Why I like homogeneous manifolds

José Figueroa-O'Farrill

Edinburgh Geometry

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Basic terminology I

- “manifold”: smooth, connected, finite-dimensional
- “Lie group”: finite-dimensional with identity 1
- G acts on M (on the left) via $G \times M \to M$, sending $(\gamma, m) \mapsto \gamma \cdot m$
- actions are effective: $\gamma \cdot m = m, \forall m \implies \gamma = 1$
- M is **homogeneous** (under G) if either
  1. G acts transitively: i.e., there is only one orbit; or
  2. for every $m \in M$, $G \to M$ sending $\gamma \mapsto \gamma \cdot m$ is surjective
  3. given $m, m' \in M$, $\exists \gamma \in G$ with $\gamma \cdot m = m'$
- γ defined up to right multiplication by the **stabiliser** of m: $H = \{\gamma \in G | \gamma \cdot m = m\}$, a closed subgroup of G
- $M \cong G/H$, hence M is a **coset manifold**
- $H \to G$ is a principal H-bundle

$$\begin{array}{ccc} H & \to & G \\ & & \downarrow \\ & & M \end{array}$$

Basic terminology II

- the action of G on M defines $G \to \mathrm{Diff}\, M$
- the differential $\mathfrak{g} \to \mathscr{X}(M)$
- evaluating at $m \in M$: exact sequence of H-modules

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow T_m M \longrightarrow 0$$

- **linear isotropy representation** of H on $T_m M$ is defined for $\gamma \in H$ as $(d\gamma\cdot)_m : T_m M \to T_m M$
- it agrees with the representation on $\mathfrak{g}/\mathfrak{h}$ induced by the adjoint representation restricted to $\mathfrak{h}$
- G/H **reductive**: the sequence splits (as H-modules); i.e., $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $\mathfrak{m}$ an $\mathrm{Ad}(H)$-module
- there is a one-to-one correspondence

$$\left\{\begin{matrix}\mathrm{Ad}(H)\text{-invariant}\\ \text{tensors on } \mathfrak{m}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}H\text{-invariant}\\ \text{tensors on } T_m M\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}G\text{-invariant}\\ \text{tensor fields on } M\end{matrix}\right\}$$

Many of our favourite manifolds are homogeneous:

Examples

- Lie groups ($H = \{1\}$)
- affine space $\mathbb{A}^n$ under the group of translations
- sphere $S^n = SO(n+1)/SO(n)$
- hyperbolic space $H^n \cong SO(n,1)/SO(n)$
- projective space $\mathbb{C}P^n = U(n+1)/U(n) \times U(1)$
- real grassmannians $SO(p+q)/SO(p) \times SO(q)$
- complex grassmannians $U(p+q)/U(p) \times U(q)$
- more exotic grassmannians $SU(n)/SO(n)$,...
- our (spatial) universe!

The principle of mediocrity

The cosmological principle

At sufficiently large scales, the Universe looks the same to all observers.

Mathematical rephrasing

As far as cosmology is concerned, the spatial universe is a 3-dimensional homogeneous manifold.

The big questions

- What (topological) manifold is it? Is it compact? Simply-connected?
- We know it's expanding: but will it do so forever? or will it eventually contract?

Homogeneous (pseudo)riemannian manifolds

- (M, g) with $g \in C^\infty(S^2T^*M)$ everywhere nondegenerate
- g has constant signature (p, q):
  - $(n, 0)$: **riemannian**
  - $(n-1, 1)$: **lorentzian**
- (M, g) homogeneous under G if G acts transitively by isometries: $\gamma^* g = g$ for all $\gamma \in G$
- $\mathfrak{g}$ acts via **Killing vector fields**: $\mathscr{L}_X g = 0$
- g_m is H-invariant inner product on $T_m M$
- in the reductive case and assuming $\pi_1(M) = \{1\}$ (or H connected), (M, g) is described algebraically by
  1. $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$
  2. $\mathfrak{h}$-invariant inner product $\langle -, - \rangle$ on $\mathfrak{m}$
- riemannian: H compact, so G/H reductive

