Cohomology of the moduli of Higgs bundles & the Hausel-Thaddeus Conjecture.

joint work w/ Davesh Maulik

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Zoom Algebraic Geometry Seminar

10. 06. 2020

Aim:

- $\mathcal{M}$ moduli of Higgs bundles on a curve

Connections to $\left\{ \begin{array}{l} \cdot \text{ Mathematical physics} \\ \cdot \text{ Representation theory} \\ \cdot \text{ Geometry} \end{array} \right.$

Aim :

- $\mathcal{M}$ moduli of Higgs bundles on a curve
  - Mathematical physics
  - Representation theory
  - Geometry
- Want to understand the structure of

$$H^*(\mathcal{M}) := H^*(\mathcal{M}, \mathbb{C})$$

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> ... At Princeton I also heard lectures from E. Witten about string theory. As a result, I concluded firstly that physics is not a branch of mathematics, and secondly, was inspired to study moduli spaces of vector bundles on curves where I can show some new results. ...
>
> — Gerd Faltings

Moduli of stable vector bundles

C Riemann surface,

$g \geq 2$

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C Riemann surface, n "rank" $\quad$ $\gcd(n,d) = 1$

d "degree"

$L \in \mathrm{Pic}^d(C)$

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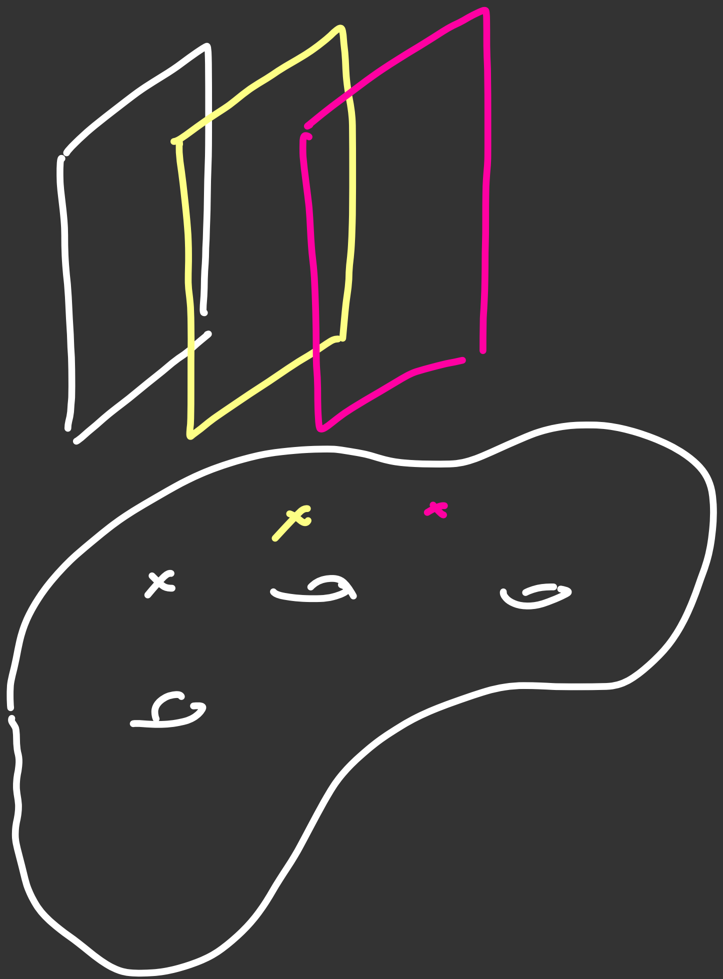

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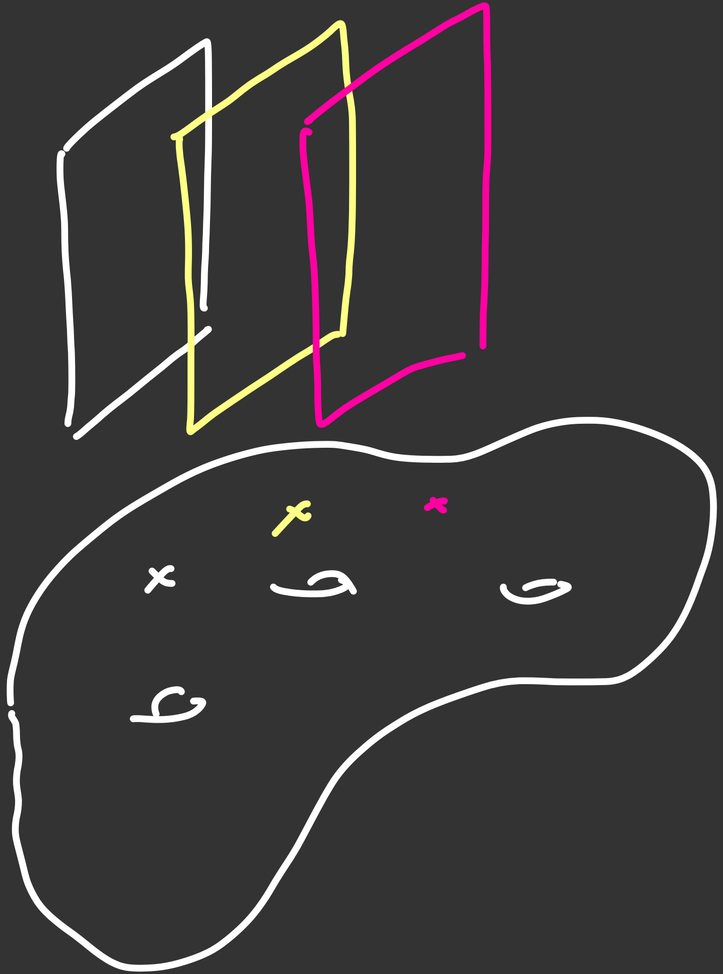

