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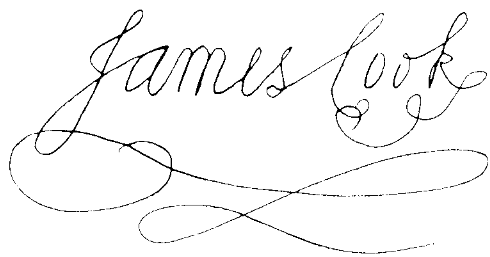

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JCMN 56

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TWO PROBLEMS

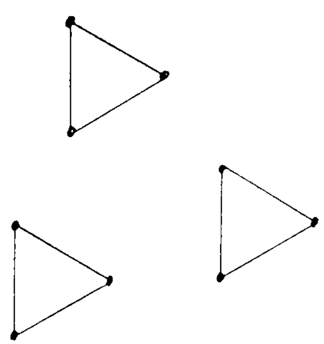
Paul Erdös
(Hungarian Academy of Sciences)

Let $x_1, x_2, \ldots x_n$ be n points in the plane; $d(x_i, x_j)$ is the distance between x_i and x_j. Assume that if two distances differ they differ by at least one. Prove that the diameter of the set $\{x_1, \ldots x_n\}$ is $\geq cn$. Perhaps for $n > n_0$ it is at least n-1.

Let $x_1, \ldots x_n$ be n points in the plane, no three on a line and no four on a circle. Prove that if $n > n_0$ these points determine at least n distances.

Readers with long memories will recall in JCMN 35, p.4069, (October 1984) the example due to Pomerance of 5 such points determining only 4 distinct distances, one occurring 4 times, one 3 times, etc. I Palànti found 8 points with only 7 distances. But see below.

NINE POINTS WITH EIGHT DISTANCES



Take almost any two complex numbers u and v. Let ω be a complex cube root of unity. The points are represented by the nine complex numbers:-

$u + v$	$u + \omega v$	$u + \omega^2 v$
$\omega u + v$	$\omega u + \omega v$	$\omega u + \omega^2 v$
$\omega^2 u + v$	$\omega^2 u + \omega v$	$\omega^2 u + \omega^2 v$

The 8 distances between them are $|u|\sqrt{3}$, $|v|\sqrt{3}$, $|u \pm v|\sqrt{3}$, $|u \pm \omega v|\sqrt{3}$ and $|u \pm \omega^2 v|\sqrt{3}$. Each of the first two occurs 9 times, and each of the other six occurs 3 times.

SYMMETRIC (OR HERMITEAN) MATRICES

Terry Tao
(6, Jennifer Avenue, Bellevue Heights, 5050, Australia)

Suppose that M is a real, square, positive definite, symmetric n×n matrix. Is it possible to find n real n-dimensional vectors (numbered 1, 2, ... n) such that the component of M in row i and column j is the inner product of the vectors numbered i and j?

If so does the result extend to complex variables and Hermitean matrices? Does it extend to linear operators in Hilbert space?

QUOTATION CORNER 36

"This book was carefully produced. Nevertheless, author, translator and publishers do not warrant the information contained therein to be free of errors. Readers are advised to keep in mind that statements, data, illustrations, procedural details or other items may inadvertently be inaccurate."

— Printed at the beginning of 'Deterministic chaos, an introduction' by H. G. Schuster.

PROBLEM IN ALGEBRA AND GEOMETRY

Terry Tao
(6, Jennifer Avenue, Bellevue Heights, 5050, Australia)

Let A, B, C, D and E be five points in the plane. Show that the determinant:-

$$\begin{vmatrix} 0 & AB^2 & AC^2 & AD^2 & AE^2 \\ AB^2 & 0 & BC^2 & BD^2 & BE^2 \\ AC^2 & BC^2 & 0 & CD^2 & CE^2 \\ AD^2 & BD^2 & CD^2 & 0 & DE^2 \\ AE^2 & BE^2 & CE^2 & DE^2 & 0 \end{vmatrix}$$

is equal to zero.

POINTS AND DISTANCES IN THE PLANE

Consider a set of n points in the plane, and the distances between them. Two problems:-

(a) Prove or disprove that the maximum distance (the diameter of the set) is not attained more than n times.

(b) Prove or disprove that the minimum distance is attained less than 3n times.

For (difficult) questions on this topic, see pages 474-477 of "A tribute to Paul Erdös" (C.U.P., 1990).

GEOMETRICAL PROBABILITY 3 (JCMN, 54, p.6017)

Jordan Tabov

Any two random points in a disc (from the uniform probability distribution) give a random line segment (ending at the two points). Find the probability that two such random line segments intersect.

We may construct the two line segments by firstly choosing 4 points (independently at random) and secondly choosing how to pair them to give two line segments.

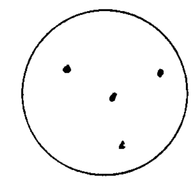


Figure 1

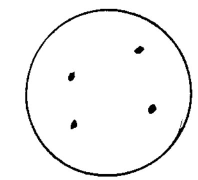


Figure 2

The four points must be either as in Figure 1, one in the triangle formed by the other three, or as in Figure 2, the vertices of a convex quadrangle. If the 4 points are as in Figure 1, the two line segments (however the 4 points are paired) will not cross. If they are as in Figure 2, then of the 3 ways of pairing the 4 points, one will give segments crossing, the other two not. These three ways are equally probable.

Referring to ARROWS IN THE TARGET (JCMN 54, p.6019 and 55, p.6032) we know that the probability of the points being as in Figure 2 is $1-35/(12\pi^2)$. The required probability, of having two intersecting line segments, is one third of this, and numerically it is 0·234827.

