JAMES COOK MATHEMATICAL NOTES

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For the cover picture we are indebted to Nathaniel Dance who painted a portrait of James Cook in 1776, to the United States Postal Service who reproduced it and to Professor J.D.E. Konhauser who sent some mint copies of the commemorative stamps. The original painting is in the Maritime Museum at Greenwich. It was just 200 years ago in May, June and July that Captain Cook was exploring the coast of Alaska.

AFTER DINNER MATHEMATICS (JCMN 14)

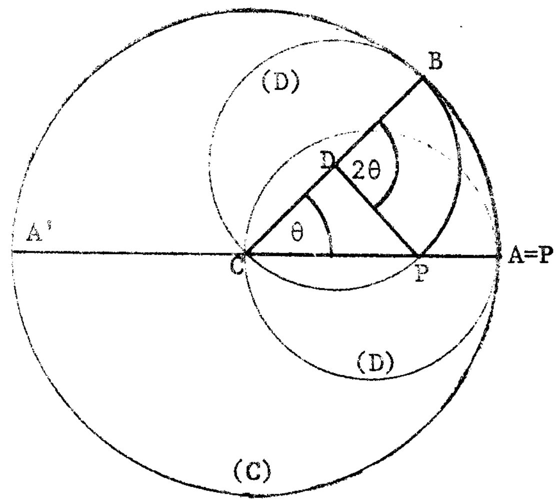
To prove the result, we use the following four facts:

(i) The kissing point of any 2 kissing circles lies on the join of their centres;

(ii) The length of an arc of a circle of radius r subtending an angle θ (in radians) at its centre is equal to $r\theta$;

(iii) The angles opposite equal sides of an isosceles triangle are equal; *and*

(iv) Any exterior angle of a triangle is equal to the sum of its opposite interior angles.



Let a point P of the rolling circle (D), with centre D and radius 1, be initially at a point A of the bigger circle (C), with centre C and radius 2, and B be the kissing point of (D) with (C) after (D) has rolled inside (C) for some time such that the radius CDB of (C) makes an angle θ with CA. It gives us the length of the arc AB of (C), as rolled over by (D), equal to 2θ, by (ii), that must be equal to the length of the arc BP of (D) with the present position of P such that $\angle BDP = 2\theta$ that fixes P on CA as the meet of (D) with CA other than C by virtue of (iii) - (iv). Thus the locus of P on (D) is the diameter AA' of (C) as (D) rolls inside (C) without slipping.

Sahib Ram Mandan.

STILL LUCKY? (JCMN 14)

The hypothesis put forward was as follows. Let $g(x)$ be continuous, strictly decreasing and convex in the interval in which it is defined, so that the inverse function $h(x)$ has the same properties, let $f(x)$ and $k(x)$ be the nearest integers to $g(x)$ and $h(x)$ respectively, then $f\{k(f(n))\} = f(n)$ for all integers n in the interval of definition. The answer is NO. Let $g(x) = 0{\cdot}7 + 5{\cdot}1/(x - 0{\cdot}2)$ then $h(x) = 0{\cdot}2 + 5{\cdot}1/(x - 0{\cdot}7)$ and it can be found that $f(3) = 3$, $k(3) = 2$, $f(2) = 4$, $k(4) = 2$.

BINOMIAL COEFFICIENTS (JCMN 12 and 13)

The question (a) from *C.A. Davis* was: if n is a positive integer and $u(k) = \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} \nu^k$ show that $u(k) = 0$ for $k = 0, 1, \dots n-1$, and find $u(n)$ and $u(n+1)$.

As we had three solutions in JCMN 13 it is gratifying to get this new one from *T. Ponnudurai* and *V. Laohakosol*.

$$(-1)^n u(k) = \sum_{\nu=0}^{n} (-1)^{n-\nu} \binom{n}{\nu} \nu^k$$

may be recognised as the n^{th} order finite difference at the origin for the function $f(x) = x^k$. For $n > k$ we know that it is zero. For the case $n = k$, we know that the n^{th} order finite difference for x^n is $n!$, and so $u(n) = (-1)^n n!$. Let $d(n)$ be the n^{th} order difference at the origin for the function x^{n+1}. Then $d(n)$ is the $(n-1)^{th}$ order difference at the origin for the function

$$(x+1)^{n+1} - x^{n+1} = (n+1)x^n + \tfrac{1}{2} n(n+1)x^{n-1} + \dots$$

(the later terms on the right may be discarded because their $(n-1)^{th}$ order differences are zero). The $(n-1)^{th}$ order difference for the second term on the right is easily written down, and so we may find

$$d(n) = (n+1)\ d(n-1) + \tfrac{1}{2}\ (n+1)!$$

To solve this recursion write it

$$d(n) - \tfrac{1}{2} n\,(n+1)! = (n+1)\{d(n-1) - \tfrac{1}{2}\ (n-1)n!\}$$

$$= \dots = (n+1)!\{d(1) - 1\}/2 = 0.$$

Therefore $$u(n+1) = (-1)^n d(n) = (-1)^n n(n+1)!/2.$$

A THEOREM OF WIELANDT AND NON-NEGATIVE MATRICES (JCMN 13 and 14)

