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The James Cook Mathematical Notes is published in 3 issues per year, dated January, May and September. The history of JCMN is that the first issue (a single foolscap sheet) appeared in September 1975, then others at irregular intervals, all the issues up to number 31 being produced and sent out by the Mathematics Department of the James Cook University of North Queensland, of which I was then the Professor. In October 1983 this arrangement was beginning to be unsatisfactory, and I changed to publishing the JCMN myself, having three issues per year printed in Singapore and posted from there. I then set a subscription price of 30 Singapore dollars per year. When in 1985 I changed to printing in Australia I kept the same price, for the Singapore dollar is a stable currency.

In October 1992 it had become clear that the paying of subscriptions by readers is an inefficient operation. Bank charges for changing currency and for international transfers, with postage, together absorb most of the initial input of money. Therefore we have abandoned subscriptions as from the beginning of 1993, issue number 60. To those who want to give something in return for the JCMN, I ask them to make a gift to an animal welfare society in their own country. The animals of the world will be grateful and so will I.

Contributors, please tell me if and how you would like your address printed.

JCMN 64, May 1994

CONTENTS

QUESTIONS TO THINK ABOUT

Shailesh Shirali
(Rishi Valley School, Andhra Pradesh, India)

(1) Let S be the set of all positive integers n such that n divides $2^n + 1$. The first few members of S include (1, 3, 9, 27, .. , 513, ...).

If n is in S then so are 3n and $2^n + 1$. To show this, take any n is in S, $2^{3n} + 1 = (2^n + 1)(2^{2n} - 2^n + 1)$, of which n divides the first factor and (because n is odd) 3 divides the second factor. To illustrate the second fact, take a simple case. $2^9+1 = 513 = 9 \times 57$, and $2^{513} + 1 = 512^{57} + 1 = 513(512^{56} - 512^{55} + \ldots - 512 + 1)$.

Is every member (except 1) of the set S expressible as either 3n or $2^n + 1$, where n is some smaller member of S? Can one characterize the elements of S in any manner?

(2) Let $x = 4^{1/3}$ and $y = 3^{1/2}$. Then the following identity holds:

$$3 \arctan \frac{y}{2x + 1} + 2 \arctan \frac{x + 1}{y} = \pi.$$

This equation is not too difficult to establish. More interesting are the questions:

From where might such an identity emerge?

Can one generate more such identities?

COMBINATORIAL NUMBER QUESTION

P. J. O'Halloran
(University of Canberra, ACT 2616, Australia)

This question arose from a problem in the Junior Mathematics Challenge for 1993.

Given any positive integer n, the problem is to construct from the numbers (1, 2, 3, 4, ... 3n) (using each number just once) a rectangular matrix of 3 rows and n columns with the property that:

First row + Second row = 3 × Third row.

It is fairly easy to show that if a solution is possible then n ≡ either 0 or 5 (mod 8).

<u>Conjecture</u> If n ≡ either 0 or 5 (mod 8) then a solution is possible.

Solutions have been found in the cases n = 8k and n = 8k+5 for all $k \le 7$.

Examples

1	2	14	5	15
11	7	10	13	12
4	3	8	6	9

1	2	3	4	15	6	19	16
14	22	18	23	21	24	20	17
5	8	7	9	12	10	13	11

DIOPHANTINE EQUATION

Thanks are due to Trevor and Nigel Tao for drawing the Editor's attention to difficulties with a Diophantine equation arising in a recent Australian Mathematical Olympiad paper.

The equation (or pair of equations) is

$$a^2 = 2n + 1 \qquad \text{and} \qquad b^2 = 3n + 1.$$

A little work on the Editor's *Peach* computer brought to light the following solutions

n	a	b	a^2	b^2
0	1	1	1	1
40	9	11	81	121
3960	89	109	7921	11881
388080	881	1079	776161	1164241
38027920	8721	10681	76055841	114083761

The purist might object that Diophantus would not approve of the use of $n = 0$ in a solution, but the inclusion of the top line in the table above makes the rule of construction for the infinite sequence easier to guess (in fact we might have gone further and put in $n=0$, $a=1$, $b=-1$) The rule is that

$$a(k) = 10a(k-1) - a(k-2) = 5a(k-1) + 4b(k-1)$$
$$b(k) = 10b(k-1) - b(k-2) = 6a(k-1) + 5b(k-1).$$

Two nice little questions now arise. Do the values of a, b and n given by the rule above all satisfy the original equations? And does the sequence include all possible solutions?

It is not hard to establish that for any n satisfying the equations above, n+1 may be expressed as a sum of two or three squares of a certain form, this may be illustrated from the figures above as follows.

$$\begin{aligned}
n + 1 &= 1 &&= 0^2 + 1^2 &&= 0^2 + 0^2 + 1^2 \\
41 &&&= 4^2 + 5^2 &&= 3^2 + 4^2 + 4^2 \\
3961 &&&= 44^2 + 45^2 &&= 36^2 + 36^2 + 37^2 \\
388081 &&&= 440^2 + 441^2 &&= 359^2 + 360^2 + 360^2 \\
38027921 &&&= 4360^2 + 4361^2 &&= 3560^2 + 3560^2 + 3561^2
\end{aligned}$$

The expression of n+1 as a sum of three squares shows an alternation of the pattern, the alternation can be shown to depend on the residue class of b mod 3, which alternates between 1 and 2. The sum always involves two adjacent squares, of which the even one is doubled, but the even one is alternately less than and greater than the odd one.

