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THE EIGHTEEN MORLEY TRIANGLES

The azimuth of a straight line.

In the plane we take a fixed straight line (for instance the x-axis of Cartesian coordinates) and we say that rotating this line through any angle α (anticlockwise) gives a line of azimuth α. Each azimuth is an angle measured modulo 180°, so that lines are not directed. In Cartesian coordinates the azimuth α of the line $y = mx + c$ is given by $\tan\alpha = m$.

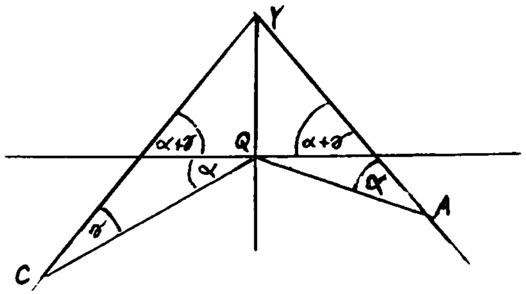
Geometry using azimuths is the essence of navigation by radio beacons with a direction-finding loop (but not a sense-detecting device), for the position of a point X is determined by two points A and B, and the azimuths of the lines AX and BX (provided that the two azimuths are unequal).

Lemma 1 Suppose that we have four points, A, C, Q and Y with

$$\text{az } CY = -\text{az } AY = \alpha + \gamma \quad \text{and} \quad \text{az } QY = 90^{\circ}$$
$$\text{az } CQ = \alpha \quad \text{and} \quad \text{az } AQ = -\gamma$$

Then $\text{az } AC = \alpha - \gamma$

Proof



Set up Cartesian coordinates with Q as origin. Let Y be $(0, y)$, A be $(r\cos\gamma, -r\sin\gamma)$ and C be $(s\cos\alpha, s\sin\alpha)$.

The information about the azimuth of CY tells us that

$$s\cos\alpha \ \sin(\alpha+\gamma) = (s\sin\alpha - y)\cos(\alpha+\gamma)$$

i.e. $\quad s\sin\gamma + y\cos(\alpha+\gamma) = 0.$

From the azimuth of AY we have

$$r\cos\gamma \ \sin(\alpha+\gamma) = (r\sin\gamma + y)\cos(\alpha+\gamma)$$

i.e. $\quad r\sin\alpha = y\cos(\alpha+\gamma)$

Therefore $r\sin\alpha = -s\sin\gamma$ and

$$\tan(\text{az } AC) = \frac{s\sin\alpha + r\sin\gamma}{s\cos\alpha - r\cos\gamma} = \frac{\sin^2\alpha - \sin^2\gamma}{\sin\alpha\cos\alpha + \sin\gamma\cos\gamma}$$

$$= \frac{\cos 2\gamma - \cos 2\alpha}{\sin 2\alpha + \sin 2\gamma} = \frac{2\sin(\alpha-\gamma)\sin(\alpha+\gamma)}{2\sin(\alpha+\gamma)\cos(\alpha-\gamma)} = \tan(\alpha-\gamma)$$

Lemma 2 Four points, A, B, C and D, are concylic if and only if $\text{az } AB + \text{az } CD = \text{az } AC + \text{az } BD$. Note that "concylic" is used here in the wide sense, to include the possibility of the four points being collinear.

Proof "Only if". Let the points be concylic. The case of collinear points is trivial, and so without loss of generality we may assume the points to be on the unit circle, and represented by complex numbers, a,b,c and d, all of modulus 1. By the "phase" of a complex number $x + iy \neq 0$ we mean the angle α (modulo 180°) such that $x\sin\alpha = y\cos\alpha$. The phase is zero when the number is real. The azimuth of AB is the phase of $a - b$, etc. Therefore

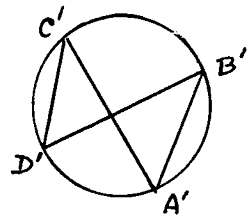
$$\text{az } AB + \text{az } CD - \text{az } AC - \text{az } BD$$

(which we want to show to be zero) is the phase of

$$\frac{(a-b)(c-d)}{(a-c)(b-d)}$$

This fraction is one of the 24 cross-ratios that can be formed from the set of four numbers a,b,c and d. We know that if x is one of the cross-ratios, then any other must have one of the values x, 1/x, 1-x, 1/(1-x), x/(x–1) and (x – 1)/x, so that one is real if and only if all the others are real.