COMTET'S "BEAUTIFUL DETERMINANT": A PROBLEM-SOLVING EXPERIENCE

CECIL ROUSSEAU

On page 203 of Louis Comtet's engaging book *Advanced Combinatorics*, there is a problem which the author titles *a beautiful determinant*. One is asked to prove that

$$\begin{vmatrix} (1,1) & (1,2) & \cdots & (1,n) \\ (2,1) & (2,2) & \cdots & (2,n) \\ \vdots & \vdots & & \vdots \\ (n,1) & (n,2) & \cdots & (n,n) \end{vmatrix} = \phi(1)\phi(2)\cdots\phi(n), \tag{1}$$

where (i,j) denotes the greatest common divisor (GCD) of i and j and ϕ denotes the Euler function

$$\phi(n) = n\prod_{p|n}\left(1-\frac{1}{p}\right).$$

[Recall that $\phi(n)$ is the number of integers k between 1 and $n-1$ such that $(k,n) = 1$.] This is a classic result. Comtet refers to papers of Smith [1875] and Catalan [1878]. This problem is also in Pólya and Szegö's *Aufgaben und Lehrsätze aus der Analysis* [Part VIII, Chapter 1, Problem 57].

Our purpose in this note is not to claim a new solution of this problem. Instead, we simply intend to describe the *process* of finding one particular solution, and in this way try to communicate to the reader a sense of serindipity and pleasure as the solution unfolds. In the spirit of Pólya's discussion of heuristics in *How to Solve It*, we shall focus on the *questions*, *experiments* and *good guesses* which lead ultimately to the solution. The resulting solution is not the shortest or most elegant, but it does offer examples of several problem-solving strategies.

Our first step is an example of **working backwards**. "What would

make the result follow immediately by induction?" The formula is obviously true for $n = 1$. Take $n > 1$ and for $i = 1, 2, \ldots, n$, let r_i denote the ith row in the $n \times n$ determinant to be evaluated. If there were scalars $c_1, c_2, \ldots, c_{n-1}$ such that

$$r_n - (c_1r_1 + \cdots + c_{n-1}r_{n-1}) = (\ 0\ \ 0\ \ 0\ \ \cdots\ \ \phi(n)\ ),$$

then the row operation

$$r_n \leftarrow (\ 0\ \ 0\ \ 0\ \ \cdots\ \ \phi(n)\ ) \tag{2}$$

would preserve the value of the determinant and the truth of (1) would follow immediately by induction when we expand the determinant by the nth row. At the same time, there must be such a linear combination if the formula is true. Let's look for it.

The second step involves **specialization.** "Is there a special case in which the desired linear combination clearly exists?" Yes, there is. Suppose that $n = p$ (a prime). Then since the first row is

$$r_1 = (\ 1\ \ 1\ \ 1\ \ \cdots\ \ 1\ )$$

and the last is

$$r_p = (\ 1\ \ 1\ \ 1\ \ \cdots\ \ p\ ),$$

using $\phi(p) = p - 1$ we see that the row operation

$$r_p \leftarrow r_p - r_1$$

yields (2). "What happens when n is composite?" Well, $n = 4$ and $n = 6$ seem like reasonable test cases. A little experimentation shows that for $n = 4$ the desired result (2) is obtained by subtracting row 2 from row 4:

$$r_4 \leftarrow r_4 - r_2.$$

For $n = 6$, we get what we are looking for by the row operation

$$r_6 \leftarrow r_6 - r_3 - r_2 + r_1.$$

Now it is time for **a bright idea**. Looking back at the preceding examples, the Möbius function comes to mind. Recall that

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n = p_1 p_2 \dots p_k \text{ (product of distinct primes)} \\ 0 & \text{if } p^2 | n \text{ for some prime } p, \end{cases}$$

and that ϕ and μ are related by

$$\phi(n) = n \sum_{d|n} \frac{\mu(d)}{d}. \qquad (3)$$

For the special cases $n = p$ (prime), $n = 4$ and $n = 6$ we have

$$\begin{aligned} \phi(p) &= p\left(1 - \frac{1}{p}\right) = p - 1 \\ \phi(4) &= 4\left(1 - \frac{1}{2}\right) = 4 - 2 \quad \text{and} \\ \phi(6) &= 6\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = 6 - 3 - 2 + 1, \end{aligned}$$

respectively. The pattern we observe in these examples is too good not to be true in general, and so we guess that the general row operation we are seeking is simply

$$r_n \leftarrow \sum_{d|n} \mu\left(\frac{n}{d}\right) r_d.$$

We can now **restate the problem** as an identity in number theory. "Is it true that the formula

$$\sum_{d|n} \mu\left(\frac{n}{d}\right)(d,k) = \begin{cases} 0 & \text{if } 1 \le k < n, \\ \phi(n) & \text{if } k = n \end{cases}$$

is true for all n?" Since $(d,n) = d$ if d is any positive divisor of n, equation (3) shows that

$$\sum_{d|n} \mu\left(\frac{n}{d}\right)(d,n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) d = \phi(n).$$

But what about $1 \le k < n$?

Perhaps we need another example of **working backwards**. Writing the formula to be proved

$$\sum_{d|n} \mu\left(\frac{n}{d}\right)(d,k) = 0 \qquad (1 \le k < n)$$

in the form

$$\sum_{m|n} m \sum_{\substack{d|n \\ (d,k)=m}} \mu\left(\frac{n}{d}\right) = 0 \qquad (1 \le k < n),$$

we see that the desired result would follow right away if, for every $m|n$,

$$\sum_{\substack{d|n \\ (d,k)=m}} \mu\left(\frac{n}{d}\right) = 0 \qquad (m|k,\ 1 \le k < n). \qquad (4)$$

The new identity (4) may be an example of wishful thinking, but let's proceed.

We are pleased to note that in order to prove (4) it suffices to consider the special case in which $m = 1$. Just make the replacements

$$d \leftarrow d/m, \qquad k \leftarrow k/m, \qquad n \leftarrow n/m,$$

and (4) reduces to

$$\sum_{\substack{d|n \\ (d,k)=1}} \mu\left(\frac{n}{d}\right) = 0 \qquad (1 \le k < n). \qquad (5)$$

It seems appropriate to **ask if we have seen it before**. Well not quite, but we certainly have seen the fundamental Möbius function identity

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d) = \begin{cases} 1 & n = 1 \\ 0 & n > 1, \end{cases} \qquad (6)$$

and are reassured by the fact that our goal (5) reduces to this identity in case $(n,k) = 1$, for then every $d|n$ automatically satisfies $(d,k) = 1$.

