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The history of the James Cook Mathematical Notes (JCMN) is that the first issue (a single foolscap sheet) appeared in September 1975, then others at irregular intervals, to number 17 in November 1978. JCMN settled into the routine of three issues per year from 1979 to 1994; but from Issue 66 (April 1995) at the start of Volume 7, it has been irregular, appearing when enough contributions are available.

The issues up to number 31 (May, 1983) were produced and sent out free by the Mathematics Department of the James Cook University of North Queensland, of which I was then the Professor. In October 1983 this arrangement was beginning to be unsatisfactory, and I started producing the JCMN myself and asking readers to pay subscriptions. In October 1992 it had become clear that the paying of subscriptions by readers is an inefficient operation. Bank charges for changing currency and for international transfers, with postage, together absorb most of the initial input of money. Therefore we abandoned subscriptions as from issue number 60 (January, 1993). I now ask readers only to tell me every two years if they still want to have JCMN. To those who want to give something in return for the JCMN, I ask them to make a gift to an animal welfare society in their own country. The animals of the world will be grateful and so will I.

Contributors, please tell me if and how you would like your address printed.

JCMN 68, January 1996

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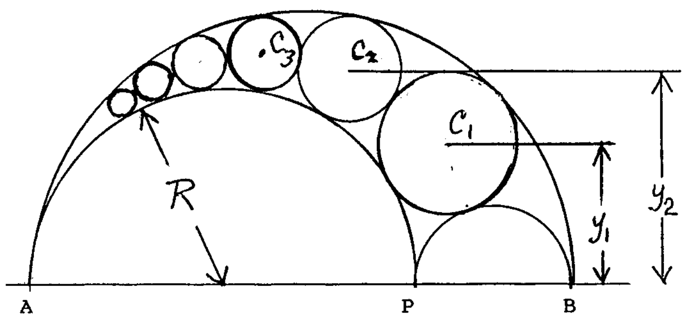
IN THE FOOTSTEPS OF PAPPUS

A. Brown

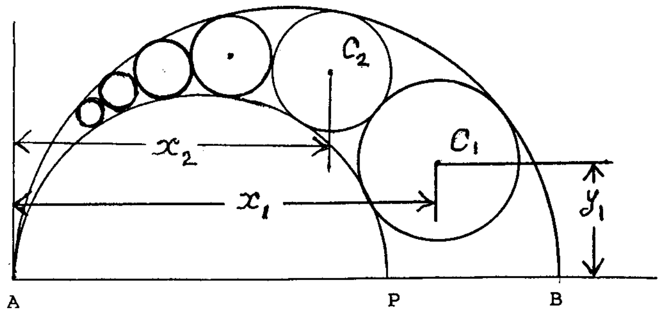


In the diagram above the semicircles on AB, AP and PB have radii 1, R and 1-R respectively. Of the sequence of smaller circles, let number n have diameter d_n, and have its centre C_n at a height of y_n above the line AB. Pappus of Alexandria proved that

$$y_n = nd_n.$$

Prove that if R is rational then so are all y_n and d_n.

As a variation on the theme, remove the constraint that the first of the small circles should touch the semicircle on PB. We then have the figure below.



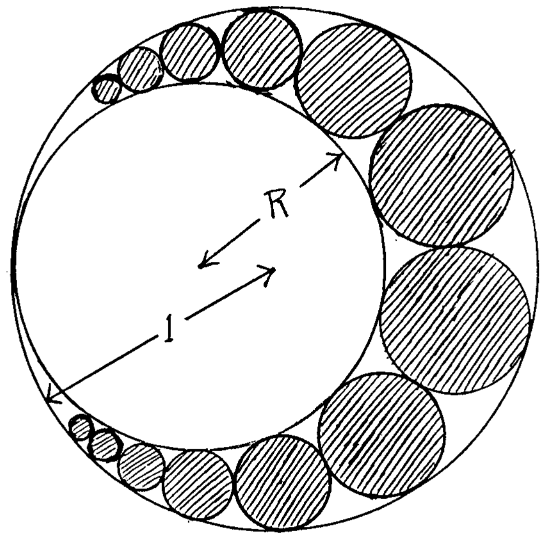
With notation as before, now use also the Cartesian coordinate x, measured from the origin A, as shown above. Prove that the coordinates (x_n, y_n) of the centre C_n of circle number n are related to the diameter d_n by:

$$x_n = k\,d_n, \qquad y_n/d_n = n - 1 + y_1/d_1.$$

where $k = \frac{1 + R}{2 - 2R}$. Also all the centres C_n are on the ellipse $4Rx(x - 1 - R) + y^2(1 + R)^2 = 0$, which has major axis 1+R and minor axis $2\sqrt{R}$.

This leads us to the result that if R is rational and the coordinates (x_1, y_1) of C_1 are rational then all x_n, y_n and d_n will be rational.

PROBLEM Suppose that we have the fixed circles of radii R and 1, as shown below. With one degree of freedom we can put one circle touching both. Starting with this circle, an infinite sequence of such circles, each touching its two neighbours, can be constructed. Investigate the sum of the areas of these circles, the shaded area below. Does the sum depend on the first circle?