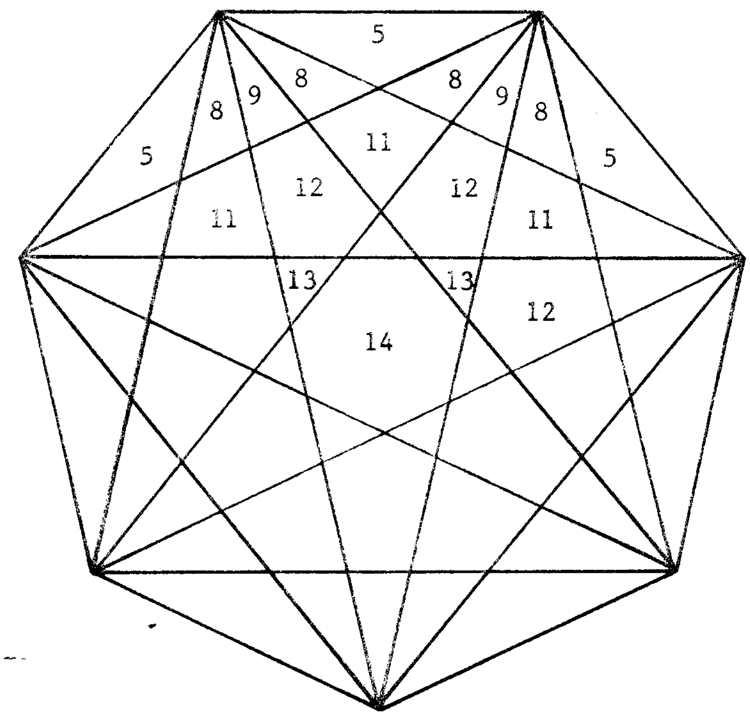
In the proof of these results in JCMN 14 use was implicitly made of the fact that if G^k is complete then G^s must be connected. This is true because any two nodes P and Q are joined by a path of length s in G^k, that is a path of length ks in G, or a path of length k in G^s.

C.J. Smyth

CONNECTION

A connected graph of n nodes has the property that for any two nodes P and Q there is an isomorphism mapping P on to Q. Prove that any two nodes are connected by at least two disjoint paths, that is paths with no nodes in common except their two ends.

COVERING WITH TRIANGLES (1)



The centre of a regular (2k+1)gon is in how many triangles made from its vertices? The answer is a simple formula but can you give a simple reason for it? I can't. The picture above illustrates the case $k = 3$, the figures in each region of the heptagon give the number of triangles covering it.

M.J.C. Baker

STILL MORE TRIGONOMETRY

For n an odd positive integer, show that

$$\sum_{j=1}^{n-1} \operatorname{cosec}^2 \left(\frac{2\pi j}{n}\right) = \left(\frac{n^2-1}{3}\right)$$

C.J. Smyth

DOUBLY STOCHASTIC MATRICES

A real square matrix is called doubly stochastic if its elements are non-negative and the sum of those in any row or any column is 1. Prove or disprove that every such matrix is reducible to diagonal form by a similarity transformation (A to $X^{-1}AX$ where X is non-singular).

H. Kestelman

AN ELECTRICAL CIRCUIT

The lattice points in three dimensions, that is points whose coordinates x, y and z are all integers, are regarded as electrical terminals. Every pair of adjacent points, that is points at unit distance apart, is connected by a one-ohm resistor. What voltage is needed to send a one-ampere current in to the circuit at (0, 0, 0) and out at (1, 0, 0)? For pure mathematicians the question may be put as follows. Find $v(0, 0, 0) - v(1, 0, 0)$ if v is such that $6v(k, m, n) - v(k-1, m, n) - v(k+1, m, n) - v(k, m+1, n) - v(k, m-1, n) - v(k, m, n+1) - v(k, m, n-1)$ is equal to 1 if $(k, m, n) = (0, 0, 0)$, equal to -1 if $(k, m, n) = (1, 0, 0)$ and is zero otherwise, and where v is bounded at infinity. Our undergraduate readers might like to start with the two-dimensional version of the problem.

