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CONTENTS

NON LINEAR SUMMATION OF SERIES

(JCMN 43, p.5039 and 44, p.5050)

A. Brown

Another recipe for summing a series is Lubkin's method (Journal of Research of the National Bureau of Standards, 48 (1952), 228-254). In his paper, Lubkin speaks of it as something that had been around the National Bureau of Standards for some time and which might be of use to a wider public. I know of it from a paper by J.E.Drummond (J. Aust. Math. Soc. Series B, 19 (1976), 416-421), which mentions it in a wider context and compares the effectiveness of some methods of this type. The version below arises from my own attempts to understand Lubkin's formula.

If $v(n) = 1/n$ and $\Delta F(n) = F(n) - F(n+1)$, then $\Delta v(n) = 1/(n(n+1))$ and this can be extended to show that

$$V(m, n) = \Delta^m v(n) = (m!)/\{n(n+1)(n+2) \ldots (n+m)\},$$

$$Z(m, n) = \sum_{j=n}^{\infty} V(m, j) = \Delta^{m-1} v(n) = V(m-1, n).$$

If we think of $V(m, j)$ as the j-th term of a series, then $Z(m, n)$ gives the sum of the tail of the series, from the n-th term onwards, and the ratio of the (n+1)-th to the n-th term is given by

$$R_{n+1} = V(m, n+1)/V(m, n) = n/(n + m + 1).$$

This ratio approaches 1 as $n \to \infty$ so we are dealing with a series which converges slowly, although it does converge for $m \geqslant 1$. We note also that

$$\frac{Z(m, n)}{V(m, n)} = \frac{n + m}{m} = \frac{(m+1)/m}{1 - R_n}, \qquad \ldots (A)$$

and $$Z(m, n) - V(m, n) = Z(m, n+1). \qquad \ldots (B)$$

From equation (A),

$$\begin{aligned}(1 - R_{n+1})\, Z(m, n+1) &= (1 + 1/m)\, V(m, n+1)\\ &= (1 + 1/m)\, V(m, n)\, R_{n+1}\\ &= (1 - R_n)\, Z(m, n)\, R_{n+1}.\end{aligned}$$

Substituting for $Z(m, n+1)$ in equation (B) leads to

$$Z(m, n) = V(m, n)\,(1 - R_{n+1})/(1 - 2R_{n+1} + R_n R_{n+1}) \qquad \ldots (C)$$

This means that for $m = 1, 2, 3, \ldots$ the summation $Z(m, n)$ can be written down from three consecutive terms of the series.

So far, the analysis is exact. For large values of n,

$$V(m, n) \sim m! / n^{m+1}$$

and this suggests that we can use equation (C) to evaluate the sum from n to infinity for any series $\sum u_n$ for which

$$u_n \sim (\text{constant})/(\text{power of } n)$$

when n is large. The next step, of course, is to forget about the argument which led to equation (C) and use this recipe for any slowly convergent series of positive terms. As in other examples in JCMN 43, the recipe turns out to be exact for a convergent geometric series.

NUMERICAL NUMBER THEORY

(JCMN 44, p.5061)

Terry Tao

We want to show that the Diophantine equation

$$x\pi + y\tan^{-1}\tfrac{1}{2} + z\tan^{-1}\tfrac{1}{4} = 0$$

has no integer solution except the trivial $x = y = z = 0$.

Define four sequences a_n, b_n, c_n, d_n, for integers $n \geq 0$ as follows.

$a_0 = 0$ $\quad b_0 = 1$ $\quad c_0 = 0$ $\quad d_0 = 1$

$a_{n+1} = 2a_n + b_n$ $\qquad c_{n+1} = 4c_n + d_n$

$b_{n+1} = 2b_n - a_n$ $\qquad d_{n+1} = 4d_n - c_n$

The relevance of these sequences is that if you let r = arctan 1/2 and s = arctan 1/4 then induction shows that

$$\tan nr = a_n/b_n \qquad \tan ns = c_n/d_n$$

and the problem is to show that r, s and π are linearly independent over the rationals.