THE LONELY RUNNER CONJECTURE

Jamie Simpson

There are n runners competing in a race. They all start at the same time and at the same point of a circular track of length n units, and they all run at different speeds. If they run for ever, can you show that for each runner there will be a time when he is at a distance at least one unit from every other runner? That this is true was first conjectured by Jorg Wills in 1968 and has since been proved for n up to and including 5. When n = 1 or 2 or 3 the conjecture is easy to prove. The n = 4 case was proved by Cusick in 1982, and the n = 5 case by Cusick and Pomerance in 1984. Their proof is not attractive and involves computer checking of a large number of cases. A much simpler, but I think unpublished, proof of this case was discovered recently by Luis Goddyn, Andras Sebo and Tarsi. The conjecture is unproved for n greater than 5.

There are a few simplifications we can make to the problem. One is that it can be shown that it is sufficient to consider the case in which the speeds of the runners are all rational numbers. Another is to note that if we have a counter-example to the conjecture then it will remain a counter-example if we subtract a constant from each runner's speed. Suppose there is one runner who is always less than one unit away from all the other runners, and we subtract his speed from the speed of each runner, so that he is now stationary. Then we get a counter-example to the following conjecture, which is equivalent to the original conjecture. There are k runners running round a circular track of length k + 1 units, all starting at the same time and place. Show that there will be a time at which all the runners are at least one unit from the starting point. In this version the

least one unit from the starting point. In this version the
n from the original problem has been replaced by k + 1, so this version is known only to hold for $k \le 4$. The subtraction process may give some of the runners negative speeds, but this can be corrected by reversing them. A runner's shortest distance from the start does not depend on whether he is running clockwise or anticlockwise round the track.

If the conjecture is true then it is best possible in the sense that we cannot replace "one unit from the starting point" with any greater distance. If the speeds of the runners are in the ratio 1 : 2 : ... : k then at some stage every runner will be at one unit away from the starting point, but there will never be a time when they are all more than one unit away. This is the only extremal example for the cases k = 1, 2 and 3, but for higher k there are others.

<u>References</u>

T. W. Cusick, *View-obstruction problems,* Proceedings of the American Mathematical Society, 84 (1982), 25-28.

T. W. Cusick and Carl Pomerance, *View-obstruction problems II,* Journal of Number Theory, 19 (1984), 131-139.

J. M. Wills, *Zur simultanen homogenen diophantischen approximation I,* Monatsh. Math., 72 (1968) 254-263.

SPHERES AND AREAS

Chris Smyth
(University of Edinburgh)

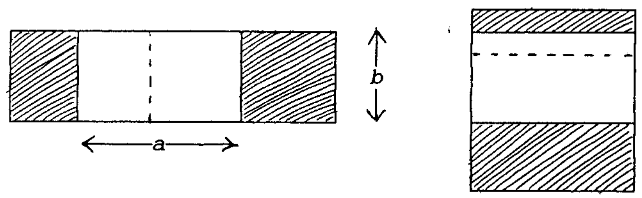
Here is a nice little bit of geometry, it must be known of course, but it was new to me.

A sphere of radius R has at the South pole a tangent plane. How much surface area, of sphere and plane, is there within a distance r from the North pole?

WRAPPING

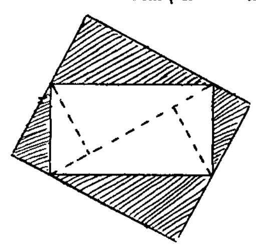
Another little bit of geometry, also to do with areas.

Consider wrapping a rectangular lamina (of size $a \times b$) with a rectangular sheet of paper of minimal area. There are three ways of doing it; the two obvious ways are with a sheet of size $(2a) \times b$ or with a sheet $a \times (2b)$, as shown below.

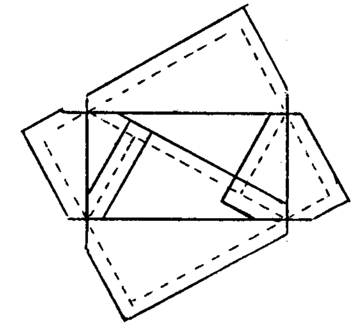


(Imagine the shaded areas folded over)

The third way is with a sheet of size $A \times B$ where $A = (a^2 + b^2)^{\frac{1}{2}}$ and $B = 2ab(a^2 + b^2)^{-\frac{1}{2}}$, folded as shown.



One illustration of the theory above is to be seen in the making of envelopes. For a rectangular $a \times b$ envelope, take the third of the wrappings above, by a sheet of size $A \times B$, and enlarge it all round, to a size of $(A+d) \times (B+d)$, giving an overlap of width d to be gummed. Four little triangular bits are wasted, they have to be cut off.



A more complicated problem arises in the wrapping of a rectangular block of $a \times b \times c$. Taking $a > b > c$, it is useful to note that the block will go in an envelope made for a lamina of $(a + c) \times (b + c)$. There is a certain inefficiency, shown by the surplus of wrapping paper found at the corners.

