JAMES COOK
MATHEMATICAL NOTES

Volume 7 Issue number 70

December 1996

Editor and distributor: B. C. Rennie
Address: 66, Hallett Road,
Burnside, SA 5066
Australia.

The history of the James Cook Mathematical Notes (JCMN) is that the first issue (a single foolscap sheet) appeared in September 1975, then others at irregular intervals, to number 17 in November 1978. JCMN settled into the routine of three issues per year from 1979 to 1994; but from Issue 66 (April 1995) at the start of Volume 7, it has been irregular, appearing when enough contributions are available.

The issues up to number 31 (May, 1983) were produced and sent out free by the Mathematics Department of the James Cook University of North Queensland, of which I was then the Professor. The arrangement was beginning to be unsatisfactory, and in October 1983 I started producing the JCMN myself and asking readers to pay subscriptions. In October 1992 it had become clear that the paying of subscriptions by readers is an inefficient operation. Bank charges for changing currency and for international transfers, with postage, together absorb most of the initial input of money. Therefore we abandoned subscriptions as from issue number 60 (January, 1993). I now ask readers only to tell me every two years if they still want to have JCMN. To those who want to give something in return for the JCMN, I ask them to make a gift to an animal welfare society in their own country. The animals of the world will be grateful and so will I.

Contributors, please tell me if and how you would like your address printed.

JCMN 70, December 1996

CONTENTS

PAUL ERDŐS

Regretfully we record the death of Paul (or Pàl) Erdös at Warsaw on 21st September, 1996. We shall not attempt to match the obituaries that will appear in other journals, but we gratefully note that Paul was a JCMN reader and contributor for many years.

FOURIER TRANSFORM (JCMN 69, p.7100)

T. C. S. Tao
(UCLA, 405 Hilgard Avenue, Los Angeles, CA 90095-1555, USA)

The question was to find the Fourier transform of the function $\frac{\log|x|}{\sqrt{|x|}}$

Lemma If s is a complex number with real part between 0 and 1, then the Fourier transform of $f(x) = |x|^{s-1}$ is

$$F(\xi) = 2\cos(\pi s/2)\,\Gamma(s)\,(2\pi)^{-s}\,|\xi|^{-s}.$$

Proof By scale invariance, we see that the Fourier transform F must be an even distribution, homogeneous of degree -s. Since s has real part between 0 and 1, this means that $F(\xi)$ must be a constant multiple of $|\xi|^{-s}$. The constant (above) can be justified formally by contour integration. We may assume that ξ is positive. The integral

$$\int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x\xi}\, dx$$

is conditionally convergent at infinity and can be broken up into two parts. The contribution of the positive real axis,

$$\int_0^\infty x^{s-1}\, e^{-2\pi i x\xi}\, dx,$$

is equal to the integral of the same function on the negative imaginary axis, i.e. $\int_0^\infty (-i)^{s-1}\, x^{s-1}\, e^{-2\pi x\xi}\,(-i)\,dx$, where $(-i)^{s-1}$ is defined as $\exp(-(s-1)\pi i/2)$. By the integral definition of the Gamma function, the above integral is just $\exp(-s\pi i/2)\,(2\pi\xi)^{-s}\,\Gamma(s)$. Similarly, the contribution to the Fourier transform from the negative real axis is $\exp(s\pi i/2)\,(2\pi\xi)^{-s}\,\Gamma(s)$. Adding the two contributions together gives the result.

Differentiating with respect to s the above Fourier transform, we see that the Fourier transform of

$$\frac{d}{ds}|x|^{s-1} = |x|^{s-1}\log|x|$$

is $\frac{d}{ds}\left(2\cos(\frac{\pi s}{2})\,\Gamma(s)\,(2\pi)^{-s}\,|\xi|^{-s}\right)$. Evaluating this explicitly at $s = \frac{1}{2}$, and using the facts that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and

$$\frac{\Gamma'(1/2)}{\Gamma(1/2)} = -\gamma - 2\log 2,$$

we obtain that the Fourier transform of $|x|^{-1/2}\log|x|$ is

$$-|x|^{-1/2}\log|x| - K\,|x|^{-1/2},$$

where $K = \frac{\pi}{2} + \log(2\pi) + 2\log 2 + \gamma = 5.372...$

DATES

Most people understand that we call this year 1996, or perhaps A.D. 1996; but our present system of dates is relatively modern. Up to about a thousand years ago most of Europe reckoned dates by the number of years from the foundation of the city of Rome, the number of a year was described as Ab Urbe Condita, abbreviated to A.U.C.

