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The James Cook Mathematical Notes came out in 3 issues per year from 1979 to 1994, but from Issue 66 (April 1995) at the start of Volume 7, it has been irregular, appearing when enough contributions are available. The history of JCMN is that the first issue (a single foolscap sheet) appeared in September 1975, then others at irregular intervals, to number 17 in November 1978, then JCMN settled into the routine of three issues per year. The issues up to number 31 (May, 1983) were produced and sent out free by the Mathematics Department of the James Cook University of North Queensland, of which I was then the Professor. In October 1983 this arrangement was beginning to be unsatisfactory, and I started producing the JCMN myself and asking readers to pay subscriptions.

In October 1992 it had become clear that the paying of subscriptions by readers is an inefficient operation. Bank charges for changing currency and for international transfers, with postage, together absorb most of the initial input of money. Therefore we abandoned subscriptions as from issue number 60 (January, 1993). I now ask readers only to tell me every two years if they still want to have JCMN. To those who want to give something in return for the JCMN, I ask them to make a gift to an animal welfare society in their own country. The animals of the world will be grateful and so will I.

Contributors, please tell me if and how you would like your address printed.

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POWER MEAN INEQUALITY

(JCMN 42, p.5020, 65, p.6370 & 66, p.7004)

P. H. Diananda

(Singapore)

The original question was about the inequality

$$(x_1 + \dots + x_n)^k - (x_1^k + \dots + x_n^k) \geq (n^k - n)(x_1 x_2 \dots x_n)^{k/n}$$

with n and k positive integers and the x_i all positive.

This was proved in the previous issue, and it was established that for negative k the inequality is reversed. For other values of k and n, the problem remains open, but a little more can be established, as follows.

Denote $(x_1 + \dots x_n)^k - (x_1^k + \dots + x_n^k)$ by LHS, and denote $(n^k - n)(x_1 x_2 \dots x_n)^{k/n}$ by RHS.

<u>Theorem 1</u> If $0 < k < 1$ and $n > 2$, then the propositions LHS $\geq$ RHS and LHS $\leq$ RHS are both untrue.

<u>Proof</u> Let $x_i = 1$ for all $i < n$, and let $x_n = x$.

LHS $= (n - 1 + x)^k - n + 1 - x^k$,

RHS $= (n^k - n)x^{k/n}$.

LHS - RHS is strictly negative when $x = 0$, and is continuous on the right, so that it is < 0 for all sufficiently small positive x.

Therefore the proposition "LHS $\geq$ RHS" is untrue.

Now consider the case of $x_1 = 1$ and $x_i = x$ for all $i > 1$.

LHS $= (1 + (n-1)x)^k - 1 - (n-1)x^k$,

RHS $= (n^k - n)x^{k-k/n}$,

LHS - RHS $= (1+nx-x)^k - 1 + x^k((n-n^k)x^{-k/n} - n + 1)$.

For all positive x, $(1+nx-x)^k > 1$, and for all sufficiently small positive x, $(n-n^k)x^{-k/n} - n + 1 > 0$,

so that LHS > RHS.

Therefore the proposition "LHS $\leq$ RHS" is also untrue.

<u>Theorem 2</u> If $k = 1/2$ and $n = 2$ then LHS $\geq$ RHS.

<u>Proof</u> Since the formula is homogeneous in the variables x_1 and x_2, it will be sufficient to consider the case where $x_1 = 1$ and $x_2 = x^4$, where $x \geq 0$.

LHS $= \sqrt{(1 + x^4)} - 1 - x^2$, and RHS $= (\sqrt{2} - 2)x$.

But $(1 + (\sqrt{2}-2)x + x^2)^2 = 1 + x^4 - (4 - 2\sqrt{2})x(1 - x)^2$

$\leq 1 + x^4$, so that

$\sqrt{(1 + x^4)} \geq 1 + x^2 + (\sqrt{2}-2)x$, i.e. LHS $\geq$ RHS.

R. A. LYTTLETON

With regret we note the death of Raymond Lyttleton at the age of 84 in May 1995. He was Professor of Theoretical Astronomy at Cambridge University.

CONGRATULATIONS

Trevor Tao and Nigel Tao both won bronze medals at the 1995 International Mathematical Olympiad competition held in July at York University, Toronto, Canada.