Take a permutation A',B',C' and D' of A,B,C and D so that A'B'C'D' are in that order round the unit circle.



Elementary geometry tells us that
az A'B' – az B'D' = angle A'B'D' = angle A'C'D' = az A'C' – az C'D'.

The cross ratio $\frac{(a'-b')(c'-d')}{(a'-c')(b'-d')}$ is therefore real

and so $\frac{(a-b)(c-d)}{(a-c)(b-d)}$ is real and the required relation between the azimuths holds.

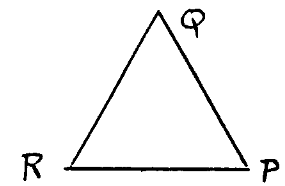
"If". Represent the points by complex numbers as before. We know that one of the cross-ratios is real, and therefore all are real. There is a bilinear transformation taking (a, b, c, d) to (0, f, 1, ∞) (for some f). Because bilinear transformations do not change cross-ratios, $f = \frac{(0-f)(1-\infty)}{(0-1)(f-\infty)}$ is real and so the four points 0, f, 1 and ∞ are collinear. Therefore the points a, b, c and d are concylic.

Bricard's Construction.

We are given three angles α, β and γ with

$$\alpha + \beta + \gamma = \pm 60^\circ \text{ (modulo } 180^\circ).$$

Take an equilateral triangle PQR with az PQ = $-\alpha - \beta - \gamma$ and az QR = $\alpha + \beta + \gamma$ and az RP = 0.



Now construct points A, B and C as follows.

A is such that	az AQ = $-\gamma$	and az AR = $-\alpha - \gamma$
B is such that	az BP = $\alpha + \beta + 2\gamma$	and az BR = $\alpha + 2\beta + 2\gamma$
C is such that	az CP = $\alpha + \gamma$	and az CQ = α

We shall show that

az AC = $\alpha - \gamma$
az BA = $-2\alpha - \gamma$
az CB = $\alpha + 2\gamma$.

These last three equations show that the triangle ABC has angles 3α, 3β and 3γ.

Proof Let Y be the intersection of AR with PC. Since az RY = –az PY the line from Y to the mid-point of RP has azimuth 90°, and since az PQ = –az RQ the line from Q to the same mid-point has the same azimuth. Therefore az QY = 90° and Lemma 1 applies, and az AC = $\alpha - \gamma$.

The azimuths of BA and CB may be deduced by means of the cyclic symmetry as follows. First we may rotate the figure, decreasing all azimuths by $\alpha + \beta + \gamma$, so obtaining points A", B", etc. Now rename these points cyclically:

P" = Q'	Q" = R'	R" = P'
A" = B'	B" = C'	C" = A'.

Also we need to re-name the angles cyclically, $\alpha = \beta'$, etc. We may now write down the construction in the new notation.

az A'Q' = az C"P" = az CP $-\alpha - \beta - \gamma = -\beta = -\gamma'$
az A'R' = az C"Q" = az CQ $-\alpha - \beta - \gamma = -\beta - \gamma = -\alpha' - \gamma'$ etc.

The first part of our theorem applies, and az A'C' $= \alpha' - \gamma'$,
az B"C" = az A'C' $= \alpha' - \gamma' = \gamma - \beta$
az BC = az B"C" $+ \alpha + \beta + \gamma = \alpha + 2\gamma$ as required.
Finally we apply the second result to the rotated picture
az B'C' $= \alpha' + 2\gamma'$.
Therefore az A"B" = az B'C' $= \alpha' + 2\gamma' = 2\beta + \gamma$
and az AB = az A"B" $+ \alpha + \beta + \gamma = \alpha + 3\beta + 2\gamma$
$= -2\alpha - \gamma$.