The 'KitKat' chocolate block that we buy in the local shops is wrapped in aluminium foil in the diagonal wrapping, the third of the methods described above. The block is not quite rectangular, but roughly the dimensions in inches are $a = 3{\cdot}75$, $b = 2{\cdot}5$, $c = {\cdot}5$. As noted above we can wrap it in a flat envelope of $4{\cdot}25 \times 3$, which from our formula above (putting $a = 4{\cdot}25$ and $b = 3$) requires a rectangle $A \times B$ where $A = 5{\cdot}202$ and $B = 4{\cdot}902$. To these dimensions there has to be added whatever is required for the overlap at the joins. The aluminium foil is not glued, it is held in place by a paper sleeve. The makers actually use a square of aluminium foil of side 5·625 for the wrapping, giving an overlap bigger in some places than in others. The calculations above indicate that the makers could save 5% of their aluminium foil by changing to a rectangular wrapping of $5{\cdot}625 \times 5{\cdot}325$ inches.

SUMS GIVEN BY ZETA FUNCTIONS

(JCMN 65, p.6360, 66, p.7010, 67, p.7030)

Recall that we defined

$$HD(k) = \sum_{n=1}^{\infty} n^{-k} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2n}\right).$$

In JCMN 67 there were given the formulae:-

$$HD(2) = \frac{11\,\zeta(3)}{4},$$

$$HD(4) = \frac{37\,\zeta(5)}{4} - 4\zeta(2)\zeta(3),$$

$$HD(6) = \frac{135\,\zeta(7)}{4} - 16\zeta(2)\zeta(5) - 4\zeta(3)\zeta(4),$$

$$HD(8) = \frac{521\,\zeta(9)}{4} - 64\zeta(2)\zeta(7) - 4\zeta(3)\zeta(6) - 16\zeta(4)\zeta(5),$$

$$HD(10) = \frac{2059\,\zeta(11)}{4} - 256\zeta(2)\zeta(9) - 4\zeta(3)\zeta(8) - 64\zeta(4)\zeta(7) - 16\zeta(5)\zeta(6).$$

Of these equations, the first and second were proved, and the next three were verified numerically to 12 decimal places. Readers were invited to guess the general formula and to find a proof. A possible answer to the first challenge is as follows:-

$$HD(2k) = \frac{2^{2k+1} + 2k + 1}{4}\zeta(2k+1) - \sum_{r=1}^{k-1} 2^{2r}\zeta(2k-2r)\zeta(2r+1).$$

This formula agrees with the numerical value of 1·500513412131 for HD(12).

BINOMIAL IDENTITY 39

Chris Rennie

Suppose that $n \le m \le m+n \le N$.

$$\text{Let } f(x) = \frac{m!\; n!\; (N-m)!\; (N-n)!}{x!\; (m-x)!\; (n-x)!\; (N-m-n+x)!\; N!}$$

$$= \binom{m}{x}\binom{N-m}{n-x}\binom{N}{n}^{-1} = \binom{n}{x}\binom{N-n}{m-x}\binom{N}{m}^{-1}.$$

for $0 \le x \le n$.

Here are two identities:-

$$\sum_{x=0}^{n} f(x) = 1, \qquad \sum_{x=0}^{n} x\, f(x) = mn/N.$$

These two equations are almost obvious if you know where f(x) comes from. There are two bags, each bag has N balls in it. In one bag are m black balls and N-m white. In the other are n black balls and N-n white. Draw at random one ball from each bag, you may get a black pair, a white pair or a mixed pair; keep on until the two bags are empty. Let x be the number of black pairs, it is a random variable taking values from 0 to n (if $n \le m$). Then f(x) is the probability of getting x black pairs. If there are x black pairs there will have to be N-m-n+x white pairs and m+n-2x mixed pairs.

Finally, find $\Sigma\, x^2 f(x)$. This will give the variance:

$$\sum_{x=0}^{n} \left(x - \frac{mn}{N}\right)^2 f(x) = \frac{mn(N-m)(N-n)}{N^2(N-1)}.$$

In the experiment from which this question arose the interest was actually in the number of matching pairs, but as the number is N-m-n+2x, there is no difficulty in finding its mean and variance from those of x.

SYMMETRIC SIMULTANEOUS EQUATIONS 3 (JCMN 67 p.7040)

FIRST SOLUTION —— A. Brown

For these equations:-
$$x(y+z) - y^2 - z^2 = a,$$
$$y(z+x) - z^2 - x^2 = b,$$
$$z(x+y) - x^2 - y^2 = c,$$

write $x = Y + Z$, $y = Z + X$, $z = X + Y$,

$a = -2A$, $b = -2B$, $c = -2C$.

The set of equations becomes
$$X^2 - YZ = A, \qquad Y^2 - ZX = B, \qquad Z^2 - XY = C,$$
which brings it back to the original problem of Alexiev (JCMN 59, p.6173, JCMN 60, p.6192, JCMN 62, p.6276, JCMN 66, p.7006 and JCMN 67, p.7038).