SUMS GIVEN BY ZETA FUNCTIONS

(JCMN 65, p.6360, JCMN 66, pp.7010-7014)

Chris Smyth
(University of Edinburgh)

The problem given in JCMN 65 and its solution in JCMN 66 were not new. Bruce C. Berndt in his book *Ramanujan's Notebooks* (published by Springer Verlag) traces the history of this equation and others to a series of letters between C. Goldbach and L. Euler in 1742 and 1743. He recounts (Part 1, page 252) how the first identity, the equation $H(2) = 2\zeta(3)$ in our notation below, has been rediscovered and published by various mathematicians in 1952, 1955 and 1982.

Euler's results may be found in his collected works, published by B. G. Teubner about 1927. In this massive work, in Vol 15 of *Opera Mathematica,* in the second volume of *Commentationes Analyticae ad theoriam serierum infinitarum pertinentes*, edited by Georg Faber, the section headed *Meditationes circa singulare serierum genus* (taken from *Novi. Comm. Acad. Sci. Petropolitanae, 20, 1775,* pp. 140-186) gives on pages 218-267 Euler's investigation into what in modern notation we would denote by $\sum_{r=1}^{\infty} r^{-m} \sum_{s=1}^{r} s^{-n}$, but which Euler denoted by $\int \frac{1}{z^m}\left(\frac{1}{y^n}\right)$, where m and n are positive integers.
We shall call it C(m, n) in the calculations below.

In particular, on page 243, we find:-

22. Interpolatione autem rite instituta hae summationes pro omnibus ordinibus ita se habebunt:

$2H(2) = 4\zeta(3)$,
$2H(3) = 5\zeta(4) - \zeta(2)\zeta(2)$,
$2H(4) = 6\zeta(5) - 2\zeta(2)\zeta(3)$,
$2H(5) = 7\zeta(6) - 2\zeta(2)\zeta(4) - \zeta(3)\zeta(3)$,
$2H(6) = 8\zeta(7) - 2\zeta(2)\zeta(5) - 2\zeta(3)\zeta(4)$,
$2H(7) = 9\zeta(8) - 2\zeta(2)\zeta(6) - 2\zeta(3)\zeta(5) - \zeta(4)\zeta(4)$,
$2H(8) = 10\zeta(9) - 2\zeta(2)\zeta(7) - 2\zeta(3)\zeta(6) - 2\zeta(4)\zeta(5)$,
$2H(9) = 11\zeta(10) - 2\zeta(2)\zeta(8) - 2\zeta(3)\zeta(7) - 2\zeta(4)\zeta(6) - \zeta(5)\zeta(5)$
etc.

unde in genera, si ponantur $m+n = \lambda$, erit

$$2H(\lambda-1) = (\lambda+1)\zeta(\lambda) - \zeta(2)\zeta(\lambda-2) - \zeta(3)\zeta(\lambda-3) - \zeta(4)\zeta(\lambda-4) - \ldots\ldots - \zeta(\lambda-2)\zeta(2).$$

(Your Editor has modernised the mathematical notation, writing H(k) for Euler's $\int \frac{1}{z^k}\left(\frac{1}{y}\right)$ or our $\sum_{m=1}^{\infty} m^{-k} \sum_{n=1}^{m} 1/n$, and $\zeta(k)$ for his $\int \frac{1}{z^k}$, now known as Riemann's zeta function. This makes the typing easier.)

The note in JCMN 66 obtained the results above, and then gave other formulae on page 7014. For some of these, proofs can be given as follows.

Consider the rectangular array of terms $r^{-m}s^{-n}$. The sum of all of them is $\zeta(m)\zeta(n)$, and the sum of the terms on the diagonal is $\zeta(m+n)$. Then, since $C(m, n) = \sum_{r=1}^{\infty} r^{-m} \sum_{s=1}^{r} s^{-n}$ is the sum of the terms in the lower triangle, and $C(n, m)$ is the sum in the upper triangle,

$$C(m, n) + C(n, m) = \zeta(m)\zeta(n) + \zeta(m+n).$$

Lemma 1 $\quad C(2, 3) = (11/2)\zeta(5) - 2\zeta(2)\zeta(3)$ and

$$C(3, 2) = 3\zeta(2)\zeta(3) - (9/2)\zeta(5).$$

Proof Consider the double sum of terms $r^{-3}s^{-2}$ (r and s from 1 to ∞), and put $t = r+s$. $\quad \zeta(2)\zeta(3) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} r^{-3}s^{-2}$