The first few values of the sequences are

n	a_n	b_n	c_n	d_n
0	0	1	0	1
1	1	2	1	4
2	4	3	8	15
3	11	2	47	52
4	24	-7	240	161
5	41	-38	1121	404
6	44	-117	4888	495
7	-29	-278	20047	-2908
8	-336	-527	77280	-31679
9	-1199	-718	277441	-203996

Induction can also show that $a_n^2 + b_n^2 = 5^n$, and that

$c_n^2 + d_n^2 = 17^n$. Now suppose that a_n and b_n have a common prime factor p. Then p would have to divide 5^n, and so $p = 5$ and 5 must divide a_n and b_n. Similarly if c_n and d_n had a common factor, 17 would divide both of them. We can see at once that this is not so. Consider a_n and b_n modulo 5, and c_n and d_n modulo 17.

n	0	1	2	3	4	5	6	7	8	9
a_n(mod 5)	0	1	4	1	4	1	4	1	4	1
b_n(mod 5)	1	2	3	2	3	2	3	2	3	2
c_n(mod 17)	0	1	8	13	2	16	9	4	15	1
d_n(mod 17)	1	4	15	1	8	13	2	16	9	4

The sequences repeat, being recursive (a_n and b_n have period 2, c_n and d_n have period 8) and so they are never 0 (i.e. divisible by 5 or 17) unless $n = 0$. Unless n is zero, a_n and b_n are coprime, and the same goes for c_n and d_n.

Suppose that $x\pi + yr + zs = 0$ for some integers x, y and z, not all zero. Then $\tan yr = \tan(-x\pi - zs) = -\tan zs$. Put $m = |y|$ and $n = |z|$. Then $\tan mr = \pm \tan ns$, and so $a_m d_n = \pm b_m c_n$. Now to show that m or n can't be zero. Suppose $m = 0$, then $c_n/d_n = \tan ns = 0$, but this means $c_n = 0$, which does not happen except in the trivial case $x = y = z = 0$. A similar proof shows that $n \neq 0$. Now we have

$$a_m d_n = \pm b_m c_n$$

but a_m/b_m and c_n/d_n are both in their lowest terms, being coprime. Therefore

$$a_m = \pm c_n \qquad b_m = \pm d_n$$

But take a look at the sequences mod 5:

n	1	2	3	4	5	6	7
a_n	1	4	1	4	1	4	1
b_n	2	3	2	3	2	3	2
c_n	1	3	2	0	1	3	2
d_n	4	0	2	1	4	0	2

Now in mod 5, $c_n = \pm a_m$ can only be 1 or 4, so c_n has to be 1, and therefore $d_n = 4$. But there is no m for which $b_m = \pm 4 = 4$ or 1, contrary to our assumption. Our result is thus proved.

A similar solution has come in from C. J. Smyth.

It is now plausible that if $x_n = \tan^{-1} a_n/b_n$ and $y_n = \tan^{-1} c_n/d_n$ then the sequence of points (x_n, y_n) for $n = 1, 2, 3, \ldots$ has in some sense a uniform distribution over the square in which $-\pi/2 < x, y < \pi/2$, so that the sequence may be used for Monte Carlo integration. The analogous problem in one dimension has been discussed by N. B. Slater, The distribution of N for which $\{\theta N\} < \phi$, Proc. Camb. Phil. Soc. 46 (1950) 525-543. See also N. B. Slater: Gaps and steps for the sequence $m\theta$ mod 1. Proc. Camb. Phil. Soc. 63 (1967) 1115-1122.

BINOMIAL IDENTITY 21

Cecil Rousseau

$$\sum_{k=m}^{n} (-\tfrac{1}{2})^k \binom{n}{k}\binom{2k}{k-m} = (-\tfrac{1}{2})^n \binom{n}{\frac{1}{2}n+\frac{1}{2}m} \text{ if } n+m \text{ is even}$$
$$= 0 \text{ if } n+m \text{ is odd.}$$

DEFINITE INTEGRAL

George Szekeres

Here is an integral which I found accidentally on the computer, (correct to 15 decimals)

$$\int_0^\infty \frac{\log x}{e^x + 1}\,dx = -\tfrac{1}{2}\log^2 2$$

I was unable to prove it, neither could I find it in Bierens de Haar's enormous collection of definite integrals, (the collection is almost 100 years old). The curious thing is that $\int_0^\infty 1/(e^x+1)dx = \log 2$, and this suggests that perhaps $\int_0^\infty \log^k x/(e^x+1)dx$ is linked in some way to powers of log 2, but for no $k > 1$ could I detect such linkage.

