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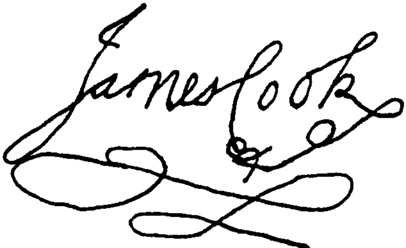

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A. P. GUINAND

Sadly we note that A.P.Guinand died on 22nd. March 1987.

FROM CAPTAIN COOK'S JOURNAL

From what I have said of the Natives of New Holland they may appear to some to be the most wretched People upon Earth; but in reality they are far more happier than we Europeans, being wholy unacquainted not only with the Superfluous, but with the necessary Conveniences so much sought after in Europe; they are happy in not knowing the use of them. They live in a Tranquility which is not disturbed by the Inequality of Condition. The earth and Sea of their own accord furnishes them with all things necessary for Life. They covet not Magnificient Houses, Household-stuff, etc.; they live in a Warm and fine Climate, and enjoy every wholesome Air, so that they have very little need of Cloathing; and this they seem to be fully sencible of, for many to whom we gave Cloth, etc., left it carelessly upon the Sea beach and in the Woods, as a thing they had no manner of use for; in short, they seem'd to set no Value upon anything we gave them, nor would they ever part with anything of their own for any one Article we could offer them. This, in my opinion, Argues that they think themselves provided with all the necessarys of Life, and that they have no Superfluities.

INTEGRAL INEQUALITY (JCMN 42, p.5023, 43, p.5028)

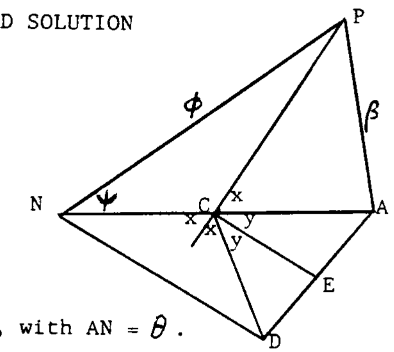
The problem was if $0 < \theta < \pi/2$ and $I = I(\theta) =$

$$\int_{\phi=0}^{\pi} \sin^2\phi \, d\phi \int_{\psi=0}^{\pi} \{1 - (\cos\theta \cos\phi + \sin\theta \sin\phi \cos\psi)^2\}^{\frac{1}{2}} d\psi$$

to find inequalities for I, such as $I \geq (4\pi/3)\cos\theta$.

THIRD SOLUTION

Diagram on the unit sphere:



Points A and N are fixed, with $AN = \theta$.
Point C is the mid-point of AN.

$$\cos\beta = \cos\theta \cos\phi + \sin\theta \sin\phi \cos\psi \quad \text{(the cosine rule)}$$

The integral I is the surface integral of $\sin\phi \sin\beta$ as P varies over the hemisphere where $0 < \phi < \pi$ and $0 < \psi < \pi$. Also I is half the integral over the whole sphere, or 2π times the integral mean of the same product.

Let C be the mid-point of AN, and let D be the mirror image of N in the great circle PC, so that the triangle DPN is isosceles. Also ACD and NCD are isosceles, for $AC = \theta/2 = NC = DC$. Let E be the mid-point of AD, then CE is perpendicular to AD, and the sine rule for the triangle ACE gives

$$\sin AE = \sin\frac{\theta}{2} \sin y$$

The three angles marked x above are all equal, and $x + y = \pi/2$.

$$\sin AD = \sin 2AE \leq 2 \sin AE = 2 \sin\frac{\theta}{2} \cos x$$

$$\sin^2 AD \leq 2(1 - \cos\theta)\cos^2 x$$

Now apply the spherical triangle inequality (see page 5056 in this issue) to the triangle ADP.

$$|\sin\phi - \sin\beta| \leq \sin AD$$

$$0 \leq \sin^2\phi - 2\sin\phi\sin\beta + \sin^2\beta \leq 2(1-\cos\theta)\cos^2 x$$

$$\sin^2\phi + \sin^2\beta - 2(1-\cos\theta)\cos^2 x \leq 2\sin\phi\sin\beta \leq \sin^2\phi + \sin^2\beta$$