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$\sim$

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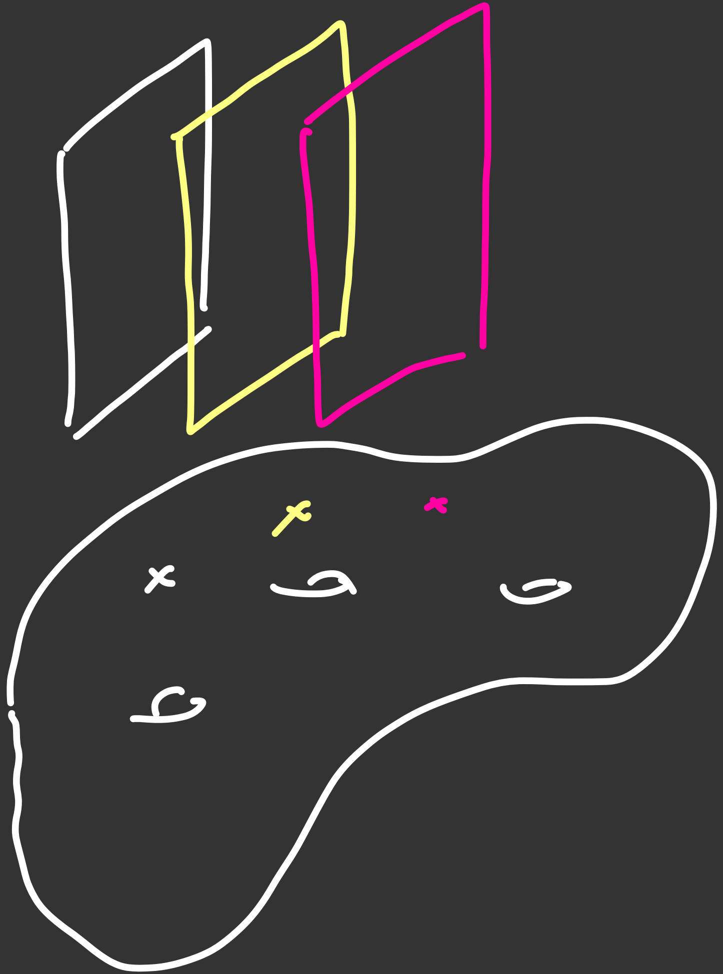

"SL_n"

$N_{n,L}$

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with :

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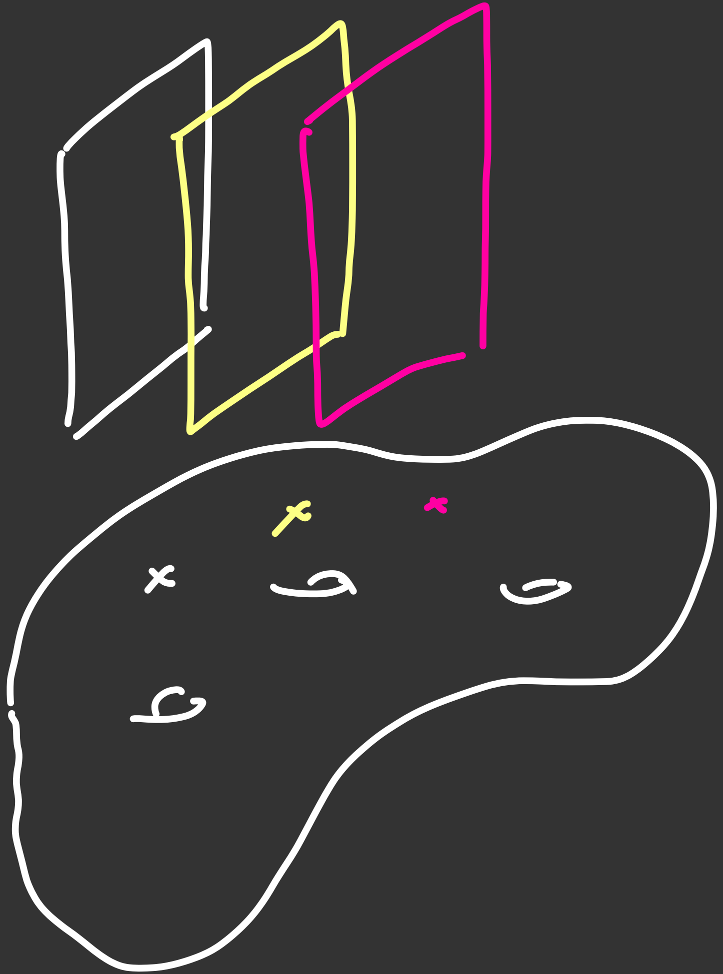

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$\Rightarrow \int_\gamma ch_k(\mathcal{U}) \in H^*(\widetilde{N}_{n,d})$

($\gamma \in H^*(C)$)

Künneth component

$$\langle \text{tautological classes} \rangle \subseteq H^*(\tilde{N}_{n,d})$$

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How about $H^*(N_{n,L})$?