Reductive homogeneous manifolds

Theorem (Ambrose–Singer, 1958; Kostant, 1960)

A simply-connected (pseudo)riemannian manifold (M, g) *is reductive homogeneous if and only if there exists a linear connection* ∇ *with torsion* T *and curvature* R *satisfying*

1. $\nabla g = 0$
2. $\nabla T = 0$
3. $\nabla R = 0$

- if $T = 0$ then (M, g) is a **symmetric space**

  CARTAN (1926)

- if $R = 0$ then (M, g) is either a Lie group with a bi-invariant metric or the round 7-sphere

  CARTAN–SCHOUTEN (1926); WOLF (1971-2)

Many of our favourite manifolds have homogeneous metrics:

Examples

- left-invariant metrics on Lie groups
- euclidean metric on affine space $\mathbb{A}^n$
- round metric on sphere $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$
- Poincaré metric on hyperbolic space $H^n \cong \mathrm{SO}(n,1)/\mathrm{SO}(n)$
- Fubini–Study metric on complex projective space $\mathbb{C}\mathrm{P}^n = \mathrm{U}(n+1)/\mathrm{U}(n) \times \mathrm{U}(1)$

The name of the game in differential geometry

- ... is finding "nice" metrics; e.g.,
  - constant sectional curvature
  - constant scalar curvature
  - **Einstein**: $\mathrm{Ric} = g$
  - extremal, etc
- These are curvature conditions $\implies$ (hard) PDEs on g
- But on a homogeneous manifold, for $X, Y \in \mathfrak{m}$ Killing

$$\nabla_X Y|_{\mathfrak{m}} = -\tfrac{1}{2}[X,Y]_{\mathfrak{m}} + U(X,Y)$$

where $U : S^2\mathfrak{m} \to \mathfrak{m}$ is defined by

$$\langle U(X,Y), Z\rangle = \tfrac{1}{2}\langle [Z,X]_{\mathfrak{m}}, Y\rangle + \tfrac{1}{2}\langle [Z,Y]_{\mathfrak{m}}, X\rangle$$

- curvature conditions (e.g., Einstein) $\implies$ **algebraic**

Homogeneous riemannian Einstein manifolds

A lot is known in low dimension.

Theorem (Böhm–Kerr, 2003)

Every simply-connected, homogeneous, riemannian manifold of dimension $\leqslant 11$ *admits a homogeneous Einstein metric.*

Theorem (Wang–Ziller, 1986)

There is a simply-connected, homogeneous, riemannian manifold of dimension 12 *which does not admit a homogeneous Einstein metric.*

But for more than century now, in Physics we work in *lorentzian* signature...

The birth of spacetime

Hermann Minkowski (1908)

Die Anschauungen über Raum und Zeit, die ich Ihnen entwicklen möchte, sind ***auf experimentell-physikalischem Boden erwachsen****. Darin liegt ihre Stärke. Ihre Tendenz ist eine radikale. Von Stund' an sollen Raum für sich und Zeit für sich völlig zu Schatten herabsinken und nur noch eine Art Union der beiden soll Selbständigkeit bewahren.*

The views of space and time that I wish to lay before you have ***sprung from the soil of experimental physics****, and therein lies their strength. They are radical. Henceforth space by itself, and time by*

In the beginning there was Newton...

- “space and time are absolute”
- universe is an $\mathbb{A}^3$ (space) bundle over $\mathbb{A}^1$ (time)
- invariant notions:
  - time differences
  - distances between simultaneous points
- relavity group = symmetry of trajectories of free particles
- Galilean group: translations, rotations, boosts
- velocities add
- inconsistent with the propagation of light!