Write the integral mean of this pair of inequalities over the sphere (or over the hemisphere). Note that $\mathrm{IM}\sin^2\phi = \mathrm{IM}\sin^2\beta = 2/3$, and $\mathrm{IM}\cos^2 x = \frac{1}{2}$.

$$4/3 - 1 + \cos\theta \leq I/\pi \leq 4/3$$

$$(4\pi/3)\cos\theta \leq (\pi/3)(1+3\cos\theta) \leq I \leq 4\pi/3$$

NON-LINEAR SUMMATION OF SERIES (JCMN 43, p. 5039)

J. B. Parker writes that in the folk-lore of numerical analysis there is the "Aitken Extrapolation"

$$(S_{n-1}^2 - S_n S_{n-2})/(2S_{n-1} - S_n - S_{n-2})$$

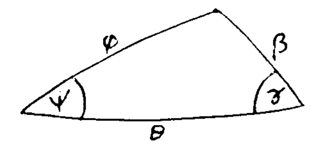
which is an estimate for the limit of a sequence $S_1, S_2, \ldots$ Denote this by A_n. Then in the notation of the article in our previous issue, where $S_n = t_1 + t_2 + \ldots + t_n$, we find

$$A_{n+2} = S_n + t_{n+1}^2/(t_{n+1} - t_{n+2})$$

which is the S_n^* given in equation (1).

FOURTH SOLUTION

J. B. Parker



The integrand $= \sin^2\phi\,(1 - (\cos\theta\cos\phi + \sin\theta\sin\phi\cos\psi)^2)^{\frac{1}{2}}$

$$= \sin^2\phi\sin\beta = \sin^3\phi\sin\psi\operatorname{cosec}\gamma,$$

but there is a formula for spherical triangles:

$$\cot\phi\sin\theta = \cos\theta\cos\psi + \cot\gamma\sin\psi$$

and so $\quad(\sin\psi\operatorname{cosec}\gamma)^2 = \sin^2\psi + \sin^2\psi\cot^2\gamma$

$$= \sin^2\psi + (\cot\phi\sin\theta - \cos\theta\cos\psi)^2$$

and the integrand may be written as

$$\sin^3\phi\,(\sin^2\psi + (\cot\phi\sin\theta - \cos\theta\cos\psi)^2)^{\frac{1}{2}}$$

One does not have to go into spherical trigonometry to establish this formula, it can be proved by ordinary methods, but I would never have spotted it without spherical trig.

The advantage in having the integrand as the square root of a sum (rather than a difference) of squares is shown by the lemma following.

Lemma Let $u(t)$ and $v(t)$ be real functions, let integration be over some set which need not be specified, let $\int u(t)dt = a$ and $\int v(t)dt = b$. Then

$$\int (u^2 + v^2)^{\frac{1}{2}} dt \geq (a^2 + b^2)^{\frac{1}{2}}$$

This inequality may be seen geometrically as follows. Suppose that there is a mass-distribution on the surface of the cone $(x^2+y^2)^{\frac{1}{2}} = z > 0$. Then the centroid is inside the cone. In this form the result is almost so obvious as not to need proving. However, a purely analytical proof may be constructed as follows. For any angle x it is clear that $(u^2 + v^2)^{\frac{1}{2}} \geq u \cos x + v \sin x$, because the difference of the squares of the two sides is $(u \sin x - v \cos x)^2$. Therefore

$$\int (u^2 + v^2)^{\frac{1}{2}} dt \geq a \cos x + b \sin x$$

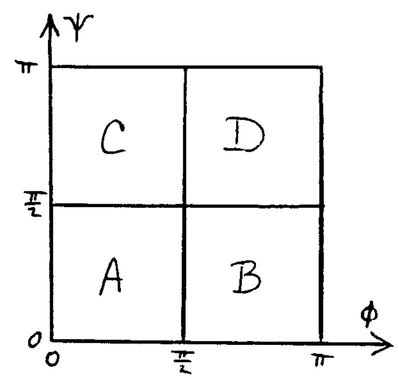
Since $(a \cos x + b \sin x)^2 + (a \sin x - b \cos x)^2 = a^2 + b^2$, we choose x so that $a \sin x = b \cos x$, and the result is obtained. The lemma applies equally to double or repeated integrals.