- The embedding gives a restriction map:

$$H^*(\tilde{N}_{n,d}) \xrightarrow{\text{res}} H^*(N_{n,L})$$

- $\langle$ tautological classes in $H^*(N_{n,L}) \rangle = \mathrm{Im}(\text{res})$

Another description of $\operatorname{Im}\left(\mathrm{res} : H^{\circ}(\widetilde{N}_{n,d}) \longrightarrow H^{\circ}(N_{n,L})\right)$

in terms of a group action $\Gamma = \mathrm{Pic}^{\circ}(C)[n]$:

Another description of $\operatorname{Im}\left(\mathrm{res} : H^*(\tilde{N}_{n,d}) \longrightarrow H^*(N_{n,L})\right)$

in terms of a group action $\Gamma = \mathrm{Pic}^0(C)[n]$:

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- $\mathrm{Im}(\mathrm{res})$
  $= \langle$ tautological classes $\rangle$
  $= \underline{H^*(N_{n,L})^\Gamma}$

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$$\nearrow$$

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Hausel-Thaddeus "generalized" this to the moduli of Higgs bundles '02. from the viewpoint of mirror symmetry.

Moduli of Higgs bundles & Hitchin fibrations

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.. In perticular, the geometric objects related to the classical mechanics, ----, which are called Hitchin's completely integrable systems, were able to explain the fundamental lemma of Langlands. ...

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C, n, d, L as before...

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$\widetilde{N}_{n,d} \subset \widetilde{M}_{n,d}$

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$\forall$ sub-Higgs bundle $\mathcal{F}\subset\mathcal{E}$, $\theta(\mathcal{F})\subset\mathcal{E}\otimes\Omega^1_C$

has a smaller slope $\mu(\mathcal{F}) < \mu(\mathcal{E})$.

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Application in the P=W conj. (de Cataldo-Hausel-Migliorini '10)

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Weight filtration

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- de Cataldo-Maulik-S '18 $\forall n, g=2$

\+ more ...

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- de Cataldo-Maulik-S '19 $\quad \forall n, g=2$
- + more ...

strategy: match the locations of the tautological classes in both filtrations.

"SL_n" "GL_n"

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κ- isotypic part

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(b) The geometry which underlies B.C. Ngô's proof of the Fundamental Lemma of the Langlands program. '08

(c) The $P=W$ conjecture for SL_n

(True for H^*_{var} • SL_2 by de Cataldo-Hausel-Migliorini '02

• SL_p by de Cataldo-Maulik-S. '20)

$\nearrow$ variant part · $\kappa \neq 0$

\ p prime

Main Theorem $\Gamma = \mathrm{Pic}^0(C)[n]$, $\Gamma \times \Gamma \longrightarrow \mathbb{G}_m$

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Group theoretic data :

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deg m

cyclic

order m

torsion line bundle

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Thm (Maulik - S)

$\forall\ \kappa \in \hat{\Gamma}$, $\exists$ natural operators

$$\begin{tikzcd} H^*(\tilde{M}'_{n/m,d}) \arrow[rr, "p_\kappa"] \arrow[dr] & & H^{*+2r}(M_{n,L})_\kappa \\ & H^*(M^\gamma_{n,L})_\kappa \arrow[ur, "\sim", "\circledast"'] & \end{tikzcd}$$

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Hausel, Thaddeus $\rightsquigarrow$ "SL_n" "PGL_n- stringy"

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- $\circledast \rightsquigarrow H^{*}(M_{n,L}) = H^{*}(M_{n,L}/\Gamma) \oplus \bigoplus_{\gamma \neq 0} H^{*+2r_\gamma}(M^{\gamma}_{n,L})_\kappa$
- $P_k\left(P_k H^{*}(\widetilde{M}'_{n/m,d})\right) = P_{k+r} H^{*+2r}(M_{n,L})_\kappa$

Hitchin fibration

$$\begin{tikzcd}
\Gamma \arrow[r, phantom, "\circlearrowright"] & \mathcal{M}_{n,L} \arrow[d, "h"] \\
& \mathcal{A}
\end{tikzcd}$$

Hitchin fibration

$$\begin{tikzcd} \mathcal{M}_{n,L} \arrow[d, "h"] \arrow[loop left, "\Gamma"'] & (\mathcal{E}, \theta) \arrow[d, mapsto] \\ \mathcal{A} & \mathrm{char.poly}(\theta) = (\overset{0}{\overset{\parallel}{a_1}}, a_2, \dots, a_n) \in \overset{\mathcal{A}}{\overset{\parallel}{\bigoplus_{i=2}^{n} H^0(C, \Omega_C^{1 \otimes i})}} \end{tikzcd}$$