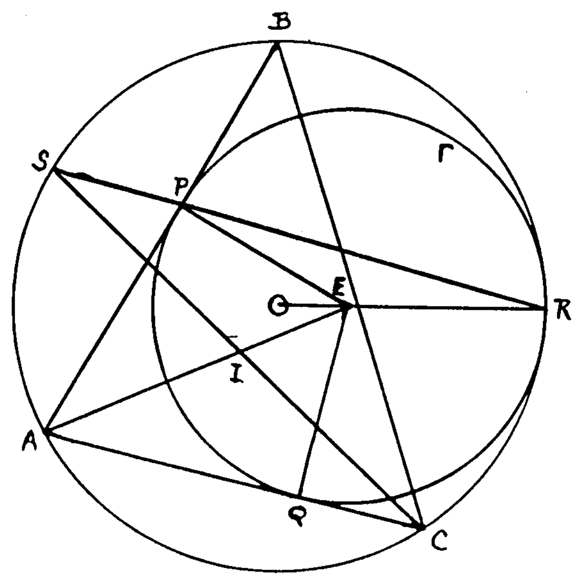
The AMO question asked for a proof that n+1 was expressible as " ... a square plus twice the succeeding perfect square". Perhaps " .... the square of a number plus twice the square of the succeeding number" would have been better. The word 'number' would have to be interpreted as possibly negative, so that, for example:

$$3961 = (-37)^2 + (-36)^2 + (-36)^2.$$

TRIANGLE PROBLEM (JCMN 63 p.6305)

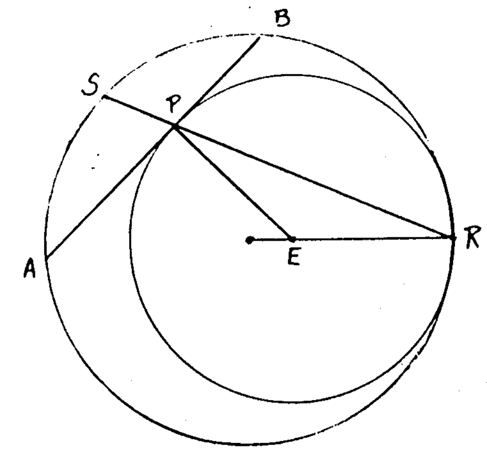
Sahib Ram Mandan, Shailesh Shirali, Esther Szekeres
& Jordan Tabov

A circle Γ touches the sides AB and AC of a triangle ABC, at points P and Q respectively, and Γ touches the circumcircle internally at R. Denote the incentre by I. Prove (or disprove) that PR must meet CI on the circumcircle.

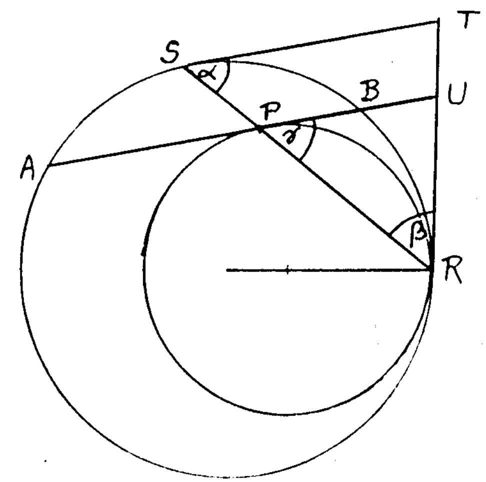


For this result (suggested in the previous issue by Nigel Tao) there are several possible proofs, and what follows below is compiled by the Editor from the contributions sent in.

Firstly observe that we may leave out parts of the original drawing. It will be sufficient to prove that S (where PR meets the circumcircle) is the mid-point of the arc AB in the drawing on the next page.



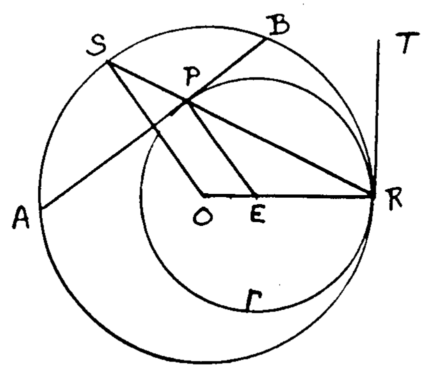
<u>First proof</u> (The way Euclid might have reasoned)



Draw the tangents at S and R, meeting at T, and let AB meet TR at U. Since ST = RT, it follows by Euclid I, 5 (the theorem known as the 'Pons Asinorum') that angle α = angle β. Similarly, because PU = RU, angle β = angle γ. Therefore the tangent ST is parallel to the chord AB. Consequently S is the mid-point of the arc AB.