Applying the lemma to our integral $I(\theta)$, we have

$$u(\phi, \psi) = \sin^3\phi \sin\psi$$
$$v(\phi, \psi) = \sin^2\phi \cos\phi \sin\theta - \sin^3\phi \cos\psi \cos\theta$$

Using the range of integration $0 < \phi < \pi$ and $0 < \psi < \pi$, we find $a = 8/3$ and $b = 0$, so that $I(\theta) \geq 8/3$.

To obtain a sharper result we may split the region of integration into four, as shown in the diagram. For all the regions a has the same value of 2/3.



$$b = \tfrac{\pi}{6}\sin\theta - \tfrac{2}{3}\cos\theta \quad \text{for A}$$
$$b = \tfrac{\pi}{6}\sin\theta + \tfrac{2}{3}\cos\theta \quad \text{for C}$$
$$b = -\tfrac{\pi}{6}\sin\theta - \tfrac{2}{3}\cos\theta \quad \text{for B}$$
$$b = -\tfrac{\pi}{6}\sin\theta + \tfrac{2}{3}\cos\theta \quad \text{for D}$$

Applying the lemma to each of the four regions gives

$$\text{Integral over A (or D)} \geq (16 + (\pi \sin\theta - 4\cos\theta)^2)^{\frac{1}{2}}/6$$
$$\text{Integral over B (or C)} \geq (16 + (\pi \sin\theta + 4\cos\theta)^2)^{\frac{1}{2}}/6$$
$$I(\theta) \geq \tfrac{1}{3}\{(16 + (\pi\sin\theta - 4\cos\theta)^2)^{\frac{1}{2}} + (16 + (\pi\sin\theta + 4\cos\theta)^2)^{\frac{1}{2}}\}$$

FIFTH SOLUTION

Vichian Laohakosol

For $0 \leq \theta \leq \pi/2$ we are interested in

$$I(\theta) = \int_0^\pi \int_0^\pi \sin^2\phi \, (1 - \cos^2\beta)^{\frac{1}{2}} \, d\phi \, d\psi$$

where $\cos\beta = \cos\theta \cos\phi + \sin\theta \sin\phi \cos\psi$.

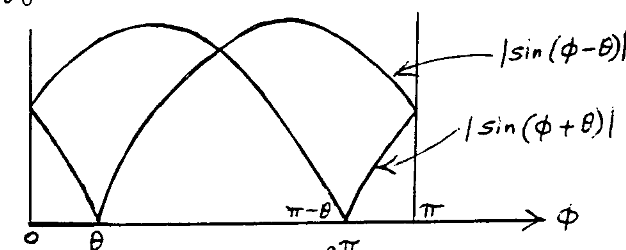
First note that $\cos(\phi + \theta) \leq \cos\beta \leq \cos(\phi - \theta)$

and so $\cos^2\beta \leq \max\{\cos^2(\phi + \theta), \cos^2(\phi - \theta)\}$

and $\sin\beta \geq \min\{|\sin(\phi + \theta)|, |\sin(\phi - \theta)|\}$

which is independent of ψ.

$$I(\theta) \geq \pi \int_0^\pi \sin^2\phi \min\{|\sin(\phi + \theta)|, |\sin(\phi - \theta)|\} d\phi$$



$$I/\pi \geq \int_0^{\pi/2} \sin^2\phi \, |\sin(\phi - \theta)| d\phi + \int_{\pi/2}^{\pi} \sin^2\phi \, |\sin(\phi + \theta)| d\phi$$

$$\text{First integral} = \int_0^\theta \sin^2\phi \sin(\phi - \theta) d\phi + \int_\theta^{\pi/2} \sin^2\phi \sin(\phi - \theta) d\phi$$
$$= 1 + (\cos 2\theta - 2\cos\theta - \sin\theta)/3$$

The second integral (by putting $\pi - \phi$ for ϕ) can be seen to have the same value. Consequently

$$I(\theta) \geq (2\pi/3)(3 + \cos 2\theta - 2\cos\theta - \sin\theta)$$

MORE ON THE INTEGRAL INEQUALITY

A. Brown

$$I(\theta) = I(2u) = 4\int_0^{\pi/2}\int_0^{\pi/2} \sin w \sqrt{Q(u, v, w)}\, dv\, dw$$

where $u = \theta/2$, and $Q = \cos^4 w + 2\cos^2 w \sin^2 w(\sin^2 v \cos^2 u + \cos^2 v \sin^2 u) + \sin^4 w(\sin^2 v \cos^2 u - \cos^2 v \sin^2 u)$ is the integral investigated in JCMN 43, p.5028-5031 and in this issue.