$$= \sum_{t=2}^{\infty}\sum_{r=1}^{t-1} r^{-3}s^{-2} = \sum_{t=2}^{\infty} t^{-2}\sum_{r=1}^{t-1} \frac{r^2+2rs+s^2}{r^3s^2} \quad \text{(where } s=t-r\text{)}$$

$$= \sum_{t=2}^{\infty} t^{-2}\sum_{r=1}^{t-1}\left(\frac{1}{rs^2} + \frac{2}{r^2s} + r^{-3}\right)$$

$$= 3\sum_{t=2}^{\infty} t^{-2}\sum_{r=1}^{t-1} r^{-1}s^{-2} + \sum_{t=2}^{\infty} t^{-2}\sum_{r=1}^{t-1} r^{-3}$$

$$= 3\sum_{t=2}^{\infty} t^{-3}\sum_{r=1}^{t-1}\left(s^{-2} + \frac{1}{rs}\right) + (C(2, 3)-\zeta(5))$$

$$= 3(C(3, 2)-\zeta(5)) + 3\sum_{t=2}^{\infty} t^{-4}\sum_{r=1}^{t-1}\left(\frac{1}{r} + \frac{1}{s}\right) + C(2, 3) - \zeta(5)$$

$$= 3C(3, 2) + C(2, 3) - 4\zeta(5) + 6(H(4) - \zeta(5))$$

$$= 3C(3, 2) + C(2, 3) - 10\zeta(5) + 6(3\zeta(5) - \zeta(2)\zeta(3))$$

Thus we have two linear equations for $C(2, 3)$ and $C(3, 2)$:

$$3C(3, 2) + C(2, 3) = 7\zeta(2)\zeta(3) - 8\zeta(5) \quad \text{and}$$

$$C(3, 2) + C(2, 3) = \zeta(2)\zeta(3) + \zeta(5),$$

and so
$$C(2, 3) = (11/2)\zeta(5) - 2\zeta(2)\zeta(3),$$
$$C(3, 2) = 3\zeta(2)\zeta(3) - (9/2)\zeta(5).$$

QED

These two results are in Euler's 1775 paper (page 260 in the edition mentioned above of the collected works), in fact he describes how $C(m, n)$ can be calculated in terms of zeta functions for any positive integer m and n. He shows that the value will always be a linear combination, with rational coefficients, of $\zeta(m+n)$ and products $\zeta(r)\zeta(s)$ with r and s positive integers and with $r + s = m + n$.

It would be rash to claim originality for the equations given on page 7014 of JCMN 66 involving

$$HD(k) = \sum_{n=1}^{\infty} n^{-k}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2n}\right),$$

but these results are not to be found in Euler's 1775 paper.

A little can now be added to the information on $HD(k)$ (to 12 decimal places) given in the previous note.

$$HD(2) = 3\cdot305656483692 = \frac{11\,\zeta(3)}{4},$$

$$HD(4) = 1\cdot682364333887 = \frac{37\,\zeta(5)}{4} - 4\zeta(2)\zeta(3),$$

$$HD(6) = 1\cdot536887024777 = \frac{135\,\zeta(7)}{4} - 16\zeta(2)\zeta(5) - 4\zeta(3)\zeta(4),$$

$$HD(8) = 1\cdot508563176864 = \frac{521\,\zeta(9)}{4} - 64\zeta(2)\zeta(7) - 4\zeta(3)\zeta(6) - 16\zeta(4)\zeta(5),$$

$$HD(10) = 1\cdot502078955654 = \frac{2059\,\zeta(11)}{4} - 256\zeta(2)\zeta(9) - 4\zeta(3)\zeta(8) - 64\zeta(4)\zeta(7) - 16\zeta(5)\zeta(6),$$

so now will someone give us the theory behind these equations?

We can offer proofs for the first two of the equations.

<u>Lemma 2</u> $HD(2) = (11/4)\zeta(3)$.