The Generalized Morley Theorem.

Consider the figure given by Bricard's construction rotated by an angle $\gamma - \alpha$. The azimuths of various lines are as follows.

Line	AB	BC	CA	PQ	QR	RP
Azimuth	-3α	3γ	0	$-2\alpha-\beta$	$\beta+2\gamma$	$\gamma-\alpha$

Line	AQ	AR	BP	BR	CP	CQ
Azimuth	$-\alpha$	-2α	$\beta+3\gamma$	$2\beta+3\gamma$	2γ	γ

.........(1)

Suppose that we are given a triangle with angles (in the usual notation, each between 0° and 180° with sum 180°) A, B and C. There are 18 ways to choose angles α, β and γ (each regarded as modulo 180°) so that $3\alpha = A$, $3\beta = B$, $3\gamma = C$ and $\alpha + \beta + \gamma \neq 0$. The family of 18 possible choices of (α, β, γ) has a transitive symmetry which may be described as follows.

Take any family (k, m, n) of residues modulo 3 such that $k + m + n \not\equiv 2$. There are clearly 18 such families; because of the 27 choices of three residue classes there are nine with sum 2. Sometimes we shall denote this family by kmn for short. Now take any possible choice of (α, β, γ) as described above. It may be seen that

$$(\alpha + k(\alpha + \beta + \gamma),\ \beta + m(\alpha + \beta + \gamma),\ \gamma + n(\alpha + \beta + \gamma)) \quad ....(2)$$

is another possible choice, and all possible choices are of this form.

We shall see that each possible choice of (α, β, γ) gives a Morley triangle, as follows. Bricard's construction gives us a figure containing a triangle similar to the given fixed ABC. Mapping it on to the given fixed triangle ABC gives a figure containing a Morley triangle PQR for ABC; this PQR may be specified by the azimuths listed above in Equation 1. If the chosen angles are replaced by (2) then the azimuth of PQ is increased by $(k - m)(\alpha + \beta + \gamma)$, of QR by $(m - n)(\alpha + \beta + \gamma)$ and of RP by $(n - k)(\alpha + \beta + \gamma)$. This shows that all the 18 Morley triangles are parallel to one another because $\alpha + \beta + \gamma = \pm 60^\circ$.

Notations. Given the triangle ABC we choose any one of the Morley triangles, denoting it by the symbol 000. If this triangle is constructed (as above) from the angles α, β and γ, then the triangle constructed from the angles given in (2) above is denoted by the symbol kmn. The three vertices of kmn are denoted by Pmn, kQn and kmR. These points are specified by their azimuths from two vertices of the triangle ABC. For instance, Pmn has azimuth $\beta + 3\gamma + m(\alpha + \beta + \gamma)$ from B and $2\gamma + 2n(\alpha + \beta + \gamma)$ from C (these values should be clear from (1) and (2)). There are 27 Morley vertices, each a vertex of two Morley triangles. For example P12 is the P vertex of triangles 012 and 112. There is no triangle 212 because $2 + 1 + 2$ is congruent to 2.

The pattern of the 18 triangles. The two Morley triangles 012 and 112, which share the vertex P12, also share two sides.

The side from P12 to 0Q2 of the first is the same line as the side from P12 to 11R of the second. More generally any side of a Morley triangle is a side of five others, as we shall see below.

From (1) we see that the side QR of triangle 000 has azimuth $\beta + 2\gamma$ and therefore the corresponding side (from kQn to kmR) of triangle kmn has azimuth

$$\beta + m(\alpha+\beta+\gamma) + 2(\gamma + n(\alpha+\beta+\gamma+)) = \beta + 2\gamma + (m-n)(\alpha+\beta+\gamma).$$