SECOND SOLUTION —— J. B. Parker

Put $S = x^2 + y^2 + z^2$ and $T = x + y + z$.

The first equation is $xT - S = a$, the others similarly.

Adding the three equations: $T^2 = a+b+c + 3S$. ...... (1)

Also $(a+S)^2 + (b+S)^2 + (c+S)^2 = (xT)^2 + (yT)^2 + (zT)^2$

$= T^2S$ which by equation (1) equals $(a+b+c)S + 3S^2$.

Therefore $a^2 + b^2 + c^2 + (a+b+c)S = 0$. .... (2)

Also, from (1) & (2), $T^2 = \dfrac{(a+b+c)^2 - 3(a^2+b^2+c^2)}{a+b+c}$

$= 2\,\dfrac{ab+bc+ca-a^2-b^2-c^2}{a+b+c}$. Put $T = R/(a+b+c)$ .... (3)

Finally, $xT = S+a = \dfrac{b(a-b) + c(a-c)}{a+b+c}$, so that the solution

is $x = \dfrac{b(a-b)+c(a-c)}{R}$, $y = \dfrac{c(b-c)+a(b-a)}{R}$, $z = \dfrac{a(c-a)+b(c-b)}{R}$,

where $R = \pm\left(2(a+b+c)(ab+bc+ca-a^2-b^2-c^2)\right)^{\frac{1}{2}}$.

$= \pm\left(6abc - 2a^3 - 2b^3 - 2c^3\right)^{\frac{1}{2}}$.

THIRD SOLUTION —— A. Brown

Write $Q(x, y, z) = 3xyz - x^3 - y^3 - z^3$. Then it is easy to verify that

$ax + by + cz = 0$, ... (1)

$bx + cy + az = Q(x, y, z)$, .. (2)

$cx + ay + bz = Q(x, y, z)$. ... (3)

From (2) and (3),

$(b-c)x + (c-a)y + (a-b)z = 0$ ....(4)

It follow from (1) and (4) that (in general)

$$\frac{x}{b(a-b)-c(c-a)} = \frac{y}{c(b-c)-a(a-b)} = \frac{z}{a(c-a)-b(b-c)} = k, \text{ say,}$$

$$= \frac{x+y+z}{2(ab+bc+ca-a^2-b^2-c^2)} = \frac{x-y}{(a-b)(a+b+c)} \quad \text{........ (5)}$$

Also, $Q(x, y, z) = bx + cy + az$

$= k\left(b^2(a-b) - bc(c-a) + c^2(b-c) - ac(a-b) + a^2(c-a) - ab(b-c)\right)$

$= k\,Q(a, b, c)$.

To evaluate k, note that

$a - b = z(x-y) + x^2 - y^2 = (x-y)(x+y+z)$.

Now use equation (5):

$a - b = \{k(a-b)(a+b+c)\}\{2k(ab+bc+ca-a^2-b^2-c^2)\}$

$= 2k^2(a-b)Q(a, b, c)$,

and so for $a \neq b$, $k^2 = \dfrac{1}{2Q(a, b, c)}$,

and $k = \pm(6abc - 2a^3 - 2b^3 - 2c^3)^{-\frac{1}{2}}$.

SPHERICAL TRIANGLES

John Parker
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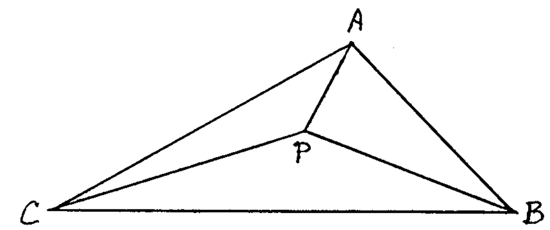
Readers will remember many references to position lines and cocked hats in previous issues of JCMN. It started with A. P. Guinand's comment in JCMN 33 (February 1984) that, given a plane triangle, the sum of squares of distances to the sides is minimized at the symmedian point; and consequently the symmedian point is (in a sense) a navigator's best estimate for position when given three position lines. Is there a corresponding result for a spherical triangle?

The idea of minimizing a sum of squares (hallowed by time, beloved by statisticians, and often very useful) may have to be abandoned for this problem; it is hard to see squares of angles leading to elegant geometry.

What can be said about the point in a spherical triangle maximizing the sum of cosines of the distances to the sides? When the triangle is small, and so nearly plane, the point will approximate to the symmedian point of the approximating plane triangle.

Is the question pure or applied mathematics? It is hard to say; although Admiralty charts are printed on flat paper, navigators do take into account the curvature of the Earth; also some navigators will tell you that for practical purposes one point inside the cocked hat is as good as another. So treat the problem as pure mathematics.

INCENTRES OF SPHERICAL TRIANGLES



In the spherical triangle ABC we put $p = \cos BC$ and $p' = \sin BC = \sqrt{(1 - p^2)}$, and similarly define $q = \cos AC$, q', r and r'. Using the notation explained in JCMN 47, pp.5136-5139, any point P has coordinates $x = \cos PA$, $y = \cos PB$, $z = \cos PC$, which may be treated as homogeneous coordinates.