<u>Proof</u> Using the notation of our previous note, consider

$$S(1,\ 1) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{1}{rs(r+s)} = H(1) = 2\zeta(3).$$

The rectangular array of terms is symmetrical, the diagonal terms add to $\zeta(3)/2$, and so the sum over the lower triangular region, where $1 \le s \le r$, is $(5/4)\zeta(3)$.

$$\text{Therefore} \quad \frac{5}{4}\zeta(3) = \sum_{r=1}^{\infty} r^{-2} \sum_{s=1}^{r} \frac{1}{s} - \frac{1}{r+s}$$

$$= 2\sum_{r=1}^{\infty} r^{-2} \sum_{s=1}^{r} \frac{1}{s} - \sum_{r=1}^{\infty} r^{-2} \sum_{s=1}^{r} \frac{1}{s} + \frac{1}{r+s}$$

$$= 2H(2) - \sum_{r=1}^{\infty} r^{-2}\left(1 + \frac{1}{2} + \ldots + \frac{1}{r} + \frac{1}{r+1} + \ldots + \frac{1}{r+r}\right)$$

$$= 4\zeta(3) - HD(2).$$

QED

<u>Lemma 3</u> $HD(4) = (37/4)\zeta(5) - 4\zeta(2)\zeta(3)$.

<u>Proof</u> $S(2,\ 2) = 2\zeta(2)\zeta(3) - 3\zeta(5)$ is the sum of the rectangular array of terms $\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} r^{-2}s^{-2}/(r+s)$. The terms on the diagonal add to $\zeta(5)/2$, and the sum of the terms in the lower triangle is therefore $\zeta(2)\zeta(3) - (5/4)\zeta(5)$.

$$\zeta(2)\zeta(3) - \frac{5}{4}\zeta(5) = \sum_{r=1}^{\infty} r^{-2} \sum_{s=1}^{r} s^{-2}/(r+s)$$

$$= \sum_{r=1}^{\infty} r^{-3} \sum_{s=1}^{r} s^{-2}\,\frac{r+s-s}{r+s} = C(3,\ 2) - \sum_{r=1}^{\infty} r^{-3} \sum_{s=1}^{r} s^{-1}/(r+s)$$

$$= 3\zeta(2)\zeta(3) - \frac{9}{2}\zeta(5) - \sum_{r=1}^{\infty} r^{-4} \sum_{s=1}^{r} \frac{r+s-s}{s(r+s)}.$$

Therefore $\frac{13}{4}\zeta(5) - 2\zeta(2)\zeta(3) =$

$$= -\sum_{r=1}^{\infty} r^{-4}\left(1 + \frac{1}{2} + \ldots + \frac{1}{r} - \frac{1}{r+1} - \ldots - \frac{1}{2r}\right)$$

$= -2H(4) + HD(4) = -6\zeta(5) + 2\zeta(2)\zeta(3) + HD(4)$, giving

$HD(4) = (37/4)\zeta(5) - 4\zeta(2)\zeta(3)$.

QED

A by-product of the calculations above for $HD(k)$ is the following set of little puzzles:-

Given the first 3 of the equations above, guess the next.

Given the first 4 of the equations above, guess the next.

Given all the 5 equations above, guess the next, and verify your guess numerically. For your convenience, here are a few figures.

$HD(12)$	=	1·500513412131
$\zeta(13)$	=	1·000122713348
$\zeta(2)\zeta(11)$	=	1·645746974518
$\zeta(3)\zeta(10)$	=	1·203252439058
$\zeta(4)\zeta(9)$	=	1·084496963929
$\zeta(5)\zeta(8)$	=	1·041155678952
$\zeta(6)\zeta(7)$	=	1·025837141401

OTHER FORMULAE

$$\sum_{n=1}^{\infty} n^{-2}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)^2 = 4{\cdot}599873743272 = \frac{17\,\pi^4}{360}.$$

$$\sum_{n=1}^{\infty} n^{-2}\left(1 + 2^{-2} + 3^{-2} + \ldots\ldots + n^{-2}\right)^2 = 2{\cdot}250337389172$$

$$= 19\,\pi^6/22680 - \zeta(3)^2.$$

$$\sum_{n=1}^{\infty} (-1)^{n+1}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2n}\right)/n = 0{\cdot}907970538300.$$

SYMMETRIC SIMULTANEOUS EQUATIONS 2 (JCMN 66, p.7006)

A. Brown

Earlier contributions considered the simultaneous equations

$x^2 - yz = a, \quad y^2 - zx = b, \quad z^2 - xy = c \quad \ldots\ldots (1)$

and as a new problem Harry Alexiev asked for a solution of the equations

$x^2 + yz = a, \quad y^2 + zx = b, \quad z^2 + xy = c. \quad \ldots\ldots\ldots (2)$

We note that if (x, y, z) is a solution of (2), then (-x, -y, -z) is also a solution, and there would be some advantage in working with the symmetric functions

$A = x + y + z, \quad B = xy + yz + zx, \quad C = xyz,$

$\alpha = a + b + c, \quad \beta = ab + bc + ca, \quad \gamma = abc,$

to make use of the symmetry of the equations.

