JAMES COOK MATHEMATICAL NOTES

Issue No. 22, Vol. 2

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An unknown artist's impression of one of the *Endeavour's* company and a New Zealand Maori haggling over the barter-price of a lobster.

COOK AND CANADA

G.F. Duff

There are some Captain Cook connections with Ontario, dating from the 1758-59 period. At that time James Cook was serving as a Master (the rank no longer exists in the Navy, he was the technical expert on navigation) in H.M.S. Pembroke under Captain John Simcoe. Cook learned surveying from Samuel Holland, a military engineer serving with the Army, who was later the Surveyor-General of Canada and mapped much of the area around Toronto about 1791. The Holland River and Holland Marsh (a very fertile vegetable garden) are named after him, this is not widely known because the present market gardeners are Dutch. Also John Simcoe (the Captain of Cook's ship) was the father of John Graves Simcoe, the first Lieut-Governor of Upper Canada in the 1790's. In the first year (1793-94) of York (which became Toronto in 1834) the vice-regal house was a large decorated tent or portable building said to have been bought from Cook's estate by the Simcoes. Lake Simcoe, into which the Holland River flows, is named after them.

COMBINATORIAL NUMBER THEORY

M. Sved

The following ideas come from work of P. Erdös.

Given N and given any prime p, suppose that a random sample of M elements is chosen out of the first N positive integers. How large does M have to be to ensure that two of the chosen numbers have the ratio p? It is known that the answer is asymptotically $\frac{pN}{p+1}$ for large N. How large must M be to ensure that some two of the chosen numbers have ratio either 2 or 3? Nothing seems to be known.

If k of the first N positive integers are chosen at random, what is the probability that they include a pair $\{a, 2a\}$?

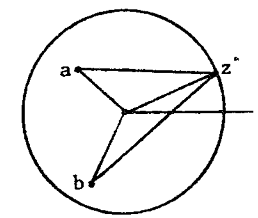
TWO POINTS IN A CIRCLE (JCMN 21, Vol.2, p.66)

G. Szekeres

Given two points, A and B, inside a circle with centre C, find P on the circle so that CP bisects the angle APB.

This is sometimes called the "circular billiard table problem" for it can be expressed in terms of a cue ball A and an object ball B on a circular billiard table with perfectly elastic cushion. The existence of such a point P is easily proved, but the interesting thing is that there is no Euclidean construction for P.

Take the circle to be the unit circle in the complex plane, and the points A and B to be the complex numbers a and b, the picture being rotated if necessary so that ab is real. The condition that P (represented by z with $|z| = 1$) makes PC bisect APB is



$$(z - a)(z - b) = z^2|z - a|\,|z - b|.$$

Squaring gives $(z - a)(z - b) = z^4(\bar{z} - \bar{a})(\bar{z} - \bar{b}) = z^2(1 - \bar{a}z)(1 - \bar{b}z)$.

Remembering that $ab = \bar{a}\bar{b}$, this can be put in the form

$$z^{-2}ab - z^{-1}(a + b) = z^2ab - z(\bar{a} + \bar{b})$$

which is the condition that $z^2ab - z(\bar{a} + \bar{b})$ be real.

Let $z = \exp i\phi$ and $a = (1/r)\exp i\theta$ and $b = (1/R)\exp -i\theta$

where r and R both > 1. The condition gives

$$\sin 2\phi = r\sin(\phi + \theta) + R\sin(\phi - \theta)$$

$$\text{or} \quad \sin\phi\cos\phi = \alpha\sin\phi - \beta\cos\phi \qquad (1)$$

where $2\alpha = (r + R)\cos\theta$ and $2\beta = (R - r)\sin\theta$.

These are given quantities, and we are supposed to construct an angle ϕ satisfying equation (1). This, in general, is impossible.