QED

<u>Second proof</u>

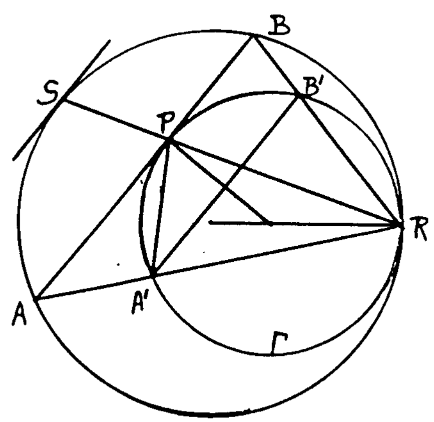


Draw RT, tangent to both circles. Since the angle between any chord of a circle and the tangent at one end equals the angle in the opposite segment, and therefore equals half the angle subtended at the centre,

$\angle PER = 2\angle PRT = 2\angle SRT = \angle SOR.$

Therefore SO is parallel to PE, which is perpendicular to AB. This means that S is the mid-point of the arc AB. QED.

<u>Third proof</u> (By transformation geometry)



The homothety with centre R, shrinking in the ratio PR/SR maps the circumcircle into the circle Γ, so that AB is

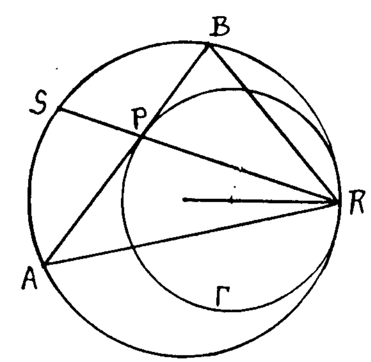
parallel to its image A′B′. Now there are two ways of completing the proof.

Looking at angles, $\angle PRB' = \angle PA'B' = \angle APA' = \angle PRA'$. This shows that the arcs AS and SB subtend the same angle at R, and so are equal. Alternatively, the tangent at S has as its the image the tangent at P, so that it must be parallel to AB, and therefore S is the mid-point of the arc AB.

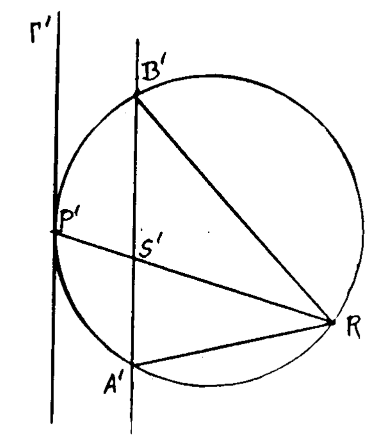
QED.

<u>Fourth proof</u> (Using inversion, which was introduced by J. Magnus in 1831)

Invert the figure from the point R.



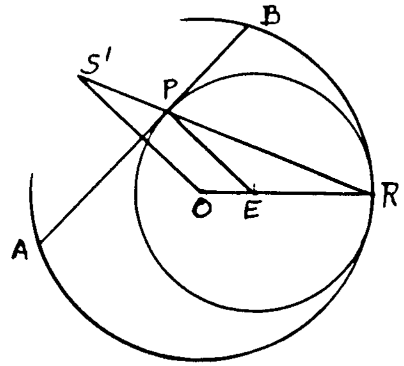
Original figure



Inverse figure

The two circles, because they touch at R, invert into parallel straight lines. Denote the image of P by P′, etc. The image of the line APB is a circle through R. $\angle A'RP' = \angle P'RB'$, and so in the figure before inversion $\angle ARP = \angle PRB$, and the two arcs are equal. QED.

Fifth Proof

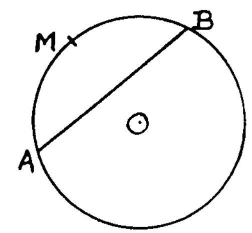


Let the perpendicular to the side AB from the centre O of the circumcircle meet the line RP at S'. Join EP. The two triangles REP and ROS' are similar (corresponding sides are parallel). But EP = ER, therefore OS' = OR, so that S' is on the circumcircle. As S'O is perpendicular to the chord AB, it follows that S' bisects the arc AB of the circumcircle.

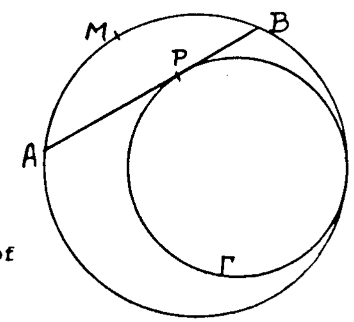
QED

Sixth Proof (Differential calculus)

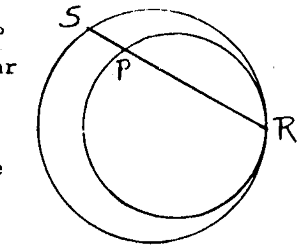
Firstly consider a moving chord AB of a fixed circle (the circumcircle of the original problem), and the mid-point M of the arc AB. Since translation of AB without rotation does not move M, the angular velocity of M round the circle is the rate of rotation of the line AB.