It is easy to check that

$2a + b + c - 2x^2 = A(y + z),$

$a + 2b + c - 2y^2 = A(z + x),$

$a + b + 2c - 2z^2 = A(x + y),$

so in the special case where A = 0 we have

$2x^2 = 2a + b + c, \quad 2y^2 = a + 2b + c, \quad 2z^2 = a + b + 2c, \ldots (3)$

and this provides a solution for (x, y, z).

In general it can be shown that

$\alpha = A^2-B, \quad \beta = A^2B-2AC-B^2, \quad \gamma = 8C^2+A^3C-6ABC+B^3 \quad \ldots\ldots (4)$

For $A \neq 0$, these two equations give A and C once B has been determined. The equation for γ becomes

$$\gamma = \frac{2(B\alpha-\beta)^2}{(B+\alpha)} + \frac{1}{2}(B+\alpha)(B\alpha-\beta) - 3B(B\alpha-\beta) + B^3$$

and hence

$0 = 2B^4 - 3\alpha B^3 + 5\beta B^2 + (\alpha^3-4\alpha\beta-2\gamma)B + (4\beta^2-\alpha^2\beta-2\alpha\gamma). \ldots. (5)$

For any B which satisfies equation (5) we can obtain corresponding values of A and C, and suitable values for x, y and z are given by the roots of the cubic

$$Z^3 - AZ^2 + BZ - C = 0.$$

(If the roots are Z_1, Z_2, Z_3, form $Z_1^2+Z_2Z_3$, $Z_2^2+Z_1Z_3$, $Z_3^2+Z_1Z_2$, and see which of these corresponds to a, which to b and which to c. This allows x, y, z to be identified.)

The equation $A^2 = B + \alpha$ gives two values for A, and gives a value of C corresponding to each of these values of A. This agrees with the idea that (-x, -y, -z) is a solution if (x, y, z) is a solution.

QUOTATION CORNER 51

Don't listen to what I say; listen to what I *mean*.

—— Richard Feynman. Quoted by Roger Penrose on page 105 of his book *Shadows of the Mind* (Oxford U.P. 1994).

QUOTATION CORNER 52

The border between good and bad passes through the heart of every one of us.

—— A. Solzhenitsyn. (Contributed by Jordan Tabov)

SYMMETRIC SIMULTANEOUS EQUATIONS

(JCMN 59, p.6173, 60, p.6192, 62, p.6276 & 66, p.7006)

These contributions are about the equations

$$x^2 - yz = a, \quad y^2 - zx = b, \quad z^2 - xy = c \quad \ldots\ldots\ (1)$$

(to be solved for x, y and z in terms of a, b and c).

Yet another possible approach is to consider the matrices:

$$X = \begin{bmatrix} x & z & y \\ y & x & z \\ z & y & x \end{bmatrix} \qquad A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$$

Denote by adj M the adjugate of any square matrix M, obtained by taking the matrix of minors, transposing it, and changing alternate signs. Recall that

$$\text{adj } M = (\det M)\, M^{-1} \quad \text{for a non-singular matrix } M.$$

$$\det \text{adj } M = (\det M)^2 \text{ if } M \text{ is } 3 \times 3, \text{ and}$$

$$\text{adj adj } M = (\det \text{adj } M)\, M\, (\det M)^{-1} = M \det M$$

$$= \pm M \sqrt{(\det \text{adj } M)}.$$

Equations (1) may be written

$$\text{adj } X = A \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (2)$$

Note that $\det A = a^3 + b^3 + c^3 - 3abc$, which may be factorised as $\det A = (a + b + c)(a + \omega b + c/\omega)(a + b/\omega + \omega c)$ (where ω denotes a complex cube root of unity). Therefore equations (1) are solved by the first column of the matrix equation

$$X = \pm (\det A)^{-1/2} \text{ adj } A, \quad \text{i.e.}$$

$$(x, y, z) = \pm(a^3 + b^3 + c^3 - 3abc)^{-1/2}(a^2-bc,\ b^2-ca,\ c^2-ab).$$

If $\det A \neq 0$ this is a solution and there is no other.