DEFINITE INTEGRAL

C. C. Rousseau

The integral above from George Szekeres can be proved as follows. Consider the identity (for $s \neq 0$)

$$(1-2^{-s})\Gamma(s+1)\zeta(s+1) = \int_0^\infty \frac{x^s}{e^x+1}\,dx \qquad \text{for } \Re(s) > -1$$

which may be proved by expanding the integrand in powers of e^{-x}. Expand both sides in powers of s.

$$\Gamma(s+1) = 1 - \gamma s + O(s^2)$$
$$\zeta(s+1) = 1/s + \gamma + O(s)$$
$$1 - 2^{-s} = \log 2\,(s - \tfrac{1}{2}s^2 \log 2 + O(s^3))$$
$$\text{Integral} = \sum_0^\infty (s^k/k!) \int_0^\infty (\log x)^k/(e^x+1)\,dx$$

Equating coefficients of s gives the result. Are more terms known in the Laurent expansion of the zeta function?

FOUR SQUARES IN ARITHMETICAL PROGRESSION

(JCMN 43, p.5034)

C. J. Smyth

Dickson's History of the Theory of Numbers tells us that it was Fermat who first stated the impossibility of four distinct integer squares in A.P. A proof of this result appears in Mordell's "Diophantine Equations", using the method of descent. Euler in fact proved a stronger result, namely that the product of four unequal positive integers in A.P. is never a square. We reconstruct a proof of the result here, also using the method of descent.

First we need a trivial variant of the standard form result for Pythagorean triples:

Lemma 1 If $A^2 + B^2 = C^2$ with A even and A, B, C > 0, then there are k, u, v > 0 with hcf (u, v) = 1 and A = 2kuv, $B = k(u^2 - v^2)$, $C = k(u^2 + v^2)$. (Thus for $4^2 + 3^2 = 5^2$, (k, u, v) = (1, 2, 1); for $8^2 + 6^2 = 10^2$, (k, u, v) = (2, 2, 1); for $6^2 + 8^2 = 10^2$, (k, u, v) = (1, 3, 1).)

Theorem: Four distinct positive integers in A.P. cannot have their product a square.

Proof: Assume that four such numbers exist. By scaling if necessary we can assume that their common difference is even (say 2d), their average a is an integer, and hcf (a, d) = 1. Choose such a set of numbers with minimal a. The numbers are a-3d, a-d, a+d, a+3d, and

$$(a^2 - d^2)(a^2 - 9d^2) = t^2 \qquad (*)$$

where a, d and t are positive integers, with a $>$ 3d and (a, d) = 1. The equation (*) may be written

$$a^4 - 10a^2d^2 + 9d^4 = t^2$$

or

$$(4d^2)^2 + t^2 = (a^2 - 5d^2)^2$$

and Lemma 1 gives

$$2d^2 = kuv$$

$$t = k(u^2 - v^2)$$

$$a^2 - 5d^2 = k(u^2 + v^2) \qquad \text{where } (u, v) = 1.$$

Lemma 2: k = 1.

Proof: First note that k divides both $2d^2$ and $2a^2$, and so divides $(2a^2, 2d^2) = 2$, therefore k = either 1 or 2. Now assume k = 2 and look for a contradiction.

$$d^2 = uv \qquad \text{and} \qquad a^2 - 5d^2 = 2(u^2 + v^2)$$

Since $a^2 - 5d^2$ is even, and a and d cannot both be even, both are odd, therefore u and v are both odd.

$$a^2 = 2u^2 + 5uv + 2v^2 = (2u + v)(u + 2v)$$

Now u and v cannot both be divisible by 3 because (u, v) = 1. If one but not both were divisible by 3 then $a^2 \equiv 2 \bmod 3$, which is impossible. Therefore neither u nor v is divisible by 3. Both are squares because $uv = d^2$ and therefore both are $\equiv 1 \bmod 3$, so that 2u + v and u + 2v are both divisible by 3. Thus

$$v, \quad (u + 2v)/3, \quad (2u + v)/3, \quad u,$$

is an A.P. of positive integers (squares in fact) with common difference (u - v)/3 which is an even positive integer. The mean is (u + v)/2 which is less than a/2 because $a^2 = (2u + v)(u + 2v) > (u + v)^2$ and we have a contradiction because our original A.P. was chosen to minimize the mean. Therefore $k = 1$. QED

Now we have $2d^2 = uv$ and $a^2 - 5d^2 = u^2 + v^2$.