Taking P to be the incentre, the angles CAP and PAB are equal. Applying the cosine rule to the triangles PCA and PBA:

$$z = \cos PC = xq + q'(1 - x^2)^{\frac{1}{2}} \cos (A/2)$$

$$y = \cos PB = xr + r'(1 - x^2)^{\frac{1}{2}} \cos (A/2)$$

Elimination of the terms with cos (A/2) gives us:

$$x(r'q - q'r) + yq' - zr' = 0,$$

Then cyclic permutation of (x, y, z), (p, q, r) and (p', q', r') leads to:

$$- xp' + y(p'r - r'p) + zr' = 0.$$

Adding these last two equations:

$$x(p' - r'q + q'r) = y(q' + p'r - r'p),$$

then from the reverse cyclic permutation:

$$z(r' + p'q - q'p) = x(p' + r'q - q'r).$$

The incentre is therefore

$$\left(1, \frac{(p' - r'q + q'r)}{q' + p'r - r'p}, \frac{(p' + r'q - q'r)}{r' - q'p + p'q}\right).$$

This expression does not seem to have the symmetry that one would expect. The cyclic permutation of (x, y, z) and (p, q, r) and (p′, q′, r′) tells us that the incentre must have coordinates:

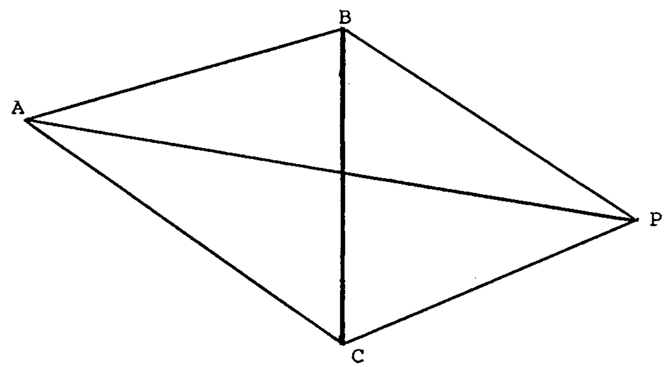
$$\left(\frac{q' + p'r - r'p}{p' - r'q + q'r'}, \ 1, \ \frac{q' - p'r + r'p}{r' + q'p - p'q} \right).$$

Is it the same point? It is, because of the identity:

$$(p' - r'q + q'r)(q' - p'r + r'p)(r' - q'p + p'q)$$
$$= (p' + r'q - q'r)(q' + p'r - r'p)(r' + q'p - p'q).$$

In fact this is essentially the TRIGONOMETRIC IDENTITY given on p.7066 below.

Now what are the coordinates of the ex-centres?



Apply the cosine law to the triangles PCB and PAB.

$$z = py + p'\sqrt{(1 - y^2)} \ \cos(90^\circ - B/2)$$
$$x = ry + r'\sqrt{(1 - y^2)} \ \cos(90^\circ + B/2).$$

Noting that $\cos(90^\circ - B/2) = \sin B/2 = -\cos(90^\circ + B/2)$, we obtain

$$p'x - (p'r + r'p)y + r'z = 0.$$

There is another symmetry that we may use, the interchanges (y, z), (q, r) and (q′, r′); this gives:

$$p'x + q'y - (p'q + q'p)z = 0.$$

From these last two equations:

$$(q' + p'r + r'p)y = (r' + p'q + q'p)z.$$

Therefore (as the coordinates are homogeneous) put

$$y = r' + p'q + q'p \quad \text{and} \quad z = q' + p'r + r'p.$$

Then $p'x = (p'r + r'p)y - r'z$

$$= (p'r + r'p)(r' + p'q + q'p) - r'(q' + p'r + r'p)$$
$$= (p'r + r'p)(p'q + q'p) - r'q'$$
$$= p'(rp'q + rq'p + r'pq) - r'q'(1 - p^2).$$

Now recall that p and p′ are the cosine and sine of an angle.

$$p'x = p'(rp'q + rq'p + r'pq - p'r'q'),$$
$$x = p'qr + pq'r + pqr' - p'q'r'.$$

Thus we have found the coordinates of the centre of the escribed circle that touches BC on the outside:

$I_a = (p'qr+pq'r+pqr'-p'q'r', \ r'+p'q+q'p, \ q'+p'r+r'p).$

By cyclic symmetry we find

$I_b = (r'+q'p+p'q, \ p'qr+pq'r+pqr'-p'q'r', \ p'+q'r+r'q).$

These two points are both on the external bisector of the angle at C of the spherical triangle, and so they must be on a great circle with C = (q, p, 1). Therefore the determinant below must be zero; but is that obvious?

$$\begin{vmatrix} p'qr+pq'r+pqr'-p'q'r' & r'+p'q+q'p & q'+p'r+r'p \\ r'+q'p+p'q & p'qr+pq'r+pqr'-p'q'r' & p'+q'r+r'q \\ q & p & 1 \end{vmatrix}$$

This is another trigonometric identity. If we take any three angles, denote their cosines by p, q and r, and denote their sines by p′, q′, and r′, then the determinant above must be zero.

