Derived autos, HK mflds & H'berg cat

Problem 1 Describe Aut(X)

X proj. variety/$\mathbb{C}$

eg cubic surface (27 lines)

Problem 2 Describe Aut($D(X)$)

- more difficult
- deeper $\because$ autos reflect:
  - string theory on X } more later!
  - rep theory }
  - birat. geom. of X

Orientation

obj	evident autos
X	f auto
	line bundle
$\mathrm{Coh}(X)$	f_*, $\otimes \mathcal{L}$
$\downarrow$ bdd complexes	shift
$\mathrm{Com}(X)$	same + $[1]$
$\downarrow$ modify morphisms	
$D^b(X)$	same + ?

Plan

(1) History

(2) Hilbert schemes

(3) Categorificat$^{\underline{n}}$

(4) New results

§1 History

X	$\mathrm{Aut}(D(X))$
ω_X / ω_X^{-1} ample	$\langle f_*, \otimes \mathcal{L}, [1] \rangle$
ell. curve ($\omega_X = \mathcal{O}$)	$\langle \text{above} + T_{\mathcal{O}} \rangle$

Def$^{\underline{n}}$: Σ spherical if

$$\mathrm{Hom}_{D(X)}(\Sigma,\Sigma) \cong H^*(S^n)$$

eg $\Sigma = \mathcal{O}$ above.

Def$^{\underline{n}}$: Spherical twist

$$T_\Sigma \in \mathrm{Aut}(D(X))$$

$$T_\Sigma : F \mapsto \left\{ \Sigma \otimes \mathrm{Hom}(\Sigma, F) \xrightarrow{ev} F \right\}$$

eg $T_{\mathcal{O}}$

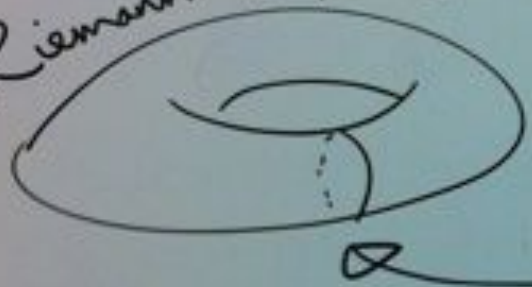
Motivation

symplectic mirror

$$D(X) \xrightarrow{\sim} \mathrm{Fuk}(M) \circlearrowleft \mathrm{GrSymAut}(M)$$

$\mathrm{GrSymAut}(M) \ni$ Dehn twist

Riemann surface



K3 surfaces S $\boxed{\omega_S = \mathcal{O}}$

- $\mathcal{O}$ spherical
- $\mathcal{O}_C$ spherical $C \hookrightarrow S$, $C \cong \mathbb{P}^1$

Examples resol^{n}s of simple sings

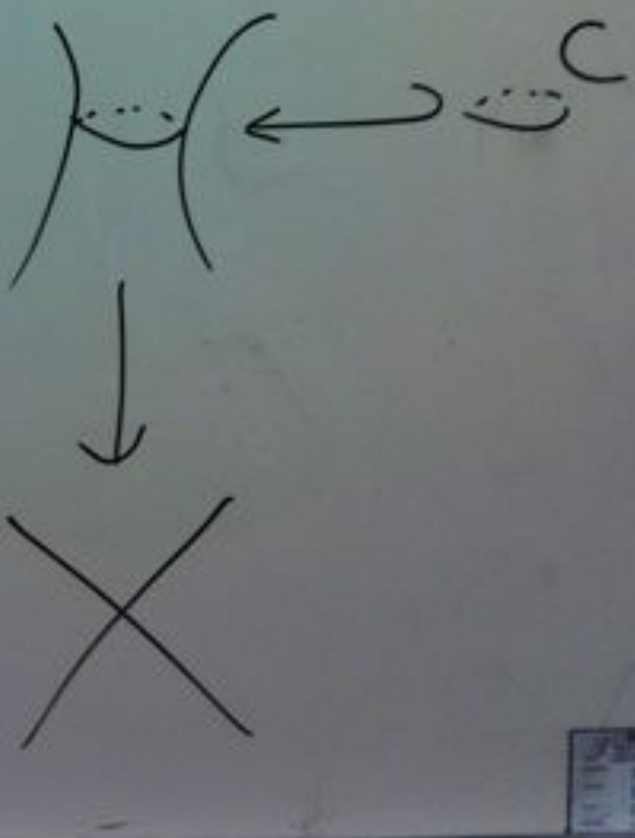
Take fin. $\Gamma \subset SL(2,\mathbb{C})$