Next, suppose that the moving chord AB is tangent at P to another fixed circle Γ. The angular rate of movement of M round the circumcircle equals the angular rate of movement of P round Γ.

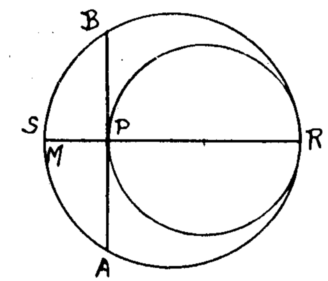


Now forget AB, consider R where the two circles touch, and S where RP meets the circumcircle. The angular rate of rotation of S round the circumcircle is twice the angular velocity of RP about R, and therefore equal to the rate of rotation of P round Γ.



Thus we have established that as P moves round Γ, the velocities at which the two points M and S move round the circumcircle are equal. But there is one position of P for which M and S coincide, therefore they always coincide.

QED



SEQUENCES WITHOUT ARITHMETIC PROGRESSIONS

(JCMN 63, pp. 6318-6321)

Don Coppersmith

(T. J. Watson Research Center, Yorktown Heights, 10598, USA)

In Terry Tao's article under this title printed in the previous issue there was put forward "Conjecture 4", that (in the notation of that article) $c(n) = O(n^{1-\epsilon})$ for some $\epsilon > 0$. This conjecture was disproved by R. Salem and D. C. Spencer in 1942. They found a sequence with

$$c(n) > n/\exp(O(\sqrt{\log n})).$$

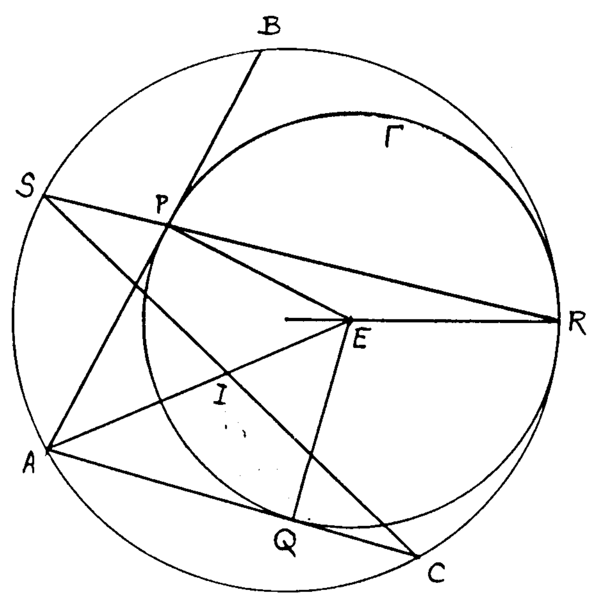
SECOND TRIANGLE PROBLEM

Shailesh Shirali

(Rishi Valley School, Andhra Pradesh, India)

A circle Γ touches the sides AB and AC of a triangle ABC, at points P and Q respectively, and Γ touches the circumcircle internally at R. Denote the incentre by I.

The wording above and the drawing below are from Nigel Tao's "Triangle Problem" on page 6305 of JCMN 63.



A second problem is this —— prove that the incentre I is the mid-point of PQ.

GENERALIZED HADAMARD INEQUALITY

(JCMN 31, p.3165, 32, p.4016 & 33, p.4032)

In 1984 the following problem appeared in JCMN 33.

It is given that W is the complex positive definite n×n Hermitean matrix given in block notation as

$$W = \begin{pmatrix} I_p & Z \\ Z^* & I_q \end{pmatrix}$$

where p+q = n, Z is rectangular p×q, and Z* is its Hermitean conjugate. Relate the eigenvalues of W to those of $I_p - ZZ^*$, and so prove that det W ≤ 1 with equality only when Z = 0.

The answer is that det W = det $(I_p - ZZ^*)$.

Proof: In block notation:

$$W\begin{pmatrix} I & 0 \\ -Z^* & I \end{pmatrix} = \begin{pmatrix} I-ZZ^* & Z \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & Z \\ 0 & I \end{pmatrix}\begin{pmatrix} I-ZZ^* & 0 \\ 0 & I \end{pmatrix}$$

where the I without subscript represents the unit matrix of size implied by the context (this makes typing easier).

Note that $\begin{pmatrix} I & 0 \\ -Z^* & I \end{pmatrix}$ and $\begin{pmatrix} I & Z \\ 0 & I \end{pmatrix}$ can both be reduced to the unit n×n matrix by the elementary row operations of adding scalar multiples of one row to another, and so they both have determinant = 1.

Therefore $\det W = \det \begin{pmatrix} I-ZZ^* & 0 \\ 0 & I \end{pmatrix} = \det(I-ZZ^*)$.