What other data (facts) are available for us to use? In addition to (6), two other basic properties of the Möbius function come to mind:

$$(r,s)=1 \Rightarrow \mu(rs)=\mu(r)\mu(s) \qquad i.e.\ \mu \text{ is multiplicative} \tag{7}$$

$$\mu(r)\neq 0 \Leftrightarrow r \text{ is square-free.} \tag{8}$$

Now we would like to factor n appropriately and so make good use of (7). Let $\alpha_p(n)$ denote the exponent of p in the prime factorization of n. Write

$$n=rs,$$

where

$$\alpha_p(r)=\begin{cases}\alpha_p(n) & \text{if } p|(n,k)\\ 0 & \text{otherwise.}\end{cases}$$

[As an example, for $n=168=2^3\cdot 3\cdot 7$ and $k=90=2\cdot 3^2\cdot 5$, we would set $r=2^3\cdot 3=24$ and $s=7$.] Then

$$d|n \text{ and } (d,k)=1 \Leftrightarrow d|s,$$

and in view of (7),

$$\sum_{\substack{d|n\\(d,k)=1}}\mu\left(\frac{n}{d}\right)=\mu(r)\sum_{d|s}\mu\left(\frac{s}{d}\right)=\mu(r)\sum_{d|s}\mu(d).$$

Now everything falls into place. If $s>1$, the result we are looking for follows immediately from (6). If $s=1$ then $r=n$ and every prime which divides n also divides k. Since $n>k$, this means that n has a square factor and so $\mu(r)=\mu(n)=0$. Thus (5) is true and our search for a solution of Comtet's problem has reached a happy conclusion.

But we shouldn't stop here. It is natural to ask if the problem we have just solved has an interesting **generalization**. Let F be an arbitrary arithmetical function and set $a_{rs}=F(m)$ where $m=(r,s)$. Looking back

at (5), we see that

$$\sum_{d|n}\mu\left(\frac{n}{d}\right)a_{dk}=\sum_{m|n}F(m)\sum_{\substack{d|n\\(d,k)=m}}\mu\left(\frac{n}{d}\right)=0 \qquad (k<n).$$

Thus, letting

$$f(n)=\sum_{d|n}\mu\left(\frac{n}{d}\right)F(d),$$

we find

$$\sum_{d|n}\mu\left(\frac{n}{d}\right)a_{dk}=\begin{cases}0 & \text{if } 1\le k<n\\ f(n) & \text{if } k=n,\end{cases}$$

so the same (induction) argument as before gives

$$\begin{vmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{vmatrix}=f(1)f(2)\cdots f(n).$$

In summary, Comtet's *beautiful determinant* provides the problem solver with a beautiful experience. It offers fertile ground for applying heuristics and discovering interesting formulas.

EDDINGTON'S CRICKET PROBLEM

R. A. Lyttleton
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This old problem might be suitable for JCMN.

The scoreboard for the first innings of Eastershire against Westershire was as follows:-

EASTERSHIRE	
Atkins	6
Bodkins	8
Dawkins	6
Hawkins	6
Jenkins	5
Larkins	4
Meakins	7
Perkins	11
Simkins	6
Tomkins	0
Wilkins	1
Extras	0
Total	60

BOWLING ANALYSIS

	O	M	R	W
Pitchwell	12·1	2	14	8
Speedwell	6	0	15	1
Tosswell	7	5	31	1

Speedwell and Tosswell each had only one spell of bowling. The game was of 6-ball overs. The score was composed entirely of singles and fours. There were no catches, no-balls or short runs. Pitchwell bowled the first over, and Speedwell the second. Atkins took the first ball.

Whose wickets were taken by Speedwell and Tosswell? Who was not out? What was the score at the fall of each wicket?

I gave it to Bradman when he was over here in 1948 and he solved it on the voyage going home. It took me a whole day to solve it. Eddington himself told me that Hardy couldn't do it.

For the benefit of those not familiar with cricket, the essentials (for this problem) are that overs are bowled from alternate ends of the pitch, no consecutive two by the same bowler; a "spell" is bowling alternate overs. There is one batsman at each end. When a ball is bowled the batsman at the opposite end may (a) be out, and be replaced by the next on the list, or (b) score 4, or (c) score 1 and change ends with the other batsman, or (d) remain. In the bowling analysis, O means overs, M means maiden overs, i.e. those in which no runs are scored, R means runs scored off the bowler, and W means wickets taken by the bowler. 12·1 overs means 12 and one ball. The innings ends when 10 wickets have fallen.

THE BOY IN THE POOL

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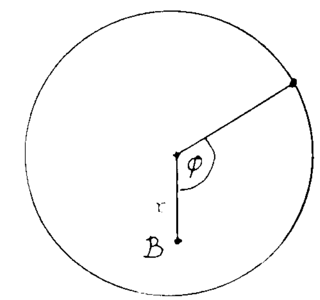
Question: A boy is in the centre of a circular swimming pool of unit radius, while his schoolteacher is at the edge. The teacher runs at unit speed, while the boy swims at speed $k < 1$. The teacher cannot swim. What is the smallest value of k that allows the boy to escape? (The boy can run faster than the teacher.)

Answer: The minimum value for k is $0{\cdot}21723 = \cos \beta$, where $\beta = 1{\cdot}35182$ is the acute angle satisfying the transcendental equation $\tan \beta = \pi + \beta$.



Proof Because of the symmetry of the circle, only two variables matter: the distance r from the boy to the centre, and the angular displacement φ (where $0 \le \varphi \le \pi$) between the boy and the teacher as seen from the centre, together they specify what we call the state (r, φ). The boy wins by reaching a state $(1, \varphi)$ for any $\varphi > 0$. The teacher wins by reaching the state $(1, 0)$, or (but in a less satisfying way) if r never reaches 1.