Lemma 3: a is odd and d is even.

Proof: Because uv is even and $(u, v) = 1$, one of u and v is even and the other is odd. Therefore $u^2 + v^2 \equiv 1 \bmod 4$. If d were odd then $a^2 \equiv 2 \bmod 4$, which is impossible. Therefore d is even and so a is odd. QED

Now we may put $d = 2d'$ and have

$$8d'^2 = uv \quad \text{and} \quad a^2 - 20d'^2 = u^2 + v^2.$$

Of u and v one must be divisible by 8 and the other odd. For some positive integers x and y either $u = 8x^2$ and $v = y^2$ or $v = 8x^2$ and $u = y^2$.

In both cases we find

$$d' = xy \qquad a^2 - 20d'^2 = 64x^4 + y^4$$

Consequently $a^2 = (10x^2 + y^2)^2 - (6x^2)^2$ and by Lemma 1

$$3x^2 = k_1u_1v_1$$
$$a = k_1(u_1^2 - v_1^2)$$
$$10x^2 + y^2 = k_1(u_1^2 + v_1^2)$$

Since k_1 divides both $3x^2$ and $3y^2$, it divides $(3x^2, 3y^2) = 3$, and k_1 is either 1 or 3.

Lemma 4: $k_1 = 1$.

Proof: If $k_1 = 3$ then $x^2 = u_1v_1$ and $10x^2 + y^2 = 3(u_1^2 + v_1^2)$ so that $x^2 + y^2 \equiv 10x^2 + y^2 \equiv 0 \bmod 3$. This is impossible because $(x, y) = 1$. Therefore $k_1 = 1$. QED

Now $3x^2 = u_1v_1$ and $10x^2 + y^2 = u_1^2 + v_1^2$.

Either u_1 or v_1 is divisible by 3, but not both. Either $u_1 = 3p^2$ and $v_1 = q^2$ or $u_1 = q^2$ and $v_1 = 3p^2$ (**)

In both cases

$$y^2 = 9p^4 + q^4 - 10p^2q^2 = (9p^2 - q^2)(p^2 - q^2)$$
$$= (q - 3p)(q - p)(q + p)(q + 3p).$$

We have an A.P. with product a square and common difference even, but we do not know if $q - 3p$ and $q - p$ are positive. Neither can be zero, and they cannot have opposite signs because y^2 is positive. If both are negative then find a modified A.P. $3p - 3q$, $3p - q$, $3p + q$, $3p + 3q$, where the product is $(3y)^2$, and all four numbers are positive.

There are two cases to consider, according to the choice made in the line marked (**) above. In each case we must show that the mean of the A.P. is less than the mean a of the original . Note that

$$a^2 = (4x^2 + y^2)(16x^2 + y^2) \geq 85, \text{ so that } a > 9.$$

Also $0 < v_1 \leq u_1 - 1$, so that $v_1^2 \leq u_1^2 - 2u_1 + 1$. Consequently $a = u_1^2 - v_1^2 \geq 2u_1 - 1$, and $u_1 \leq \frac{1}{2}a + \frac{1}{2} < 5a/9$.

Case 1. $u_1 = 3p^2$, $v_1 = q^2$.

First note that $q^2 = v_1 < u_1 = 3p^2 < 9p^2$, so that $q - 3p$ is negative, and we use the modified A.P. with mean $= m = 3p$. $m^2 = 9p^2 = 3u_1 < 5a/3 < 5a^2/27 < a^2/4$, and so $m < a/2$.

Case 2. $u_1 = q^2$, $v_1 = 3p^2$.

Here $q^2 = u_1 > v_1 = 3p^2 > p^2$, so that $q - p$ is positive. Mean $= m$ and $m^2 = q^2 = u_1 < 5a/9 < 5a^2/81 < a^2/16$, so that $m < a/4$.

Afterthought — It is easy to prove an extension of the result of Euler given above, as follows.

Theorem Four integers in arithmetical progression cannot have their product a square except in the following three cases: (i) One of them is zero.