The second part of the calculation uses the fact that W is positive definite. Take any p-dimensional column vector x.

$$(x^*, -x^*Z)\begin{pmatrix} I & Z \\ Z^* & I \end{pmatrix}\begin{pmatrix} x \\ -Z^*x \end{pmatrix} > 0, \quad (\text{if } x \neq 0)$$

i.e. $x^*(I - ZZ^*)x > 0$, showing that I − ZZ* is positive definite. Because $x^*(I - ZZ^*)x \le x^*x$, all the eigenvalues of I − ZZ* are between 0 and 1. If det W = 1 then all the eigenvalues of I − ZZ* are 1, and so ZZ* = 0. From this it follows that $\|Z^*x\| = 0$ for all x, and Z = 0. QED

The roots of this contribution date from 1983, when "Kestelman's Last Problem" appeared in JCMN 31, (H. Kestelman posted it from London a week before he died). It asked for a proof of:-

Theorem G Let A be the positive definite Hermitean matrix $A = \begin{pmatrix} P & N \\ N^* & Q \end{pmatrix}$ where P is p×p, Q is q×q, N is p×q and N* is the Hermitean conjugate of N. Then det A ≤ det P det Q, with equality only when N = 0.

The proof given in JCMN 32 was in two parts, as follows.

(i) Because P and Q are positive definite Hermitean they have positive definite Hermitean square roots, S and T. Then A = MWM, where $M = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$, $W = \begin{pmatrix} I & Z \\ Z^* & I \end{pmatrix}$ and $Z = S^{-1}NT^{-1}$.

Therefore det A = det P det Q det W, and W is positive definite. It remains to be proved only that det W ≤ 1.

(ii) The eigenvalues of W have sum equal to the sum of the diagonal elements (the "spur" or "trace" of W), the eigenvalues have arithmetic mean = 1 and therefore geometric mean ≤ 1, with equality only when they are all equal.

Therefore det W = product of eigenvalues ≤ 1, with equality only when W = I.

From part (i) of the JCMN 32 proof and the first part of the proof given on the previous page, we find:-

If $A = \begin{pmatrix} P & N \\ N^* & Q \end{pmatrix}$ is any non-singular Hermitean matrix (not necessarily positive definite) then

$$\det A = \det Q \det(P - NQ^{-1}N^*).$$

Proof: Using the notation above, $\det W = \det(I-ZZ^*)$ $= \det(I - S^{-1}NT^{-1}T^{-1}N^*S^{-1})$, where we must be a little careful over the definitions of S and T; every Hermitean matrix has a square root, though it is not generally Hermitean. But the above identity will hold, recall that P and Q must be non-singular, therefore also S and T. Consequently det A = det MWM = det P det Q det W = $\det Q \det(P - NQ^{-1}N^*)$.

A quaint identity, but is it useful? or beautiful?

To the reader who is puzzled by the heading "Generalized Hadamard Inequality", there is a connection between Kestelman's Theorem G and the familiar Hadamard inequality (the modulus of a determinant cannot exceed the product of lengths of the vectors represented by the rows or columns). It is as follows.

Lemma 1: If C is Hermitean symmetric and positive definite, then 0 < det C ≤ (the 1, 1 element) × (the 1, 1 minor) (Apply Theorem G with p = 1 and q = n-1)

Lemma 2: If C is Hermitean symmetric and non-negative definite, then det C ≤ (product of diagonal elements); there is equality only when either C is diagonal or one diagonal element is zero. (Use lemma 1 with induction on n)

Hadamard's Inequality: For any complex square matrix B, put C = B*B, it is Hermitean and non-negative definite. Each diagonal element of C is the square of the length of the vector representing a column of B, and det C is the square of |det B|. Apply Lemma 2, this gives Hadamard's inequality for the columns of matrix B. For the result on rows, use BB* instead of B*B. The case of equality is easily seen, either B is singular or C is diagonal (in which case the columns or rows of B are orthogonal).

QUOTATION CORNER 46

Professor —— was in 1992 awarded an honorary DSc from the University of ——, but one of his less publicised achievements has been to increase the external grant income of his Department (Pathology) from £1·5 million p.a. when he took over in 1987 to approximately £6 million p.a. It thus brings in more outside funds than the Department of Engineering and nearly as much as the Department of Physics.

—— From Peterhouse, A Record, 1990-93.

BINOMIAL IDENTITY 38

$$\sum_{r=1}^{n} (-1)^{n-r}\binom{n}{r}\binom{mr}{n+1} = n(m-1)m^n/2.$$

HERMITEAN MATRICES

Terry Tao
(Mathematics Department, Princeton Univ. NJ 08544, USA)

Let A and B be invertible $n \times n$ Hermitean matrices. If their arithmetic and harmonic means are both positive definite, does it follow that A and B are both positive definite?

In other words, if $A + B$ and $A^{-1} + B^{-1}$ are both positive definite, are A and B necessarily both positive definite?