To see this write down the equation for $x = \cos\phi$:

$$x^4 - 2\alpha x^3 + (\alpha^2 + \beta^2 - 1)x^2 + 2\alpha x - \alpha^2 = 0 \qquad (2)$$

Because of the squaring only two of the solutions of (2) will yield a solution to the original problem, but in general no solution of (2) is constructible by Euclidean methods. Only in very special cases (such as $r = R$ or $\theta = 0$ or $\pi/2$) will the solution of (2) be reducible to two or three quadratics. To take a special example, let $r = 2$, $R = 6$, $\theta = \pi/3$; then $\alpha = 2$, $\beta = \sqrt{3}$ and the equation becomes $x^4 - 4x^3 + 6x^2 + 4x - 4 = 0$, and with the substitution $x = y + 1$ it becomes

$$y^4 + 8y + 3 = 0.$$

This is typically an equation whose Galois group is S_4, that is one which requires a genuine cubic for its solution. Those readers still engaged in teaching complex numbers might find all this quite instructive.

TRIANGLES FROM FOUR LINES

C.F. Moppert

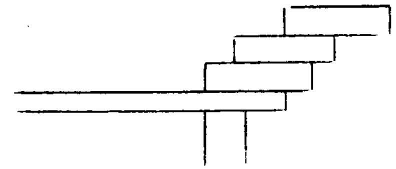
Given any four lines in the Euclidean plane, any three of them form a triangle. Each triangle has an orthocentre, the intersection of the altitudes. Are the four orthocentres collinear?

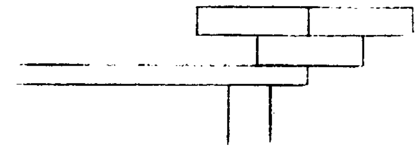
BINOMIAL IDENTITY NUMBER NINE

J.B. Parker

$$\sum_{s=0}^{\infty} (b/2)^s \binom{2s}{s}/(s!) = e^b \sum_{s=0}^{\infty} (b/2)^{2s}(s!)^{-2}$$

PILES OF BRICKS





Our contemporary "Parabola" from the University of New South Wales has been discussing how to put bricks on the edge of a table to give the maximum overhang, with particular reference to the method shown in the left hand picture above, which gives

$$\frac{1}{2}(1 + \frac{1}{2} + \frac{1}{3} + \dots \frac{1}{n})$$

brick lengths of overhang with n bricks, for instance 11/12 using three bricks. There are better ways, for instance the picture on the right shows how to get one length overhang with three bricks.

What is the best that can be done with 4 or 5 or 6 ...? or asymptotically with n for large n?

ZEROS OF A DETERMINANT

Show that if $a_1, \dots a_n$ are unequal and all positive then the determinant

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ a_1 & a_2 & \dots & a_n \\ a_1^2 & a_2^2 & \dots & a_n^2 \\ \dots & \dots & \dots & \dots \\ a_1^{n-2} & a_2^{n-2} & \dots & a_n^{n-2} \\ a_1^r & a_2^r & \dots & a_n^r \end{vmatrix}$$

as a function of the real variable r has no zero except $r = 0, 1, \dots r - 2$.

VISITING ON A CIRCLE (2)

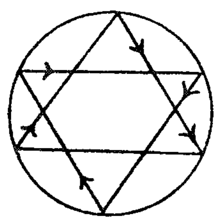
F.J.M. Salzborn

The question asked in the last issue (page 67) has trivially the answer NO. For example the six numbers (2, 2, 2, 2, 2, 2) have their sum a multiple of six, but there is no way of visiting the m = 6 points on a trip that has each stop a distance two from the previous stop. The problem is better modified as follows.

Consider a circle with m points (labelled 1, 2, ... m) equally spaced, the distance between adjacent points being one, travelling round the circle. We consider round trips $i_1 \to i_2 \to \dots i_n \to i_{n+1} = i_1$ going clockwise any number of times round the circle. Let $d(i_k, i_{k+1})$ be the distance between two consecutive visiting points. Clearly $\sum d(i_k, i_{k+1})$ is a multiple of m.

Problem: Given m positive integers $a_1, \dots a_m$, all < m, with their sum a multiple of m, can one find a set of trips visiting each point of the circle exactly once and such that $\{a_1, \dots a_m\}$ is the set of distances travelled (not necessarily in the same order)?

