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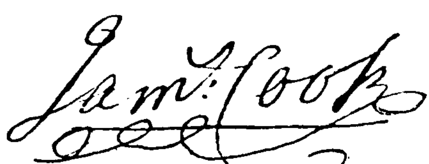

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CONTENTS

PROBLEMS

P. Erdös

(a) Let $a(1) \geq a(2) \geq \ldots$ be an infinite sequence of positive real numbers with $\sum a(n) = \infty$. For integer k let n_k be the integer for which

$$a(1) + a(2) + \ldots + a(n_k - 1) < k \leq a(1) + \ldots + a(n_k - 1) + a(n_k)$$

Find a necessary and sufficient condition for the convergence of $\sum_{k=1}^{\infty} (a(n_k))^2$.

This came up in a paper (not yet published) of mine with Sichely and Ivo (two young Hungarian mathematicians). The necessary and sufficient condition turns out to be the convergence of $\sum a_n^2$. The proof is not difficult.

(b) Let $X_1, \ldots, X_n$ be n distinct points in the plane in general position, i.e., no three on a line and no four on a circle.

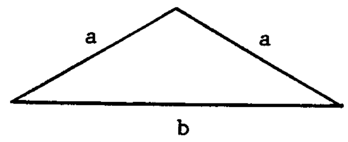
How many distinct distances must the points determine?

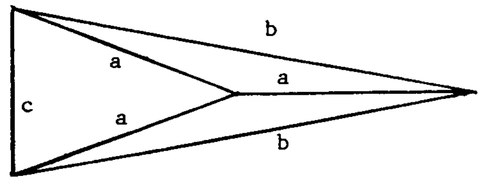
Let f(n) be the largest integer such that the points determine at least f(n) different distances. Estimate f(n) as well as you can. I could not do much here; could not even prove $f(n)/n^2 \to 0$ or $f(n) > n/2$.

(c) Consider n points in the plane, no three on a line and no four on a circle.

Is it possible to choose the points so that they determine $n-1$ distinct distances, the i-th distance occurring $n-i$ times?

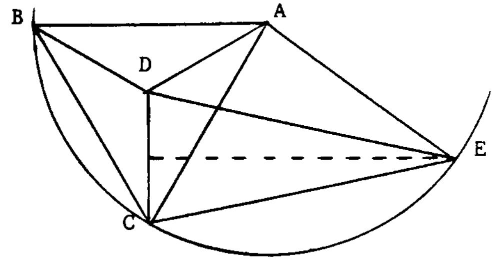
For $n = 3$ and $n = 4$ this is trivial:



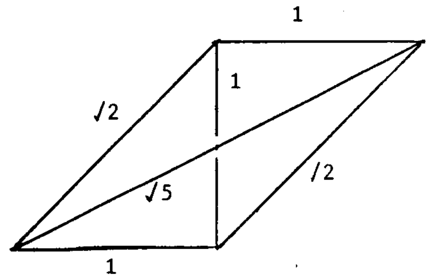


I thought that it could not be done for $n > 4$ but Pomeranic gave a nice construction for $n = 5$ and two Hungarian students gave a construction for $n = 6$ which I do not remember. Perhaps for $n > 6$ it is no longer possible. Here is the construction of Pomeranic.

AB = BC = CA = AE
≠ AD = BD = CD
≠ DE = CE
≠ BE



If we further require that no three of the points be equidistant from another (as B, C and E are equidistant from A in the figure above) then I am not sure that 5 such points are possible. For 4 points it is possible.



THE QUESTING PAIRS OF CAMELOT

Marta Sved

It had become a habit that the knights of Camelot went on their quests in pairs. This had many advantages; company eased the boredom and the strain of the long treks, and the help they could give to each other in coping with the gigantic tasks was invaluable. It also guarded the virtuous humility of the knights, for if an achievement were shared by two of them they were less likely to succumb to the mortal sin of pride.

However, it turned out one day at the round table meeting that next day all the knights, 2n of them, were to set out on their journeys. At the same time a message was received heralding the arrival of a party of visiting knights.

- Guests are as important as quests - announced King Arthur - It is important that half of us stay here to receive the visitors -

- The task is easy - said Merlin - Just choose n out of 2n who should stay here, and there are $\binom{2n}{n}$ ways of making the choice -

Sir Gawain and Sir Bauduin addressed the king in unison:

- With your approval, your Majesty, we want to stay together, on a quest, or at home. We have achieved such rapport that we find it hard to give up our partnership -

Queen Guinevere, always thoughtful and sensitive, said:

- The two knights raise an important point. The question is not only who should stay and who should go, but which pairs should stay together and which should be broken up -

Merlin was quick with his considerations:

- Suppose that we allow 2i pairs to stay together and break up the rest of the partnerships -

- Why an even number 2i? - interrupted Sir Mordred.

- Clear - said Merlin - We want to have half of our knights here. Some pairs we shall divide and of the undivided pairs as many have to stay as be allowed to go -

- Go on, Merlin - said the king.