RANDOM FIGURES

Let x and y be independent random variables, each having a Gaussian distribution with mean zero and variance one. The pair (x, y) can be regarded as specifying a point in the plane. A sample of n such points determines a convex hull of area $A(n)$. Can you find the distribution of $A(n)$? That is hard, but start with $A(3)$, and see what you can say about $A(n)$ for large n, an asymptotic expression perhaps, or just an estimate for the order of magnitude.

J.B. Parker

A PARTY GAME - THE STARTER CAN WIN (JCMN 14)

In JCMN14 a game was described in which players take it in turns to add either O or X to a sequence, with the object of not making two identical adjacent segments of length two or more. The first player unable to comply with this condition loses a point.

There are only a finite number of possible sequences before some player must lose a point, and no infinite sequences. In fact a simple computer tree search reveals that there are precisely 72 such finite sequences (starting with O, as we can clearly assume). The largest is OXOOXXOOOXXXOOXXOX (length 18), and the shortest OXXOX.

As a curiosity, note that if the longest sequence is reversed, and O and X interchanged, we obtain the longest sequence again. This is not very surprising because ... (no, I leave the reasoning to you!).

In the two-person game the player who starts can win the first point by the simple strategy of always playing O. He is then never forced to play X. If the other player doesn't make any unnecessary losing moves, the only two possible games are OOOXOOO and OXOOOXO, followed by either X or O.

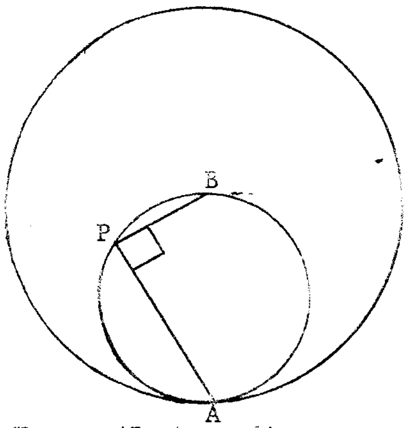
C.J. Smyth

The longest sequence, as given above, was also found by *J.B. Parker*, but without the help of a computer.

AFTER DINNER MATHEMATICS (JCMN 14)

When I was a lad and went to school,
We used a compass, a pencil and rule
Er, Euclid was for us a breeze
With theorems, props and Q.E.D's.
But now new maths has hit the kids
the old geom has hit the skids.
So how the hell can I work out
what a circle in a circle is all about,
when all JC will let me use
is 'school geometry' - I ask yous!

[Old maths. proof]



Point P is rotating about A (assuming no slipping) and hence it is moving in the direction PB where

$\angle APB = \pi/2$.

"The diameter of a circle subtends an angle of $\pi/2$ at any point on the circumference".

Hence AB is a diameter of the small circle.

$\therefore$ B is the centre of the large circle.

$\therefore$ PB is on a diameter of the large circle.

Q.E.D.

R.B. Potts

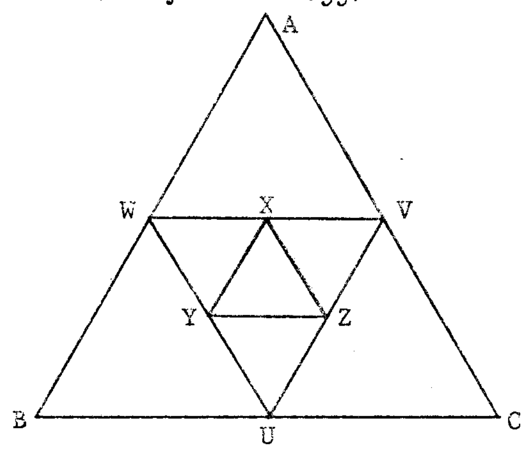
Other solutions came from *H.O. Davies* and *V. Laohakosol*.

EDITORIAL POLICY

In any circle there are those who find it hard to write in prose, so always trying to be fair we put some verse in here and there. And if your mathematics rhymes as people find it does at times your theorems will be better than those common ones that merely scan.

A NEST OF TRIANGLES (JCMN 13)

by Sahib Ram Mandan (Indian Institute of Technology)



We follow an example in Coxeter's book, "*The Real Projective Plane*" pp 203-204.