In A.U.C. 707 (described nowadays as 47 B.C.) Julius Caesar, who then held the position of Pontifex Maximus in Rome, engaged the Egyptian astronomer Sosigenes of Alexandria to set the calendar in order. Previously the number of days in a year had been decided each year by the magistrates or priests, and the months had got out of step with the seasons, the end of December A.U.C. 707 was in late autumn instead of mid-winter (by our modern reckoning it would have been October 12th). To put things right, the year A.U.C. 708 (46 B.C.) was given two extra months, making for the year a total of 445 days, it became known as the "year of confusion". (A reference may be found in the Letters of Cicero). From then on (from A.U.C. 709 or 45 B.C.) the numbers of days in each month were fixed, and every fourth year was to be a leap year, differing from other years by having a repetition of February 23rd, the sixth day before the Kalends of March. This gave an average of 365¼ days per year, the modern estimate is that it should be 365·2422, the number of mean solar days to the tropical year.

Also Julius Caesar ordered that the year should start at the beginning of January. The previous confusion about when the year should start can be explained by the old Roman calendar, it was said to date from Romulus, and consisted at first of only 304 days, in 10 lunar months from March to December, with a blank period in the winter; the two Etruscan months of January and February were added in the reign of Numa, originally January at the beginning of the year and February at the end, but these two months were interchanged about A.U.C.452.

The resulting "Julian Calendar" started operating at the beginning of A.U.C.709 (45 B.C.) and was in use in most of Europe until the adoption, gradually over the 8th, 9th and 10th centuries, of our present A.D. system of dating. There were changes in the names of the months, Quintilis and Sextilis becoming July and August, and small adjustments in the lengths of the months, the original plan of Julius Caesar was to make the number of days in each month, January, February, etc. to be 31, 29, 31, 30, 31, 30, 31, 30, 31, 30, 31, 30, respectively. except for February in Leap years. The other change in the Julian calendar in that time was that the beginning of the year changed from January 1st to March 25th; I have not been able to find out how, why or when that change came about.

A monk called Dionysius Exiguus, who was in Italy between about A.U.C 1246 - 1290, or A.D. 493 - 537, had the idea that instead of counting years from the foundation of Rome we should count them from the birth of Jesus Christ. He thought that the birth was on 25th December in the year A.U.C. 753. In this proposed change there was to be no year zero (the Roman numeral system did not have a notation for zero) and so A.U.C 753 became 1 B.C.; and A.U.C. 754 became 1 A.D. St Matthew's Gospel is clear on the date of Christ's birth and the massacre of the Holy Innocents being in the reign of Herod the Great, and modern historians mostly agree on dating the death of Herod as in late March or early April in A.U.C. 750 = 4 B.C., so that Dionysius Exiguus was mistaken in his historical research. It is too late now to correct his mistake, and no doubt we must continue to describe this year (A.U.C. 2749) as A.D. 1996, but people keen on celebrating the second millenium might consider doing so this year or next year, and not in four or five years time.

Thanks are due to Donald Simpson for his help with Roman history.

DISTRIBUTION FREE METHODS IN NAVIGATION

John Parker
(Oak Tree Cottage, Reading Road, Padworth Common, RG74QN)

Navigation is the art, or science, of conducting a vessel from one place to another. The navigator's task is carried out via intermediate position determinations (e.g. pinpoints or multi-position-line fixes). A well known instance is the so-called "cocked hat", which is the triangle formed by three position lines. For good reasons navigators are taught to try to choose the three lines such that the resulting triangle is as near as possible equilateral. The adopted fix is taken somewhere inside the triangle. The (easily constructed) incentre is a good enough choice, but now that we have quick, expertly programmed methods of constructing the symmedian point (JCMN 40, p.4008) this rough choice, though adequate, is outmoded.

People became interested not only in the position itself but also in its error. The classical approach was to obtain a region (centred on the fix) such that one can say that there is some assigned probability (usually 0·5 or 0·95) that the unknown true position lies inside the region. This raises problems about which we must say a wee bit. This early approach made two assumptions; firstly that the navigator's position lines conformed to the (zero mean) Gaussian law, and secondly that the parameters of the three distributions were known *a priori*. Very often it was assumed that the several lines shared the same parameter (i.e. they had the same standard deviation). In all cases the error zones are ellipses. The possibility that all lines are subject to a common, unknown, systematic error is ruled out in distribution free theory. In air navigation this is acceptable since the random errors swamp the generally small systematic ones. At sea this is often not the case, but the effect of unknown systematic error can be catered for by choosing position lines which give a "good geometry", i.e. the directions of the lines are reasonably evenly distributed in azimuth.