- Of the 2i pairs who stay together we have $\binom{2i}{i}$ ways of choosing those free to go, and of course, there are 2^{n-2i} ways in which it can be decided which of the partners of each separated pair is to stay or to go -

- This tells us that

$$\binom{2n}{n} = \sum_{i=0}^{[n/2]} \binom{n}{2i}\binom{2i}{i} 2^{n-2i} \quad -$$

- Perhaps I am simple minded - said King Arthur - I would go about it differently. Why not choose i pairs $(2i \leqslant n)$ to stay here? This can be done in $\binom{n}{i}$ ways. We still need n - 2i knights from the broken up partnerships to stay here, so we select out of the remaining n - i pairs our n - 2i pairs which will be broken up. And having decided that we make 2^{n-2i} choices for the partners we want here. Hence we have

$$\binom{2n}{n} = \sum_{i=0}^{[n/2]} \binom{n}{i}\binom{n-i}{n-2i} 2^{n-2i} \quad -$$

- This is splendid! - exclaimed Merlin - Moreover the version of your majesty is more general. It works even if we want to keep here k knights instead of n. We then have

$$\binom{2n}{k} = \sum_{i=0}^{[k/2]} \binom{n}{i}\binom{n-i}{k-2i} 2^{k-2i} \quad -$$

- General! General Merlin, carve these formulae in a stone for future generations - said the king.

- Yes - answered Merlin - I remember a question about it. Binomial Identity 18, on page 4052 of JCMN 34 -

Readers will recall that identity 18(b) was as above and that 18(a) was

$$\sum_{i=0}^{[n/2]} \binom{2i}{i}\binom{n}{2i} 2^{n-2i} = \binom{2n}{n}; \quad \text{from the}$$

multiplicative identity $\binom{n}{2i}\binom{2i}{i} = \binom{n}{i}\binom{n-i}{n-2i}$ it may be seen that 18(a) is the special case of 18(b) when $k = n$.

SERIES EXPANSION (JCMN 33, p.4029)

J. B. Parker

To expand $(2-2(1-x^2)^{\frac{1}{2}})^{\frac{1}{2}}$ in a power series, put $x = \cos\theta$, where $0 < \theta < \pi/2$. The expression becomes

$$(2-2\sin\theta)^{\frac{1}{2}} = 2^{\frac{1}{2}}(\cos\theta/2 - \sin\theta/2)$$
$$= (1+x)^{\frac{1}{2}} - (1-x)^{\frac{1}{2}}$$

$$= 2\sum\binom{1/2}{2r+1} x^{2r+1} = \sum \frac{(4r)!}{(2r)!(2r+1)!} 2^{-4r} x^{2r+1}$$

$$= x + \frac{1.3}{2.4}(x^3/3) + \frac{1.3.5.7}{2.4.6.8}(x^5/5) + \ldots$$

As a corollary it may be shown that

$$1 + \sum_1^\infty \frac{(4r)!}{(2r)!(2r+1)!} x^r \quad \text{and} \quad 1 - \sum_1^\infty \frac{2(4r-2)!}{(2r)!(2r-1)!} x^r$$

are inverses of one another (for x of modulus less than 1/4).

ADDING NUMBERS (JCMN 34, p.4055)

Esther and George Szekeres

The problem from C.J. Smyth was to find whether if $\{1,2,\ldots,3n\}$ is partitioned into three equal sets it is possible to choose one number out of each set so that one of these numbers is the sum of the other two. The answer is YES, and it is a consequence of the following more powerful result.

Theorem. Suppose that the set $\{1,2,\ldots,N\}$ is partioned into three sets so that it is impossible to choose one number out of each set with one of these three numbers equal to the sum of the other two. Then one of the three sets has no more than $N/4$ members.

Proof. Let the sets (each arranged in increasing order) be

$$A = \{x_1, x_2, \ldots, x_a\}$$
$$B = \{y_1, y_2, \ldots, y_b\}$$
$$C = \{z_1, z_2, \ldots, z_c\}$$

where (as we may assume without loss of generality) $x_1 < y_1 < z_1$. Clearly $x_1 = 1$, $y_1 \geq 2$, $z_1 \geq 4$ and $a + b + c = N$.

Let r be the minimum difference between successive members of C, and let k be the smallest suffix for which $z_{k+1} - z_k = r$.

Lemma. We shall show that $y_1 \leq r$.

Proof. Suppose that $y_1 > r$, then we shall find a contradiction as follows. Note that r and $y_1 - r$ are both in A.

$z_k + r - y_1 = z_k - (y_1 - r) \notin B$ and $= (z_k + r) - y_1 \notin A$, therefore $\in C$,

$z_k - y_1 \notin A$ and $= (z_k + r - y_1) - r \notin B$, therefore $\in C$.

We have two elements of C differing by r, so contradicting our choice of k as minimal.

Having established the lemma, we can now reduce the theorem to the following four cases. Case 1, $r \geqslant 4$; case 2, $r = y_1 = 3$; case 3, $r = 3$ and $y_1 = 2$; and case 4, $r = y_1 = 2$. Let S be the sequence of length N in which element number t is the set (A, B or C) containing t.

<u>Case 1</u> ($r \geqslant 4$). Remembering that $z_1 \geqslant 4$, it follows that every C is preceded in S by three or more elements either A or B. Therefore $3c \leqslant a + b$, and so $4c \leqslant a + b + c = N$, and $c \leqslant N/4$.

