, C, D ((2); (4), p. 140). A, B, C, D map lines $\lambda_{\epsilon\alpha}, \lambda_{\epsilon\beta}, \lambda_{\epsilon\gamma}, \lambda_{\epsilon\delta}$ on F; all points of, say, $\lambda_{\epsilon\beta}$ are mapped on B but its different points are uniquely linked to the different directions in π at B. There are six other lines

$$\lambda_{\beta\gamma}, \quad \lambda_{\gamma\alpha}, \quad \lambda_{\alpha\beta}, \quad \lambda_{\alpha\delta}, \quad \lambda_{\beta\delta}, \quad \lambda_{\gamma\delta}$$

on F, mapped in π by the joins

$$AD, \quad BD, \quad CD, \quad BC, \quad CA, \quad AB;$$

with this notation lines on F are skew when their binary suffixes share a letter, incident when there is no shared letter. Each line meets three others, and the Hessian duads of their triads of intersections are basic in the geometry. The directions at A of a, a' in π map the Hessian duad on $\lambda_{\epsilon\alpha}$ of its intersections with $\lambda_{\beta\gamma}, \lambda_{\gamma\delta}, \lambda_{\beta\delta}$. The two types of condition imposed on the sextics of P in π, four involving pairs of nodal tangents and the other six involving intersections with joins of nodes, all map *the same type of condition* on F. The curves of the pencil P on F – we use the same letters to label curves on F as label their maps in π – thus cut each of the ten lines λ_{ij} at the Hessian duad of the triad of intersections with other lines; P has 20 base points.

All five pencils $\alpha, \beta, \gamma, \delta, \epsilon$ in π map pencils of conics on F; the conics of i meet those four lines whose binary suffix includes i. Conics in different pencils have a single intersection; conics in the same pencil are skew. The conics of α that are mapped by a and a' cut the Hessian duads on $\lambda_{\alpha\beta}, \lambda_{\alpha\gamma}, \lambda_{\alpha\delta}, \lambda_{\alpha\epsilon}$. Each of Σ, Σ' consists of five conics, one from each pencil; reducible curves, of order 10, with 10 double points. In any of the five pencils three of the conics are line-pairs.

12. Every transformation of S turns the system of cubics through A, B, C, D into itself ((4), p. 49) so that it maps a linear transformation in [5]; F is invariant under a group S of *projectivities* isomorphic to the symmetric group of degree 5 and imposing all 5! permutations on $\alpha, \beta, \gamma, \delta, \epsilon$. The cubics, being adjoint and of order less by 3, cut the canonical series on any sextic with nodes at A, B, C, D but no other multiple points; hence the prime sections of F cut the canonical series on all those curves of P that are

non-singular: these, W in particular, are canonical curves. They are, as mapped by 4-nodal sextics, intersections of F with quadrics. Each of them is invariant under S^+, a group of 60 projectivities imposing even permutations on $\alpha, \beta, \gamma, \delta, \epsilon$; but W is invariant under the whole group S. The other member of P invariant under S is Π, the product of the ten lines on F. P includes a pair R, R' of rational curves, each with six nodes.

Every curve other than $\Sigma, \Sigma', \Pi, R, R'$ in P is non-singular, and so canonical: this is proved by using the Zeuthen–Segre invariant ((3), p. 315; (1), p. 185). If a pencil of curves of genus p on F has σ base points, and if δ is the number of double points of these curves, then

$$I = \delta - \sigma - 4p$$

is the same for every pencil on F. But, for each pencil i, $\delta = 3$ and $\sigma = p = 0$ so that $I = 3$. Hence, for P,

$$3 = \delta - 20 - 24,$$

$$\delta = 47,$$

to which $\Sigma, \Sigma', \Pi, R, R'$ contribute 10, 10, 15, 6, 6.

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