The calculation is not as hard as it looks. Subtracting q times the third column from the first, and p times the third from the second, the determinant is reduced to a 2 × 2 determinant. By using the fact that p and p′ have the sum of their squares equal to 1, and similarly q and q′, the determinant may be put in the form:

$$\begin{vmatrix} q'(pr-p'r'-q) & p'(q-pr+p'r') \\ q'(q'r'+p-qr) & p'(qr-q'r'-p) \end{vmatrix}$$

which is clearly zero.

TRIGONOMETRIC IDENTITY

Prove that if A, B and C are any angles then:

$$\cos A \cos B \sin(A-B) + \cos B \cos C \sin(B-C) + \cos C \cos A \sin(C-A) + \sin(A-B)\sin(B-C)\sin(C-A) = 0.$$

ORTHOCENTRES AGAIN

(JCMN 47, p.5136, 50, p.5204, 51, p.5220 & 58, pp.6127-6129)

Take a plane triangle ABC, with centroid G and orthocentre H. Taking G as origin, let **p**, **q** and **r** be the position vectors of the vertices A, B and C, then **p** + **q** + **r** = 0.

Rotation of 180° about the mid-point of BC is the mapping **x** → **q** + **r** − **x**, which maps ABC to the congruent triangle A′CB, where A′ has position vector **q** + **r** − **p** = −2**p**. Similarly define B′ and C′, they have position vectors −2**q** and −2**r**.

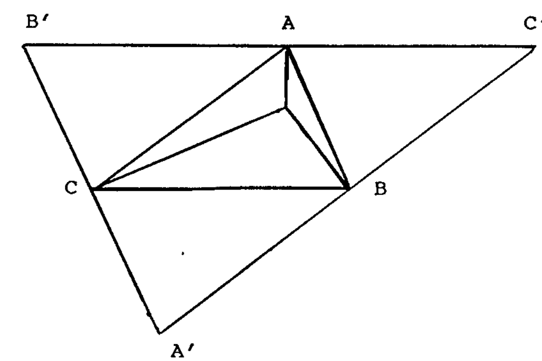


Figure 1

Note that B′C′ is parallel to CB, so that AH is the perpendicular bisector of the side B′C′ of the triangle A′B′C′, and therefore passes through the circumcentre of A′B′C′; and similarly for BH and CH. Therefore H is the circumcentre of A′B′C′. In trilinear coordinates with ABC as triangle of reference, A′ is (−bc, ca, ab), B′ is (bc, −ca, ab) and C′ is (bc, ca, −ab); their circumcircle is

$$a^4x^2 + b^4y^2 + c^4z^2 + (a^2 + b^2 + c^2)(bcyz + cazx + abxy) = 0.$$

where as usual we denote the sides of the triangle ABC by a, b and c.

Recall how the note from J.D.E. Konhauser and A. Brown in JCMN 51 answered the questions from JCMN 50 —— why is H called the 'orthocentre' of ABC? and of what is it the centre? That note gave two answers, H is a centre of similitude of the 9-point circle and the circumcircle of ABC, and H is the centre of the orthocircle, which is the unique circle with the property that the triangle is self-polar with respect to it, see the drawing in JCMN 58. Now we have found a third answer —— H is the centre of the circumcircle of A'B'C'.

In fact there is a fourth answer, quite simple, but nobody thought to mention it. The orthocentre H of any acute-angled triangle ABC is the incentre of the orthic (or pedal) triangle DEF (i.e. the triangle whose vertices are the feet of the perpendiculars from A to BC, from B to CA and from C to AB). See fig. 2. If ABC is obtuse-angled then H is the centre of one of the escribed circles of the orthic triangle DEF. See fig. 3. The angles marked with a dot are all equal, and the angles marked with two dots are all equal.

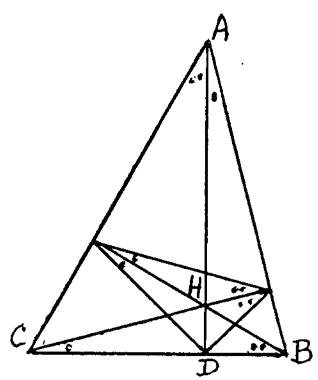


Figure 2

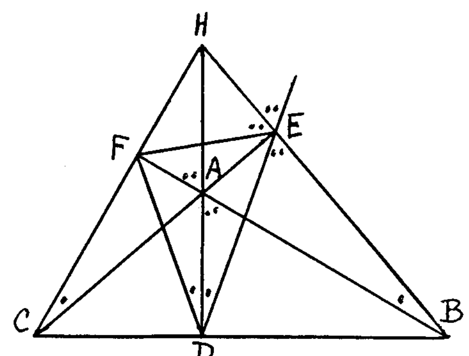


Figure 3

If (in Fig. 1) **v** is the position vector of the circumcentre O of ABC then **-2v** is the position vector of the circumcentre of A'B'C', which is H. Similarly the mid-points of BC, CA and AB, with position vectors -**p**/2, -**q**/2 and -**r**/2 have circumcentre with position vector -**v**/2, which is the nine-point centre of ABC.