<u>Case 2</u> ($r = y_1 = 3$). For any t consider $z_t - 1$ and $z_t - 2$. Neither can be in B because 1 and 2 are in A, and neither can be in C because $r = 3$. Therefore every C in S is preceded by two A's, and so $2c \leqslant a$. It follows that $b + 3c \leqslant a + b + c = N$ and so either b or $c \leqslant N/4$.

<u>Case 3</u> ($r = 3$ and $y_1 = 2$). Consider $S = \{A, \ldots, C, X, Y, C, \ldots\}$. Neither X nor Y can be C (because $r = 3$) and neither can be B (because 1 is in A), and neither can be A (because 2 is in B). Therefore there is a contradiction.

<u>Case 4</u> ($r = y_1 = 2$). For convenience write m for z_k. Then $m + 2$ is in C. The two numbers $m - 1$ and $m + 1$ cannot be in B because $1 \in A$ and cannot be in C because $r = 2$. Therefor they are both in A. Also

$$3 = 1 + 2 \notin C \quad \text{and} = (m+2) - (m-1) \notin B, \text{ therefore } \in A.$$

Also $m - 2 \notin C$ (choice of k) and $\notin A$, therefore $\in B$.

The sequence S therefore looks like

$$S = \{A, B, A, \ldots, B, A, C, A, C, \ldots\} .$$

We shall prove by induction that all the odd numbers are in A. Suppose not, then there is a smallest odd number $2k + 3$ not in A (where $k \geqslant 1$). Now consider first the

case where $2k + 1 < m$.

$$m - (2k+1) \notin B \quad \text{and} = (m-2) - (2k-1) \notin C, \text{ therefore } \in A,$$

$$2k + 3 = m + 2 - (m - 2k - 1) \notin B \quad \text{and} = (2k+1) + 2 \notin C$$

therefore $\in A$.

The other case, where $2k + 1 > m$, is dealt with similarly as follows:

$$(2k+1) - m \notin B \quad \text{and} = (2k-1) - (m-2) \notin C, \text{ therefore } \in A,$$

$$2k + 3 = (m+2) + (2k+1-m) \notin B \quad \text{and} = 2k + 1 + 2 \notin C,$$

therefore $\in A$.

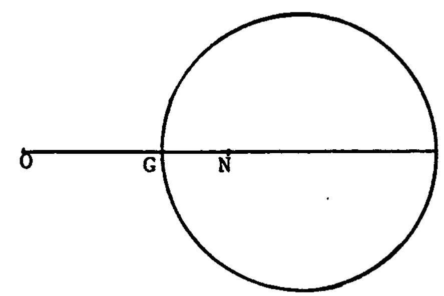
This has established that all the odd numbers are in A. It follows that $a \geqslant N/2$ and $b + c \leqslant N/2$, so that either b or $c \leqslant N/4$.

SYMMEDIAN POINT (JCMN 33, p.4030)

A.P. Guinand

Suppose that, for some unknown triangle, we know the Euler line, with its landmarks, the circumcentre O, the centroid G, the nine-point centre N and the orthocentre H. Then what can be said about the position of the symmedian point K?

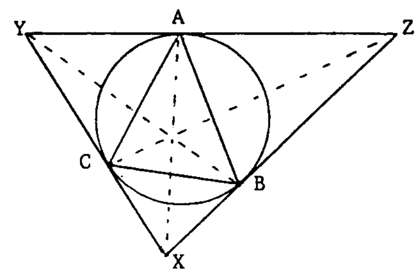
I conjecture that the region in which K might be found is the "critical circle" with GH as diameter.



SYMMEDIAN POINT AGAIN

(JCMN 32, p. 4008, 33, p. 4030, 34, pp. 4062, 4063)

A.P. Guinand



The tangents to the circumcircle at the vertices A, B and C form another triangle XYZ in perspective with ABC; the centre of perspective is the symmedian point.

Proof. The circumcircle is $ayz + bzx + cxy = 0$ in trilinear coordinates, and the tangents $az + cx = 0$ at B and $bx + ay = 0$ at C meet at X with coordinates $(-a, b, c)$, collinear with A(1, 0, 0) and the symmedian point (a, b, c).

I find it hard to believe that this result doesn't lurk somewhere in the ancient literature, but that is one of the hazards of this topic.

QUOTATION CORNER 16

From the Diary of Samuel Pepys, 9th June, 1663.

"... and then comes Creed and he and I talked about Mathematiques and he tells me of a way found out by Mr Jonas Moore, which he calls Duodecimall arithmetique, which is properly applied to measuring, where all is ordered by inches, which are 12 to a foot; which I have a mind to learn."