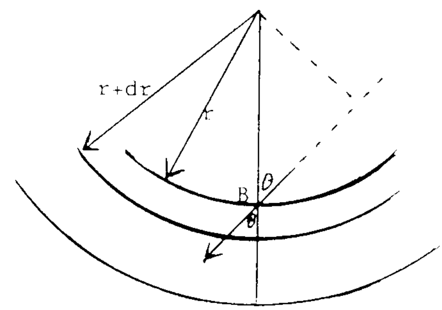
We may assume that both travel at maximum speed, for if one thinks he may obtain an advantage by slowing down, the other may slow down proportionately and nullify the benefit.

The angular speed of the boy is $(k/r)\sin \theta$ where θ is the angle his track makes with the outward radius. Note that if $r < k$ this can be made greater than that of the master (whose angular speed is 1). This means that the boy can control φ whenever he is inside the central circle of radius k. But for the boy to escape he must leave this disk sooner or later. Therefore I will consider the boy's path only after he has left this disk for the last time. Therefore we have an initial state of (k, φ) where the boy has chosen the φ.

Now that the boy is outside the disk his angular velocity is always less than that of the teacher. This means that φ will always decrease. Because the boy wants to avoid the state $\varphi = 0$, he will seek to maximize φ, and so will choose the initial position $\varphi = \pi$.

By a similar argument we can assume that the boy always moves outward. If he doubles back and reduces r, sooner or later he must return to the first value of r, but this time will have a worse value of φ, as φ always decreases. Hence there is no benefit in going inwards.

Now we can solve the problem. Suppose that the state is (r, φ) and the boy is moving at an angle θ to the radius. We may assume from the reasoning above that θ is an acute angle. Then after an infinitesimal time $(dr/k) \sec \theta$, the state will have changed to

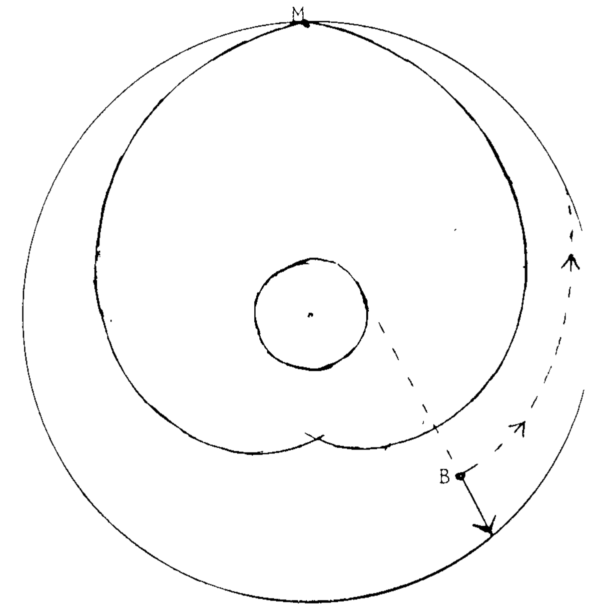


$(r + dr, \varphi + (dr/r)\tan \theta - (dr/k) \sec \theta)$. Now, because $r + dr$ is fixed, the boy will choose θ to maximize the new φ. By elementary calculus or geometry it can be seen that he chooses θ so that $k = r \sin \theta$. Geometrically this means that the optimum direction for the boy is tangential to the inner circle of radius k. From here it is a routine matter to confirm the answer given above.

It is of interest to look at a related question, suppose the boy does not have the required speed ratio, i.e. he cannot swim at 0·21723... times the master's running speed. What initial configurations then allow him to escape?

To answer this we need a little familiarity with involutes; an involute is the locus of a point on a line rolling on a circle (called the base circle of the involute), or of a knot on a piece of string being unwound from a fixed circle. Two knots on the piece of string trace "parallel" involutes.

There are two involutes through any point outside the base circle. Much of the calculation above still applies, and the optimum path for the boy is tangential to the circle of radius k (the speed ratio). With this as base circle, draw the two involutes through the point M where the master is. These two involutes bound the region from which the boy can escape to the bank. To see this, imagine the boy to be the knot on a piece of string being unwound from the inner circle (of radius k) rotating with unit angular velocity (so that he moves at speed k in a straight line). Now change to the rotating frame of coordinates in which the master is at rest. The inner circle and the point of attachment of the string to it are at rest, and so the boy moves along an involute parallel to the involute shown, and so reaches the bank ahead of the master.



This problem has several obvious generalizations, e.g. if the pool were elliptic or rectangular, etc. or if the teacher could swim, or if there were two teachers, etc., but these problems are more difficult.

GEOMETRICAL INEQUALITY (JCMN 55, p.6047)

J. F. Rigby and Terry Tao

The problem in the previous issue was the case $\lambda = 3$ of the more general problem:-

If $\lambda > 0$, find the bounds of the function $f(a, b, c, \lambda) =$

$$\frac{b-c}{b+c+(\lambda-1)a} + \frac{c-a}{c+a+(\lambda-1)b} + \frac{a-b}{a+b+(\lambda-1)c}$$

where a, b and c are the sides of any triangle. The answer, we shall see, is that the bounds are $\pm B(\lambda)$ where

$$B(\lambda) = \{(\lambda+4)\sqrt{\lambda(\lambda+4)} - 4(\lambda+1)\sqrt{\lambda+1}\}/\{\lambda(\lambda-2)(\lambda+2)\}$$
$$= (\lambda-2)(1-2/\lambda)/\{(\lambda+4)\sqrt{\lambda(\lambda+4)} + 4(\lambda+1)\sqrt{\lambda+1}\}$$

It may be noted that zero and negative values of λ are of no interest, for the function f is then unbounded. The value $\lambda = 2$ makes the function f identically zero, and so from now on we may take $\lambda \neq 2$. The case of $\lambda = 1$ was solved by D.S. Mitrinovic and W. Janous in Crux Mathematicorum, vol 12, 1986, p.11, and is quoted in the book "Recent Advances in Geometric Inequalities" (Kluwer, 1989) by the former. Also the lower bound must be minus the upper bound, because an odd permutation of the three sides reverses the sign of f.