This vector algebra has located the four important points on the Euler line of the triangle ABC:-

H = Orthocentre, position vector = -2**v**
N = Nine-point centre position vector = -**v**/2
G = Centroid position vector = 0
O = Circumcentre position vector = **v**,

proving Euler's result that the lengths HN, NG, GO are in the ratio 3:1:2.

In fact it is now clear that the orthocentre of A'B'C' is also on the Euler line, with position vector 4**v**.

****

The question from JCMN 51 —— of what is the orthocentre the centre? —— may also be asked about spherical triangles.

There are three answers, two of them different from those in the plane case. Firstly the orthocentre H is the centre of perspective of the original triangle ABC and its polar triangle A'B'C', because the great circle AA' is the altitude, the perpendicular from A on to the opposite side BC, and so on.

Secondly, can we find a circle, other than the orthocircle, of which H is the centre? The geometry on the sphere is very similar to plane projective geometry (see ANALYTIC GEOMETRY FOR

SPHERICAL TRIANGLES, JCMN 47, October 1988, pp. 5136-5139). The theorem of Desargues applies; the triangles ABC and A′B′C′ are in perspective, and so the intersections of corresponding sides must be on a great circle. In fact the intersection of BC with B′C′ is a point at 90° distance from A and from A′ and therefore also from the orthocentre H. Similarly the two other such intersections, of CA with C′A′ and of AB with A′B′; these three points are all at 90° from H, so that H is the centre (or pole) of the great circle through these three points.

A third answer is that, as in the plane case, there is an orthocircle with respect to which the triangle ABC is self-polar, and the orthocentre is the centre of this circle. To explain this we need the methods of the article from JCMN 47 mentioned above. For readers who do not have it on their bookshelves, here is a summary.

Let three unit vectors $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$, be the position vectors of the vertices A, B, C, of a spherical triangle. Any vector represents a point on the unit sphere, in the sense that when the vector is normalised to unit length it is the position vector of the point. Then the vector products $\underline{\beta}\times\underline{\gamma}$, $\underline{\gamma}\times\underline{\alpha}$, $\underline{\alpha}\times\underline{\beta}$, represent the vertices A′, B′, C′, of the polar triangle.

The vector $\underline{\alpha} + \underline{\beta} + \underline{\gamma}$ represents the centroid, which is the intersection of the medians, or the centre of gravity of unit masses at the vertices. A point represented by any vector $\underline{u}$ may also be represented by the three coordinates $x = \underline{u}.\underline{\alpha}$, $y = \underline{u}.\underline{\beta}$, $z = \underline{u}.\underline{\gamma}$, and we treat these coordinates as homogeneous, so that $\underline{u}$ need not be of unit length. The circumcentre O of ABC is the point (1, 1, 1). Let the cosines of the sides of the triangle ABC be p, q, r.

The altitude from A to BC is the great circle AA′, having equation $qy = rz$, and the orthocentre is the intersection of the three altitudes. The altitudes of the spherical triangle project into the altitudes of the plane triangle. The median from A is the great circle:

$$(rx-qx-y+z)(p+1) + (qy-rz)(q+r) = 0$$

and the side BC is $x(1-p^2) + y(pq-r) + z(rp-q) = 0$. The coordinates of the following few points may be noted:

A: (1, r, q) B: (r, 1, p) C: (q, p, 1)

A′: (1, 0, 0) B′: (0, 1, 0) C′: (0, 0, 1)

Mid-point of BC: (q+r, 1+p, 1+p).

Centroid, G: (1+q+r, p+1+r, p+q+1).

Orthocentre, H: (qr, rp, pq).

Circumcentre, O: (1, 1, 1).

Foot D of altitude from A: $(q^2+r^2-2pqr,\ r-rp^2,\ q-qp^2)$

Since (Theorem 1 of the JCMN 47 article) any linear homogeneous equation in (x, y, z) represents a great circle on the sphere, it follows that if we project the sphere radially on to the tangent plane at H then such a linear equation will represent a straight line in the plane, i.e. the coordinates (x, y, z) may be regarded as homogeneous coordinates on the tangent plane. The projection of A′B′C′ is the triangle of reference in the plane. Any small circle on the sphere projects into a conic in the plane, but a small circle with H as centre will project into a circle in the plane, also with H as centre. The great circle with H as pole, or the line at infinity on the plane, has equation $\frac{x}{p-qr} + \frac{y}{q-rp} + \frac{z}{r-pq} = 0$.

The geometry of poles and polars with respect to conics in the plane carries over to the geometry on the sphere, and so we can say that the triangle ABC is self-polar with respect to the orthocircle on the sphere, which is the projection of the orthocircle of the plane triangle.