\- C.J. Smyth

CAPTAIN COOK AND THE MOON

C.F. Moppert

This article is concerned with the problem of determining longitude, or rather with the history of the methods used for this purpose. The story of Captain James Cook illustrates this history particularly well, as he applied at one time or other all the methods that were available in his time. They were

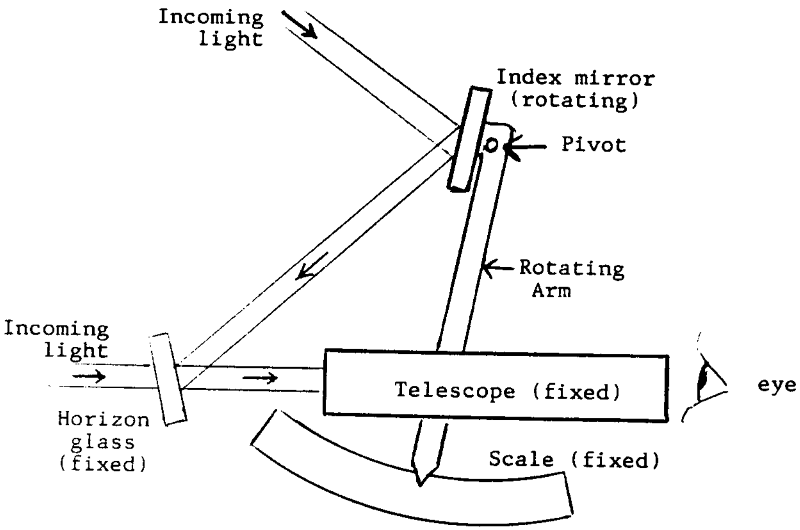
(1) observing an eclipse of the sun,
(2) observing the passage of the moons of Jupiter,
(3) measuring the angle between the moon and either the sun or a star,
(4) using the chronometer.

The fact that Cook used not only (3) and (4), which are possible at sea, but also (1) and (2), which require a land-based observatory, shows that he was much more than a competent navigator.

The ideal instrument for measuring angles is the sextant. The immediate predecessor of the sextant was the "quadrant", developed to a high degree of precision by John Hadley, a vice-president of the Royal Society, in 1731 (three years after Cook's birth). The quadrant measured angles up to 90° and its use was for measuring altitudes above the horizon. The change to a sextant, capable of measuring angles up to 120°, is credited to Captain (later Admiral) John Campbell, and was motivated by the requirements of the "lunar distance" method of finding longitude.

The only reason why I dare to contribute to the vast literature on Cook and on navigation is that I own a sextant and have done some work with it. It was given to me as a present on my 60th birthday by my sister and my brother-in-law. It cost about \$600. Hadley's quadrant in Cook's time cost some 6 guineas (his pay as a lieutenant for a month).

On my sextant the scale is on a circle of 6 inches (15 cms.) radius, and I can make measurements to an accuracy



This diagram is to illustrate the optical principle of the sextant. The two images are superimposed and seen together through the telescope; the angle between the two things observed is read from the position of the rotating arm on the scale. The scale reads zero when the two mirrors are parallel. The angular displacement of the arm and the index mirror is half the reading shown on the scale.

of two minutes of arc. Instruments in astronomical observatories can measure to one hundredth of a second of arc, but for a navigational instrument such as a sextant that can be used at sea, it is hard to get better accuracy than one minute of angle.

Let us look at the problem of determining longitude. At any point (apart from the poles) there is local

noon, the time when the Sun is highest. Greenwich mean time is defined so that, on the average, noon at Greenwich is at 12 o'clock. I won't go into the question of why this is so only "on the average". If you find that local noon is at 6 p.m. Greenwich mean time, then your longitude is 90° West of Greenwich, because the Earth rotates once in 24 hours, or 15 degrees per hour. In principle, then, you can find your longitude if you carry a clock giving G.M.T.

Now we may say, as Mr. Johann Werner first said in 1514, that we have a wonderful clock hanging in the sky, its hand being the Moon and its face the other heavenly bodies. The Moon takes about 28 days to move right round the sky, so that in one hour the angle between the Moon and the Sun (or any other suitably placed star) changes by about half a degree. If we can measure the angle to an accuracy of half a minute we have time to an accuracy of one minute, that is to say, longitude to an accuracy of 15 minutes of angle. On the equator 15 minutes of angle is 15 nautical miles (the nautical mile being 6080 feet or 1852 metres).

If. This "if" includes several obstacles. The main one - we must have tables predicting the position of the Moon (predicting the positions of the Sun or stars is relatively easy). The first adequate tables were calculated by Tobias Mayer in Goettingen and published in 1753. Mayer used the theory developed by Leonhard Euler in St. Petersburg. Euler was born in 1707; it did not take long for mathematical discoveries to bear fruit. It was Nevil Maskelyne the Astronomer Royal, who started the Nautical Almanac containing lunar distance tables for use by navigators.

Now let us look at some of the practical problems of reading our celestial "clock" by measuring the angular distance between Sun and Moon. Of course both must be above the horizon and they must not be too close together, for it is hard to see the Moon when it is less than 40° from the Sun.