As a, b and c are the sides of a triangle they may be represented by $a = y+z$, $b = z+x$ and $c = x+y$, with x, y and z non-negative; conversely any such x, y and z give a, b and c satisfying the triangle inequalities, and therefore the sides of some triangle. The triangle is degenerate when one of x, y and z is zero. Making this substitution we find

$$f(y+z,\ z+x,\ x+y,\ \lambda) = (\lambda-2)^2(y-x)(x-z)(z-y)/\text{Polynomial}$$

where the polynomial is in the four variables and has all coefficients non-negative. We may assume that $z \le y \le x$; this makes f non-negative. If $z > 0$ then consider replacing x, y and z by $x-z$, $y-z$ and 0, respectively, making the triangle degenerate. The polynomial is made smaller, and therefore f (if non-zero) is increased in magnitude. This shows that the bounds of f are attained only in the case of degenerate triangles. Now, therefore, we shall consider degenerate triangles; without loss of generality we may put $a = 1-s$, $b = 1+s$ and $c = 2$, where $0 < s < 1$.

$$f(1-s,\ 1+s,\ 2,\ \lambda) = (\lambda-2)(1-2/\lambda)s(1-s)(1+s)/\{(\lambda+2)^2-s^2(\lambda-2)^2\}$$

This expression (as a function of s) is zero at the ends of the

unit interval and positive inside; it has derivative zero when

$$s^4(\lambda-2)^2 - 2s^2(\lambda^2+8\lambda+4) + (\lambda+2)^2 = 0.$$

As a quadratic in s^2 this has one root in the unit interval and the other > 1. The smaller root is given by

$$s^2(\lambda-2)^2 = \lambda(\lambda+4) + 4(\lambda+1) - 4\sqrt{\lambda(\lambda+1)(\lambda+4)}$$

and the positive value of s (which is the one we want) is given by $s(\lambda-2) = \sqrt{\lambda(\lambda+4)} - 2\sqrt{\lambda+1}$ (note how both sides change sign at $\lambda = 2$). Now (because this is the only maximum in the interval) we may find the bound by substituting this value of s in $f(1-s, 1+s, 2, \lambda)$, using the obvious simplifications:-

$$(\lambda+2)^2 - s^2(\lambda-2)^2 = 4\sqrt{\lambda(\lambda+1)(\lambda+4)} - 4\lambda$$

and $$(\lambda-2)^2(1-s^2) = 4\sqrt{\lambda(\lambda+1)(\lambda+4)} - 12\lambda.$$

Then $B(\lambda)$ is found to have the value quoted above. The bounds are attained in the case of the degenerate triangle given by the value of s found above.

The function $$\lambda B(\lambda) = \{(\lambda+4)\sqrt{\lambda(\lambda+4)} - 4(\lambda+1)\sqrt{\lambda+1}\}/(\lambda^2-4)$$
$$= (\lambda-2)^2/\{(\lambda+4)\sqrt{\lambda(\lambda+4)} + 4(\lambda+1)\sqrt{\lambda+1}\}$$

has the interesting property that it is unchanged if we replace λ by $4/\lambda$. As a consequence, it is a function of $v = \lambda + 4/\lambda$; call it $F(v)$. This $F(v)$ may be found to be given by;-

$$(v+4)(v-4)F^2 = v^2+28v+88 - 8(v+5)\sqrt{v+5}.$$

or $$F^2 = (v-4)^2/\{v^2+28v+88 + 8(v+5)\sqrt{v+5}\}$$

(Disappointingly these expressions seem to have no simple square roots)

The inequality $-1 < \Sigma(b-c)/a < 1$ for triangles may be obtained from our result by taking the limit of $\lambda B(\lambda)$ as λ tends to infinity (or to zero, because of the property noted in the previous paragraph). Alternatively take the limit of $F(v)$ as v tends to infinity, clearly = 1. The bounds in this case are not attained even by degenerate triangles, for putting (as before) $a = 1-s$, $b = 1+s$, $c = 2$, the sum, f, is found to be $-s$. The values $s = \pm 1$ cannot be allowed; they give very degenerate triangles, with one side zero, and they make the given sum meaningless because of dividing zero by zero.

POSITION LINES (JCMN, 41, p.4218 and 55, p.6033)

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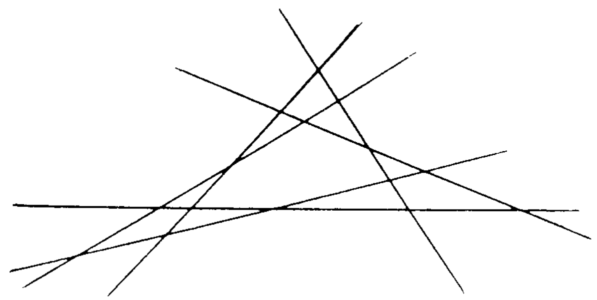
The case of 3 position lines (giving a cocked hat) and the case of 4 (giving an arrowhead) have been discussed in earlier issues. Now consider n position lines. It may be proved by induction that they divide the plane into $1+n(n+1)/2$ regions, where we are ignoring cases of probability zero, such as two of the lines being parallel or three being concurrent. Some of these regions are bounded and some unbounded.



<u>Lemma 1</u> There are 2n unbounded regions and $(n-1)(n-2)/2$ bounded regions.

<u>Proof</u> A large circle, containing all the points of intersection, is divided into 2n arcs, one for each unbounded region.

<u>Lemma 2</u> The probability of the true position, P, being in any one of the unbounded regions is 2^{-n}.

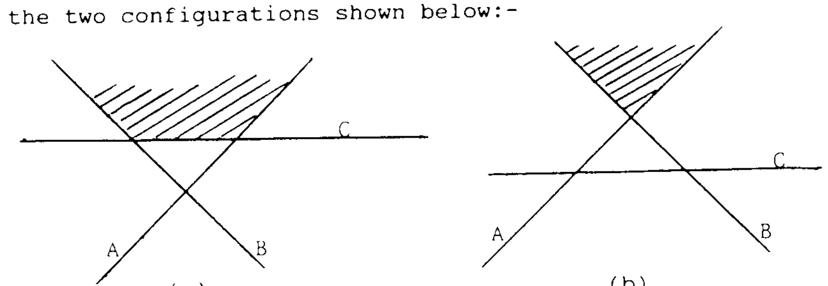
<u>Proof</u> Irrespective of the errors in the position lines, there is one region that contains the North point at infinity, i.e. it contains all but a finite length of the ray going North from the true position P (or in fact from any other point). P will be in this region if all the position lines go South of P. These n events are independent, each with probability 1/2. As our "North" was arbitrary, the same holds for any other direction.