Similarly the azimuths of the other two sides of kmn are:

$$\text{PR} \quad : \quad \gamma - \alpha + (n-k)(\alpha+\beta+\gamma)$$

$$\text{and PQ} \quad : \quad -2\alpha - \beta + (k-m)(\alpha+\beta+\gamma).$$

From these formulae we may calculate the azimuth of any side of any of the 18 Morley triangles. For example, the side from P12 to 0Q2 in triangle 012 has azimuth

$$-2\alpha - \beta + (0-1)(\alpha+\beta+\gamma) = \beta + 2\gamma.$$

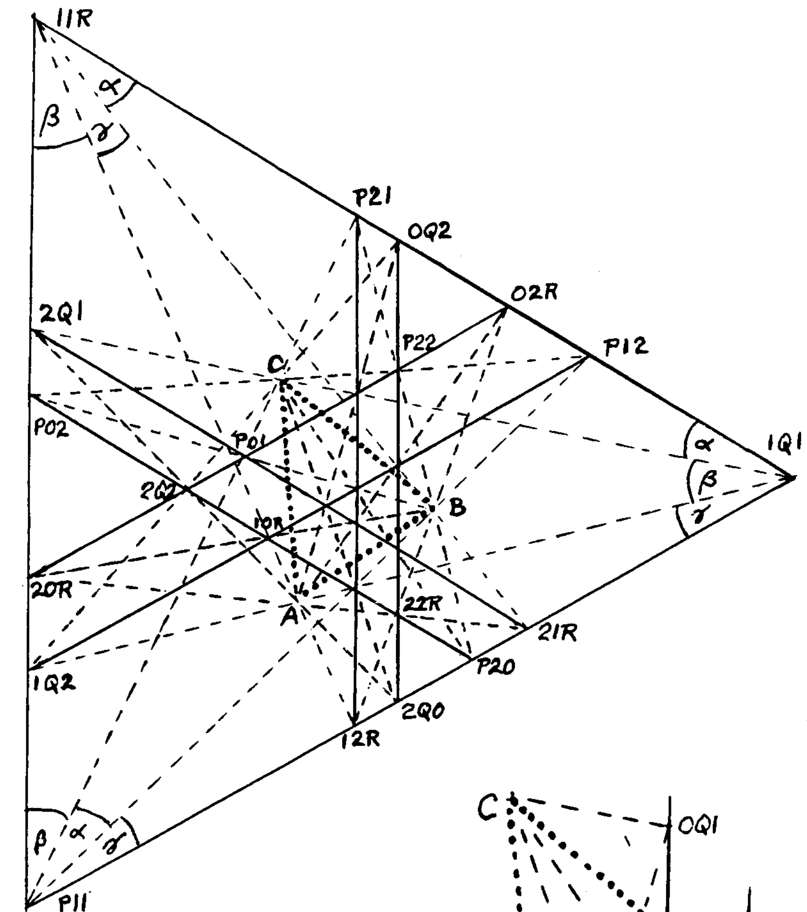
Likewise the side from P12 to 11R in triangle 112 has azimuth

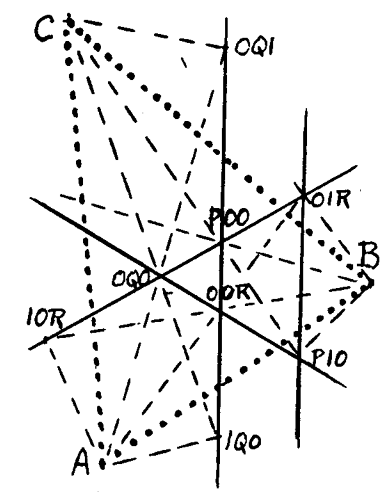
$$\gamma - \alpha + (2-1)(\alpha+\beta+\gamma) = \beta + 2\gamma .$$

These results may be conveniently set out in a table:

Point	to	Point	in	Triangle
P12		11R		112
11R		1Q1		111
1Q1		P21		121
P21		02R		021
02R		0Q2		022
0Q2		P12		012

The azimuths of all these lines are equal to $\beta + 2\gamma$, so that all six points are on a line. There are two other lines parallel to this one, each line being a side of six Morley triangles. One line contains the points P10, 0Q0, 00R, P01, 2Q1 and 21R in the triangles 010, 000, 001, 201, 211 and 210. The other contains points P20, 1Q0, 10R, P02, 2Q2 and 22R in the triangles 120, 100, 102, 202, 222 and 220. The combinatorial relations of the points, lines and triangles are best shown in the picture.