MARCH 1769.					[35]
	Distances of ☽'s Center from Stars, and from ☉ west of her.				
Days.	Stars Names.	12 Hours.	15 Hours.	18 Hours.	21 Hours.
		° ′ ″	° ′ ″	° ′ ″	° ′ ″
1	Spica ♍	65. 0. 32	66. 31. 19	68. 2. 18	69. 33. 33
2		77. 13. 27	78. 46. 13	80. 19. 18	81. 52. 38
3	Antares.	44. 9. 57	45. 45. 7	47. 20. 38	48. 56. 30
4		57. 1. 6	58. 39. 7	60. 17. 31	61. 56. 17
5		70. 15. 43	71. 56. 43	73. 38. 4	75. 19. 48
6		83. 53. 47			
10	The Sun.		38. 47. 19	40. 28. 14	42. 9. 11
11		50. 33. 46	52. 14. 31	53. 55. 11	55. 35. 45
12		63. 56. 54	65. 36. 43	67. 16. 22	68. 55. 52
13		77. 10. 48	78. 49. 14	80. 27. 29	82. 5. 32
14		90. 12. 44	91. 49. 33	93. 26. 10	95. 2. 34
15		103. 1. 24	104. 36. 30	106. 11. 23	107. 46. 3
16		115. 36. 23	117. 9. 48	118. 43. 0	120. 15. 58
14	Aldebaran.	19. 15. 16	20. 52. 18	22. 30. 14	24. 8. 51
15		32. 27. 8	34. 7. 13	35. 47. 20	37. 27. 29
16		45. 47. 36	47. 27. 21	49. 6. 57	50. 46. 26
17		59. 1. 41	60. 40. 17	62. 18. 45	63. 57. 3
18	Pollux.	30. 41. 53	32. 16. 3	33. 50. 23	35. 24. 50
19		43. 17. 24	44. 51. 50	46. 26. 12	48. 0. 28
20	Regulus.	18. 53. 31	20. 25. 44	21. 58. 13	23. 30. 50
21		31. 14. 12	32. 46. 43	34. 19. 8	35. 51. 29
22		43. 31. 30	45. 3. 8	46. 34. 37	48. 5. 59
23		55. 40. 43	57. 11. 14	58. 41. 37	59. 11. 53
24		67. 41. 27			
24	Spica ♍	13. 43. 18	15. 11. 44	16. 40. 17	18. 8. 54
25		25. 32. 24	27. 1. 1	28. 29. 37	29. 58. 12
26		37. 20. 49	38. 49. 17	40. 17. 44	41. 46. 11
27		49. 8. 53	50. 37. 31	52. 6. 12	53. 34. 57
28		60. 59. 52	62. 29. 10	63. 58. 37	65. 28. 12
29		72. 58. 23	74. 28. 57	75. 59. 43	77. 30. 41
30	Antares.	39. 37. 11	41. 9. 28	42. 42. 2	44. 14. 55
31		52. 4. 4	53. 38. 54	55. 14. 5	56. 49. 38

A page from the 1769 Nautical Almanac. For a sequence of times at intervals of 3 hours the authors had selected one star to the East and one to the West of the Moon, so as to ensure that whenever the moon was up and the sky clear it should be possible to make a lunar distance observation to find longitude. The four pages for the month of March would have been used by Cook while sailing across the South Eastern Pacific from Cape Horn to Tahiti during his first Pacific voyage. Note the astronomical convention that the day goes from noon to noon, so that "21 hours" on 1st March means what nowadays we call 9 o'clock in the morning of 2nd March.

Therefore the clock is out of order for about ten days at the time of the full moon and a week at the time of the new moon. Also, of course, there is the requirement of not too much cloud or fog, and so this "clock" has limitations. Some of these difficulties were solved by the way the Nautical Almanac (see the copy of one page from one of the editions that Cook used) gave lunar distances from various stars near the ecliptic.

The British parliament set up in 1714 the "Board of Longitude", consisting of 22 commissioners and charged with the duty to give a prize to anyone finding a practicable method of obtaining longitude at sea. The prize was £10,000 for an accuracy of one degree, £15,000 for an accuracy of 40 minutes of angle, and £20,000 for an accuracy of 30 minutes. The first man to win part of the award was Tobias Mayer in 1765 for his lunar tables; that he got it only after his death is perhaps a minor point. During this period John Harrison built his first marine chronometer, and in 1736 the Board of Longitude made arrangements for him to sail with the chronometer to Lisbon to test it. In 1761 he built an entirely different chronometer and also had it tested. For comparison the moons of Jupiter were used. The final test was Cook's second voyage using Kendal's copy of Harrison's latest chronometer. The results were so good that Harrison was given the maximum prize of £20,000.

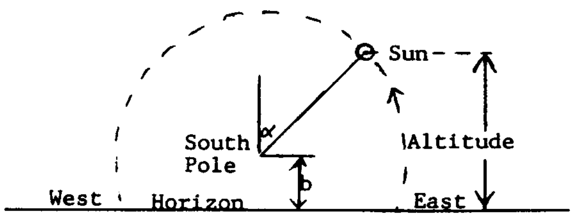
In Cook's first Pacific voyage (1768-1771) when he charted the East coast of Australia, he had only the method of lunar distances for finding longitude. Lunar distances were given in the Nautical Almanac until 1905, but now they are almost forgotten because good chronometers are so easily available.

One of Cook's calculations of longitude by lunars would take about four hours, using seven-figure logarithm tables. Of course we could do it much more quickly now with a pocket calculator, but it is still complicated. My sextant is easier to use than Cook's, for it has a better telescope and has a micrometer instead of a vernier for measuring fractions of a degree.