(ii) The common difference is zero.

(iii) The mean is zero.

Proof Consider if possible such an A.P., cases (i) and (ii) having been excluded, the four integers are unequal and all non-zero. Arranging them in increasing order, there are just three possibilities:

(a) All four are positive.

(b) The first two are negative and the others positive.

(c) All four are negative.

Firstly (a) is impossible by Euler's result above.

Secondly consider (b), let the A.P. be

$$p - q,\ p,\ p + q,\ p + 2q$$

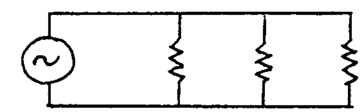
Consider the new A.P. $-3p,\ q - p,\ p + 2q,\ 3p + 3q$. All four numbers are positive. The common difference $2p+q$ is non-zero because it is twice the mean of the first A.P.. The product is a square because it is 9 times the product of the first A.P., this is impossible by Euler's result above.

Thirdly consider (c). Change all the signs and reverse the order. This gives (a) which we have seen to be impossible.

MODERN COOKING

(JCMN 43, p.5042)

The question was about heating a number of cups of coffee in a microwave oven. My physicist son tells me that a microwave oven heats food in two ways, by inducing currents in accordance with Maxwell's equations and by exciting a resonant frequency of water molecules. Therefore as a model we take the alternating current circuit theory problem of n identical components, each of unknown complex reactance, in parallel with a power source of unknown characteristics.



Let the generator output voltage and current be represented by the complex numbers V and C, where as usual the complex time factor $e^{i\omega t}$ is omitted. Suppose that they are related by $V + AC = B$, where A and B are unknown complex constants. Each cup of coffee is regarded as a component with complex reactance R, so that the n cups are like a single two-terminal component of reactance R/n. We therefore have

$$V = CR/n \qquad \text{and} \qquad V + AC = B$$

with solution $C = nB/(R+nA)$ and $V = RB/(R+nA)$. The mean rate of heat production in each cup is the real part of $\frac{1}{2}V\bar{C}/n = \frac{1}{2}R|B|^2|R + nA|^{-2}$. The time taken for the coffee to become suitably hot for drinking is therefore proportional to $|R+nA|^2$. Putting $R = p + iq$ and $A = a + ib$, the time is proportional to $(p + na)^2 + (q + nb)^2$ which is a quadratic in n, in which the square term and the constant term both have their coefficients positive. From the data (the first three below)

Number of cups, n	1	2	3	4	5
Time in seconds	30	45	70	105	150

the method of finite differences shows us that five cups take just five times as long as one cup.

QUOTATION CORNER 24

Fill the cup and fill the can:
  Have a rouse before the morn:
Every moment dies a man,
  One and one sixteenth are born.

— Alfred Lord Tennyson, amended by Charles Babbage.

R. B. Potts told us the story of how Babbage wrote to Tennyson asking him to correct the statistical error in the published version of the poem "The Vision of Sin". Tennyson had written this verse with the fourth line "Every moment one is born." Babbage added the opinion that although the value 17/16 was not as accurate as might be wished, it was probably good enough for the purposes of poetry.

BINOMIAL IDENTITY 20 (JCMN 44, p.5059)

Cecil Rousseau

If $n < m$ then $\sum_{n+1}^{m} 2^{m-r}\binom{r-1}{n} = \sum_{n+1}^{m}\binom{m}{r}$.

Proof: Consider the polynomial $2^{m-1}\sum_{1}^{m}\left(\frac{1+x}{2}\right)^{r-1}$

$$= 2^{m-1}\left\{1 - \left(\frac{1+x}{2}\right)^m\right\}\Big/\left\{1 - \frac{1+x}{2}\right\} = \frac{2^m}{1-x} - \frac{(1+x)^m}{1-x}.$$

The coefficient of x^n in the LHS is $2^{m-1}\sum_{n+1}^{m} 2^{1-r}\binom{r-1}{n}$,

and in the RHS is $2^m - \sum_{0}^{n}$ Coefficient of x^r in $(1+x)^m$

$$= \sum_{0}^{m}\binom{m}{r} - \sum_{0}^{n}\binom{m}{r} = \sum_{n+1}^{m}\binom{m}{r}.$$

BINOMIAL IDENTITY 22

C. J. Smyth

Let a square matrix have i, j entry $\binom{i-1}{j-1}$. Show that its inverse has i, j entry $(-1)^{i+j}\binom{i-1}{j-1}$.