Have min. resol^n $\widehat{\mathbb{C}^2/\Gamma}$

$$\downarrow$$

of quot. sing $\mathbb{C}^2/\Gamma$

eg $\Gamma = \mathbb{Z}/2$



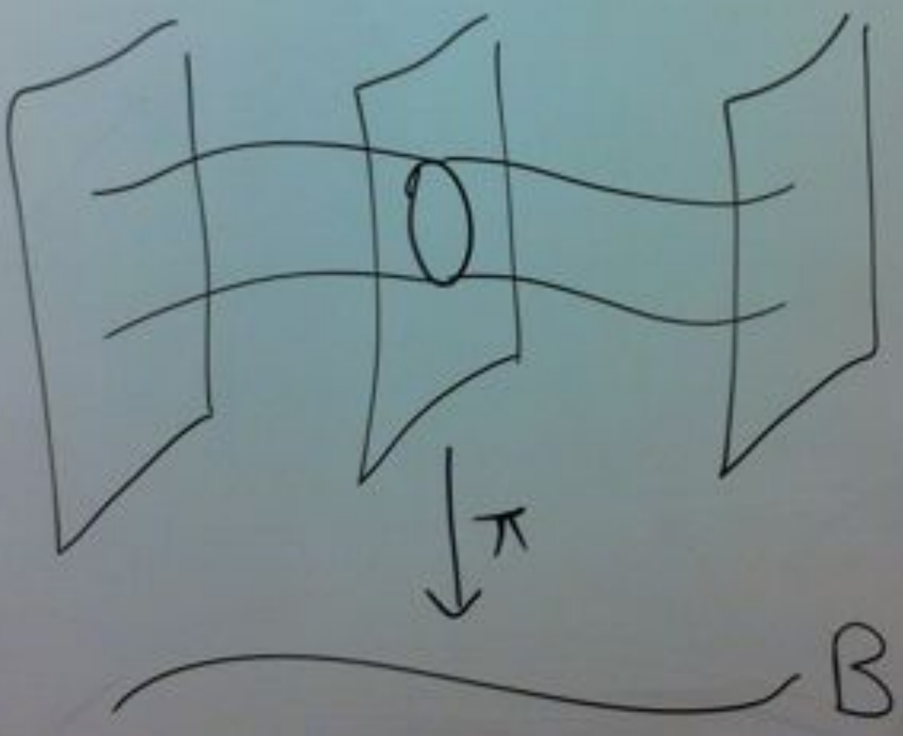
$\mathcal{O}_C$ spherical

__Hope__ $\mathrm{Aut}(D(S)) = \langle \text{above} + \text{twists} \rangle$

· v. difficult.

__CY3 X__ $\boxed{\omega_X = 0}$

eg $S \times B$ base curve



$\exists$ __relative__ spherical twist

Given by $P : D(B) \longrightarrow D(X)$

$P \dashv Q$, adjoint.

<u>Def:</u> $T_P := \{PQ \to 1\}$

<u>Thm</u> [...] P "spherical"

$\Longrightarrow T_P \in \mathrm{Aut}(D(X))$

<u>X 4-fold</u>

$\mathrm{Aut}(D(X)) = \langle \, ? \, \rangle$

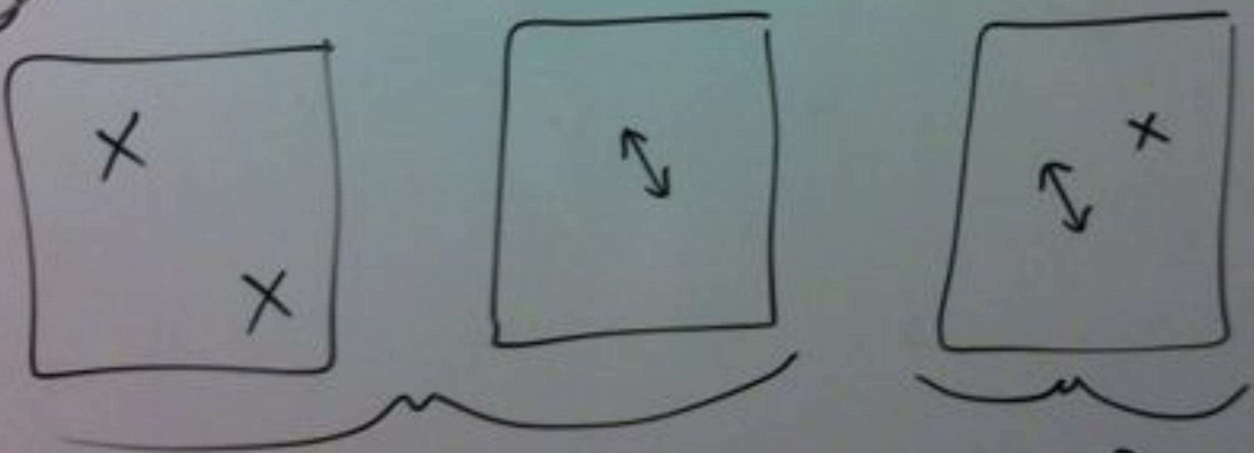
Addington / ~

§2 Hilbert schemes

S surface

$S^{[n]} \equiv \mathrm{Hilb}^n S := \{$ideal sheaves of closed subschemes of S, length $n\}$