I shall now describe to you my own experiences in some detail. On the 12th March, 1984 at 18.10 by the clock I measured in Warrandyte the angle between the Sun and the Moon with my sextant and found the value $109^{o}39'$ (or in decimals of a degree, 109.65^{o}). From the Nautical Almanac I calculated the angle at hourly intervals :-

GMT	0700		0800		0900
Angle	109.01		109.54		110.08
Difference		0.53		0.54	

The angle of 109.39^{o} corresponds to GMT of 08.12. How can we calculate from this the longitude of Warrandyte? We need to find the local solar time at Warrandyte at the moment of the observation



The Sun moves every day in a circle round the South Pole, and the local time (before or after noon) is the hour angle α in the figure above. It can be found if we know the latitude b and measure the altitude of the Sun, by solving the spherical triangle whose vertices are the Sun, the South Pole and the Zenith, on the celestial sphere.

The finding of longitude by lunars consists of the following main steps:

1. Find the latitude.
2. Measure a lunar distance, noting the time on a watch (for this purpose no great accuracy is needed; Cook would have used a "deck watch", accurate within a half a minute per day).
3. Compare the watch with local time by measuring the altitude of the Sun, or by observing sunrise or sunset.

4. Find the GMT of your lunar distance measurement, from the Nautical Almanac.
5. Find the local time of your lunar distance measurement.
6. Subtracting the last two results gives longitude.

Actually, it is not so simple. Corrections have to be made for refraction, for the difference between mean time and true time (something like 5 or 10 minutes), and for lunar parallax. The parallax correction is due to the Moon's position (in astronomical tables) being given as direction from the centre of the Earth. The direction from any point on the surface of the Earth will differ by an amount of up to one degree of angle (which is the apparent radius of the Earth as seen from the Moon).

EXPANSION FROM RAMANUJAN

G. Szekeres

In a partial solution of a problem proposed in the Journal of the Indian Math. Soc., Ramanujan considered $y = f(n)$ defined by

$$e^{n}/2 = 1 + n/1! + n^{2}/2! + \dots + n^{n-1}/(n-1)! + n^{n}y/n!$$

He noted that y had an expansion in powers of 1/n of the form

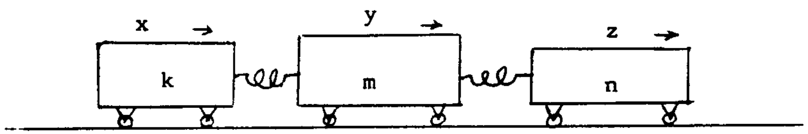
$$1/3 + (4/135)n^{-1} - (8/2835)n^{-2} - (16/8505)n^{-3} + \dots$$

Ordinary mortals would expect half-integer powers to appear in the expansion, but to Ramanujan it seemed obvious that only the integer powers could survive. Is it obvious?

NUMBER THEORY AND MECHANICS

G. Berszenyi and A. Zajta

When teaching the mechanics of vibration it seems natural to set the example of three trolleys connected by equal springs and running on a frictionless railway line, asking students to calculate the normal modes of vibration.



Taking the masses of the trolleys as k, m and n, the displacements as x, y and z, the rate of the springs as r, and the (radian) frequency of the vibration as w, the equations to be solved are

(1) $w^2kx + r(y-x) = 0$

(2) $w^2my + r(x-2y+z) = 0$

(3) $w^2nz + r(y-z) = 0$

In order to simplify the typing of the question, to ease the marking and to minimise the computation required, we try to ensure that all the given parameters (k, m, n and r) and all the answers (w, x, y and z) are (+ or −) integers. Now, in order to set a good question, we have the Diophantine problem of solving (1), (2) and (3) in integers. We shall find infinitely many solutions.

The system clearly has three normal modes, one of them the trivial solution of zero frequency, so that eliminating x, y and z gives a quadratic in w^2,

(4) $kmn\,w^4 - (km + mn + 2kn)w^2 r + (k+m+n)r^2 = 0$

which has integer solutions in w^2 and r if and only if the discriminant is a perfect square, i.e. there is an integer a such that

(5) $(m(k-n))^2 + (2kn)^2 = a^2.$

In view of the usual parametrization of Pythagorean triples it follows that a solution is given by

(6) $m = k + n$ and $a = k^2 + n^2.$

The solutions of the quadratic (4) are

(7) $w^2/r = (k+n)/(kn)$ or $2/(k+n).$

We have to choose k, n and r so that the values of w^2 given by (7) are perfect squares. To do this, take any integers b and c, then choose k and n so that $kn = 2b^2$ and put $r = kn(k+n)c^2$. The roots of (4) are then

$$w = \pm\, c(k+n) \quad \text{or} \quad \pm\, 2bc.$$

Finally, integer x, y and z may be chosen because the ratios x : y : z are known rational numbers.

CORRECTION

In JCMN 33, in the article "A Model for Company Finance" there are two numerical misprints. On p.4036, line 13, the "has \$550 in cash" should become "has \$500 in cash". On p.4037, also on line 13, the "cash holding of \$1100" should be changed to "cash holding of \$1000". Is the line number 13 unlucky? The thanks of the Editor are due to the readers who pointed out these errors.

