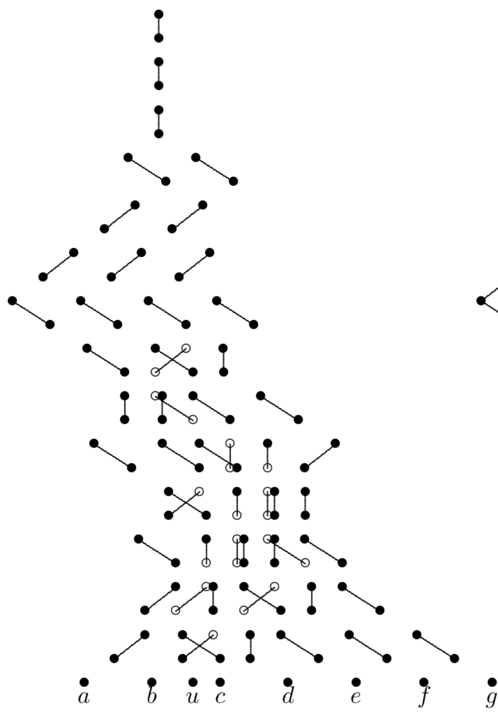
Roots pairing for $\mathbb{E}_8$.



Hasse diagram of $\mathbb{E}_8$.

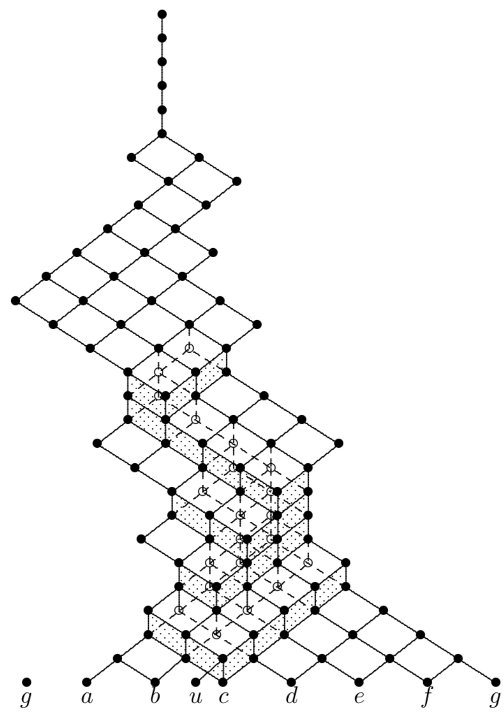


TABLE 1. The root lattices $H_2^-(X) \cap K_X^\perp$

Smith type of $X_\mathbb{R}$	M	$(M-1)$	$(M-2)$	$(M-3)$	$(M-4)$	$(M-2)Ia$	$(M-2)Ib$
Topology of $X_\mathbb{R}$	$\mathbb{RP}^2\#4\mathbb{T}^2$	$\mathbb{RP}^2\#3\mathbb{T}^2$	$\mathbb{RP}^2\#2\mathbb{T}^2$	$\mathbb{RP}^2\#\mathbb{T}^2$	$\mathbb{RP}^2$	$\mathbb{RP}^2 \amalg \mathbb{K}$	$(\mathbb{RP}^2\#\mathbb{T}^2) \amalg \mathbb{S}^2$
$H_2^-(X) \cap K_X^\perp$	E_8	E_7	D_6	$D_4 + A_1$	$4A_1$	D_4	D_4

Smith type of $X_\mathbb{R}$	M	$(M-1)$	$(M-2)$	$(M-3)$
Topology of $X_\mathbb{R}$	$\mathbb{RP}^2 \amalg 4\mathbb{S}^2$	$\mathbb{RP}^2 \amalg 3\mathbb{S}^2$	$\mathbb{RP}^2 \amalg 2\mathbb{S}^2$	$\mathbb{RP}^2 \amalg \mathbb{S}^2$
$H_2^-(X) \cap K_X^\perp$	0	A_1	$2A_1$	$3A_1$