<u>Theorem</u> The probability of P being in the union of all the bounded regions is $1 - 2n/2^n$.

<u>Proof</u> This follows at once from Lemma 1 and Lemma 2.

The formula in this theorem includes the results in JCMN 55 about cocked hats and arrowheads (n = 3 or 4). The meaning of these results should perhaps be emphasised.

They are the probabilities that we have to assign when we know the directions of the position lines, and know nothing else (though in fact the directions are not relevant). One might ask if the probabilities are altered by our having the additional information of the position lines, without of course knowing the true position P. To clarify this question, take n = 3. Suppose position line A runs NE-SW, B runs NW-SE and C runs E-W. Before we see the position lines, we ascribe probability 1/8 to P being in the unbounded region that contains an infinite line running North, i.e. being North of all three position lines. Now suppose we have the position lines, they can be in either of the two configurations shown below:-



(a) (b)

If we know which one of these is the actual configuration of the position lines, what probability do we then ascribe to P being in the shaded area? Is it still 1/8 in both cases? Remember that we are assuming no knowledge of the error distributions, except the symmetry condition. Use the notation "NNN" to denote the true position being in the shaded area, and "a" to denote the cocked hat being as in (a) above. Can we say that $P(NNN|a) = 1/8$? No. Because $P(NNN|a) = P(a|NNN)P(NNN|)/P(a|)$ and $P(NNN|) = 1/8$ and $P(a|) = 1/2$, this is equivalent to saying that $P(a|NNN) = 1/2$. Let x, y and z be the errors of the three position lines. The assertion would be that $P(x + y \geq \sqrt{2}z|) = 1/2$; and we cannot say this about three independent random variables about which we know only that all are positive.

A further possible question is: suppose we know both the cocked hat and the error distributions; does that affect the probability of P being in the cocked hat? Yes. For clearly if the cocked hat is very small compared with the errors, then P is very unlikely to be in the cocked hat.

PSEUDO-RANDOM NUMBERS

My Hitachi 'Peach' computer offers a sequence of 'random numbers', to be regarded as uniformly distributed on the unit interval. I sometimes wonder what they are. The sequence has period 8388608, which is 2 to the power of 23. The first few and a few in the middle, in hexadecimal notation (that is in the scale of 16, with A for 10, B for 11, C for 12, D for 13, E for 14 and F for 15) are:-

1	·97500F0	(·591065)	2	·353EE84	(·207991)
3	·8D0C2D0	(·550967)	4	·A2C7EF0	(·635863)
5	·1AA6FB4	(·104110)	6	·CE6BE50	(·806334)
7	·0AFBF95	(·042907)	8	·F4304B0	(·953862)
9	·5AB2A80	(·354289)	10	·8E76A10	(·556498)
11	·721E850	(·445778)	12	·035C66F	(·013129)
13	·8B1F140	(·543443)	14	·881B1C0	(·531664)
15	·50FFD18	(·316403)	16	·7768428	(·466435)
17	·1A583F4	(·102909)	18	·FAD5590	(·979818)
19	·24EBDD8	(·144224)	20	·482FDE8	(·281980)
21	·3C8A2B4	(·236483)	22	·1DE1546	(·116720)
23	·73EEA98	(·452860)	24	·8F4F3A0	(·559803)
...	...	...	...	...	...
4194301	·28C1E34	(·159208)	4194302	·39F3AC4	(·226374)
4194303	·5F83218	(·373095)	4194304	CFC7520	(·811635)

(Now we start the second half of the list)

4194305	·17500FA	(·091065)	4194306	·B53EE90	(·707991)
4194307	·0D0C2DF	(·050967)	4194308	·22C7EE4	(·135863)
4194309	·9AA6FC0	(·604111)	4194310	·4E6BE40	(·306334)
4194311	·8AFBFA0	(·542907)	4194312	·74304A0	(·453862)
4194313	·DAB2A70	(·854289)	4194314	0E76A07	(·056498)
4194315	·F21E860	(·945778)	4194316	·835C660	(·513129)
...	...	...	...	...	...

The decimal equivalents are given in brackets, but I suspect that decimal notation is used by the computer only as a means of communicating with us humanoids, so that the decimal form is no more than the 6-decimal-place approximation to the value of the hexadecimal expression (remember that 6 hexadecimal digits are as good as 7 decimal digits). From the computer manual it appears that a single-precision floating-point number is stored in 4 bytes of memory, and (if positive) its logarithm (to base 2) must be between -128 and +127.

Can anyone guess how these numbers are generated?

ACUTENESS OF RANDOM TRIANGLES IN A DISC

(JCMN 54, p.6023)

Using 4 different definitions of a 'random triangle', it was asked what was the probability of the triangle being acute. Here we shall look at case (b), where the triangle is formed by three random points from the uniform distribution inside a circle. The probability is 1-3p where p is the probability of one particular angle being obtuse; in this case we shall find that $p = 3/8 - 4/(3\pi^2) = 0{\cdot}2399$, and $1-3p = 4/\pi^2 - 1/8 = 0{\cdot}2803$.

The calculation starts by recalling Lemma 1 from GEOMETRICAL PROBABILITY on page 6012 of JCMN 54.