Hitchin fibration

$$a_i = \mathrm{tr}(\wedge^i \theta) \in H^0(C, \Omega_C^{1\otimes i})$$

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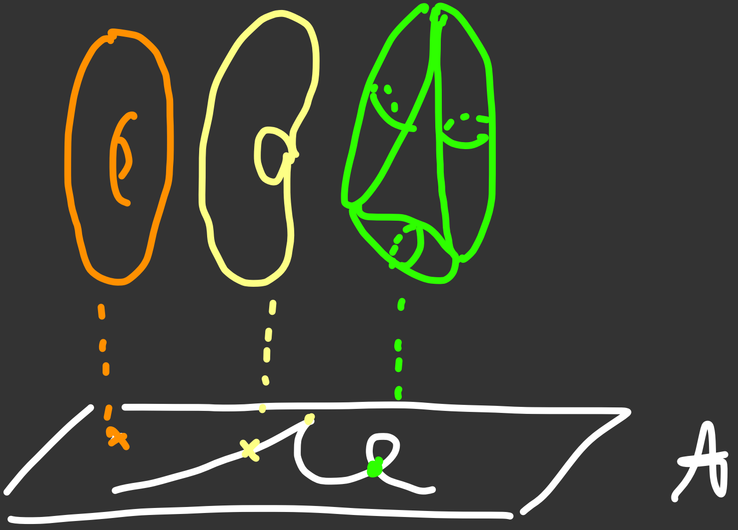
$$a_i = \mathrm{tr}(\wedge^i \theta) \in H^0(C, \Omega_C^{1 \otimes i})$$

- h is proper, Lagrangian



Hitchin fibration

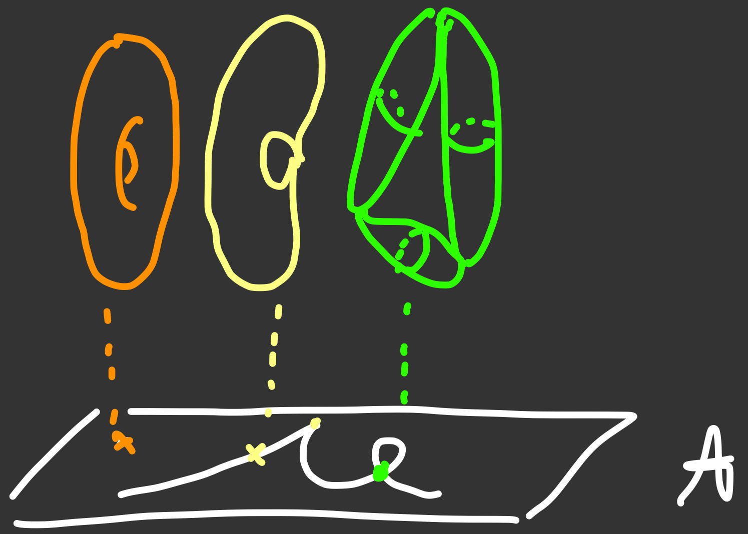
$$a_i = \mathrm{tr}(\Lambda^i \theta) \in H^0(C, \Omega_C^{1 \otimes i})$$

$$\begin{tikzcd} \Gamma \circlearrowright \mathcal{M}_{n,L} \arrow[d, "h"] & (\mathcal{E}, \theta) \arrow[d, mapsto] \\ \mathcal{A} & \mathrm{Char.poly}(\theta) = (\overset{0}{\overset{\shortparallel}{a_1}}, a_2, \dots, a_n) \in \bigoplus_{i=2}^{n} H^0(C, \Omega_C^{1 \otimes i}) \end{tikzcd}$$

- h is proper, Lagrangian

- κ-decomposition at sheaf level

$$Rh_* \underline{\mathbb{C}} = \bigoplus_{\kappa \in \hat{\Gamma}} (Rh_* \underline{\mathbb{C}})_\kappa$$



A key step in establishing the main theorem is :

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<u>Thm</u> (Maulik-S) $K \longleftrightarrow \gamma$ via Weil pairing

then $\exists$ natural operator

★ $$c_K : \left(Rh_* \underline{\mathbb{C}}\right)_\kappa \xrightarrow{\sim} \left(Rh^\gamma_* \underline{\mathbb{C}}\right)_\kappa [-2r]$$

$$\begin{tikzcd}
\mathcal{M}^\gamma_{n,L} \arrow[d, "h^\gamma"] & \\
\mathcal{A}^\gamma \arrow[r, hook, "\text{codim } r"] & \mathcal{A}
\end{tikzcd}$$

Proof of ☆ :

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Step1 : Examine ☆ over a general point of $\mathcal{A}^{\gamma}$

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Given:

$a \in \mathcal{A}^{\gamma}$ general

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Step1: Examine ☆ over a general point of $\mathcal{A}^{\heartsuit}$

(we deal with Hitchin fibers with mild singularities)

Data: $a \in \mathcal{A}^{\heartsuit}$ general $\rightsquigarrow$

$$\begin{tikzcd}
\widetilde{C}_a \arrow[r, "\text{norm.}"] \arrow[d] & C_a \arrow[d] \arrow[r, phantom, "\subset"] & T^*C \arrow[dl] \\
C' \arrow[r, "\pi"] & C &
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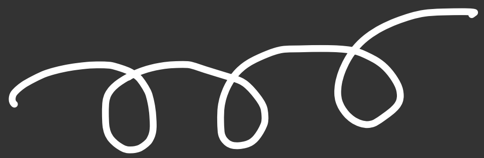
- C_a



- $\widetilde{C}_a$ smooth

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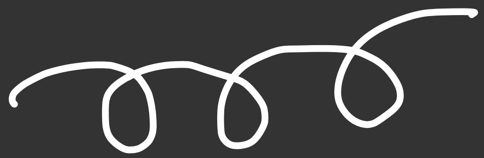
Spectral curves for h & h^γ