For example, if m = 6 then {2, 2, 2, 2, 2, 2} requires two trips, one round the even-numbered points and one round the odd-numbered points.



BINOMIAL IDENTITY NUMBER TEN

$$n^2\binom{2n}{n}\sum_{r=0}^{n-1}\frac{1}{(2r+1)(2n-2r-1)}\binom{2r}{r}\binom{2n-2r-2}{n-r-1} = 2^{4n-3}$$

BINOMIAL IDENTITY NUMBER EIGHT (JCMN 21, Vol.2, p.78)

$$\sum_{r=0}^{n} 1\Big/\binom{n}{r} = (n+1)\sum_{r=0}^{n} 2^{-r}/(n-r+1)$$

Similar solutions from *A.P. Guinand* and *J.B. Parker*:

Denote the left hand side by S_n and the right hand side by T_n.

$$S_n = (1/n!)\sum_0^n r!(n-r)!$$

$$= 1 + (1/n!)\sum_1^n (r-1)!(n-r)!((n+1)-(n-r+1))$$

$$= 1 + ((n+1)/n!)\sum_0^{n-1} s!(n-s-1)! - (1/n!)\sum_0^{n-1} s!(n-s)!$$

$$S_n = 1 + \frac{n+1}{n}S_{n-1} - (S_n - 1)$$

$$S_n = 1 + \frac{n+1}{2n}S_{n-1}$$

Now take the other side

$$2^nT_n/(n+1) = \frac{2^n}{n+1} + \frac{2^{n-1}}{n} + \dots + \frac{4}{3} + \frac{2}{2} + \frac{1}{1}$$

$$= \frac{2^n}{n+1} + \frac{2^{n-1}}{n}T_{n-1}$$

$$T_n = 1 + \frac{n+1}{2n}T_{n-1}$$

The two sequences satisfy the same recurrence relation and have the same initial value, $S_0 = T_0 = 1$, and so they must be equal.

QUOTATION CORNER (5)

"Mathematics are most important madam! I don't want to have you like our silly ladies. Get used to it and you'll like it. It will drive all the nonsense out of your head." —— Prince Nicholas Bolkonski to his daughter in Tolstoy's *War and Peace*, book 1, chapter 5.

PRODUCT OF SINES (JCMN 21, Vol.2, p.79)

$$\sin n\theta = 2^{n-1} \sin\theta \ \sin(\theta + \pi/n) \ \ldots \ \sin(\theta + (n-1)\pi/n)$$

Proof from *J.B. Parker*

If $0 < \theta < \pi/n$ all factors are positive and the logarithm (or more precisely one value of the complex logarithm) of the RHS is

$$(n-1)\ln 2 + i\textstyle\sum_0^{n-1}(\theta + \pi j/n) - n\ \ln(2i) + \sum_0^{n-1} \ln(1 - \exp - 2i(\theta + \pi j/n))$$

$$= -\ln 2 + in\theta + \tfrac{1}{2}i\pi(n-1) - ni\pi/2 - \textstyle\sum_1^{\infty} \frac{1}{s} \exp(-2i\theta s)\sum_0^{n-1}\exp-(2\pi ijs/n)$$

The last sum is zero unless s is a multiple of n, in which case the sum is n.

$$\ln \text{RHS} = -\ln 2 + in\theta - i\pi/2 - \textstyle\sum_1^{\infty} \frac{1}{ns} \exp(-2i\theta ns)n$$

$$= -\ln 2 + in\theta - i\pi/2 + \ln(1 - \exp - 2i\theta n)$$

$$= \ln \sin n\theta$$

Proof from *E.C.G. Sudarshan*

$$2(i^n)(\text{RHS}) = \exp\ i(n\theta + (n-1)\pi/2)\ \exp\ i(n\theta + (n-1)\pi/2)$$

$$(1 - \exp 2i\theta)(1 - \exp - 2i(\theta + \pi/n)) \ \ldots \ (1 - \exp - 2i(\theta + (n-1)\pi/n))$$

but $(1 - z)(1 - wz) \ldots (1 - w^{n-1}z) = 1 - z^n$ when $w = \exp - 2i\pi/n$

Therefore $\text{RHS} = \frac{1}{2} e^{in\theta} i^{-1}(1 - \exp - 2in\theta) = \sin n\theta$

