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Frontiers of rationality Longyearbyen (Spitsbergen), July 16, 2014

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Sébastien Boucksom (CNRS-Université Pierre et Marie Curie) and Salvatore Cacciola (Roma Tre University)

• Introduction

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As we all know, knowledge of the behavior of these maps often says a lot about the geometry of X itself. In particular, there are some closed subsets, associated to L, that govern, asymptotically, this behavior.

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(introduced in 2000 by Nakamaye, and in 2006 by Ein, Lazarsfeld, Mustață, Nakamaye and Popa)

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Just to mention a few instances, we recall the fundamental papers of Takayama, Hacon and McKernan on the birationality of pluricanonical maps

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Great, but how do we compute them? To this goal we use another notion, present in the mentioned papers, the one of restricted volume.

For every subvariety $Z \subseteq X$ not contained in $\mathbf{B}_+(L)$, it is easily seen that the restriction of L to Z is big. But there is more: the space of sections of $|mL|_Z|$ that extend to X has maximal growth. In fact if we denote by $H^0(X|Z, mL)$ the image of the restriction map $H^0(X, mL) \to H^0(Z, mL_{|Z})$ and set $d = \dim Z$, then we claim that

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 $\lim_{m \to +\infty} \operatorname{vol}_{X|Z}(L + \frac{1}{m}A) = 0$ for every ample A and for every irreducible component Z of $\mathbf{B}_+(L)$.



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Proof:

We can assume that L is big. Let Z be an irreducible component of $\mathbf{B}_+(L)$. Recall that dim $Z \ge 1$ by a well known result of Zariski.

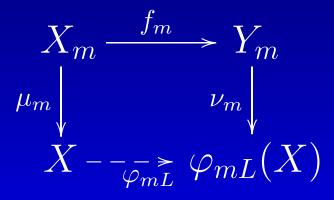
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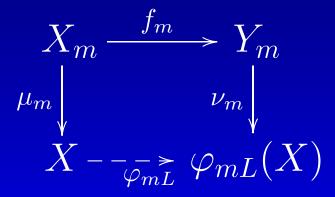
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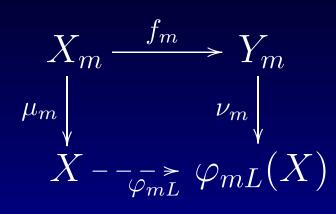
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where μ_m is the normalized blow-up of X along the base ideal of |mL|,



$$\begin{array}{cccc} X_m & \xrightarrow{f_m} & Y_m \\ \mu_m & & \nu_m \\ & & & \\ X - \xrightarrow{-\varphi_{mL}} & \varphi_{mL}(X) \end{array}$$

 ν_m is the normalization of $\varphi_{mL}(X)$ and $f_m: X_m \to Y_m$ is the induced birational morphism.

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with A_m ample on Y_m , F_m effective and such that $\operatorname{Supp}(F_m) = \mu_m^{-1}(\mathbf{B}(L)) \subseteq \mu_m^{-1}(\mathbf{B}_+(L))$.

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If $x \notin \text{Exc}(\mu_m)$ and $\mu_m(x) \notin \mathbf{B}_+(L)$, then μ_m, φ_{mL} and ν_m are isomorphisms in a neighborhood of x, contradicting the fact that f_m is not an isomorphism in a neighborhood of x.

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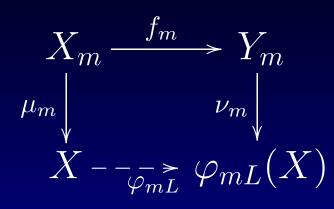
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Now let Z_m be the strict trasform of Z on X_m , so that $Z_m \subseteq \mu_m^{-1}(\mathbf{B}_+(L))$ is an irreducible component. Of course $Z_m \not\subseteq \operatorname{Supp}(F_m)$, since $Z \not\subseteq \mathbf{B}(L)$. Hence Z_m is an irreducible component of $\operatorname{Exc}(f_m)$



$$\begin{array}{cccc} X_m & \xrightarrow{f_m} & Y_m \\ \mu_m & & \nu_m \\ & & & \\ X & \xrightarrow{- \varphi_{mL}} & \varphi_{mL}(X) \end{array}$$

and then it is contracted by f_m , whence also by $\nu_m \circ f_m$ and we deduce that Z is contracted by φ_{mL} .

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Graded linear series

Graded linear series

Let \mathcal{L} be a line bundle on a variety Z. A graded linear series W is a sequence of subspaces $W_m \subseteq H^0(Z, m\mathcal{L}), m \in \mathbb{N}$ such that $W_m \otimes W_s \subseteq W_{m+s}$ for every $m, s \ge 0$.

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Lemma

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Lemma

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Lemma

There exists C > 0 such that $\dim W_m \le Cm^{\kappa(W)}$. This lemma follows from a deep result of Kaveh-Kovanskii (using Okounkov bodies), but we will give a simple proof ispired by a paper of Di Biagio-Pacienza.

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Now the proof of the Lemma, that is $\dim W_m \leq Cm^{\kappa(W)}$. We can assume that the base filed is uncountable, since the estimate is invariant under base field extension.

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dim $T = \kappa(W)$ and $\Psi_m(T) = \Psi_m(Z)$. (here we use that the base field is uncountable, as we impose countably many conditions). We now claim that the restriction map

 $W_m \to H^0(T, m\mathcal{L}_{|T})$ is injective for $m \gg 0$

(and this will prove the Lemma).

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Now I would like to briefly come back to the fact, mentioned in the beginning, that, given a big line bundle L on a variety X, then the maps φ_{mL} are an isomorphism on $X - \mathbf{B}_+(L)$ for $m \gg 0$. Using the same method of proof of the previous theorem, we can prove

Another result (folklore) Theorem (Boucksom, Cacciola, -)

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So the two theorems work for Q-divisors. What about real divisors? (I must say that ELMNP's theorem, in its "continuity version", also holds for real divisors)

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(a) the maps φ_{mD} do not exist; (b) the restricted volume $vol_{X|Z}(D)$ is not defined (it is defined, by ELMNP, only when $Z \nsubseteq \mathbf{B}_+(D)$). (note that it does not seem to work to use $\lfloor mD \rfloor$).

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(a) the maps φ_{mD} do not exist; (b) the restricted volume $vol_{X|Z}(D)$ is not defined (it is defined, by ELMNP, only when $Z \not\subseteq \mathbf{B}_+(D)$). (note that it does not seem to work to use $\lfloor mD \rfloor$). Nevertheless we are able to generalize to \mathbb{R} -divisors using a recent idea of Birkar (used to prove Nakamaye's theorem on any scheme).

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Now the two theorems above go through.