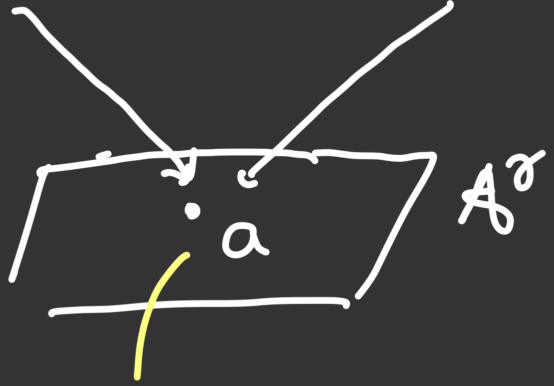
- C_a



- $\widetilde{C}_a$ smooth

$h^{-1}(a) = \overline{\mathrm{Prym}(C_a/C)}$

$h^{\sigma\,-1}(a) = \mathrm{Prym}(\tilde{C}_a/C)$

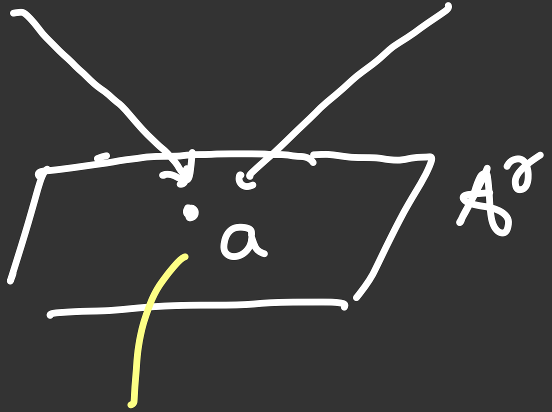


$(C_a, \tilde{C}_a)$

$\}$ Union of m copies of abelian varieties

$m = \deg(C' \to C)$

$h^{-1}(a) = \overline{\mathrm{Prym}(C_a/C)}$ $\qquad$ $h^{\gamma\,-1}(a) = \mathrm{Prym}(\tilde{C}_a/C)$



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- Affine part of $\mathrm{Prym}(C_a/C)$ has dim $=$ # nodes $= r$

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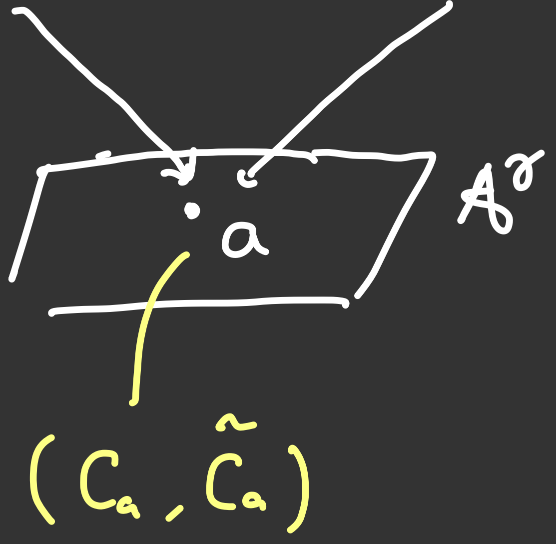

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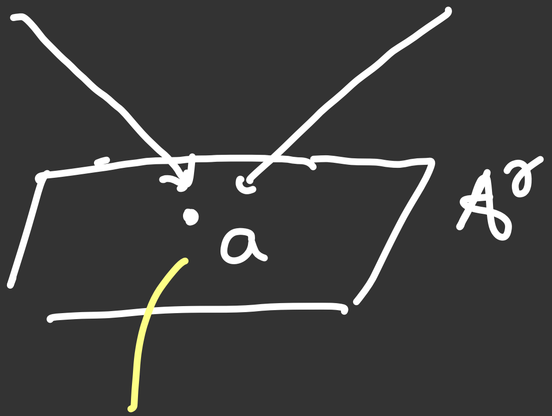

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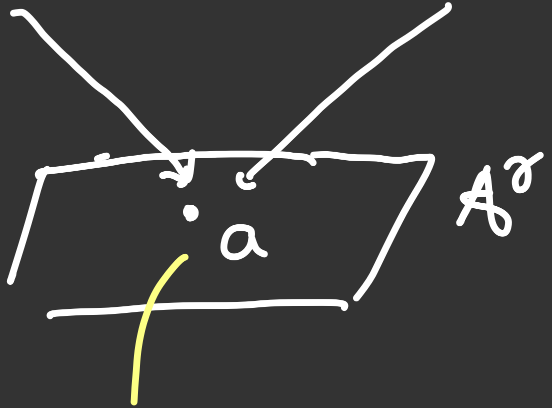
- $\overline{\mathrm{Graph}(u)} : H^*(h^{-1}(a)) \longrightarrow H^{*+2r}(h^{\gamma -1}(a))$

$$h^{-1}(a) = \overline{\mathrm{Prym}(C_a/C)} \qquad h^{\gamma\,-1}(a) = \mathrm{Prym}(\tilde{C}_a/C)$$

$\wr$

Union of m copies of abelian varieties



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Step 2 : Extend the correspondence to the boundary.

(ok, spread)

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(???)

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(???)

DIFFICULTY : $\tilde{C}_a$, C_a can have bad singularities and several components ....

Hard to compute " $H^{*}\left(\overline{\mathrm{Prym}(C_a/C)}\right)$ " etc ..