<u>Lemma 1</u> Let B and C be random points from the uniform distribution in the unit circle with centre O. The joint probability of B being at distance between r and r+dr from O and of the perpendicular distance from O to BC being between p and p+dp is

$$\frac{4(1 - 2p^2 + r^2)\, r\, dr\, dp}{\pi \sqrt{r^2 - p^2}}$$

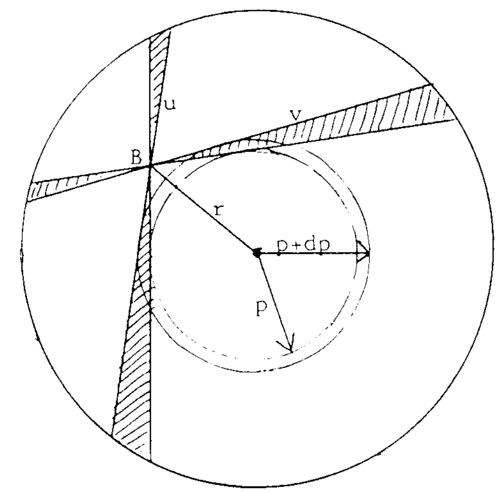


Figure 1

We shall use both the result and the notation of this lemma. Let p and r be as defined above. Apologies are due for the use of the symbol p with two different meanings, taken from the two contributions above.

Draw a new diagram with BC horizontal and with B above and to the left of O.

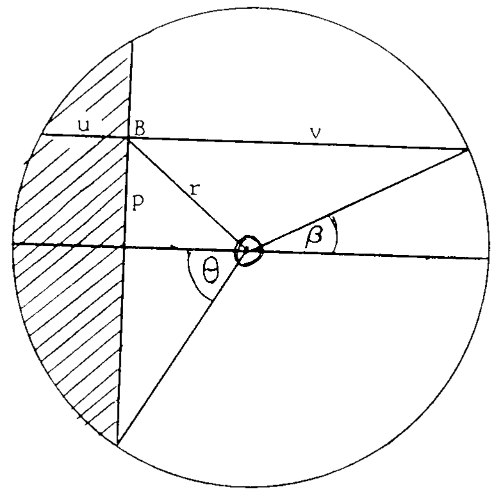


Figure 2

Note the geometrical identities:

$u = \sqrt{(1 - p^2)} - \sqrt{(r^2 - p^2)}$ $\qquad v = \sqrt{(1 - p^2)} + \sqrt{(r^2 - p^2)}$

$u^2 + v^2 = 2 - 4p^2 + 2r^2$ $\qquad v^2 - u^2 = 4\sqrt{(r^2-p^2)}\sqrt{(1-p^2)}$

$\cos\theta = \sqrt{(r^2-p^2)}$ $\qquad \sin\theta = \sqrt{(1+p^2-r^2)}$

Shaded area $= \theta - \sin\theta\cos\theta$ $\qquad p = \sin\beta$

The angle B of the triangle will be obtuse if either A is to the left of B and C to the right, or vice versa. The probability of A being to the left of B is $(1/\pi)$ times the shaded area, and the probability of C being to the right of B is (recalling that C must be in one of the thin triangles of Figure 1) $v^2/(u^2+v^2)$. The probability of B being obtuse (given r and p) is therefore

$$\frac{v^2}{u^2+v^2}\;\frac{\theta - \sin\theta\cos\theta}{\pi} + \frac{u^2}{u^2+v^2}\;\frac{\pi - \theta + \sin\theta\cos\theta}{\pi}$$

This may be expressed (using the geometrical identities above) as

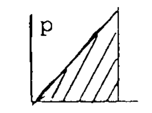
$$\frac{1}{2} - \frac{2\sqrt{(r^2-p^2)}\sqrt{(1-p^2)}(\pi/2-\theta)}{\pi(1-2p^2+r^2)} + \frac{2(r^2-p^2)\sqrt{(1-p^2)}\sqrt{(1+p^2-r^2)}}{\pi(1-2p^2+r^2)}$$

Lemma 1 tells us that this expression must be multiplied by

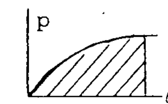
$$(4/\pi)(1-2p^2+r^2)\ r\ dr\ dp\ /\sqrt{(r^2-p^2)}$$

and integrated over the region where $0 < p < r < 1$ to give what we want, the other p, the probability of angle B being obtuse. This gives a long formula which we write for the moment as 1/2 - First integral - Second integral. Now we have to evaluate these two integrals.

<u>First integral</u> $\quad (8/\pi^2)\iint\sqrt{(1-p^2)}(\pi/2-\theta)\ r\ dr\ dp$



Consider integration with respect to r for each p. Change the variable from r to θ. The interval of integration for θ is $\beta = \arcsin p < \theta < \pi/2$, and $r\ dr = -\sin\theta\cos\theta\ d\theta$. The first integral is therefore



$$(8/\pi^2)\iint\sqrt{(1-p^2)}(\pi/2-\theta)\cos\theta\sin\theta\ d\theta\ dp$$

over the set where $0 < p < \sin\theta$ and $0 < \theta < \pi/2$.

Now integrate with respect to p, recalling that

$$\int_0^{\sin\theta}\sqrt{(1-p^2)}\ dp = (\theta + \sin\theta\cos\theta)/2.$$

Integral $= 1/(2\pi^2)\int_0^{\pi/2}(2\theta + \sin 2\theta)(\pi - 2\theta)\sin 2\theta\ d\theta$ which without difficulty may be evaluated as $1/16 + 1/(\pi^2)$.

<u>Second integral</u> $\quad (8/\pi^2)\iint\sqrt{(r^2-p^2)}\sqrt{(1-p^2)}\sqrt{(1+p^2-r^2)}\ r\ dr\ dp$ over the set where $0 < p < r < 1$.

First consider the integration with respect to r for each p, the interval of integration being from p to 1. This inner integral is $\quad (8/\pi^2)\sqrt{(1-p^2)}\int_p^1\sqrt{(r^2-p^2)}\sqrt{(1+p^2-r^2)}\ r\ dr$

Change the variable from r to θ, with $2r\ dr = -\sin 2\theta\ d\theta$. The inner integral becomes $(1/\pi^2)\sqrt{(1-p^2)}\int_\beta^{\pi/2} 1 - \cos 4\theta\ d\theta$, which equals $(1/4\pi^2)\cos\beta\,(2\pi - 4\beta + \sin 4\beta)$. Therefore

Second integral $= (1/4\pi^2)\int_0^1(2\pi - 4\beta + \sin 4\beta)\cos\beta\ dp$.