PROBLEMS Does a spherical triangle have an interesting great circle like the Euler axis of a plane triangle? Recall that the centroid, circumcentre and orthocentre are not on a great circle except when the triangle is isosceles (proved in Theorem 2 of the article in JCMN 47).

Are the the mid-points of the sides and the feet of the altitudes of a spherical triangle on a conic?

ERRATUM

Recall the article ANALYTIC GEOMETRY FOR SPHERICAL TRIANGLES in JCMN 47 (October 1988) pages 5136-5139.

The coordinates of the centroid G of the spherical triangle ABC were given correctly as (1+q+r, p+1+r, p+q+1) on page 5137, but wrongly as (q+r, p+r, p+q) on page 5138. The misprint did not upset the proof of the Theorem 2, that if the circumcentre, orthocentre and centroid are on a great circle then the triangle is isosceles.

WANTED INEQUALITY

(JCMN 64, p.6321 & JCMN 67, p.7046)

If $f(\theta)$ has period 2π and if

$$x = \int_0^{2\pi} |f|^2\, d\theta, \qquad y = \int_0^{2\pi} |f'|^2 d\theta, \qquad z = \left|\int_0^{2\pi} e^{i\theta} |f|^2 d\theta\right|,$$

then what inequalities connect x, y and z?

In JCMN 67 there was the statement "Numerical evidence suggests an inequality (1)"

$$4y(x - z) \geq z^2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (1)$$

But what is suggested by numerical evidence may not be true. In this case it was untrue, as may be shown by the example of $f(\theta) = 4 + 4\cos\theta + \cos 2\theta$. It give $x = 49\pi$, $y = 20\pi$ and $z = 36\pi$, not satisfying the inequality (1). The same example disproves the other suggested inequality:

$$x + y \geq 2z. \qquad \ldots\ldots\ldots\ldots\ldots\ldots \qquad (2)$$

QUOTATION CORNER 54

Whatever you do, do it with conviction.

—— Sir Thomas Beecham.

KIRKMAN'S SCHOOLGIRLS

In 1850 T. P. Kirkman, writing in the *Lady's and Gentleman's Diary*, introduced his problem about schoolgirls. There are 15 girls and every day they go for a walk, three abreast, in a column of five ranks. Thus each girl has two companions (i.e. those in the same rank of three) each day; can the pattern of their walks be arranged so that in a week each girl has each of the other fourteen as a companion just once? The answer is that it is possible. This problem was solved in 1862, and since then the various solutions have been analysed and classified.

An elaboration of the problem may be of interest. The traditional old English school term was of 13 weeks, and so the girls had 91 walks in a term, and altogether 455 triples of girls walked together in a rank of three. But 455 is equal to the binomial coefficient $\binom{15}{3}$, which is the number of different triples that can be chosen from 15 girls.

Can the walks be arranged so that each of the 455 possible triples comes together just once in a term? This is a question about what in the language of the subject are called "balanced incomplete block designs". The question can be made more difficult by asking that in each week the walks should satisfy Kirkman's condition of each girl walking once with each of the others.

HALF INTEGER GAUGES

Long ago and far away (note that the Editor is not claiming this little bit of probability theory to be socially relevant) aircraft were being built in accordance with what was called "stressed skin construction". The required strength and stiffness were obtained largely from the aircraft skin, which was of aluminium sheet. The sheets were produced in a rolling mill, and their thicknesses were described by the Birmingham Wire Gauge scale, for instance:

Gauge number	8	9	10	11	12	13	14	15
Thickness, ins.	·168	·148	·134	·120	·109	·096	·083	·072

(The thicknesses are roughly in geometric progression)

In the aircraft industry it was only the integer-numbered gauges that were used; to obtain sheet of any other thickness a special order had to be placed with the rolling mill.

In the designing of aircraft, certain calculations were made to find what thickness of metal was required for each part of the skin; the designer then specified the (largest) gauge number that would give the required thickness; for instance, if the calculations indicated 0·090 inches then the designer would call for 13-gauge sheet (0·096" thick).

The thickness actually used therefore was more than had been calculated to be necessary, by a random amount of up to 12%. It might reasonably be supposed that on the average the skin of the aircraft was about 6% thicker than was needed according to the stress calculations.

The suggestion was made that the rolling mills and the factory stores should make available sheets of half-integer as well as integer gauge numbers, for instance 8½ gauge (0·156"), 9½ gauge (0·141"), etc. If this were done then (it was claimed) the amount of unnecessary metal used would on the average be 3% instead of 6%. Such a saving might be enough to justify the extra work involved in having twice as many different thicknesses of sheet metal available in the factory.

It would be interesting to have some comment on the belief that the use of half-integer gauges would save 3% of the metal without increasing the risk of structural failure.

For those not familiar with engineering, the calculation of stresses in aircraft is far from accurate, for instance there is practically no data on what turbulence might be found in a heavy thundercloud. For such reasons engineers use a large "factor of safety" in various stages of design; so that in a new aircraft, flown in accordance with the rules, in reasonably good weather, the actual local stress in any part of the skin probably never reaches as much as one tenth of the ultimate strength of the metal.

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