$$Q(u, v, w) = Q(\pi/2-u, \pi/2-v, w)$$

so that $I(\theta) = I(2u) = I(\pi - 2u) = I(\pi - \theta)$ which shows that $I(\theta)$ (if differentiable) has a stationary value at $\theta = \pi/2$. Is this a maximum or a minimum? There are difficulties in differentiating under the integral sign because the Q in the integrand has a zero at $v = u$, $w = \pi/2$, so that its square root is not differentiable. However, a careful investigation shows that $I(\theta)$ has a local minimum at $\theta = \pi/2$. Whether it is a lower bound for the interval remains to be determined.

$$I(\pi/2) = 2\int_0^{\pi/2} \sin w\,(1 + \cos^2 w)\,E(k)\,dw$$

where $k = (1 - \cos^2 w)/(1 + \cos^2 w)$ and $E(k) = \int_0^{\pi/2} (1 - k^2 \sin^2 x)^{\frac{1}{2}}\,dx$, which is the complete elliptic integral of the second kind. A numerical integration using tables of E(k) gave $\frac{1}{4} I(\pi/2) = 0{\cdot}93182$.

It is possible to expand E(k) as a power series in k^2 (though strictly this is permissible only for $k^2 < 1$) and then to obtain expressions for integrals of the type

$$P_{2n} = \int_0^{\pi/2} \sin w\,(1 + \cos^2 w)\,k^{2n}\,dw$$

and this gives a slowly converging series for $I(\pi/2)$. The first eight terms were calculated and Lubkin's method was used to approximate to the sum of the "tail" (from the seventh term onwards). This gave $\frac{1}{4}I(\pi/2) = 0{\cdot}93183$, agreeing reasonably well with the numerical integration.

STILL MORE ON THE INTEGRAL INEQUALITY

The Editor's "Peach" computer has produced the following values for the double integral:

θ	0	·1	·2	·3	·4	·5
$I(\theta)$	4·189	4·184	4·169	4·145	4·113	4·075
θ	·6	·7	$\pi/4$	·8	·9	1·0
$I(\theta)$	4·032	3·987	3·948	3·942	3·897	3·854
θ	1·1	1·2	1·3	1·4	1·5	$\pi/2$
$I(\theta)$	3·817	3·784	3·758	3·740	3·730	3·727

The programming of the Monte Carlo method brought up a pure mathematical problem (see Numerical Number Theory, p.5061) The calculations may be a little suspect until that problem is solved, but by running all night, using a hundred thousand points, we obtained reasonable agreement with A. Brown's two estimates above for the case of $\theta = \pi/2$, getting 0·93183 for $\frac{1}{4}I(\pi/2)$. In finding the other values above, ten thousand points seemed enough to give the third decimal place with low probability of error, (more precisely, to give confidence that the error is less than one in the last decimal place).

SPHERICAL TRIANGLE INEQUALITY (JCMN 43, p.5032)

If the sides of a spherical triangle are a, b and c, then

$$\sin a \leq \sin b + \sin c$$

If the triangle is non-degenerate (the points are not on a great circle, all the sides and angles are strictly between 0 and π) then the triangle inequality becomes strict, with $<$ instead of $\leq$.

Lemma If $u > 0$ and $|u-v| \leq 1$ and $|u+v| \leq 1$, then the quadratic $f(x) = ux^2 + 2vx + u + 2 \geq 0$ in the closed interval $[-1, 1]$ and > 0 in the open interval $(-1, 1)$.

Proof Case 1, when $|v| > u$. Note that $f(1) = 2u+2v+2 \geq 0$ and $f(-1) = 2u-2v+2 \geq 0$. The minimum of the function f is at $x = -v/u$, outside the closed interval $[-1, 1]$. Therefore f is monotonic in the interval and the results follow.