And Maxwell said “Let there be light”

- Maxwell’s equations describe the propagation of electromagnetic waves (e.g., light)
- They are not Galilean invariant, but Poincaré invariant
- Poincaré group: isometries of a flat homogeneous lorentzian manifold
- **Minkowski spacetime**: a very accurate model of the universe (at some scales)
- It is consistent with quantum theory (RQFT) and is *spectacularly* successful:

$$\left(\frac{g-2}{2}\right) = \begin{cases} 1\,159\,652\,182.79(7.71) \times 10^{-12} & \text{theory} \\ 1\,159\,652\,180.73(0.28) \times 10^{-12} & \text{experiment} \end{cases}$$

Einstein's theory of general relativity

- Equivalence principle: gravitation is geometric!
- Minkowski spacetime is flat: geodesics which start parallel remain parallel
- Replace it with (four-dimensional) lorentzian (M, g)
- subject to Einstein equations:

$$\mathrm{Ric} = T$$

where T models the "matter" in the universe

GR mantra

Spacetime tells matter how to move, matter tells spacetime how to curve

Supergravity

- result of ongoing effort to marry GR and quantum theory
- many supergravity theories, painstakingly constructed in the 1970s and 1980s
- "crown jewels of mathematical physics"
- the formalism could use some improvement!
- The geometric set-up:
  - (M, g) a lorentzian, spin manifold of dimension $\leqslant 11$
  - some extra geometric data, e.g., differential forms $F, \ldots$
  - a connection $D = \nabla + \cdots$ on the spinor bundle
- $g, F, \ldots$ are subject to Einstein-like equations

Eleven-dimensional supergravity

- Unique supersymmetric theory in $d = 11$

  NAHM (1979), CREMMER+JULIA+SCHERK (1980)
- (bosonic) fields: lorentzian metric g, 3-form A
- Field equations from action (with $F = dA$)

$$\underbrace{\frac{1}{2}\int R\,\mathrm{dvol}}_{\text{Einstein–Hilbert}} - \underbrace{\frac{1}{4}\int F \wedge \star F}_{\text{Maxwell}} + \underbrace{\frac{1}{12}\int F \wedge F \wedge A}_{\text{Chern–Simons}}$$

- Explicitly,

$$\begin{aligned} d \star F &= \tfrac{1}{2} F \wedge F \\ \mathrm{Ric}(X, Y) &= \tfrac{1}{2}\langle \iota_X F, \iota_Y F\rangle - \tfrac{1}{6} g(X, Y)|F|^2 \end{aligned}$$

together with $dF = 0$

Homogeneous supergravity backgrounds

- A triple (M, g, F) where $dF = 0$ and (g, F) satisfying the above PDEs is called an **(eleven-dimensional) supergravity background**.
- A background (M, g, F) is said to be **homogeneous** if some Lie group G acts transitively on M preserving both g and F.
- There is by now a huge catalogue of eleven-dimensional supergravity backgrounds:
  - Freund–Rubin: $AdS_4 \times X^7$, $AdS_7 \times X^4$,...
  - pp-waves
  - branes: elementary, intersecting, overlapping, wrapped,...
  - Kaluza–Klein monopoles,...
  - ...
- Many of them are homogeneous or of low cohomogeneity.

Supersymmetry

- Eleven-dimensional supergravity has local supersymmetry
- manifests itself as a connection D on the spinor bundle
- D is **not** induced from a connection on the spin bundle
- the field equations are encoded in the curvature of D:

$$\sum_i e^i \cdot R^D(e_i, -) = 0$$

- geometric analogies:
  - $\nabla\varepsilon = 0 \implies \mathrm{Ric} = 0$
  - $\nabla_X\varepsilon = \frac{1}{2}X \cdot \varepsilon \implies$ Einstein
- a background (M, g, F) is **supersymmetric** if there exists a nonzero spinor field ε satisfying $D\varepsilon = 0$
- such spinor fields are called **Killing spinors**

Killing spinors

- Not every manifold admits spinors: so an implicit condition on (M, g, F) is that M should be **spin**
- The spinor bundle of an eleven-dimensional lorentzian spin manifold is a real 32-dimensional symplectic vector bundle
- The Killing spinor equation is

$$D_X \varepsilon = \nabla_X \varepsilon + \tfrac{1}{12}(X^\flat \wedge F) \cdot \varepsilon - \tfrac{1}{6}\iota_X F \cdot \varepsilon = 0$$

  which is a linear, first-order PDE:
  - linearity: solutions form a vector space
  - first-order: solutions determined by their values at any point
- the dimension of the space of Killing spinors is $0 \leqslant n \leqslant 32$
- a background is said to be ν**-BPS**, where $\nu = \frac{n}{32}$