Enlargement of the middle part of the picture above.

COMBINATORIAL PROBLEM GENERALIZED
(JCMN 38, p.4149)

R.N. Buttsworth

The conjecture in the previous issue is true. It is the case $k = 1$ of the following more general result.

Theorem. Let n be any positive integer and k any non-negative integer. Suppose that we have a set of n elements and a class of "special" subsets such that the intersection of any two has no more than k elements. Let A(r) be the number of special subsets that have exactly r elements. Then

$$n(n-1)\ldots(n-k) \geq \sum_{r=k+1}^{\infty} r(r-1)\ldots(r-k)\,A(r)$$

and the result is best possible in the sense that it cannot be replaced by the strict inequality.

Proof. Consider all the $\binom{n}{k+1}$ subsets of size $k+1$. They fall into two classes.

Class 1 : Any such set that is contained in a special subset.

Class 2 : All others.

No member of Class 1 can be contained in two distinct special subsets, and each special subset (with r elements) contains exactly $\binom{r}{k+1}$ subsets of Class 1. The total number of subsets of Class 1 is therefore $\sum_{r=k+1}^{\infty} \binom{r}{k+1} A(r)$, or less.

Therefore $\binom{n}{k+1} \geq \sum_{r=k+1}^{\infty} \binom{r}{k+1} A(r)$ and multiplying both sides by k! gives the result.

To show that the result is best possible take as special subsets all those with $k+1$ elements. Then

$$\sum r(r-1)\ldots(r-k)\,A(r) = (k+1).k.\ \ldots\ 2.1\binom{n}{k+1}$$
$$= n(n-1)\ \ldots\ (n-k).$$

Stop press.

A similar solution has come in from Jamie Simpson.

THE TOMBOLA PROBLEM

J.B. Parker

The game of Tombola (= Bingo = Housey-Housey) is perhaps familiar. Each player has a card with 15 of the numbers from 1 to 99, keen players often taking more than one card. The caller takes at random (without replacement) the numbers from 1 to 99, and announces them to the players. This continues until one of the players wins by having all the numbers on a card called. The number of calls made before the game ends is a random variable between 15 and 99. What is the median of its distribution?

The problem is related to that of finding a good approximation to $\frac{(n-a)!(n-b)!}{n!(n-a-b)!}$ which has a simple interpretation as the probability that two subsets (of a and b elements) chosen independently from a set of n elements will be disjoint.

QUOTATION CORNER

Drivers must be over 21 or under 70 years of age.

From a QANTAS brochure on care hire.

k - FOLD REAL FUNCTIONS

H. Burkill and B.C. Rennie

Let k be a positive integer or ∞. We then call a real valued function f on an interval k-fold if it takes all the values exactly k times. In JCMN 31, p.3180 Marta Sved gave an example of a 2-fold function on $\mathbb{R}$. This function has an infinite number of discontinuities and Marta Sved asked whether, for $k = 2, 3, \ldots$, every k-fold function on $\mathbb{R}$ must be of this kind. We here give half an answer to this question and explore related problems, leaving many unsolved.

<u>k = 2</u>.

In this case we have obtained just one general result.

<u>Theorem 1</u>. Let I be a non-degenerate interval in $\mathbb{R}$ (open, half-open or compact). Then every 2-fold function on I has at least one discontinuity.

Proof.

Suppose that the function $f : I \to \mathbb{R}$ is both 2-fold and continuous.

Take two distinct points $a, b \in I$, with $a < b$, say, such that

$$f(a) = f(b).$$

Since f cannot be constant on [a,b], at least one of

$$\sup_{a \leqslant x \leqslant b} f(x), \qquad \inf_{a \leqslant x \leqslant b} f(x)$$

differs from $f(a)(=f(b))$. Suppose that this is true for the supremum, so that there exists $p \in (a,b)$ such that

$$f(p) = \sup_{a \leqslant x \leqslant b} f(x) > f(a) = f(b).$$

There must now be another point $q \in I$ such that $f(q) = f(p)$.