Thus there are in general two solutions (x, y, z) for any (a, b, c). Now consider the exceptional cases, those with $\det A = 0$, i.e. $(a + b + c)(a + \omega b + c/\omega)(a + b/\omega + \omega c) = 0$.

or $$a^3 + b^3 + c^3 = 3abc \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (3)$$

We shall see that in these cases there is in general no solution (x, y, z), but among these exceptional cases there are exceptions with infinitely many solutions.

Suppose that (3) holds and there is a solution (x, y, z). We shall see that this imposes further conditions on the parameters (a, b, c). There is X, as defined above.

Because of the algebraic identities

$$(\det X)^2 = \det \text{adj } X \quad \text{and} \quad \text{adj adj } X = X \det X,$$

we have $\det X = 0$ and $\text{adj } A = \text{adj adj } X = X \det X = 0$.

$$a^2 - bc = 0, \quad b^2 - ca = 0, \quad c^2 - ab = 0.$$

Therefore $a^3 = b^3 = c^3 = abc$. These are the extra condition imposed on (a, b, c) by the existence of a solution (x, y, z) when $\det A = 0$. The values of (a, b, c) satisfying these equations are in the four sets listed below. For each such (a, b, c) there are infinitely many solutions (x, y, z), as explained in JCMN 60, pp.6192-6194 and JCMN 62, p.6276. These solutions may be summarised as follows:-

(a, b, c) = (0, 0, 0). For this value there are infinitely many solutions of (1):

$(x, y, z) = (t, t, t)$ or $(t, \omega t, t/\omega)$ or $(t, t/\omega, \omega t)$ for any t.

$(a, b, c) = (k, k, k)$ for any complex $k \neq 0$. For each of these values there are infinitely many solutions of (1). They may be written either in the form

$$(x, y, z) = 2\sqrt{(k/3)}(\cos\varphi, \cos(\varphi-2\pi/3), \cos(\varphi+2\pi/3))$$

for any φ,

or as $(x, y, z) = \sqrt{k/3}\,(p + \frac{1}{p}, \frac{p}{\omega} + \frac{\omega}{p}, \omega p + \frac{1}{\omega p})$

for any $p \neq 0$.

$(a, b, c) = (k, \omega k, k/\omega)$ for any complex $k \neq 0$. For each of these values there are infinitely many solutions of (1):

$(x, y, z) = \sqrt{k/3}\,(p + \frac{1}{p}, p + \frac{\omega}{p}, p + \frac{1}{\omega p})$ for any $p \neq 0$.

$(a, b, c) = (k, k/\omega, \omega k)$ for any complex $k \neq 0$. For each of these values there are infinitely many solutions of (1):

$(x, y, z) = \sqrt{k/3}\,(p + \frac{1}{p}, p + \frac{1}{\omega p}, p + \frac{\omega}{p})$ for any $p \neq 0$.

SYMMETRIC SIMULTANEOUS EQUATIONS 3

Solve the equations

$$x(y + z) - y^2 - z^2 = a$$
$$y(z + x) - z^2 - x^2 = b$$
$$z(x + y) - x^2 - y^2 = c$$

for (x, y, z) in terms of (a, b, c).

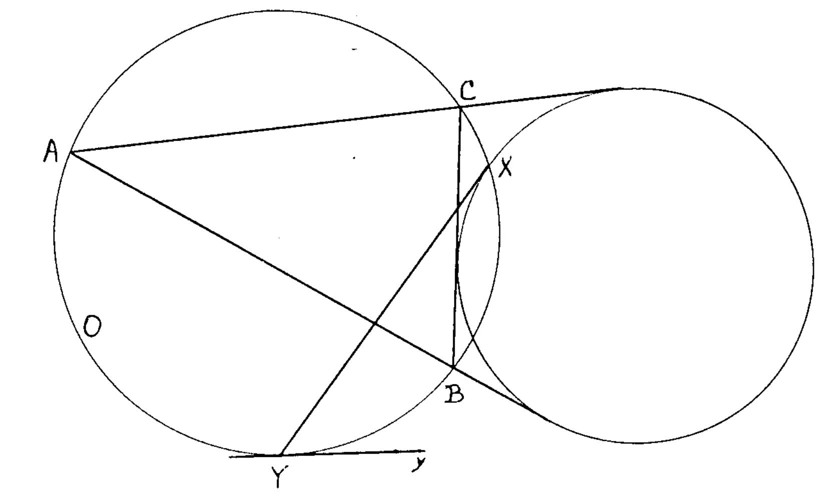
PROBLEM ON CIRCLES

(JCMN 65, p.6366, JCMN 66, p.7009)