Supersymmetries generate isometries

- The **Dirac current** V of a Killing spinor ε is defined by

$$g(V, X) = (\varepsilon, X \cdot \varepsilon)$$

- Fact: V is Killing and $\mathscr{L}_V F = 0 \implies \mathscr{L}_V D = 0$
- ε Killing spinor $\implies$ so is $\mathscr{L}_V \varepsilon$
- This turns the vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where
  - $\mathfrak{g}_0$ is the space of F-preserving Killing vector fields, and
  - $\mathfrak{g}_1$ is the space of Killing spinors

  into a Lie superalgebra

  JMF+Meessen+Philip (2004)
- It is called the **Killing superalgebra** of the supersymmetric background (M, g, F)

The homogeneity conjecture

Empirical Fact

Every known ν-BPS background with $\nu > \frac{1}{2}$ is homogeneous.

MEESSEN (2004)

Theorem

Every ν-BPS background of eleven-dimensional supergravity with $\nu > \frac{3}{4}$ is homogeneous.

JMF+MEESSEN+PHILIP (2004)

What we proved is that the ideal $[\mathfrak{g}_1, \mathfrak{g}_1]$ of $\mathfrak{g}_0$ generated by the Killing spinors spans the tangent space to every point of M: local homogeneity

Generalisations

Theorem

Every ν-BPS background of type IIB supergravity with $\nu > \frac{3}{4}$ is homogeneous.
Every ν-BPS background of type I and heterotic supergravities with $\nu > \frac{1}{2}$ is homogeneous.

JMF+HACKETT-JONES+MOUTSOPOULOS (2007)

The theorems actually suggest a stronger version of the conjecture: that the symmetries which are generated from the supersymmetries already act (locally) transitively.

What good is it?

If the homogeneity conjecture were true, then classifying homogeneous supergravity backgrounds would also classify ν-BPS backgrounds for $\nu > \frac{1}{2}$.

This would be **good** because

- the supergravity field equations for homogeneous backgrounds are algebraic and hence simpler to solve than PDEs
- we have learnt **a lot** (about string theory) from supersymmetric supergravity backgrounds, so their classification could teach us even more

Homogeneous supergravity backgrounds

A homogeneous eleven-dimensional supergravity background is described algebraically by the data $(\mathfrak{g}, \mathfrak{h}, \gamma, \varphi)$, where

- $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $\dim \mathfrak{m} = 11$
- γ is an $\mathfrak{h}$-invariant lorentzian inner product on $\mathfrak{m}$
- φ is an $\mathfrak{h}$-invariant 4-form $\varphi \in \Lambda^4 \mathfrak{m}$

subject to some algebraic equations which are given purely in terms of the structure constants of $\mathfrak{g}$ (and $\mathfrak{h}$).

Skip technical details

Explicit expressions

Choose a basis X_a for $\mathfrak{h}$ and a basis Y_i for $\mathfrak{m}$. This defines structure constants:

$$[X_a, X_b] = f_{ab}{}^c X_c$$
$$[X_a, Y_i] = f_{ai}{}^j Y_j + f_{ai}{}^b X_b$$
$$[Y_i, Y_j] = f_{ij}{}^a X_a + f_{ij}{}^k Y_k$$

If M is reductive, then $f_{ai}{}^b = 0$. We will assume this in what follows.

The metric and 4-forms are described by $\mathfrak{h}$-invariant tensors γ_{ij} and φ_{ijkl}.

We raise and lower indices with γ_{ij}.