Change the variable from p to β. Second integral $=$ $(1/8\pi^2)\int_0^{\pi/2}(1 + \cos 2\beta)(2\pi - 4\beta + \sin 4\beta)\ d\beta = 1/16 + (1/3\pi^2)$.

Collecting our results, the probability of the angle B being obtuse is $\quad$ 1/2 - first integral - second integral

$$= 3/8 - 4/(3\pi^2) = 0.2399051.$$

QED

ACUTENESS OF RANDOM TRIANGLES IN A SQUARE

(JCMN 54, p.6023)

Consider the question (c) in the article noted above, 3 random points from the uniform distribution in a square form a triangle, what is the probability that it be acute-angled?

Firstly we must establish :-

Theorem 1 Suppose that x, y and z are independent random variables from the uniform distribution on the unit interval (0, 1). Then (x-y)(x-z) is a random variable in the interval (-1/4, 1) with a probability density, i.e. the probability that the expression take a value between p and p+dp, divided by dp:-

$2\log((1+\sqrt{1+4p})/(1-\sqrt{1+4p})) - 4\sqrt{1+4p}$ if $-1/4 < p < 0$ (1)

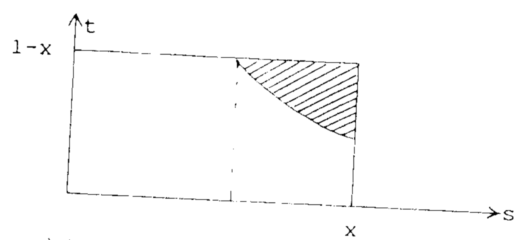
and $4\sqrt{p} - 4 - 2\log p$ if $0 < p < 1$ (2)

The integrated forms of these are the probabilities:-

$P(-1/4 < (x-y)(x-z) < p < 0) =$
$\sqrt{1+4p}(1-8p)/3 + 2p\log((1+\sqrt{1+4p})/(1-\sqrt{1+4p}))$ (1′)

and $P(0 < (x-y)(x-z) < p) =$
$(8/3)p\sqrt{p} - 2p - 2p\log p$ (2′)

Proof Consider (1′), and for clarity put $q = -p > 0$. Negative values of (x-y)(x-z) occur when either $y < x < z$ or $z < x < y$, and each has probability 1/6, and in either case the probability density of x is $6(1-x)(1+x)dx$. We want to find, for each q in the interval (0, 1/4), the probability that $q < -(x-y)(x-z) < 1/4$. We need consider only the case where $0 < y < x < z < 1$, the other case is the same. Firstly take x to be fixed while y and z are random variables, and note that under this constraint $s = x-y$ and $t = z-x$ are uniformly distributed in the intervals (0, x) and (0, 1-x) respectively; we need the probability that $st > q$.



It is (shaded area)/(area of rectangle), easily found to equal

$1 - (q + q\log(x(1-x)/q))/(x(1-x))$

Now we must take the mean of this over x in the interval (0, 1) with density $2(1-x)(1+x)dx$. This gives the result (1′) above, and differentiation gives (1).

A similar calculation gives (2′) and (2). QED

Theorem 2 The probability of the random triangle being acute is

$$53/150 - \pi/40 = 0\cdot274794$$

Proof Consider the triangle with vertices (x, u), (y, v) and (z, w) where the 6 random variables u, v, w, x, y and z are independent and uniform in the interval (0, 1). The cosine of the angle at (x, u) has the same sign as the random function p+q, where $p = (x-y)(x-z)$ and $q = (u-v)(u-w)$. But p+q is just the sum of two independent random variables from the distribution described in Theorem 1. To evaluate the probability of p+q being negative we may separate the following cases.

Case 1 with p and q both positive has probability 4/9, and the contribution that it makes is 0.

Case 2 with p and q both negative has probability 1/9, and the contribution from this case is 1/9.

Case 3 with p positive and q negative has probability 2/9, and the contribution from this case is the integral of the product:-

(probability that $(x-y)(x-z) < -p$) ×
(probability that $p < (x-y)(x-z) < p+dp$).

Case 4 with p negative and q positive makes the same contribution as case 3.

We therefore find, for the probability of the angle at the point (x, u) being obtuse, the value:-

$$1/9 + 2\int_0^{1/4} (\sqrt{1-4p}(1+8p)/3 - 2\log((1+\sqrt{1-4p})/(1-\sqrt{1-4p}))) \times (4\sqrt{p}-4-2\log p)\,dp$$

Changing the variable to $x = \sqrt{1-4p}$, this becomes 1/9 +

$$\int_0^1 (6x^2+4x^4+(3x-3x^3)\log((1-x)/(1+x)))(\sqrt{1-x^2}-2-\log(1/4-x^2/4))dx/3$$

The evaluation of this integral is elementary. Anybody in your first year calculus class should be able to do it, but might complain that you set hard questions. The answer is

$$1/9 + (\pi/40+47/150)/3 = \pi/120 + 97/450 = 0\cdot241735.$$

The probability of the triangle being acute-angled is therefore 1 - 3(the value found above) =

$$53/150 - \pi/40 = 0\cdot274794.$$

QED

ACUTENESS OF RANDOM TRIANGLES 2

Let E be a measurable plane set of finite area. A random triangle is found by taking 3 points from the uniform probability distribution on E. Let A(E) be the probability that such a random triangle be acute-angled. It might be conjectured that A(E) attains its maximum of $4/\pi^2 - 1/8 = 0{\cdot}2803$ when E is the inside of a circle.

Apologies are due for inaccuracies of some Monte Carlo estimates given in a previous contribution (ACUTENESS OF RANDOM TRIANGLES, JCMN 54, p. 6023) We now have exact values for A(E) of 0·2803 when E is a disc and 0·2748 when E is a square, and a Monte Carlo estimate gives 0·252 when E is an equilateral triangle.

ANALYSIS PROBLEM

If a continuous function of the real variable has derivative zero at the rationals, then must it be constant?