TRIANGLE GEOMETRY

J.B. Parker

Let A, B and C be the angles of a triangle, and consider the three complex numbers $(u, v, w) = (-1, e^{iC}, e^{-iB})$. In the complex plane they form an acute-angled triangle with angles $(B+C)/2$, $(C+A)/2$ and $(A+B)/2$.

$$\begin{aligned} |uv + vw + wu|^2 &= (uv + vw + wu)(\bar{u}\bar{v} + \bar{v}\bar{w} + \bar{w}\bar{u}) \\ &= 3 + (v\bar{w} + w\bar{v}) + (u\bar{w} + w\bar{u}) + (u\bar{v} + v\bar{u}) \\ &= 3 - 2(\cos A + \cos B + \cos C) \\ &= 1 - 8 \sin A/2 \sin B/2 \sin C/2. \end{aligned}$$

This is one way of showing that the last expression above is positive except for equilateral triangles.

Another use of the same algebra is to consider the triangle represented by the points u^2, v^2 and w^2; it has angles A, B and C. The point $-uv-vw-wu$ is the centre I of the inscribed circle, because the vector IA is represented by the complex number $u^2 + uv + vw + wu = (u+w)(u+v)$ of which the square is $(u^2 - w^2)(u^2 - v^2)\ \dfrac{u+w}{u-w}\ \dfrac{u+v}{u-v}$, an expression in which the last two factors are pure imaginary.

We can verify that this point is the centre of the inscribed circle and not of one of the three escribed circles, as follows. By the identity above we know that the point is inside the circumcircle. A typical excentre is $vw + wu - uv$ and the square of its distance from the origin, by a similar calculation is

$$3 + 2\cos A + 2\cos B - 2\cos C = 1 + 8 \sin C/2 \cos A/2 \cos B/2 > 1,$$

so that the three excentres are distinguished from the incentre by all being outside the circumcircle.

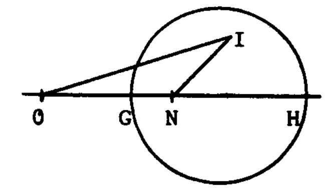
The algebra above therefore also establishes that in any triangle : $OI^2/R^2 = 3 - 2(\cos A + \cos B + \cos C)$.

The fact that the three complex numbers u,v and w are the trilinear coordinates of one of the circular points at infinity is surely a coincidence, isn't it?

MORE TRIANGLE GEOMETRY

A.P. Guinand

The three following comments arise from John Parker's article above.



(a) The points OGNH (circumcentre, centroid, nine-point centre and orthocentre) are spaced along the Euler line as shown, with the lengths OG, GN and NH in the ratio 2 : 1 : 3. It is known (JCMN 30, p.3127) that the incentre I must be in the circle on GH as diameter. John Parker's note above suggests a simple proof, as follows.

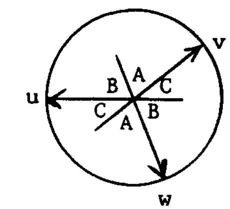
$OI = |vw + wu + uv| = |1/u + 1/v + 1/w| = |\bar{u} + \bar{v} + \bar{w}| = |u + v + w|$

and $2IN = |u^2 + v^2 + w^2 + 2vw + 2wu + 2uv| = |u + v + w|^2$.

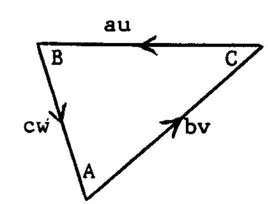
Because $OI \leq 1$ it follows that $2IN \leq OI$, which means that I is inside the circle shown.

(b) What about $(u, v, w) = (-1, e^{iC}, e^{-iB})$ being trilinear coordinates of a circular point at infinity? Coincidence, yes - but not inexplicable.

The complex numbers u, v and w are unit vectors in the plane like this:

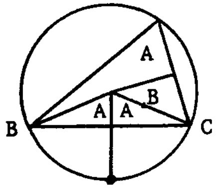


If we draw a triangle with sides parallel to the three vectors then the angles are A, B and C, and so the sides are proportional to a, b and c. Consequently we find that u, v and w must satisfy the equation $au + bv + cw = 0$. By taking the complex conjugate and multiplying by uvw it follows



that they also satisfy $avw + bwu + cuv = 0$. These last two equations shown that (u, v, w) is on the line at infinity and the circumcircle respectively; therefore the point (u, v, w) is one of the circular points.

Is is obvious that $ayz + bzx + cxy = 0$ is the circumcircle? Yes, for it is a conic through the three vertices and the point where $x = R(\cos A - 1)$ and $y = z = R(\cos B + \cos C)$ and the two similar points, easily verified.



(c) Suppose that instead of giving u, v and w the specific values as above, we just take them to be any three complex numbers of unit modulus. Then $-vw - wu - iv$ is a tritangent centre (centre either of the inscribed circle or of one of the three escribed circles) of the triangle with vertices u^2, v^2 and w^2. If K is the point $-u^2 - v^2 - w^2$ then it can be shown (A.P.Guinand, Euler lines, tritangent centers, and their triangles, American Math.Monthly, 91 (1984)) that

$$4\, IN(IK + IN) \leq 9\, OI^2$$

Interpreting this inequality in terms of complex numbers, we have the following:

<u>Theorem</u> If u, v and w are complex numbers of unit modulus, then:

$$|u + v + w|^2 + 2|u^2 + v^2 + w^2 - vw - wu - uv| \leq 9.$$

It looks like a simple bit of algebra but probably is not, for it is equivalent to some fairly subtle and complicated geometry.