Case 2, when $|v| \leq u$.

$$u f(x) = u^2x^2 + 2uvx + u^2 + 2u = (ux + v)^2 + u^2 - v^2 + 2u > 0$$

Apply the lemma with $u = \sin a \sin b$, $v = \cos a \cos b$ and $x = \cos C$. By the cosine rule $\cos c = v + ux$.

$$\cos^2 c = (ux + v)^2 = u f(x) - u^2 + v^2 - 2u \geq v^2 - u^2 - 2u,$$

$$\sin^2 c \leq 1 - v^2 + u^2 + 2u$$

$$= \sin^2 a + \cos^2 a - \cos^2 a \cos^2 b + \sin^2 a \sin^2 b + 2 \sin a \sin b$$

$$= \sin^2 a + \cos^2 a \sin^2 b + \sin^2 a \sin^2 b + 2 \sin a \sin b$$

$$= (\sin a + \sin b)^2.$$

Note that for a non-degenerate triangle $u f(x) > 0$ and the inequality becomes strict.

OTHER SPHERICAL TRIANGLE INEQUALITIES

A. Brown

If the sides of a non-degenerate spherical triangle ABC are a, b and c, then they satisfy three triangle inequalities,

$$a < b + c \qquad b < c + a \qquad c < a + b$$

These follow from the fact that the shortest curve between two points on a sphere is a great circle. The sides also satisfy a fourth inequality

$$a + b + c < 2\pi$$

This may be proved by considering the triangle A_1BC, where A_1 is the point diametrically opposite to A, the sides are a, π-b and π-c, and therefore $a < (\pi - b) + (\pi - c)$.

An intuitive understanding of the four inequalities comes from thinking about a loop of string with knots at points A, B and C, the distances between the knots being BC = a, etc. If the a, b and c satisfy the four inequalities then the loop of string will not pass over a unit sphere, and if the string is laid on the sphere and pulled tight (pulling at the knots) then the string forms a spherical triangle with sides a, b and c. The four inequalities are necessary and sufficient for the existence of a spherical triangle with these three sides.

The relation $\sin a < \sin b + \sin c$ follows analytically from the four inequalities above. To prove this note that each of the sides a, b and c must be between 0 and π. Also

$$\sin\frac{b+c}{2} - \sin\frac{a}{2} = 2\sin\frac{b+c-a}{4}\cos\frac{a+b+c}{4} > 0$$

and $$\cos\frac{b-c}{2} - \cos\frac{a}{2} = 2\sin\frac{a+b-c}{4}\sin\frac{a-b+c}{4} > 0.$$

Therefore $\quad 0 < \sin\frac{a}{2} < \sin\frac{b+c}{2}$

and $\quad 0 < \cos\frac{a}{2} < \cos\frac{b-c}{2}$

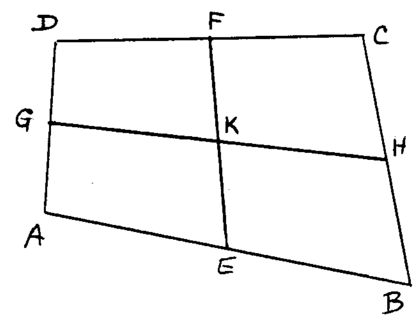
Multiplication gives $\quad 0 < \frac{1}{2}\sin a < \frac{1}{2}(\sin b + \sin c)$

which is the required inequality.

HYPERBOLOID AREAS

Jordan Tabov

Let ABCD be a plane quadrilateral, and let the segments joining the mid-points of opposite sides divide it into four quadrilaterals AEKG, CFKH, BHKE and DGKF, as shown.



It is well known that the sum of the areas of the first two of them equals the sum of the areas of the other two.

Is this result also true for skewed quadrilaterals? More precisely: every skew quadrilateral ABCD in R^3 determines a unique parabolic hyperboloid $\mathcal{H}$ which contains all four sides of ABCD. The segments GH and EF joining the mid-points of the opposite sides lie in $\mathcal{H}$. So we have again four quadrilaterals, now curved surfaces on $\mathcal{H}$. Are their areas connected as in the plane case?

CALCULATING PRIMES

Alastair Rennie

The following Basic program uses only addition, with no multiplication or division, to calculate prime numbers.

```
10 DEFINT P,Q
20 INPUT"How many primes";N
30 DIM P(N+1),Q(N)
40 P(1)=1
50 FOR I=1 TO N
60 P(I+1)=P(I)
70 Q(I)=P(I)+P(I)
80 PRIME=0
90 P(I+1)=1+P(I+1)
100 FOR J = 2 TO I
110 IF Q(J)=P(I+1) THEN Q(J)=Q(J)+P(J):PRIME=1
120 NEXT J
130 IF PRIME=1 THEN 80
140 PRINT P(I+1);:NEXT I
150 END
```

What are we doing to celebrate the number of the year (A.D.) being the 300th prime?