OLD FASHIONED PROBLEM

Evaluate the infinite continued fraction

$$\frac{1}{2+}\ \frac{2}{3+}\ \frac{3}{4+}\ \frac{4}{5+}\ \cdots \quad (= 0{\cdot}39221119\ldots)$$

The first few of the successive convergents are

Numerator	1	3	15	87	597
Denominator	2	8	38	222	1522

TWOS AND THREES

It had been the custom for the knights of the Round Table to go out on their quests in pairs. Sir Gareth and Sir Bedivere had been pursuing a band of robbers on a lonely stretch of the Fosse Way through the Mendip Hills when Sir Gareth had been wounded by an arrow. The robbers had escaped because Sir Bedivere had turned back to bandage Sir Gareth's wound and take him back to Camelot.

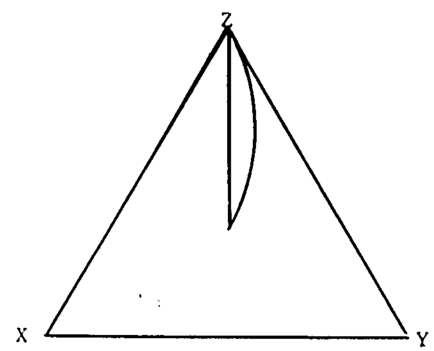
Queen Guinevere had looked after Sir Gareth until he was fit again, then she went to tell the whole story to King Arthur. "I see the difficulty," he said, "Sir Bedivere was put in the position of either abandoning his duty or abandoning his companion. It might be better if the knights went on their expeditions in groups of three instead of in pairs, then in a case of injury there would be one to see to the wounded and one to finish the task they were doing. So could we partition the 6n knights into 2n trios instead of the present 3n pairs?"

"Of course," answered the Queen, "we know that there are $6^{-2n}(6n)!/(2n)!$ ways of arranging the 6n knights in trios; but suppose that a trio were to contain both members of an existing pair, the third member of the trio might feel unhappy at being an outsider. So I think we ought to exclude all such arrangements. But I don't know how many possibilities there are with this restriction" "Of course Merlin would be able to tell us, but I'm afraid he's away in Babylonia at some meeting of magicians."

POLYNOMIAL INEQUALITY (JCMN 60, p.6222)

This problem was about the three norms $\|f\|_1$, $\|f\|_2$ and $\|f\|_\infty$ for a real function on the unit interval.

If no restriction is placed on the functions then the answer is simple: $\|f\|_1 \le \|f\|_2^2/\|f\|_1 \le \|f\|_\infty$, these are the best possible inequalities. The first of these inequalities comes from the Cauchy-Schwarz inequality, and the second from $|f(x)|^2 \le |f(x)|\,\|f\|_\infty$. The inequalities may be represented graphically by using $x = \|f\|_1$, $y = \|f\|_2$ and $z = \|f\|_\infty$ as trilinear coordinates with an equilateral triangle of reference, as shown below, the possible region for the point (x, y, z) is bounded by the median x = y and by the circular arc $y^2 = xz$ between the points (1, 1, 1) and (0, 0, 1).

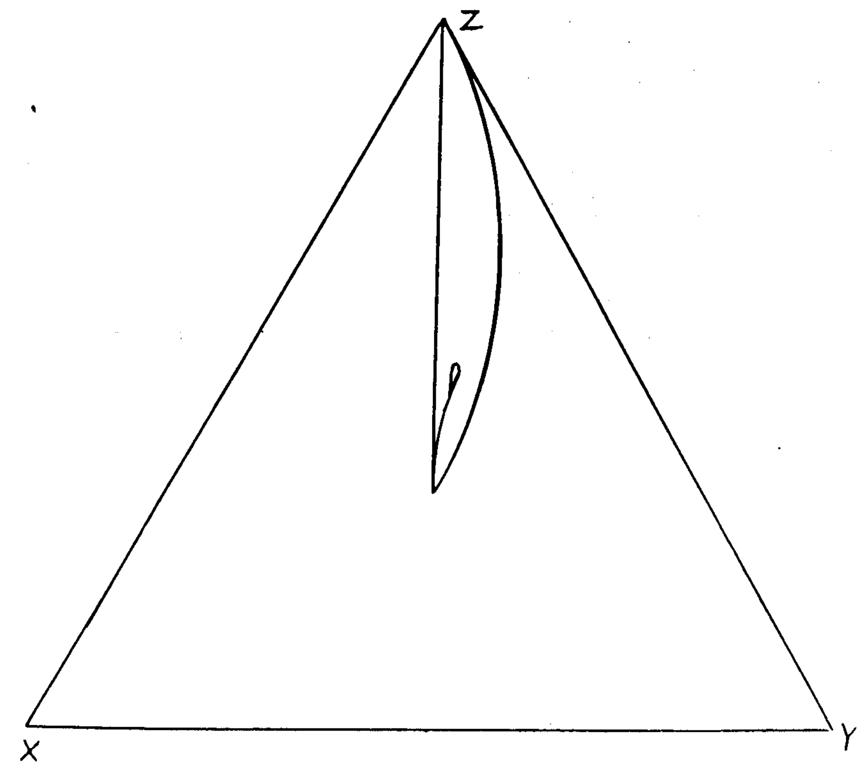


The problem in JCMN 60 asked what could be said when the functions are constrained to be polynomials of degree n; much more difficult! Of course the case n = 0 is trivial, the function is a constant, the three norms are equal, and the only point attained is (1, 1, 1).