(i) If $q \in (a,b)$, say $p < q < b$, then $f(x) \leqslant f(p) = f(q)$ for all $x \in [p,q]$ and, since f does not take any values more than

twice, $f(x) < f(p) = f(q)$ for all $x \in (p,q)$. Let u be any point of (p,q), so that

$$f(u) < f(p) = f(q).$$

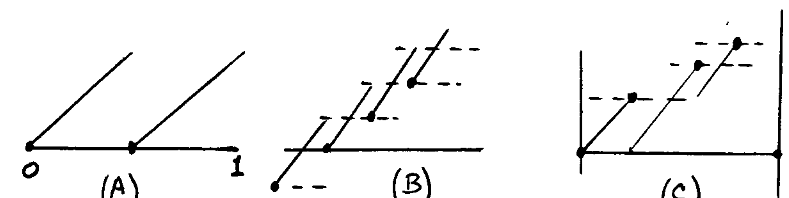
Put
$$X = \max(f(a), f(u)).$$

Then all values in $(X, f(p))$ are taken in each of the intervals $(a,p), (p,u), (u,q), (q,b)$ and this contradicts the hypothesis that f is 2-fold.

(ii) If $q > b$, then all values in $(f(a), f(p))$ are taken in each of the intervals $(a,p), (p,b), (b,q)$ and we again have a contradiction.

(iii) If $q < a$, a contradiction is obtained as in (ii).

When I is half-open, the conclusion of theorem 1 is best possible, as is shown by example A below which exhibits a 2-fold function on $[0,1)$ with just one discontinuity. Examples B and C show 2-fold functions on $\mathbb{R}$ and $[0,1]$ respectively, each with infinitely many discontinuities.



The function in (B) is, of course, given by

$$f(x) = 2x - \lfloor x \rfloor,$$

where $\lfloor x \rfloor$ denotes the integral part of x.

For a given k, denote by $\lambda(k)$ the minimum number of discontinuities which a k-fold function on an open interval must have; and denote by $\mu(k), \nu(k)$ the corresponding numbers

for half-open and compact intervals, respectively.
We have shown that

$$1 \leq \lambda(2) \leq \infty, \quad \mu(2) = 1, \quad 1 \leq \nu(2) \leq \infty.$$

$3 \leq k < \infty$.

Theorem 1 hold for all types of intervals; the next theorem specifically excludes open intervals.

Theorem 2. Let $k \in \{3,4,\dots\}$ and let I be a non-degenerate compact or a half-open interval. Then every k-fold function on I has at least one discontinuity.

Proof.

Suppose that the function f is k-fold and continuous on I.

(i) Let I be compact. Then f assumes its supremum Y at k points, say $a_1, a_2, \dots, a_k$, where

$$a_1 < a_2 < \dots < a_k.$$

Since f is not constant in $[a_i, a_{i+1}]$. the infimum of f in this interval differs from $f(a_i)(= f(a_{i+1})$ and is therefore attained at some point b_i in (a_i, a_{i+1}). If

$$X = \max_{1 \leq i \leq k-1} f(b_i),$$

every value in (X,Y) is attained in each of the $2(k-1)$ intervals

$$(a_1, b_1), (b_1, a_2), \dots, (a_{k-1}, b_{k-1}), (b_{k-1}, a_k).$$

As $k > 2$, $2(k-1) > k$ and we have a contradiction.

(ii) Let I be a half-open interval; we may clearly take I to be a bounded interval of the form $[a,b)$.

The function $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$\phi(x) = \frac{x}{|x| + 1} \quad (x \in \mathbb{R})$$

is continuous, strictly increasing and bounded. Hence the function $g : [a,b) \longrightarrow \mathbb{R}$ given by

$$g(x) = \frac{f(x)}{|f(x)| + 1} \quad (a \leq x \leq b)$$

is bounded, it is such that $g(x) = g(y)$ if and only if $f(x) = f(y)$, and it is continuous at a point if and only if f is continuous at that point. We may therefore assume that our original function f is bounded.