Ngô's idea (in his proof of the FL):

try to show that both sides of ☆ have "full supports"

$\bigoplus IC(L_\alpha)[-i_\alpha]$ $\quad$ L_α local system on an open subset of $\mathcal{A}^{\heartsuit}$

$\rightsquigarrow$ general fibers govern the "boundary".

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- de Cataldo-Heinloth-Migliorini:

  there are a lot of new supports whose generic pts lie in $\mathcal{A}^\heartsuit \setminus \mathcal{A}^{\heartsuit,\mathrm{ell}}$

What does it mean ?

$$(Rh_* \underline{\mathbb{C}})_x \quad \& \quad (Rh_*^\delta \underline{\mathbb{C}})_x[-2r]$$

Both have contributions NOT detected by general Hitchin fibers,

but they should still match (for some reason ...)

?

Thm (Chaudouard-Laumon, de Cataldo)

$\deg(K_C)$

$\parallel$

D effective divisor with $\deg D > 2g-2$

$(\Rightarrow D \ncong K_C)$

$$\begin{tikzcd}
\mathcal{M}^{D}_{n,L} = \{ (\mathcal{E}, \theta) \mid \theta : \mathcal{E} \to \mathcal{E} \otimes \mathcal{O}_C(D) \} \arrow[d] \\
\mathcal{A}^{D}
\end{tikzcd}$$

, then there are NO supports whose generic points $\in \mathcal{A}^{D} \setminus \mathcal{A}^{D,\mathrm{ell}}$

Finally, we solve our problem by connecting Hitchin fibrations associated with D (deg $> 2g-2$) & K_C .

③ ④

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$= \{ K_C\text{-Higgs bundle whose Higgs field has at worst a simple pole at } p \}$.

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$D = K_C + p$

$\mathcal{M}^{D}_{n,L} = \{$ D-Higgs bundles $\}$

$= \{$ K_C-Higgs bundle whose Higgs field has at worst a simple pole at p $\}$.

$$\begin{tikzcd}
\mathcal{M}_{n,L} \arrow[r, hook] \arrow[d] & \mathcal{M}^{D}_{n,L} \arrow[d] \\
\mathcal{A} \arrow[r, hook] & \mathcal{A}^{D}
\end{tikzcd}$$

(not Cartesian !!)

$$\begin{tikzcd}
\mathcal{M}_{n,L} \arrow[r, hook] \arrow[d] & \mathcal{M}^D_{n,L} \arrow[d] \\
A \arrow[r, hook] & A^D
\end{tikzcd}$$

We think of $H^{\cdot}(\mathcal{M}_{n,L})$ as a vanishing cohomology $H^{\cdot}(\mathcal{M}^D_{n,L}, \varphi_f)$.

$$\begin{tikzcd}
\mathcal{M}_{n,L} \arrow[r, hook] \arrow[d] & \mathcal{M}^{D}_{n,L} \arrow[d] \arrow[dr, "f"] & \\
\mathbb{A} \arrow[r, hook] & \mathbb{A}^{D} \arrow[r, "g"] & \mathbb{A}^{1}
\end{tikzcd}$$

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Thm (Maulik-S)

$\exists$ a regular function $f : \mathcal{M}^{D}_{n,L} \longrightarrow \mathbb{A}^{1}$ passing through $\mathbb{A}^{D}$.

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$\exists$ a regular function $f : \mathcal{M}^D_{n,L} \longrightarrow \mathbb{A}^1$ passing through $\mathbb{A}^D$.

(1) $\mathcal{M}_{n,L} = \mathrm{Crit}(f) \left(= \{ df = 0 \} \right)$

$$\begin{tikzcd}
\mathcal{M}_{n,L} \arrow[r, hook] \arrow[d] & \mathcal{M}^D_{n,L} \arrow[d] \arrow[rd, "f"] & \\
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We think of $H^\bullet(\mathcal{M}_{n,L})$ as a vanishing cohomology $H^\bullet(\mathcal{M}^D_{n,L}, \varphi_f)$.

Thm (Maulik - S)

$\exists$ a regular function $f : \mathcal{M}^D_{n,L} \longrightarrow \mathbb{A}^1$ passing through $\mathbb{A}^D$.

(1) $\mathcal{M}_{n,L} = \mathrm{Crit}(f) \left(= \{ df = 0 \} \right)$

(2) $\varphi_f \left(IC_{\mathcal{M}^D_{n,L}} \right) = IC_{\mathcal{M}_{n,L}}$

We went to show ☆ . . .

We want to show ☆ . . .

Show ☆ for $\deg(D) > 2g-2$

We want to show ☆ ...