BINOMIAL IDENTITY 20

If $n < m$ then $\quad \sum_{r=n+1}^{m} 2^{m-r}\binom{r-1}{n} = \sum_{r=n+1}^{m}\binom{m}{r}$

HOLIDAY PROBLEM (JCMN 43, p. 5034)

Seven people each made three visits to a holiday resort. During these visits each met each of the others. Prove that at one time there were three of them at the resort.

Solution Suppose that there were never more than two of the people there together. This will lead to a contradiction. For each of the $\binom{7}{2} = 21$ pairs choose a time when they met. These 21 times are all distinct. Arrange them in order, $t_1 < t_2 < \dots < t_{21}$. At least two people must have arrived before t_1. At least one must have arrived between t_1 and t_2 because one of the pair that met at t_2 was not there at t_1. Similarly one must have arrived between t_2 and t_3, etc. This makes at least 22 arrivals, contradicting the given fact that there were precisely $7 \times 3 = 21$ arrivals.

THOUGHTS ON THE HOLIDAY PROBLEM

Sahib Ram Mandan

Suppose that there were three people together on each visit. Number the people 1, 2, ... 7. For example suppose that 1's first visit was shared with 2 and 5, and then 2, 3 and 7 were together, then 3, 5 and 4, etc., as shown in the array:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 5 & 6 & 7 & 1 & 4 \\ 5 & 7 & 4 & 2 & 6 & 3 & 1 \end{pmatrix}$$

This is the 7-point geometry of Fano, with 7 lines, 3 points on each line and 3 lines through each point. Each entry in the array represents a point, and each column represents a line. Every 2 points are joined by one line, every 2 lines meet in one point. It can be regarded as the plane projective geometry over the Galois field of residue classes modulo 2.

NUMERICAL NUMBER THEORY

I wanted to evaluate the double integral of the "Integral Inequality" (JCMN 42, p.5023, 43, p.5028, and in this issue) using a Monte Carlo numerical integration. To obtain a pseudo-random sequence of values of ϕ and ψ (actually only their sines and cosines, the angles themselves not being needed) I took $\phi = 2n\tan^{-1}\frac{1}{2}$ and $\psi = 2n\tan^{-1}\frac{1}{4}$ for $n = 1, 2, 3, \dots$ with the necessary modification to make $\sin\phi$ always positive. The method will work only if the Diophantine equation

$$x\pi + y\tan^{-1}\tfrac{1}{2} + z\tan^{-1}\tfrac{1}{4} = 0$$

has no solution in integers except the trivial $x = y = z = 0$. Can this be proved?

The numbers ½ and ¼ in the equation above are important (apart from being the only fractions provided on my typewriter) for if ¼ were changed to 1/3 the equation would become soluble.

SPHERICAL TRIANGLES

Spherical triangles have much in common with plane triangles. The cosine and sine rules for the latter emerge as limiting cases of those for the former as the size of the triangle becomes small relative to the sphere. Nevertheless we (this is the editorial "we") know much less about the geometry of spherical triangles.

Are the altitudes concurrent? Are the medians concurrent? If so, calling the points of intersection the orthocentre and centroid respectively, do these two points and the circumcentre lie on a great circle? This question is natural because the corresponding points of a plane triangle are on the Euler axis. Are the mid-points of the sides and the feet of the altitudes concyclic? In plane triangles these six points are on the nine-point circle. Is there a formula for the circum-radius or the in-radius in terms of the sides and angles?

FUNCTIONAL EQUATION

R. L. Agacy

Let f(x) be a real function of the real variable such

that $f(f(x)) = x$

and $f(x+y) = f(x) + f(y)$

and $f(xy) = f(x)\,f(y)$

for all real x and y.

Prove that $f(x) = x$.

What can you say if f is a complex function of the complex variable with the same three properties?

BACK NUMBERS

The JCMN for its first eight years (1975-1983) was published by the Mathematics Department of the James Cook University. Issues 1-31 of this period have been reprinted in three paperback volumes, on sale for 10 Australian dollars each, postage included, from

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Since 1983 the JCMN has had no connection with the James Cook University. For issues from 32 onwards the back numbers are available from me at

66, Hallett Road,
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Basil Rennie