eg



Hilb^2 $\qquad\qquad$ Hilb^3

Hyper-Kähler varieties X

Fact S K3 $\Longrightarrow$

$\mathrm{Hilb}^n(S)$ HK

· Berger classificat$\underline{\text{n}}$:

· Holonomy $\subset Sp(n)$

· $4 \mid \dim_{\mathbb{R}} X$

· $\exists\, \omega$ alg. symplectic

New autos:

<u>Def</u>: $\mathcal{E}$ <u>$\mathbb{P}^n$-object</u> if

$$\mathrm{Hom}(\mathcal{E},\mathcal{E}) \simeq H^*(\mathbb{P}^n)$$

Have $\mathbb{P}^n$-twist, $\mathbb{P}^n$-functors

eg $\mathcal{O}_L$, $L \overset{\text{Lag.}}{\hookrightarrow} X$, $L \cong \mathbb{P}^n$

Correspondences

Take curve $C \hookrightarrow S$

$$\begin{tikzcd} & P_i \arrow[dl] \arrow[dr] & \\ S^{[i]} & & S^{[i+1]} \end{tikzcd}$$

$$P_i := \left\{ (I, I') \;\middle|\; I \supset I', \; \mathrm{Supp}(I/I') = \{pt\} \subset C \right\}$$

<u>IDEA</u>: add $pt \in C$

Nakajima-Grojnowski

Nat. correspondences give operators:

$$
\underbrace{
\begin{tikzcd}
H_\bullet(S^{[i-1]}) \arrow[r, bend left, "p^H"] & H_\bullet(S^{[i]}) \arrow[l, bend left, "Q^H"] \arrow[r, bend left, "p^H"] & H_\bullet(S^{[i+1]}) \arrow[l, bend left, "Q^H"]
\end{tikzcd}
}_{\text{Poincaré duals}}
$$

Get $[P^H, Q^H] = \mathrm{id} \times \text{const}$

H'berg alg $[p, q] = 1$

§3 Categorificat^{n}

Decategorification

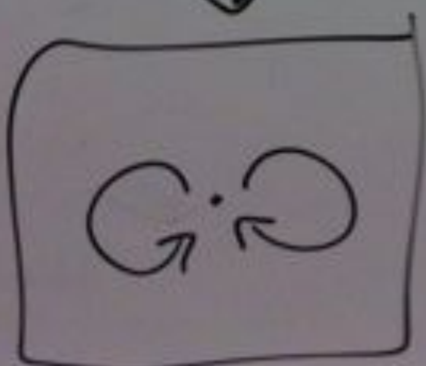
eg FinSet := category (sets, maps)

$$\begin{array}{l} \text{FinSet} \\ \big\downarrow \ \text{iso classes} \\ \mathbb{N}_0 \end{array}$$

eg

2-category w/ one obj

$\big\downarrow$



1-category w/ one obj (monoid).

Take $\underset{\text{algebra}}{A} \rightarrow \underset{\text{ring}}{R}$

View as $\underset{\text{1-cat}}{A} \xrightarrow{\text{f-tor}} \underset{\text{1-cat}}{\mathrm{Aut}}(R)$

Categorify to $\underset{\text{2-cat}}{A} \xrightarrow{\text{f-tor}} \underset{\text{2-cat}}{\mathrm{Aut}}\,\underset{\text{1-cat}}{(R)}$

Cautis-Licata

Sim. for $D(\mathrm{Hilb}^{\bullet}(S))$

- Case $S = \widehat{\mathbb{C}^2/\Gamma}$
- harder:

$$\left\{\begin{array}{c} \text{rings} \\ \text{f}^{\underline{n}}\text{s} \\ \text{equality} \end{array}\right\} \longrightarrow \left\{\begin{array}{c} \text{categories} \\ \text{f}^{\underline{n}}\text{tors} \\ \text{nat. isos} \end{array}\right\}$$