Proof by addition rules

The right hand side is a trigonometric polynomial of order n, denote it by $f(\theta)$, and because $f(\theta + \pi/n) = -f(\theta)$ the function f is of period $2\pi/n$, and f must be of the form $B \cos n\theta + C \sin n\theta$. The product of the n factors can in $n - 1$ steps be reduced to a sum by applying the addition rules. For instance the first two factors are

dealt with by putting

$$2 \sin\theta \ \sin(\theta + \pi/n) = \cos \pi/n - \cos(2\theta + \pi/n)$$

At each stage we may reject the term that gives a trigonometric polynomial of order less than n (for these terms ultimately give zero)

$$\text{RHS} \to -2^{n-2}\cos(2\theta + \pi/n)\ \Pi_2^{n-1} \sin(\theta + r\pi/n)$$

$$\to -2^{n-3}\sin(3\theta + (1+2)\pi/n)\ \Pi_3^{n-1} \sin(\theta + r\pi/n)$$

$$\to 2^{n-4}\cos(4\theta + (1+2+3)\pi/n)\ \Pi_4^{n-1} \sin(\theta + r\pi/n)$$

and ultimately

$$\pm \tfrac{\sin}{\cos}(n\theta + (1 + 2 + \ldots + (n-1))\pi/n)$$

$$= \pm \tfrac{\sin}{\cos}(n + \tfrac{n-1}{2}\pi)$$

where we have + when $n \equiv 0$ or 1 (mod 4) and *sin* when $n \equiv 1$ or 3 (mod 4). Checking the four cases verifies the answer.

POLYNOMIALS FROM A RECURSION (JCMN 21, Vol.2, p.66)

The first problem was to show $\phi_m(x)$ to be a polynomial with integer coefficients if it is defined by $\phi_0(x) = \phi_1(x) = 1$ and $\phi_{m-2}\phi_m = \phi_{m-1}^2 + x^{2m-3}(1 - x)$.

Solution from *J.B. Parker.*

Put $f_m(x) = \phi_m(x)/x^m$, so that $f_0 = 1$, $f_1 = 1/x$ and $f_2 = (1 + x - x^2)/x^2$.

Then $f_{m-2}f_m - f_{m-1}^2 = 1/x - 1 = f_{m-3}f_{m-1} - f_{m-2}^2$ and therefore

$$\frac{f_m + f_{m-2}}{f_{m-1}} = \frac{f_{m-1} + f_{m-3}}{f_{m-2}} = \ldots = \frac{f_2 + f_0}{f_1} = \frac{1 + x}{x}$$

so that $\phi_m(x) = (1 + x)\phi_{m-1}(x) - x^2\phi_{m-2}(x)$

It follows by induction that $\phi_m(x)$ is a polynomial with integer coefficients. In fact:

$$\phi_m(x) = \frac{(1-\alpha)\beta^m - (1-\beta)\alpha^m}{\beta - \alpha} =$$

$$= (\beta^m + \ldots + \alpha^m) - x(\beta^{m-1} + \beta^{m-2}\alpha + \ldots + \alpha^{m-1})$$

where α and β are the roots of the quadratic $\theta^2 - \theta(1+x) + x^2 = 0$, so that $\alpha + \beta = 1 + x$ and $\alpha\beta = x^2$.

The second problem was to find the roots of these polynomials, the only answer to come in has been that from the proposer.