If f(x) does not tend to a limit as $x \to b-$, let

$$\alpha = \liminf_{x \to b-} f(x), \quad \beta = \limsup_{x \to b-} f(x),$$

so that $\alpha < \beta$. Then f takes every value (strictly) between α and β infinitely many times in the interval $[a,b)$, which is impossible.

It follows that f(x) tends to a limit as $x \to b-$. Call the limit ρ and extend the definition of f to [a,b] by putting $f(b) = \rho$. Then f is continuous on [a,b] and f takes all its values, expect ρ, k times. At least one of

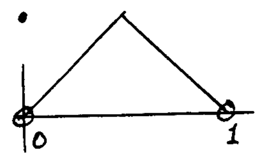
$$\inf_{a \leq x \leq b} f(x), \quad \sup_{a \leq x \leq b} f(x)$$

is not ρ. Using the argument of (i) we therefore again arrive at a contradiction.

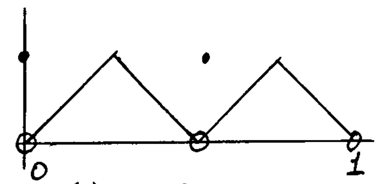
Theorem 3. Let I be a half-open interval. Then, for each $k \in \{2,3,4,\dots\}$, there is a k-fold function on I with $\lfloor \frac{1}{2}k \rfloor$ discontinuities.

Proof.

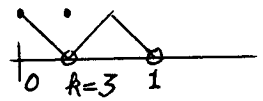
The two types of functions, for even and for odd k, are illustrated below. Example A is simpler than D, but its generalizations have unnecessarily many discontinuities.



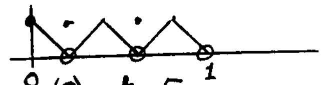
(D) $k=2$



(E) $k=4$



(F) $k=3$

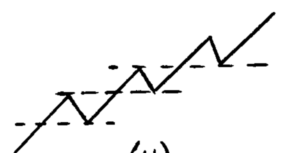


(G) $k=5$

Theorem 4. Let I be an open interval. Then, for each $k \in \{3,5,7,\dots\}$, there is a continuous k-fold function I.

Proof.

We may assume that $I = \mathbb{R}$. The case $k = 3$ illustrates the general construction.



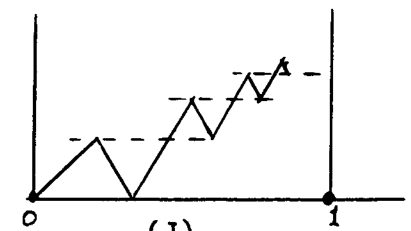
(H)

For any $k \in \{2,3,\dots\}$, the function f on $\mathbb{R}$ given by

$$f(x) = kx - (k-1)\lfloor x \rfloor \quad (x \in \mathbb{R})$$

provides an example of a k-fold function on $\mathbb{R}$ with infinitely many discontinuities. After theorem 4 this is significant only for even k.

A variant of example H shows that there is a 3-fold function on a compact interval with one discontinuity. Curiously, the construction is not readily generalized to the cases $k = 5,7,\dots$ However, for any $k \in \{3,4,5,\dots\}$, example C can be adapted to yield a k-fold function on [0,1] with infinitely many discontinuities.



(I)

Theorems 2-4 and example I provide us with all the desired information about 3-fold functions:

$$\lambda(3) = 0, \quad \mu(3) = 1, \quad \nu(3) = 1.$$

For larger values of k our knowledge is much less complete:

When $k = 5,7,9,\dots$,

$$\lambda(k) = 0, \quad 1 \leqslant \mu(k) \leqslant \tfrac{1}{2}(k-1), \quad 1 \leqslant \nu(k) \leqslant \infty;$$

and, when $k = 4,6,8,\dots$,

$$0 \leqslant \lambda(k) \leqslant \infty, \quad 1 \leqslant \mu(k) \leqslant \tfrac{1}{2}k, \quad 1 \leqslant \nu(k) \leqslant \infty.$$

Thus the major mystery is $\lambda(k)$ for even k greater than 2.