Homogeneous Hodge/de Rham calculus

The G-invariant differential forms in $M = G/H$ form a subcomplex of the de Rham complex:

- the de Rham differential is given by

$$(d\varphi)_{jklmn} = -f_{[jk}{}^{i}\varphi_{lmn]i}$$

- the codifferential is given by

$$(\delta\varphi)_{ijk} = -\tfrac{3}{2} f_{m[i}{}^{n}\varphi^{m}{}_{jk]n} - 3U_{m[i}{}^{n}\varphi^{m}{}_{jk]n} - U_{m}{}^{mn}\varphi_{nijk}$$

where $U_{ijk} = f_{i(jk)}$

Homogeneous Ricci curvature

Finally, the Ricci tensor for a homogeneous (reductive) manifold is given by

$$\begin{aligned} R_{ij} = &-\tfrac{1}{2} f_i{}^{k\ell} f_{jk\ell} - \tfrac{1}{2} f_{ik}{}^{\ell} f_{j\ell}{}^{k} + \tfrac{1}{2} f_{ik}{}^{a} f_{aj}{}^{k} \\ &+ \tfrac{1}{2} f_{jk}{}^{a} f_{ai}{}^{k} - \tfrac{1}{2} f_{k\ell}{}^{\ell} f^{k}{}_{ij} - \tfrac{1}{2} f_{k\ell}{}^{\ell} f^{k}{}_{ji} + \tfrac{1}{4} f_{k\ell i} f^{k\ell}{}_{j} \end{aligned}$$

It is now a matter of assembling these ingredients to write down the supergravity field equations in a homogeneous Ansatz.

Methodology

Classifying homogeneous supergravity backgrounds of a certain type involves now the following steps:

- Classify the desired homogeneous geometries
- For each such geometry parametrise the space of invariant lorentzian metrics $(\gamma_1, \gamma_2, \dots)$ and invariant closed 4-forms $(\varphi_1, \varphi_2, \dots)$
- Plug them into the supergravity field equations to get (nonlinear) algebraic equations for the γ_i, φ_i
- Solve the equations!

Homogeneous lorentzian manifolds I

- Their classification can seem daunting!
- We wish to classify d-dimensional lorentzian manifolds (M, g) homogeneous under a Lie group G.
- Then $M \cong G/H$ with H a closed subgroup.
- One starts by classifying Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$ with
    - codimension d
    - Lie subalgebras of closed subgroups
    - leaving invariant a lorentzian inner product on $\mathfrak{g}/\mathfrak{h}$
- This is hopeless except in very low dimension.
- One can fare better if G is semisimple.

Definition

The action of G on M is **proper** if the map $G \times M \to M \times M$, $(\gamma, m) \mapsto (\gamma \cdot m, m)$ is proper. In particular, proper actions have compact stabilisers.

Homogeneous lorentzian manifolds II

What if the action is not proper?

Theorem (Kowalsky, 1996)

If a simple Lie group acts transitively and non-properly on a lorentzian manifold (M, g)*, then* (M, g) *is locally isometric to (anti) de Sitter spacetime.*

Theorem (Deffaf–Melnick–Zeghib, 2008)

If a semisimple Lie group acts transitively and non-properly on a lorentzian manifold (M, g)*, then* (M, g) *is locally isometric to the product of (anti) de Sitter spacetime and a riemannian homogeneous space.*

This means that we need only classify Lie subalgebras corresponding to *compact* Lie subgroups!

Some recent classification results

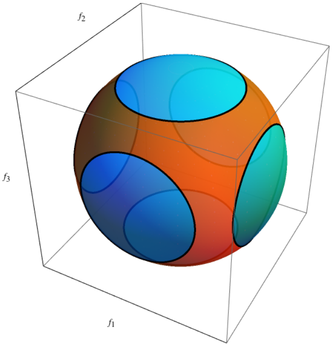
- Symmetric eleven-dimensional supergravity backgrounds

  JMF (2011)
- Symmetric type IIB supergravity backgrounds

  JMF+Hustler (in preparation)
- Homogeneous M2-duals: $\mathfrak{g} = \mathfrak{so}(3,2) \oplus \mathfrak{so}(N)$ for $N > 4$