The case n = 1 is not difficult, the point must be on a one-dimensional locus, consisting of the arc of the hyperbola

$$4x^2 - 3y^2 + z^2 - 2xz = 0$$

from the point (1, 1, 1) to (1, $2/\sqrt{3}$, 2), and there joined to a little loop with an equation of the fourth degree. The double point corresponds to the functions f(x) = 2x − 1 and f(x) = x, which both have $\|f\|_1 = 1/2$, $\|f\|_2 = 1/\sqrt{3}$ and $\|f\|_\infty = 1$. See the drawing below.



QUANTUM MECHANICS

This little note might (following Coxeter and Greitzer) be entitled "Quantum mechanics revisited", or (following Klein) "Elementary quantum mechanics from an advanced standpoint", but the Editor is a firm believer in short titles — they make the Index simpler.

Once the pure mathematical scene has been set, classical (i.e. non-relativistic) quantum mechanics, as expounded in most text books, is based on three axioms:

Axiom 1 When an observable P is measured the expectation of the result is $\langle\psi|P|\psi\rangle$, where ψ is the "wave function" or "wave vector" or "state vector".

Axiom 2 The equation of motion is $i\hbar d\psi/dt = H\psi$, where H is the Hamiltonian. (This is the Schrödinger picture, the Heisenberg picture, in which ψ is constant but the observables change with time, is essentially equivalent). ($2\pi\hbar$ is Planck's constant)

Axiom 3 The result of measuring any observable P is to project the wave vector orthogonally on to the eigenspace of P that corresponds to the eigenvalue that was the result of the observation.

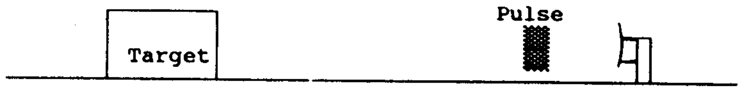
Some physicists would add another, the exclusion principle, but this seems a little less fundamental, perhaps it could be regarded as an observed fact of particle physics rather than as a law of quantum mechanics.

These axioms are all forced on us by experiment, but in somewhat indirect ways. In particular, axiom 3 depends mainly on observations of the spin of a particle or the polarization of a photon, that is on an observable that has only two eigenvalues. But what about observables with continuous spectra?

Consider a system consisting of a particle (or rigid body) constrained to move on a straight line, and suppose that we measure the position. Let Q be this observable, the Cartesian coordinate, it may be regarded as a physical quantity or as a Hermitean linear operator or as a random variable. There is a conjugate momentum P, and the classical Hamiltonian is $H = \frac{1}{2}P^2/m$, where m is the mass. There are theoretical reasons for supposing that the coordinate and the momentum satisfy a commutation relation $QP-PQ = i\hbar$, and that also they satisfy Heisenberg's uncertainty principle, regarding P and Q as random variables, the product of their standard deviations must be $\geq \frac{1}{2}\hbar$ (see JCMN 54, p. 6009).

There are difficulties in applying Axiom 3 (above) to such a system. If we regard the result of the observation as a value for the coordinate, then projecting the state vector on to the eigenspace would mean giving the particle infinite energy, and we know that this does not happen in the ordinary science laboratory. But if, on the other hand, we regard the result as a probability distribution for the coordinate, then what can the axiom mean by "the eigenspace"?

One way in which we might hope to resolve this difficulty is to say that the observation does not determine the coordinate (for of course no experiment is quite accurate), but just finds the coordinate to be in some interval (a, b). Then Axiom 3 asserts that the wave function is projected orthogonally into the corresponding eigenspace, which consists of all functions zero outside (a, b). But this would mean that the wave function after the experiment would (almost always) have simple discontinuities at the points a and b. This would necessitate the Fourier transform being $O(1/x)$ at infinity, and therefore the expectation of the square of the momentum, i.e. the expectation of the kinetic energy, being infinite. In fact we know that if we use radar to find the position of a train on a straight railway line, the train does not at that moment vanish in a red flash and a puff of smoke. So this trick fails us, and it looks as if Axiom 3 needs to be changed.



Consider the system sketched above. The experimenter sends a wave pulse which is reflected from the target. The reflected pulse is detected, and the experimenter can estimate the position of the target from the elapsed time, but can also estimate the velocity of the target by the Doppler effect (comparing the frequencies of the transmitted and received signals). The wave function describing the position of the target will certainly be changed by the experiment, but in what way? Our answer (our replacement for Axiom 3) should conform to the old axiom in applications to polarization of a photon, but should allow for experiments like the one above, where both Q and P are estimated.

One worrying thought about the experiment described above is that the radar pulses will exert a pressure on the target, so that the perturbation of the wave function by the experiment will depend on which side of the target the experimenter is.