G. Szekeres writes as follows.

Readers equipped with the tables of Abramovicz-Stegun, Chapter 22, are now in a position to determine the roots of $\phi_m(x)$. To bring the polynomials to the standard form of these tables, set

$$\psi_m(x) = x^m\phi_m(1/x)$$

Then $\psi_0(x) = 1$, $\psi_1(x) = x$ and $\psi_2(x) = x^2 + x - 1$,

and $\psi_{m+1}(x) = (x+1)\,\psi_m(x) - \psi_{m-1}(x)$ for $m \geq 1$.

A little manipulation of the relevant table 22.7 shows that $\psi_m(x)$ is that polynomial of degree m for which

$$\psi_m(2\cos\theta - 1) = (\sin(m+1)\theta - \sin m\theta)/\sin\theta .$$

If you don't believe the tables, verify it directly from the recursion which requires

$$\sin(m+2)\theta - \sin(m+1)\theta = 2\cos\theta(\sin(m+1)\theta - \sin m\theta) - \sin m\theta + \sin(m-1)\theta.$$

A little elementary trigonometry will convince you that this is indeed correct. Consequently the roots of $\psi_m(x)$ are given by $x = 2\cos\theta - 1$ where θ satisfies $\sin(m+1)\theta = \sin m\theta$. These θ are the angles $\theta = (2k+1)\pi/(2m+1)$. Hence the roots of $\psi_m = 0$ are

$$2\cos\frac{2k+1}{2m+1}\pi - 1 \quad (k = 0, 1, \ldots, m-1)$$

The roots of $\phi_m(x) = 0$ are:

$$(2\cos\frac{2k+1}{2m+1} - 1)^{-1} \text{ for } k = 0, 1, \ldots, m-1 \text{ with } 3k \neq m-1.$$

This shows that $\phi_m(x)$ is of degree $m-1$ instead of m whenever $m-1$ is divisible by 3, a fact which can be verified from the recursions above.

BINOMIAL IDENTITY NUMBER SIX (JCMN 21, Vol.2, p.77)

$$\sum_{r=n}^{n+m-1} \binom{n+m-1}{r} x^r (1-x)^{n+m-1-r} = \sum_{r=0}^{m-1} \binom{n+r-1}{r} x^n (1-x)^r$$

$$= \sum_{r=n}^{\infty} \binom{m+r-1}{r} x^r (1-x)^m \quad \text{if } |x| < 1.$$

J.B. Parker writes that these can be worked out by pure algebra. For the first identity put $\phi_{n,m}(x)$ for RHS - LHS.

Then $\phi_{n-1,m}(x) + (1-x)\phi_{n,m-1}(x) = \phi_{n,m}(x)$, and a two-dimensional induction shows that $\phi = 0$. For the second identity

$$\sum_{r=n}^{\infty} \binom{m+r-1}{r} x^r (1-x)^m = 1 - \sum_{r=0}^{n-1} \binom{m+r-1}{r} x^r (1-x)^m .$$

Now use the identity

$$\sum_{0}^{n-1} \binom{n+m-1}{m+r}(1-x)^r x^{n-r-1} = \sum_{0}^{n-1} \binom{m+r-1}{r} x^r$$

which is obtained from the first one by interchanging m with n and x with 1 - x.

$$\text{L.H.S.} = 1 - \sum_{0}^{n-1} \binom{n+m-1}{m+r}(1-x)^{m+r} x^{n-r-1}$$

Now change the dummy variable from r to s = n - r - 1.

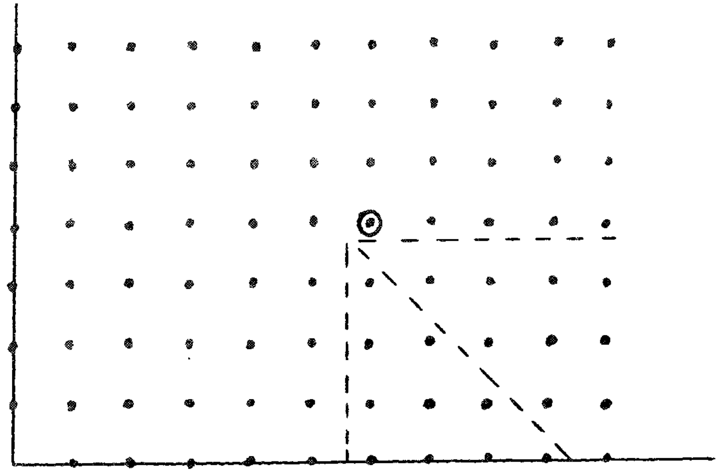
$$\text{L.H.S.} = 1 - \sum_{0}^{n-1} \binom{n+m-1}{n+m-s-1}(1-x)^{m+n-s-1} x^{s}$$