QUOTATION CORNER 25

He was not interested in rigour, which for that matter is not of first-rate importance in analysis beyond the undergraduate stage, and can be supplied, given a real idea, by any competent professional.

— J. E. Littlewood on S. Ramanujan. (From the book: "Littlewood's Miscellany" edited by B. Bollobas, C.U.P. 1986)

SPHERICAL TRIANGLES (JCMN 44, p. 5061)

Esther Szekeres

Macquarie Library has a book "Spherical Trigonometry", first edition by Todhunter, 1859, revised by Leathem in 1911. It contains answers to most of the questions raised in JCMN 44, but everything is done by spherical trigonometry.

Here is a Euclidean proof that the medians of a spherical triangle intersect in the centroid G.

Consider ABC both as a spherical triangle (on a sphere with centre O) and as a plane triangle. Let M_1 be the mid-point of the arc BC, and let m_1 be the mid-point of the chord BC, then O, m_1 and M_1 are collinear. Let g be the centroid of the plane triangle, and let G be where Og meets the sphere. The four points A, O, m_1 and M_1 are in a plane, and this plane contains O and g, and therefore also G. This proves that the arc AM_1 contains G. Similar reasoning applies to the other medians BM_2 and CM_3, so that our result is proved.

The altitudes are concurrent because there is an analogue of Ceva's Theorem that holds between the sines of the arc lengths. Call this intersection the orthocentre H. A. Hart (Quarterly Journal, Vol IV, 1861) discovered that there is a circle with centre on the great circle through H and G, touching the incircle and the three escribed circles. This circle is an analogue of the nine-point circle of a plane triangle, but it does not in general pass through the mid-points of the sides.

If the radius of Hart's circle is ρ and that of the circumcircle is R, then $\tan\rho = \frac{1}{2}\tan R$, showing another analogy with the nine-point circle of a plane triangle.

Another difference from plane triangles is that the circumcentre, centroid and orthocentre are not in general on a great circle, as one might have expected from the existence of the Euler line of a plane triangle.

I sometimes wonder if the properties of plane triangles will be forgotten and waiting to be rediscovered in a hundred years time, like those of spherical triangles now.

QUOTATION CORNER 26

Most of the bills for publications have to be paid in sterling but much of the revenue comes from overseas subscriptions, in dollars. The subscription rates have to be decided in May or June for the following winter. The sterling subscription is decided by estimating publishing costs but the dollar subscription rate must also be based on an estimate of the dollar exchange rate for the following January. Last May we anticipated an exchange rate of 1·83 dollars to the pound by next January. Already by early November the dollar has fallen to 1·77. If the dollar falls much further by January then we face the prospect of making a loss on all our publications.

— From the Treasurer's 1987 Report to the London Mathematical Society, printed in LMS Newsletter No. 146 (January 1988)

PROBLEM ON ARITHMETIC PROGRESSIONS

C. J. Smyth

There was a trick used in "Four squares in arithmetic progression" pp. 5074-5078 above, of noting that if (a, b, c, d) is in AP then so is (-3b, -a, d, 3c). It has a parallel for three variables. If (a, b, c) is in AP then so is (-2a, c, 4b). This suggests the following problem.

Let $n \geqslant 3$, and let $k_1, k_2, \ldots k_n$ be non-zero real numbers, not all equal, with the property that if $(a_1, a_2, \ldots a_n)$ is in AP, then $(k_1a_1, k_2a_2, \ldots k_na_n)$ after re-ordering is also in AP. By scaling we can assume that at least half the k's are positive, and that the least positive k_i is 1.