TABLE 3. Count of real lines on X

Smith type of $X_\mathbb{R}$	M	$(M-1)$	$(M-2)$	$(M-3)$	$(M-4)$	$(M-2)Ia$	$(M-2)Ib$
Topology of $X_\mathbb{R}$	$\mathbb{RP}^2\#4\mathbb{T}^2$	$\mathbb{RP}^2\#3\mathbb{T}^2$	$\mathbb{RP}^2\#2\mathbb{T}^2$	$\mathbb{RP}^2\#\mathbb{T}^2$	$\mathbb{RP}^2$	$\mathbb{RP}^2\amalg\mathbb{K}$	$\mathbb{RP}^2\#\mathbb{T}^2\amalg\mathbb{S}^2$
# real lines on $X_\mathbb{R}$	240	126	60	26	8	24	24
# hyperbolic lines on $X_\mathbb{R}$	128	70	36	18	8	16	16
# elliptic lines on $X_\mathbb{R}$	112	56	24	8	0	8	8

Smith type of $X_\mathbb{R}$	M	$(M-1)$	$(M-2)$	$(M-3)$
Topology of $X_\mathbb{R}$	$\mathbb{RP}^2\amalg4\mathbb{S}^2$	$\mathbb{RP}^2\amalg3\mathbb{S}^2$	$\mathbb{RP}^2\amalg2\mathbb{S}^2$	$\mathbb{RP}^2\amalg\mathbb{S}^2$
# real lines on $X_\mathbb{R}$	0	2	4	6
# hyperbolic lines on $X_\mathbb{R}$	0	2	4	6
# elliptic lines on $X_\mathbb{R}$	0	0	0	0

TABLE 4. Count of real tritangent section

Arrangement of $C_\mathbb{R}$ in $Q_\mathbb{R}$	$\langle 4\vert 0\rangle$	$\langle 3\vert 0\rangle$	$\langle 2\vert 0\rangle$	$\langle 1\vert 0\rangle$	$\langle 0\vert 0\rangle$	$\langle \vert\vert\vert \rangle$	$\langle 1\vert 1\rangle$
total number	120	64	32	16	8	24	24
hyperbolic	64	36	20	12	8	16	16
elliptic	56	28	12	4	0	8	8

New examples of surgery invariant counts in real algebraic geometry

V. Kharlamov

(work in progress, joint with S. Finashin)

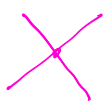
Introduction

Turning-point: Welschinger, C.R.A.S., 2003

"counting real rat curves on real rat. surfaces"

D, $c_1 D - 1$ pts, $c_1 = c_1(TX)$

$(-1)^{\#\,\text{sol. pts}}$

 cross-point  solitary node

$$\sum_{A \in [D]} (-1)^{\#\, s.p(A)}$$

does not depend on pts

preserved under equiv. def.

and equiv. isomorphisms

not preserved under "wall-crossing"

(under Morse-Lefschetz transf.)

stronger invariance property:
to be preserved under wall-crossing

Basic example: signed count of real lines on a cubic surface

27 over $\mathbb{C}$,

top. $X_{\mathbb{R}}$: $\#7\mathbb{R}p^2$, $\#5\mathbb{R}p^2$, $\#3\mathbb{R}p^2$, $\mathbb{R}p^2$, $\mathbb{R}p^2 \sqcup S^2$

27 15 7 3 3

B. Segre: hyperbolic and elliptic

$27 = 15_h + 12_e$, $15 = 9+6$, $7 = 5+2$, $3 = 3+0$

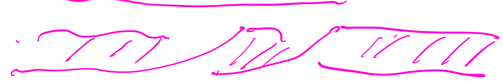
$\#h - \#e = 3$

2010s by Okonek-Teleman & Finashin & me

Counting lines on hypersurfaces $\deg = 2n-1$ in $\mathbb{R}p^{n+1}$

sign. c. $= \#h - \#e = (2n-1)!!$

$\ell \subset S$ hyperbolic, if M. band(ℓ) = "flat"

elliptic : 