Take U V W as triangle of reference, X = (0, 1, p), Y = (q, 0, 1), Z = (1, r, 0), and the coordinates of the lines BC, CA and AB as (0, 1, u), (v, 0, 1) and (1, w, 0). These three lines form a triangle unless uvw = -1. The conditions of perspectivity of the three pairs of triangles

UVW, XYZ; UVW, ABC; XYZ, ABC

are pqr = 1, uvw = 1 and pqr = uvw respectively.

This implies the given result that if two of the pairs are perspective then so is the third.

Calling a triangle XYZ cevian to another triangle UVW when the former is inscribed in the latter and the two triangles are in perspective, and then calling UVW anticevian to XYZ, Guinand's theorem may be stated as follows.

Any cevian triangle of a given triangle is perspective to any anticevian triangle.

THE FRIENDSHIP THEOREM

Suppose that in some society every two people have exactly one common friend. Prove (or disprove) that there must be somebody who is everybody's friend.

FINDING AN INTEGRAL

In Jeffreys and Jeffreys the associated Legendre function is defined, when m and n are positive integers with m < n, by

$$P_n^m(x) = 2^{-n}(n-m)!\ (m!n!)^{-1}\,(1-x^2)^{m/2}(d/dx)^{m+n}(x^2-1)^n$$

Other definitions are also used (see Whittaker and Watson, or Hobson's *Spherical and Ellipsoidal Harmonics*) but let's stick to this one.

Evaluate $\int_{-1}^{1} P_n^m(x)\,dx$.

J.B. Parker

DOES IT CONVERGE?

For $x > 0$ define the sequence of numbers $S_n(x)$ by

$$S_n(x) = (1 + x(1 + x^2(1 + x^3(\ldots(1 + x^n)^{\frac{1}{2}}\ldots)^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}}$$

For what x does it converge? What can you say about the limit?

V. Laohakosol.

DIOPHANTINE PROBLEM

Find all solutions of $(m+n\omega)^q = r$, where m, n, q, and r are integers, $q > 1$, $mn \neq 0$, $\omega^3 = 1$ and $\omega \neq 1$.

C.J. Smyth

INTELLIGENCE TEST

Why is the knob controlling the outlet of water from my (imported) washing machine marked dwnd ?

RANK ONE MATRICES (JCMN 13 and 14)

The question from JCMN 13 was "The square matrix A has rank one. Show that it is similar to a diagonal matrix if and only if its square is non-zero." The first of the solutions in JCMN 14 was as follows.

A must be of the form uv' where u and v are non-zero column vectors. The only solution of the eigenvalue equation $uv'x = Ax = \lambda x$ is $\lambda = v'u$ with eigenvector u. Since $A \neq 0$, if it is similar to a diagonal matrix then $v'u \neq 0$ and therefore $A^2 = uv'uv' = v'uA \neq 0$. Conversely if $A^2 \neq 0$ then u is not orthogonal to v. Take a new basis with u as first element and the others all orthogonal to v, then A becomes diagonal, with only the top left element non-zero.

R.B. Potts writes as follows. I found the published answer on Rank One Matrices unconvincing. It seems to imply that u and v are real vectors, personally I don't like using the word 'orthogonal' for $u'v = 0$ when u and v may be complex. There is of course the difficulty that a vector u may be orthogonal to itself, so that Gram-Schmidt is not obvious.

Perhaps other readers share these doubts of Professor Potts. One remedy might be to replace v by $\bar{v}$ in the reasoning above, putting $A = u\bar{v}'$, so that the eigenvalue is $\bar{v}'u$. Then if the word orthogonal is understood in the Hermitian sense there is no difficulty over Gram-Schmidt orthogonalization. Alternatively we could be obstinate and insist that if $u'v \neq 0$ in complex n-dimensional space there is a basis with u as first element and the other $n-1$ elements chosen from the x for which $x'v = 0$. However we the editor agree on the unwisdom of using the word orthogonal to mean $x'y = 0$ in complex space, and we will try to avoid it.

THIRD TIME LUCKY (JCMN 13, 14)

For this problem (related to pressing the $1/x$ key repeatedly on a calculator), we had $10^9 \le n \le 10^{10}$, $f(n)$ nearest integer to $g(n) = 10^{19}/n$. We showed that $f(f(f(n))) = f(n)$ and, for $n < \theta = 10^{19/2}$, $f(f(n)) = n$ in the last issue. We shall now find all possible values of $f(f(n)) - n$, and the least n with $f(f(n)) \neq n$. We have $g(n) - \frac{1}{2} < f(n) < g(n)+\frac{1}{2}$, so

$$g(g(n)+\tfrac{1}{2}) - \tfrac{1}{2} < f(f(n)) < g(g(n)-\tfrac{1}{2}) + \tfrac{1}{2},$$

which gives $|f(f(n)) - n| < \frac{1}{2}(1+n^2/10^{19}) + \varepsilon$, where $|\varepsilon| < 10^{-8}$.