Fourier-Mukai

$$\begin{array}{ccc} & P & \text{corresp} \\ \swarrow & & \searrow \\ X & & Y \end{array} \rightsquigarrow \Phi_P : D(X) \to D(Y)$$

Case $\Gamma = \mathbb{Z}/2$

Fourier-Mukai gives f$^\wedge$tors:

$$\begin{tikzcd}
D(S^{[i-1]}) \arrow[r, bend left, "P"] & D(S^{[i]}) \arrow[l, bend left, "Q"] \arrow[r, bend left, "P"] & D(S^{[i+1]}) \arrow[l, bend left, "Q"]
\end{tikzcd}$$

adjoint (up to shifts)

$$PQ \simeq QP \oplus [1] \oplus [-1]$$

cf. quantum H'berg rel$^\wedge$:

$$[p, q] = h + h^{-1}$$

Above is consequence of:

Thm $\exists$ f$^{\underline{n}}$tor $\mathcal{H}_\Gamma \longleftarrow$ 2-cat (Khovanov)

$$\begin{array}{c} \mathcal{H}_\Gamma \\ \downarrow \\ \mathrm{Aut}\left(D\left(\mathrm{Hilb}^\bullet\left(\widehat{\mathbb{C}^2/\Gamma}\right)\right)\right) \end{array}$$

Proof Algebraic.

Decategorification gives N-G action of $\mathfrak{h}_\Gamma$ on homology.

$\uparrow$ H'berg alg

Idempotent modif$^{\underline{n}}$

$$P^2 = P^{2+} \oplus P^{2-}$$

sim. Q^2

Reflection el$^{\underline{t}}$

Unique Weyl group el$^{\underline{t}}$ acts by

$$\underbrace{\cdots P^{2+}Q^{2-} \oplus P^{2-}Q^{2+} \longrightarrow PQ \longrightarrow \underline{1}}_{f^{\underline{n}}\text{tor}}$$

General K3s

Any spherical $\mathcal{E} \in D(S)$.

Setup:

$\otimes\mathcal{E}$ $\quad$ $P_{\mathcal{E}}$ $\leftarrow$ nat. constr^n

$\|$ $\quad$ $\|$

$$\begin{tikzcd}
D(pt) \arrow[r, bend left, "P"] & D(S) \arrow[l, bend left, "Q"] \arrow[r, bend left, "P"] & D(S^{[2]}) \arrow[l, bend left, "Q"]
\end{tikzcd}$$

adjoints

Lemma $P^2 = P^{2+} \oplus P^{2-}$

$Q^2 = Q^{2+} \oplus Q^{2-}$

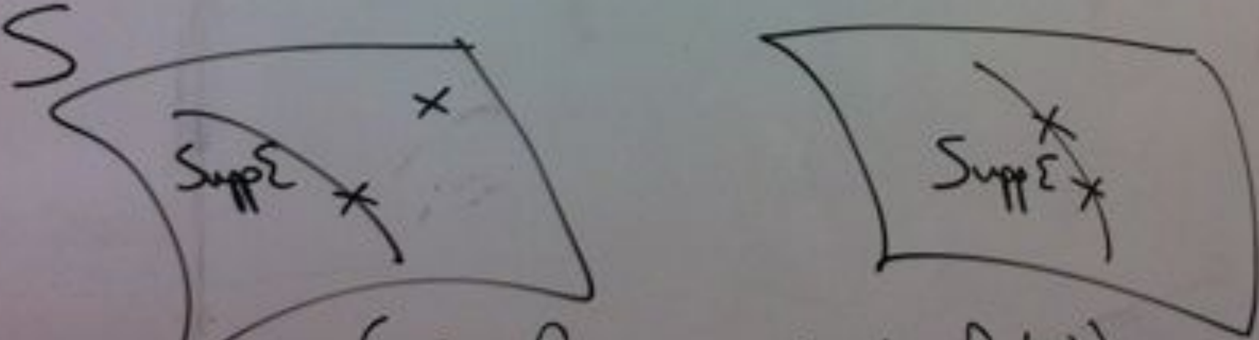
$\mathbb{P}^2$-objects

Thm For hyper-Kähler $H := S^{[2]}$

$$T := \left\{ P^{2+}Q^{2-} \oplus P^{2-}Q^{2+} \longrightarrow PQ \longrightarrow 1 \right\} \in \mathrm{Aut}(D(H))$$

Rmk New?

Rmk Behavior:



(pairs of xs $\rightsquigarrow$ pts of H)

$T \sim$ family twist $\qquad$ $T \sim \mathbb{P}^2$-twist

Proof. Geometric rather than algebraic

(CL use equiv w/ module cat)