In terms of Pin-structures

$S \subset \mathbb{R}p^3$ on $\mathbb{R}p^3$ unique (up to rev.) Spin-structure

$\Rightarrow$ unique (up to reversing) Pin^--str. on S

$$TS \oplus \det TS = \mathrm{in}^* T\,\mathbb{R}p^3$$

$$q(\ell) = \mathrm{hol}\,\theta + 2$$

$$q : H_1(S; \mathbb{Z}/2) \to \mathbb{Z}/4$$

$$q(x+y) = q(x) + q(y) + 2xy \mod 4$$

$q(w_1 S_{\mathbb{R}}) = 1$ or

hyp. $\iff q(\ell) = q(w_1)$

ell $\iff q(\ell) = q(w_1) + 2$

$$\#h - \#e = \sum_{A_{\mathbb{R}} \subset S_{\mathbb{R}}} i^{\,q(A_{\mathbb{R}})-1}$$

all real rat. curves in divisor classes D : $-KD = 1$

X a real dP-surface in a given complex def. class of X (fixed K_X^2)

Assume that for $\forall$ real X in this class, $\exists$ a Pin^--str. θ_X (resp. a pair of opp. str.) on $X_{\mathbb{R}}$ such that

(1) preserved under equiv. def. and equiv. isom.

(2) q vanishes on all real van. cycles

$\mathcal{L}_m = \{ x \in Pic_{\mathbb{R}} X = Ker(1+conj_* : H_2(X) \circlearrowleft);$
$xK = -m \}$

$$N_{m,\kappa} = \sum_{A \in \mathcal{E}_{\mathbb{R}}(m,\kappa,\mathfrak{X})} i^{\,g(A_{\mathbb{R}}) - m^2} \, \underline{w(A)}$$

r. rat. curves through a $\underline{\mathfrak{X}}$ and with $-AK = m$
κ real pts, $\frac{1}{2}(m-\kappa-1)$ pairs of c.c.

$$w(A) = (-1)^{\#\text{cross-pts}}$$

Th 1 Under above assumptions, this count is surgery invariant.

Th 2 These assumptions hold in the cases $K^2 = 1, 3$ and $K^2 = 2$ (a pair)

Case of $K^2 = 1$

$Eff(X)$ semi-group of effective div. cl.
$Eff_{\mathbb{R}}(X)$ real eff. classes

$Eff(X)$ generated by lines & $-K$
line = embedded (-1)-curve
$\Leftrightarrow D^2 = -1, \ -DK = 1, \ D \in Ker(1+conj_*)$

Th 3 $N_{1,0} = 8, \ N_{2,1} = 30, \ N_{3,0} = 160, \ N_{4,1} = 1800 \ldots$
The only non-zero $N_{m,\kappa}$ are those with $m \equiv \kappa - 1 \ (2)$ and $\kappa = 0$ or 1. There exists an explicit algorithm based on J. Solomon's

real WDVV for calculating all the $N_{m,k}$.

<u>Remark</u> double-covering $X \to Q \subset P^3$ (Q quad. cone) ramified in a sextic $= Q \cap$ Cubic and the vertex of Q.

$N_{1,0}$ = twice # 3-tangent hyp. sections to $Q \cap C$ completed by counting in $-K$.

$N_{2,1}$ = twice # 6-tangent sections by quadrics, tangent to $Q \cap C$ and compl. by $-2K$.

$N_{1,0}$ in elementary way

- bijection $\{l\} \leftrightarrow \{e\}$ roots, (-2)-vectors in $K^\perp \subset H_2(X)$

over $\mathbb{R}$ $\{l_{\mathbb{R}}\} \leftrightarrow \{e\}$ roots in $\underbrace{K^\perp \cap \{1+\mathrm{conj}_* = 0\}}_{``\Lambda_{\mathbb{R}}"}$

- existence of a standard basis in this root system representable by real vanishing cycles.

$$\sum_{\text{lines}} i^{q(L_{\mathbb{R}})-1} = \sum_{\text{roots}} (-1)^{q(e)}$$

Rule: g, f, e roots $g = f + e$, $fe = 1$

$$\Rightarrow (-1)^{q(f)} + (-1)^{q(g)} = 0$$

$$\Rightarrow \text{Count of lines} = 2\,\mathrm{rk}\,\Lambda_{\mathbb{R}}$$

The input of $-K$ in our count in the first layer ($N_{1,0}$)

$= \chi(X, \mathbb{R}) - 1$ due to Lefschetz trace f-la

$\Rightarrow$ total count $= 2\,\mathrm{rk}\,\Lambda_{\mathbb{R}} + (8 - 2\,\mathrm{rk}\,\Lambda_{\mathbb{R}}) =$

$= 8.$ $\boxtimes$