It might be objected that in non-relativistic mechanics the speed of light is infinite, so that there is no radar, and the example above has to be modified; the experimenter must have an apparatus that shoots out small perfectly elastic test particles and measures their time of flight and their velocities going and returning (a little unrealistic, but a thought-experiment of which Einstein might have approved). The problem of how the observation changes the wave function of the target remains the same.

FLYWHEELS IN QUANTUM MECHANICS

Books and lecture courses in quantum mechanics mostly assert that to every observable there is a corresponding Hermitean linear operator, operating on the Hilbert space of wave functions. However there is one simple mechanical system for which this idea leads to difficulties (see JCMN 54, p.6011).

Consider the system consisting of a rigid body able to rotate about a fixed axis (for brevity we shall call the system a flywheel, but by adding a force of gravity to the system it may become a pendulum)). The coordinate used by the experimenter to describe the state of the system is the angle θ of rotation from some fixed position, but with 0° and 360° meaning the same thing, or more generally θ meaning the same as $\theta + 360°$ or (if we use radians) $\theta + 2\pi$. There is no Hermitean operator with exactly this set of real numbers as spectrum. Some other approach is needed.

To set up a suitable theory, take as the space of wave functions the complex Hilbert space $L^2(0, 2\pi)$, which is isomorphic to the space ℓ^2 of all complex sequences like $(\dots c_{-1}, c_0, c_1, c_2, \dots)$ with $\sum_{r=-\infty}^{\infty} |c_r|^2$ convergent. The isomorphism is given explicitly, if ψ is a function in the first space, by $\psi(\theta) = \sum_{r=-\infty}^{\infty} c_r e^{ir\theta}$.

The coordinate that we use to describe the position of the flywheel is the linear operator R defined (for any wave function ψ in the L^2 space) by $(R\psi)(\theta) = e^{-i\theta}\psi(\theta)$. In the space ℓ^2, when R operates on (... c_{-1}, c_0, c_1, ...) it gives the sequence with c_{r+1} in position r, i.e. R just moves all the components one place to the left. This operator R is unitary, its Hermitean conjugate R* maps $\psi(\theta)$ to $e^{i\theta}\psi(\theta)$, or in ℓ^2 moves each component one place to the right, so that R* is the inverse of R. R has a continuous spectrum, consisting of all complex numbers of unit modulus; the set of eigenvalues is empty.

Now, we must find a "momentum" conjugate to this coordinate R. Define the operator P in L^2 by $(P\psi)(\theta) = -i\hbar\psi'(\theta)$, or in ℓ^2 by P(... c_r, ...) = (... $\hbar r c_r$, ...). Clearly P is Hermitean and its eigenvalues are the integer multiples of $\hbar$. It seems natural to call it the angular momentum.

It is easy to verify the commutation relation

$$RP - PR = \hbar R.$$

Can we find a form of Heisenberg's uncertainty relation for these two observables R and P?

A little thought shows that any uncertainty principle must be rather different from the familiar

(Standard deviation of coordinate) ×

(standard deviation of momentum) $\geq \frac{1}{2}\hbar$,

because the system can be in (or nearly in) an eigenstate of angular momentum (so that the standard deviation is zero), but

it is hard to see in what sense the standard deviation of the coordinate could be infinite, or even very large, or even moderately large.

We saw in JCMN 54, pp. 6007-6011, that the usual form of Heisenberg's uncertainty principle is a physical illustration of a certain analytic inequality for complex functions. So here, if we cannot find the uncertainty principle, it would be interesting to look for the analytic inequality that is surely there, waiting to be found.

Consider elements of ℓ^2, i.e. complex sequences such as $c = (\ldots c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots)$. To each such element there are three real positive parameters:

$$x = \Sigma\,|c_r|^2, \qquad y = \Sigma\, r^2|c_r|^2 \qquad \text{and} \qquad z = |\Sigma\, \bar{c}_r c_{r-1}|.$$

What are the inequalities that hold between them?

Trivially $0 \leq z < x$, and all the inequalities must be homogeneous. The physics suggests that z can be nearly equal to x, but only when y is large. The ratio y/x can be regarded as the variance (the square of the standard deviation) of the angular momentum P. The ratio z/x is probably related to the accuracy of measurement of the coordinate R, being nearly equal to 1 if the error is small, that is if the wave function ψ is large in a small arc of the unit circle and small elsewhere.

QUOTATION CORNER 47

Dr. Ian Bennett, chief executive of *Life Be in it*, says: "More than 150,000 Australians will need treatment for skin cancer this year and skin cancer rates are still rising round the world. What we are launching is a special vending machine which offers individuals resealable bottles of the *Life Be in it 15 plus* sunscreen."

—— *Eastern Courier Messenger* (January 12, 1994) (An advertising paper distributed in Adelaide suburbs)

Readers with long memories will recall how some 30 years ago there came on the market, heavily advertised, the skin creams that block ultra-violet light, a triumph of technology. The chemical industry has done well out of that project, not only has there been the predictable increase in the diseases (such as rickets and osteoporosis) of ultra-violet deficiency, but also now according to this claim there is an increase in skin cancer.

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