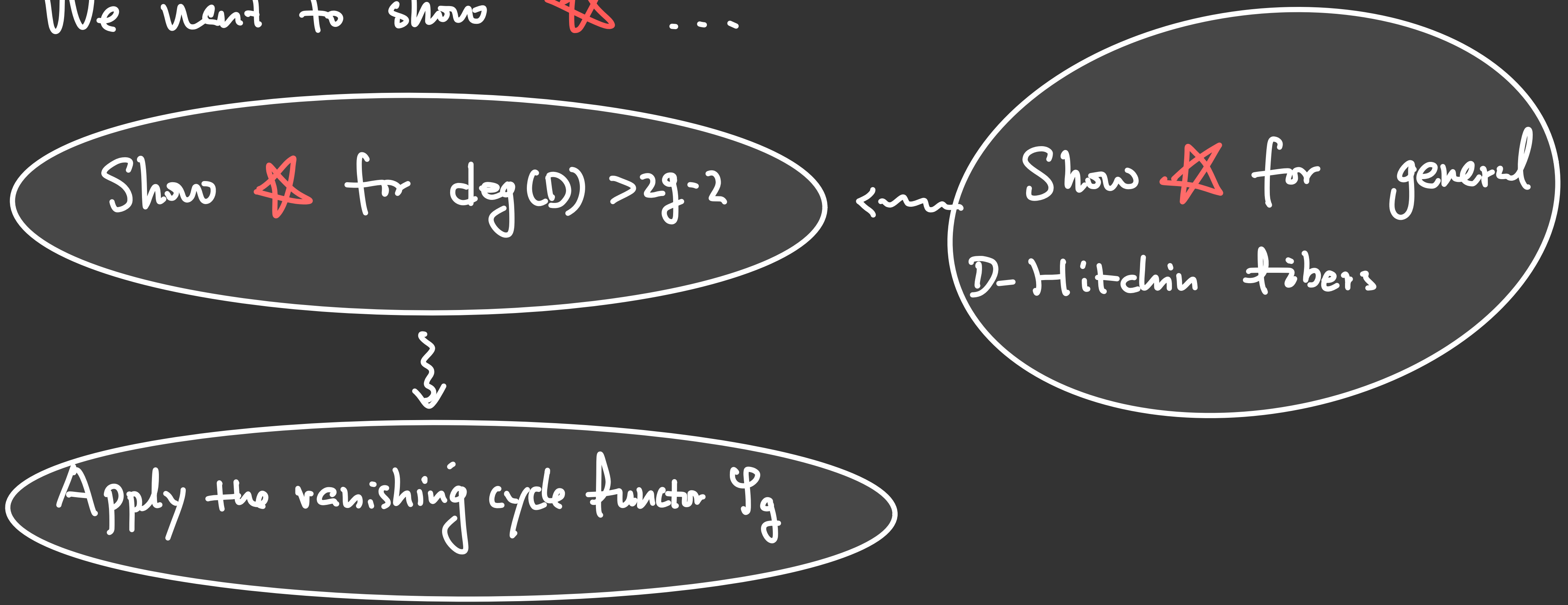
Show ☆ for $\deg(D) > 2g-2$

Show ☆ for general D-Hitchin fibers

mild singularities

$(C_a, \tilde{C}_a) = ($  $,$  $)$

We want to show ☆ . . .