Since $1 + n^2/10^{19} \le 11$, $f(f(n)) - n$ has the possible values $0,\pm1,\pm2,\pm3,\pm4,\pm5$.

All these values can occur. For instance $f(f(999999999d)) = 9999999990$ for digit d = 5, 4, 3, 2, 1, 0, and $f(f(987654321d)) = 9876543220$ for d = 5, 6, 7, 8, 9.

To find the least n: $f(f(n)) \neq n$, first note that for $n = n_0 + k$, $g(n) - g(n+1) \doteq 1 - 2k/\theta$ for $k \ll \theta$. Here $n_0 = [\theta] = 3162277660$. So $f(n) = f(n+1) + 1$ until there is an n_1 : $m + \frac{1}{2} > g(n_1) > g(n_1+1) > m - \frac{1}{2}$ for some integer m. For then we have $f(n_0+k) = n_0 - k$ and $f(f(n_0+k)) = n_0 + k$ ($n_0 + k = n_0, n_0 + 1, \ldots, n_1$) and $f(n_1+1) = f(n_1)$, $f(f(n_1+1)) = n$. Hence we must find the first $n = n_1 + 1$ with fractional part of $g(n) > \frac{1}{2}$.

Now $g(n_0+k) = \theta/(1+(k-\delta)/\theta)$, where $\delta = \theta - n_0 = 0{\cdot}168379\ldots$

$$= n_0 - k + 2\delta + (k-\delta)^2/\theta + \varepsilon, \text{ where } |\varepsilon| < k^3/10^{19}.$$

Then $2\delta + (k-\delta)^2/\theta + \varepsilon = \frac{1}{2}$ gives $k \doteq \delta + \sqrt{\theta(\frac{1}{2}-2\delta)} = 22720{\cdot}56$,

and indeed $f(n_0 + 22720) = f(n_0 + 22721)$, $f(f(n_0 + 22721)) = n_0 + 22720$.

Hence the least n : $f(f(n)) \neq n$ is $n = 3162300381$. (The next one is 3162338311, corresponding to $k \doteq \delta + \sqrt{\theta(\frac{3}{2}-2\delta)} = 60650{\cdot}76$).

C.J. Smyth

A MACROSCOPIC THEORY OF FROTH

Suppose that the bubbles are small and that their arrangement is in a pattern that is isotropic and can be described by a single parameter b, the mean bubble diameter. We ignore gravity and inertia, the only forces are surface tension and air pressure. On the macroscopic scale let the vector field u be the velocity of the air, and v of the bubbles, they will in general be unequal because the air can diffuse through a bubble surface. By considering the forces at the boundary of an arbitrary rectangular region it follows that the air pressure p, the bubble size b and the surface tension T are connected by a relation $p - kT/b =$ constant, where k is some constant depending on the microstructure. The diffusion of air through the bubbles is at velocity $u-v$ which must be equal to $-\lambda b$ grad p, for some λ depending on the microstructure and on the diffusion properties. Eliminating the pressure p between the two equations above, we find $v = u + \lambda b \,\mathrm{grad}(kT/b)$, and putting $b = \exp(-\beta)$ this can be written $v = u + \lambda kT \,\mathrm{grad}\, \beta$. Now suppose that there is conservation of bubbles, the number per unit volume varies as b^{-3} or $\exp 3\beta$ and the conservation equation is

$$(\partial/\partial t) \exp 3\beta + \mathrm{div}(v \exp 3\beta) = 0$$

or $$\exp(3\beta)\,(3\partial\beta/\partial t + \mathrm{div}\, v + 3v.\mathrm{grad}\,\beta) = 0$$

or $$3\, D\beta/Dt + \mathrm{div}\, v = 0$$

where D/Dt indicates rate of change following the bubbles. The equation found earlier for v gives

$$D\beta/Dt + (\lambda kT/3)\,\mathrm{div}\,\mathrm{grad}\,\beta = 0\ .$$

This equation tells us that where the bubbles are biggest they will be getting bigger and where they are smallest they will be getting smaller, so that we should not expect a uniform froth at rest to be stable. What actually happens when a bubble gets very small is that it vanishes, making its neighbours a little bigger. Therefore we should not make the assumption that bubbles are conserved. Some change in the model is required for a good theory.

B.C. Rennie

Your editor would like to hear from you anything connected with mathematics or with James Cook.

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