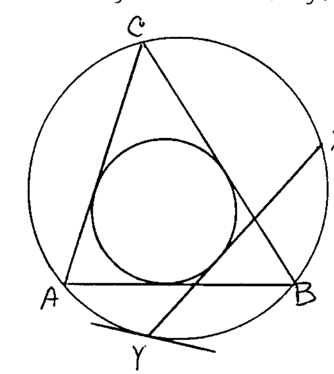
Jordan Tabov

(Bulgarian Academy of Sciences, 1113, Sofia, Bulgaria)

The note by Sahib Ram Mandan in the previous issue might need a little amplification.

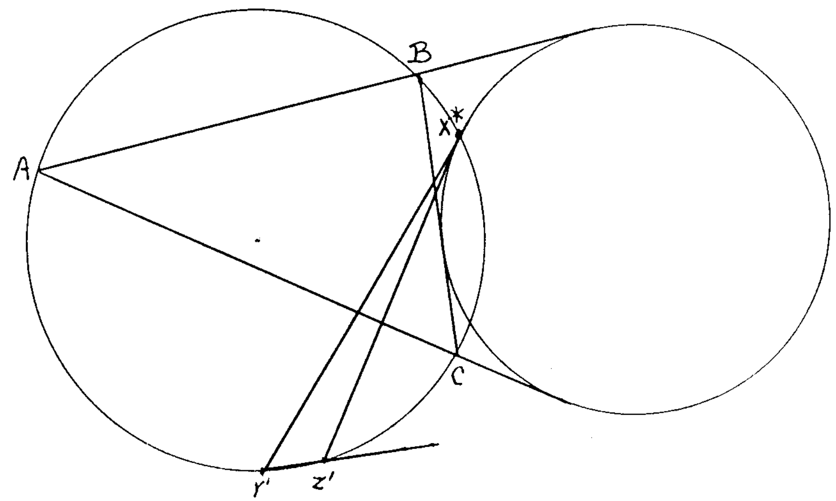


Recall the reasoning: "The tangent y to (O) at Y may be regarded as the line YZ joining two coincident points Y and Z on the circle (O). We may regard XYZ as a degenerate triangle inscribed in the circle (O)."



If we apply a similar reasoning to a different figure (see the sketch) it seems to lead to the obviously wrong conclusion that the tangent at Y should touch the inscribed circle.

The reason why the "degenerate triangle" idea leads to the right answer in the first case becomes clear when we analyse the limiting process.



Take X* on the circumcircle, close to X and outside the escribed circle. From X* there are two tangents X*Y′ and X*Z′ to the escribed circle; they give a triangle inscribed in the circumcircle, so that Y′Z′ must touch the escribed circle. In the limit X* becomes X and Y′Z′ becomes the tangent at Y.

In the second case, with the inscribed circle, the method fails because we cannot get two neighbouring tangents meeting on (or anywhere near) the circumcircle, they meet near their points of tangency.

BOOK REVIEW

Shadows of the mind, sub-titled *A search for the missing science of consciousness* by Roger Penrose, Oxford University Press, 1994, xvi + 457 pages. Australian price $49·95.

We are reminded of how Lord Kelvin at the Royal Institution on Friday evening, April 27, 1900, gave a lecture entitled *Nineteenth century clouds over the dynamical theory of heat and light*.

His "Cloud I" was what he called the "relative motion of ether and ponderable bodies" and it was the problem of explaining the results of the Michaelson-Morley experiment of 1887, the only suggestion available at the time was that rigid bodies moving through the ether suffered a "Fitzgerald contraction". His "Cloud II" was the Maxwell-Boltzmann law of equi-partition of energy in statistical mechanics. A form of this theorem had been stated, but without satisfactory proof, by Waterston in 1846, in a paper rescued from oblivion by Lord Rayleigh in 1892; and different versions of the law were given by Maxwell and Boltzmann from 1859 onwards. The doctrine seemed to contradict the kinetic theory of gases by leading to wrong values (wildly wrong) for specific heats of gases. Should physics condemn the experimental evidence of thermodynamics as wrong? or discard the beautifully simple and highly successful kinetic theory of gases? or suspect some subtle error in the available proofs of the Maxwell-Boltzmann law? Of course Kelvin could not guess the answers to either of these two problems, but they were to emerge very soon afterwards.