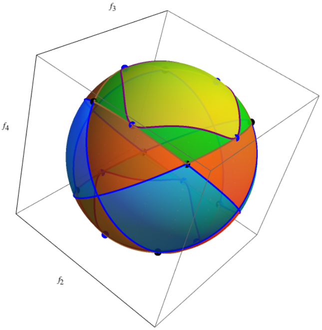
  JMF+Ungureanu (in preparation)

A moduli space of AdS_5 symmetric backgrounds



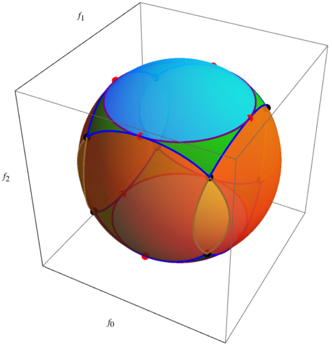
- (black line) $AdS_5 \times S^2 \times S^2 \times T^2$
- (blue) $AdS_5 \times S^2 \times S^2 \times H^2$
- (orange) $AdS_5 \times S^2 \times S^2 \times S^2$

A moduli space of AdS_3 symmetric backgrounds



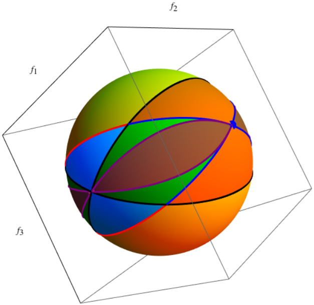
- $AdS_3 \times S^2 \times T^6$
- $AdS_3 \times \mathbb{C}P^2 \times T^4$
- $AdS_3 \times T^4 \times S^2 \times S^2$
- $AdS_3 \times \mathbb{C}P^2 \times S^2 \times T^2$
- $AdS_3 \times \mathbb{C}P^2 \times H^2 \times T^2$
- $AdS_3 \times \mathbb{C}P^2 \times H^2 \times H^2$
- $AdS_3 \times \mathbb{C}H^2 \times S^2 \times S^2$
- $AdS_3 \times \mathbb{C}P^2 \times S^2 \times H^2$
- $AdS_3 \times \mathbb{C}P^2 \times S^2 \times S^2$

A moduli space of AdS_2 symmetric backgrounds



- (red dot) $\mathrm{AdS}_2 \times S^2 \times T^7$
- (black dot) $\mathrm{AdS}_2 \times S^5 \times T^4$
- (crimson line) $\mathrm{AdS}_2 \times T^5 \times S^2 \times S^2$
- (brown line) $\mathrm{AdS}_2 \times S^5 \times H^2 \times T^2$
- (blue line) $\mathrm{AdS}_2 \times S^5 \times S^2 \times T^2$
- (light blue square) $\mathrm{AdS}_2 \times H^5 \times S^2 \times S^2$
- (yellow square) $\mathrm{AdS}_2 \times S^5 \times H^2 \times H^2$
- (orange square) $\mathrm{AdS}_2 \times S^5 \times S^2 \times H^2$
- (green square) $\mathrm{AdS}_2 \times S^5 \times S^2 \times S^2$

And one final gratuitous pretty picture



- $\mathrm{AdS}_2 \times S^3 \times T^6$
- $\mathrm{AdS}_2 \times S^3 \times S^2 \times T^4$
- $\mathrm{AdS}_2 \times \mathbb{C}P^2 \times H^3 \times T^2$
- $\mathrm{AdS}_2 \times \mathbb{C}P^2 \times T^5$
- $\mathrm{AdS}_2 \times \mathbb{C}P^2 \times S^3 \times T^2$
- $\mathrm{AdS}_2 \times \mathbb{C}H^2 \times S^3 \times S^2$
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- $\mathrm{AdS}_2 \times \mathbb{C}P^2 \times S^3 \times S^2$

Summary and outlook

- With patience and optimism, some classes of homogeneous backgrounds can be classified
- In particular, we can "dial up" a semisimple G and hope to solve the homogeneous supergravity equations with symmetry G
- Checking supersymmetry is an additional problem, perhaps it can be done at the same time by considering homogeneous supermanifolds
- The proof of the homogeneity conjecture remains elusive...

Thank you for your attention