GEOMETRIC INEQUALITY

$8 \cos A \cos B \cos C < 1$ for non-equilateral triangles.

CAMELOT AND THE GAMES

Marta Sved

When King Arthur announced the invitation to send teams to compete in archery and javelin-throwing (with n in each team) the m knights sitting at the Round Table received the news in an argumentative mood.

– They want us to send 2n knights – said Sir Bauduin – There are $\binom{m}{2n}$ ways of choosing them out of the m of us – said Sir Gawain.

– And not distinguish between javelin-throwing and archery? – interjected Sir Mordred in his usual sarcastic manner.

King Arthur said soothingly – Having our 2n knights we can select the n for archery in $\binom{2n}{n}$ ways, and so we have $\binom{m}{2n}\binom{2n}{n}$ possibilities to think about –

Sir Archibald raised an objection, saying – Some of us could well compete in both events –

Bedlam broke out. King Arthur rose from his seat to silence the augument. – I expect impeccable conduct from my knights. Clearly there must be some of us who will not be able to compete in even one event. Besides, there are other important things to be done. I am not sure that I can spare 2n of you for that tournament –

– We must send at least n knights – said Merlin. – Consider all the possibilities. Suppose that we send a party of 2n–d where $0 \leqslant d \leqslant n$. Then from these we choose d to compete in both events, and from the remaining 2n–2d we have to select n–d for archery, the rest for javelin-throwing –

– Alternatively – said Sir Lancelot – We could select n archers out of the 2n–d first, and from these n select d to compete in both events –

The king agreed. - Since there are $\binom{m}{2n-d}$ ways of making the first selection, we have

$$\sum_{d=0}^{n} \binom{m}{2n-d}\binom{2n-d}{d}\binom{2n-2d}{n-d} = \sum_{d=0}^{n} \binom{m}{2n-d}\binom{2n-d}{n}\binom{n}{d}$$

possibilities to consider. Agreed? -

All the knights nodded in agreement, but Merlin remained unmoved. The king turned to him. - Do you not agree Merlin? -

- Oh, I agree, your Majesty. - The two formulae are perfectly correct. But we could select n archers out of the m knights, and then again select n out of the m for javelin-throwing, thereby including all possibilities. It gives us the identity

$$\binom{m}{n}^2 = \sum_{d=0}^{n} \binom{m}{2n-d}\binom{2n-d}{d}\binom{2n-2d}{n-d} = \sum_{d=0}^{n} \binom{m}{2n-d}\binom{2n-d}{n}\binom{n}{d} -$$

BINOMIAL IDENTITY 17 (JCMN 34, p.4052)

Marta Sved

$$\sum_{r=0}^{n} \binom{n}{r}^2 \binom{m+r}{2n} = \binom{m}{n}^2 \quad \text{for } m > n > 0.$$

In order to prove this, first note by the Vandermonde convolution formula (which incidentally has a very simple combinatorial proof),

$\binom{m+r}{2n} = \sum_{d=0}^{r} \binom{r}{d}\binom{m}{2n-d}$. The left-hand side of the formula being investigated is therefore

$\sum_{r=0}^{n} \binom{n}{r}^2 \sum_{d=0}^{r} \binom{r}{d}\binom{m}{2n-d}$ which by interchanging the two summations may be written

$$\sum_{d=0}^{n} \binom{m}{2n-d} \sum_{r=d}^{n} \binom{n}{r}^2 \binom{r}{d}$$

But by the multiplication formula $\binom{n}{r}\binom{r}{d} = \binom{n}{d}\binom{n-d}{n-r}$,

so that the sum may be written

$$\sum_{d=0}^{n} \binom{m}{2n-d}\binom{n}{d} \sum_{r=d}^{n} \binom{n}{r}\binom{n-d}{n-r},$$

but from the convolution formula

$$\sum_{r=d}^{n} \binom{n}{r}\binom{n-d}{n-r} = \binom{2n-d}{n}$$

and the sum is

$$\sum_{d=0}^{n} \binom{m}{2n-d}\binom{n}{d}\binom{2n-d}{n},$$

which was shown to be $\binom{m}{n}^2$ by Merlin in the Camelot story printed above.

LITERARY COMPETITION

At the local university the Vice-Chancellor has ordained that all rules and regulations are to be re-written to eliminate sexist words and phrases. Suggestions are wanted for appropriate non-sexist substitutes for "bachelor" and "master" in the titles of degrees.

EDITORIAL

Contributions will be welcomed. They should be written so as to be clear to all mathematicians.

Since Issue 32 (October 1983) the JCMN has been published by me (the Editor). Issues 1 to 31 were published by

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and they are on sale for $10 (Australian) per volume (including postage by surface mail); cheques for these volumes should be made payable to the James Cook University.

My address is either at the University (address above) or at home (see page 4066).

Basil Rennie