Cloud I was explained away by Einstein's special relativity in 1905, and Cloud II was explained away by Max Planck's announcement of his quantum theory on December 14th, 1900. Nineteenth century physics had been built on the two foundations of Euclidean space and Newtonian mechanics, it was when Planck questioned the second and Einstein questioned the first that twentieth century physics began.

Sir Roger Penrose in this book does for the theoretical physics of the twentieth century what Lord Kelvin did for that of the nineteenth. He points out how the current theories, although they have behind them a century of great achievements, can be seen to have cracks in their foundations.

The book is in two parts, the first is called *Why we need new physics to understand the mind,* with the sub-title *The non-computability of conscious thought.* It is about mathematical logic, algorithms, Turing machines, Gödel, robots, artificial intelligence, computability, and consciousness. Don't expect this reviewer to give a neat summary of all that in one paragraph.

The second part of the book is *What new physics we need to understand the mind,* with the sub-title *The quest for a non-computational physics of mind.*

The adjective "non-computational" to describe physics may be puzzling, it does not mean scientists being able to throw away their slide-rules and computers, it means the behaviour of a system not being "computable" in the sense of mathematical logic.

The author gives a careful description of the usual basis for quantum theory, with wave functions satisfying Schrödinger's equation, and with physical observables corresponding to operators on the Hilbert space of wave functions, and with the mysterious (but firmly established) doctrine that the act of observing a physical quantity changes the wave function discontinuously into an eigenfunction (or, more precisely, projects the wave vector into the eigenspace); this phenomenon

is described as "the R operation".

This second part of the book goes on to consider more subtle things like quantum entanglement and quantum coherence, with mention of many recent proposals for modifying the theory of the basic R operation.

There is a photograph and a description of a tiny creature called a paramecium. The typical paramecium lives in a pond and swims using hundreds of tiny legs, she can find food, can retreat from danger, and can avoid obstacles in the water. There is disagreement among the experts as to whether she has the ability to learn. Probably she is weak on long division, and on the whole the average paramecium seems to have a lot in common with the average undergraduate.

Many people put forward the view that intelligence is related to the number of neurons in the brain, the average undergraduate is said to have about a hundred thousand million, but the paramecium is what biologists call a single-celled organism, with no brain and no neurons at all. How can she think? In fact there is emerging a new subject called "nanobiology" dealing with the very small things in biology, and one of the questions being discussed is how the microtubules in the cytoskeleton may be the parts that do the thinking for a paramecium. If you want clarification of this suggestion you will have to read the book yourself. But allow yourself plenty of time, I found it hard going.

The book has a good index and has a frighteningly large bibliography covering 26 pages.

B.C.R.

WANTED INEQUALITY

(JCMN 64, p.6351)

The following question was asked in the article FLYWHEELS IN QUANTUM MECHANICS in JCMN 64.

If (... c_{-1}, c_0, c_1, c_2, ...) is a complex sequence and

$$x = \sum_{r=-\infty}^{\infty} |c_r|^2, \qquad y = \sum_{r=-\infty}^{\infty} r^2|c_r|^2, \qquad z = \left|\sum_{r=-\infty}^{\infty} \bar{c}_r c_{r-1}\right|$$

then what inequalities are there between the positive real numbers x, y and z?

An equivalent question is obtained if instead of starting with a complex sequence we start with a complex function $f(\theta)$ of period 2π, and put:

$$x = \int_0^{2\pi} |f|^2 d\theta, \quad y = \int_0^{2\pi} |f'|^2 d\theta, \quad z = \left|\int_0^{2\pi} e^{i\theta}|f|^2 d\theta\right|.$$

Numerical evidence suggests an inequality

$$4y(x - z) \geq z^2, \qquad \text{............. (1)}$$

in which there is equality in the cases of $f(\theta) = 1$ and of $f(\theta) = 1 + \cos\theta$.

Another suggestion is

$$x + y \geq 2z \qquad \text{.............. (2)}$$

which is a tangent plane (or you might call it a supporting half-space) for the quadric cone represented by the inequality (1).

QUOTATION CORNER 53

A zoo-keeper thinks he is a naturalist.
A naturalist thinks he is a biologist.
A biologist thinks he is a biochemist.
A biochemist thinks he is a physicist.
A physicist thinks he is a mathematiciam.
A mathematician thinks he is God.
God thinks he is a zoo-keeper.

—— Anonymous.

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