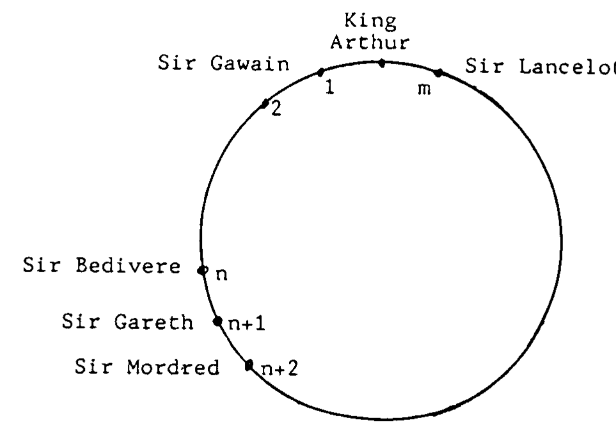
Show that either $n = 3$ and $(k_1, k_2, k_3) = (-2, 4, 1)$ or $n = 4$ and $(k_1, k_2, k_3, k_4) = (-1, -3, 3, 1)$, or the same reversed in both cases.

MYSTERIOUS MESSAGE

On Sunday, 16th August I left my computer doing some calculations, and when I went back to look at it the answers were on the screen, with the word "Ready" (It is a good machine and when it has finished one job always offers to start another) But underneath was IOU78999999. Our black cat, Squeak, who was asleep on the table, is the only witness, but is not able to shed any light on the affair.

ANOTHER MOMENT IN CAMELOT

Marta Sved



King Arthur looked around the table. It was the right moment. All of his m knights were there, filling up all the places at the Round Table. It was just the time for discussing his plans:

— I know that all of you like to go on quests and visits to meet challenges, — he said — but in the coming weeks we will have many things to settle here, and I want at least n+1 of you to stay here with me. —

Looking to his right, he added:

— Let us make sure that n+1 of you, for example those on my right, remain in Camelot. —

As the knights counted to n+1, the King turned to the last of them, and said:

— Sir Gareth, I entrust to you the task of adding a few more to the n+1 required to stay. Toss a coin for each of the other m-n-1 of you; if the coin shows head the knight stays here, and if tail he is free to go —

Sir Gareth answered:

— Yes, your Majesty. There will be 2^{m-n-1} possible outcomes, and each of these provides n+1 or more knights to stay here. —

However, Sir Mordred, sitting on the right of Sir Gareth, objected, as always.

— If I may say so, your Majesty, this does not account for all the possibilities, for in each of the outcomes all the knights on my left are to stay here. If I were to choose n to stay from the n+1 on my left, and toss a coin for the m-n-2 on my right, this would give another $\binom{n+1}{n} 2^{m-n-2}$ arrangements, each with myself and at least n others staying, and each arrangement different from the ones proposed before, since now one of the knights on my left would be free to go.—

There was now an outcry from all the knights seated to the right of Sir Mordred:

— What if I ... ? —

King Arthur ordered silence:

— I see your point! The points of all of you! I like to be fair. We will get all the possibilities if every knight to the right of Sir Bedivere, except Sir Lancelot, does his share of coin-tossing. —

— Yes, indeed — said Sir Lancelot — there is no one on my right for whom I could toss, but I would still have to choose

the n knights to stay from the m-1 on my left. —

Sir Gawain, sitting at the King's right, who could not expect to do any coin tossing or selecting, played his part by summarising the result:

— This will give us

$$2^{m-n-1} + \binom{n+1}{n} 2^{m-n-2} + \binom{n+2}{n} 2^{m-n-3} + \dots + \binom{m-1}{n} \text{ arrangements.—}$$

At this moment Merlin walked in with Queen Guinevere.

— What a bewildering formula! — exclaimed the Queen.

Merlin added:

— Why, if all you want is to have at least n+1 knights attending the King at his Round Table, then you just choose n+1 knights, or n+2, or n+3 and so on, perhaps even all the m of you. This gives altogether

$$\binom{m}{n+1} + \binom{m}{n+2} + \binom{m}{n+3} + \dots + \binom{m}{m} \text{ arrangements. —}$$

He strained his eyes, looking into the future, and exclaimed:

— This is just BINOMIAL IDENTITY 20, which is to appear in the October issue of the James Cook Mathematical Notes in the 1987th year of our Lord:

$$\sum_{r=n+1}^{m} 2^{m-r} \binom{r-1}{n} = \sum_{r=n+1}^{m} \binom{m}{r} \quad \text{for } n < m.$$

EDITOR'S NOTE

Held over to the next issue there is a note from A. Brown on the equal median problem for spherical triangles. If two medians are equal, then must the triangle be isosceles? (Answer - no!)

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Basil Rennie