$$= 1 - \sum_{0}^{n-1} \binom{n+m-1}{r} x^r (1-x)^{m+n-1-r}$$

$$= \sum_{n}^{n+m-1} \binom{n+m-1}{r} x^r (1-x)^{m+n-1-r}$$

which is the left hand side of the first identity. The proposer of the problem, *G. Bode,* obtained it from different ways of finding the probability that when you toss a penny (with probability x for head and 1 - x for tail) you will reach n in your score of heads before you reach m tails.

In the picture on the following page, a sequence of tosses is seen as a random walk among the lattice points of the plane, a head is one step to the right and a tail is one step up. To reach n = 6 heads before reaching m = 4 tails is to cross any one of the three dotted lines in your infinite sequence of tosses of the coin. The first formula is the probability of crossing the diagonal line, the second that of crossing the vertical line, and the third that of crossing the infinite horizontal line.

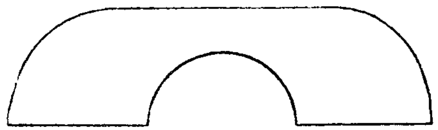


The probabilistic argument of course establishes the identities only for 0 < x < 1, but analytic continuation finishes the job.

FAREWELL TO SECRETARY

On behalf of readers we send good wishes to Lyn Seager who has resigned from the Mathematics Department to go into business with her husband. When Lyn first came, the JCMN was done on wax stencils with rather crude drawings, and the pages stapled in one corner; she it was who persuaded the Copying Section and the Printery of the University to produce the more elegant copy which you see before you. Now Mrs. Michele Askin has taken over and will try to keep up the good work. May Ron and Lyn Seager have a prosperous future.

GEOMETRIC INEQUALITY (JCMN 19, p.32 and 21, p.79 of Volume 2)



This was about the largest area of a table that can be carried round a corner in a corridor of unit width. *J.M. Hammersley* mentioned the problem in his 1968 article on "Soft Intellectual Trash" (in Bull. I.M.A. wasn't it?) and he writes to say that the value $2/\pi + \pi/2$ is not the best possible, for it can be improved by rounding off the corners of the semicircle. Also a note from *V. Klee* says that for publications on this problem one should see SIAM Review, 11, (1969) pp. 75-78 and 12, (1970) pp. 582-586.

NEW BOOK

Sir Horace Lamb (1849 - 1934) was one of the great mathematicians of his period, his book on Hydrodynamics is still (I think) in print, and a gap in the history of applied mathematics has been filled by the publication of "A Profile of Horace Lamb" by *R. Radok* and *S. Radok*. The book contains a portrait photograph, a bibliography, and abstracts of all Lamb's mathematical papers, with contents pages of his books, verbatim accounts of his more important public lectures, nine pages of biography, and a comprehensive index.

Soft covers, 102 pages, 11½ × 8 inches or 29 × 21 cms. Price $9 (Australian) or £4.50 (Sterling) or $10 (U.S.A.) or approximate equivalents, sea mail postage included. All currencies are accepted in any form. Published by the Mathematics Department, James Cook University of North Queensland, Post Office James Cook, 4811, Australia.

Your editor would like to hear from you anything connected with mathematics or with James Cook.

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ADDENDUM

To the note on the previous page about Horace Lamb might be added the comment from L. Bode that in the Journal of Fluid Mechanics (Volume 90, 1979, pp. 202-207) is an article "One Hundred Years of Lamb's Hydrodynamics" by L. Howarth.