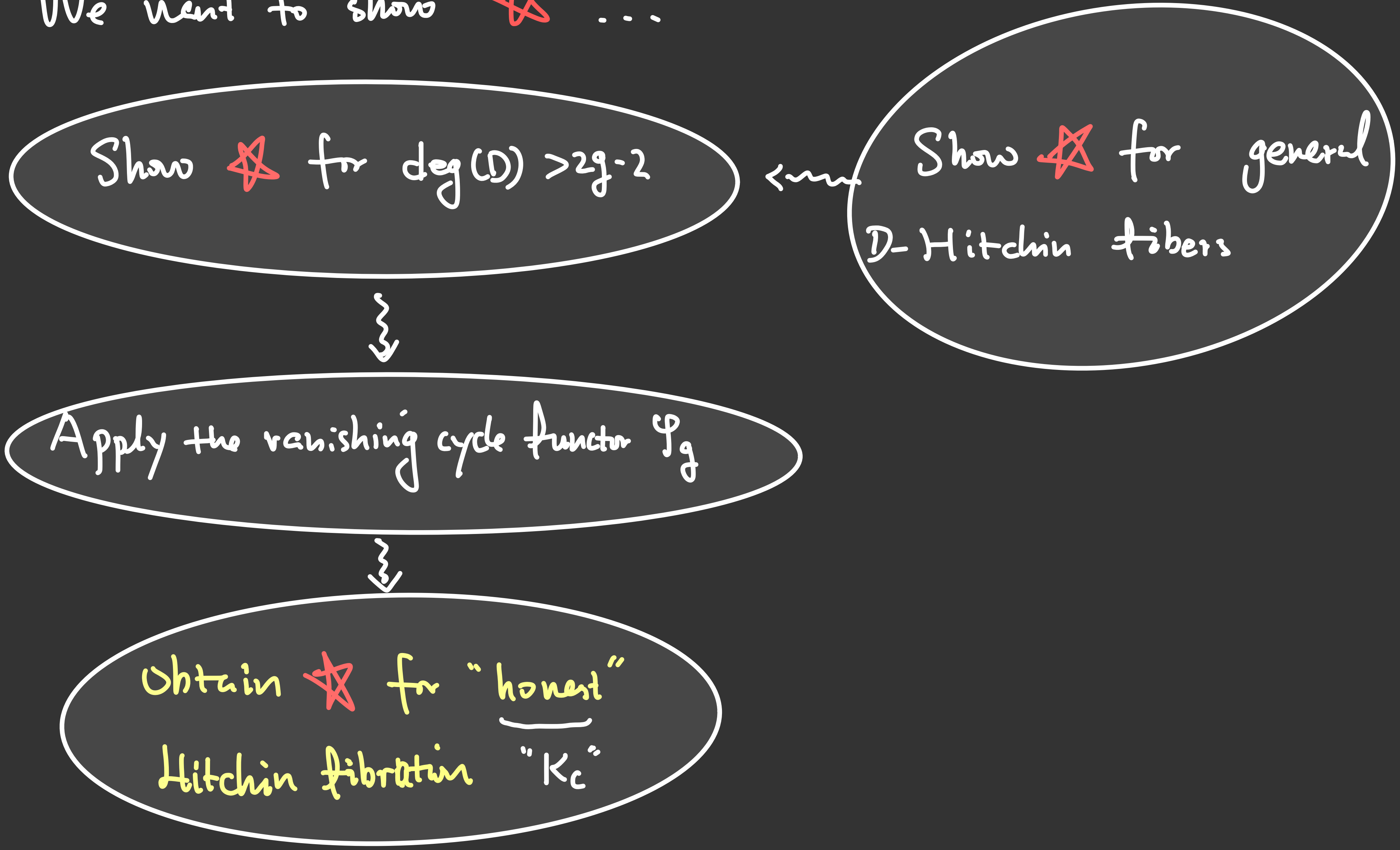


Show ☆ for deg(D) >2g-2

Show ☆ for general D-Hitchin fibers

Apply the vanishing cycle functor φ_g

We want to show ☆ . . .



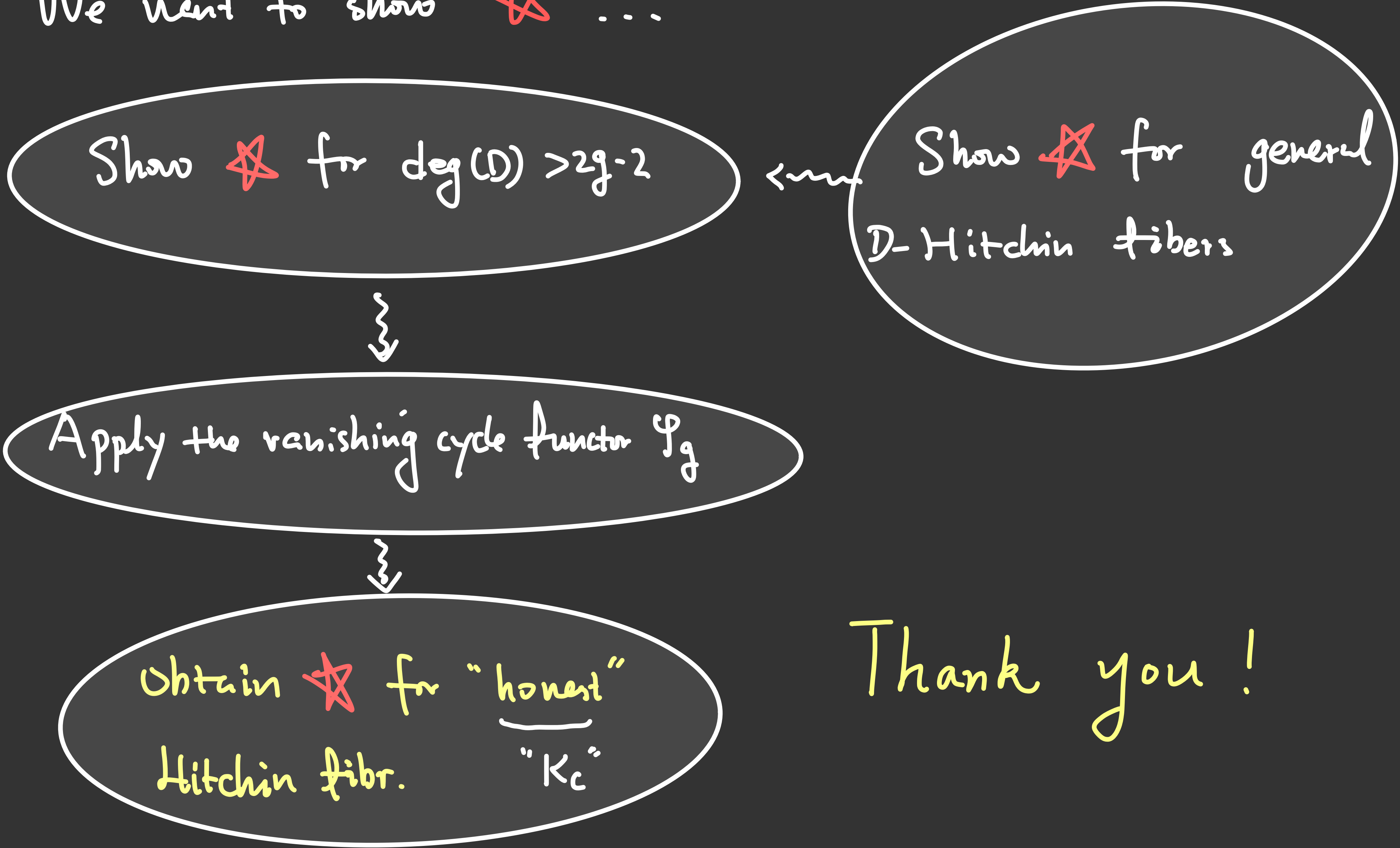
Show ☆ for general D-Hitchin fibers

Show ☆ for $\deg(D) > 2g-2$

Apply the vanishing cycle functor φ_g

Obtain ☆ for "honest" Hitchin fibration "K_c"

We want to show ☆ . . .



Show ☆ for deg(D) > 2g-2

Show ☆ for general D-Hitchin fibers

Apply the vanishing cycle functor φ_g

Obtain ☆ for "honest" Hitchin fibr.

"K_c"

Thank you!