$k = \infty$.

Constant functions show that $\lambda(\infty) = \mu(\infty) = \nu(\infty) = 0$. However, it is also easy to find non-constant continuous functions on open or half-open intervals which take all their values infinitely often; the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : (0,1] \to \mathbb{R}$ given by

$$f(x) = \sin x \ (x \in \mathbb{R}), \quad g(x) = \sin(1/x) \quad (0 < x \leqslant 1)$$

provide obvious examples. It is much less evident that there exists a non-constant continuous function on a compact interval which takes all its values infinitely many times.

In fact, a continuous function on [0,1] has been constructed which is such that any non-vertical straight line meets its graph either not at all or in an uncountable set. (J. Gillis, Note on a conjecture of Erdös, Quart.J.Math.Oxford 10 (1939), 151-154.) If one is merely interested in horizontal lines, as we are in the present context, then the function and its construction can be considerably simplified.

Theorem 5. a k-fold function (k finite) with only a finite number of discontinuities can have only simple discontinuities.

Proof.

Let f be a k-fold function on the interval I with only a finite number of discontinuities. Let $c \in I$ (where c is not the left end point of I) and suppose that

$$\lim_{x \to c-} f(x)$$

does not exist; thus

$$\alpha = \liminf_{x \to c-} f(x) < \limsup_{x \to c-} f(x) = \beta .$$

Take any real number γ such that

$$\alpha < \gamma < \beta$$

Then, given $\varepsilon > 0$, there exist $u, v \in (c-\varepsilon, c)$ such that

$$f(u) < \gamma \quad \text{and } f(v) > \gamma .$$

Also, since f has only a finite number of discontinuities, there exists $J = (c-\delta, c)$ such that f is continuous in J.

Now take $u_1 \in J$ such that $f(u_1) < \gamma$. Then take $u_2 \in (u_1, c)$ such that $f(u_2) > \gamma$, take $u_3 \in (u_2, c)$ such that $f(u_3) < \gamma$, etc.

Since f is continuous in each interval $[u_i, u_{i+1}]$, for $i = 1,2,\ldots$, there exists $x_i \in (u_i, u_{i+1})$ such that $f(x_i) = \gamma$. The x_i are distinct and so f takes the value γ infinitely often, which is impossible. N.B. The hypothesis that f has only a finite number of discontinuities can clearly be relaxed to c being an isolated discontinuity.

PILLOW PROBLEMS

For those who have not read the book of C.L. Dodgson (or Lewis Carrol) it should be explained that a pillow problem should be thought out in bed with the eyes tightly shut. Here are two.

Can Morley's theorem be extended to spherical triangles?

Take a tetrahedron ABCD; we trisect (internally) the angles between each pair of faces and (as in Morley's theorem) define four points PQRS as follows. S is defined by the plane SBC making with ABC half the angle that it makes with DBC, SCA making with ABC half the angle that it makes with ADC, and SAB making with ABC half the angle that it makes with DAB. P, Q and R are defined similarly.

Is PQRS regular?

QUOTATION CORNER

"The unreasonable effectiveness of mathematics in science...."

Eugene Wigner

"What is now proved was once only imagined."

William Blake

The quotations above are from C.J. Smyth who comments that the second does not refer to the Riemann Hypothesis.

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EDITORIAL

The JCMN for its first eight years, 1975-1983, was published by the Mathematics Department of the James Cook University of North Queensland, address:

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Volume 2 (Issues 18-24)
Volume 3 (Issues 25-31)

These volumes are available for $10 (Australian) each, including postage, from the Head of the Mathematics Department. I should explain that I am now Head of Department, but will retire at the end of December and leave the University. Since Issue 32 (October 1983) I have edited and published JCMN. In 1986 my wife and I plan to leave Townsville and go to

66 Hallett Road, Burnside,
South Australia, 5066, Australia.

but this issue (39, dated February, 1986) is being prepared in Townsville in December 1985 (address as on page 4162).

Basil Rennie