Both assumptions can easily be challenged but there are areas in modern navigation, Kalman Filter work being a prime example, where it is necessary to make them.

Distribution free methods do not need either assumption. Their use was first noted by H. E. Daniels in his pioneering article (*The Theory of Position Finding*, Journal of the Royal Statistical Society, 13B, p.126, 1951) where error zones for the general n-line fix were presented. A simpler approach to this method is given by Terry Tao (JCMN 56, p.6076).

In the particular case n = 3 (Cocked Hat) this leads to the famous Admiralty Manual of Navigation theorem, based on minimal assumptions and described in JCMN 55, p.6033. It states that the probability that the true position lies inside the cocked hat is 1/4. Recent developments are due to our Editor who (in my view) has wrapped up the entire n = 3 case. Though the phase space can be regarded as broken down into 7 regions (three corner regions, three side regions and the interior of the hat), the navigator is seldom interested in any except the interior, though as a rough guide it could be mentioned (JCMN 56, p.6076) that the other six regions each carry a probability of 1/8. But Basil goes much further. Choosing the incentre as a point of reference, he delineates the several probabilities of the true position being within each of the six sectors of the interior of the cocked hat; these are all equal to 1/24. The choice of the incentre as point of reference has already been noted.

However, navigators can, and should, query the validity of these results. Because of random position line errors, the "size" of a cocked hat, as measured by its area, or, preferably, by the inradius, varies appreciably from occasion to occasion, even if the observations are all taken by the same observer and the environmental conditions stay the same. The "luck of the draw" will sometimes lead to a small hat, and an unfortunate (random) collection of "pluses and minuses" will lead to a big one. It is not correct to estimate a probability of 1/4 of being inside the hat in both cases.

The crux of the matter is this. With no information at all, and making no assumptions beyond those listed in JCMN 55, p.6033 (incidentally, blunders, though not systematic errors, are accommodated within these assumptions) one can confidently assert the 1/4 probability result plus the more refined breakdown developed in JCMN 69, p.7085. But the situation is transformed as soon as the navigator achieves his single hat, one of all the possible hats that could have been attained but were not. The navigator must make his inferences, if any, on the basis of his actual cocked hat, possibly using his (often very rough) knowledge of the likely observational errors in his three lines.

In the Admiralty Manual of Navigation theorem there is no such thing as a "big" or "small" hat. A "big" hat simply means that the (*a priori* unknown) intrinsic errors of the lines are large, and conversely for a "small" hat.

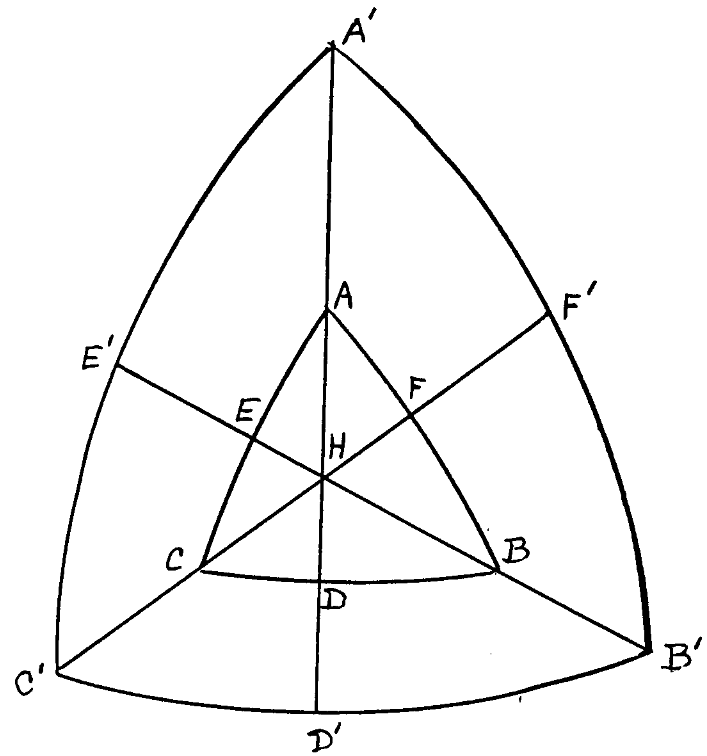
It is right to ditch the theorem after, though not before, the navigator has resolved and plotted his three position line observations.

SPHERICAL TRIANGLES

(JCMN 68 p. 7072)

In JCMN 68 the question was asked whether the mid-points of the sides and the feet of the altitudes of a spherical triangle are on a conic. The answer is YES. There is an analytic proof as follows.

Let ABC be a spherical triangle and let A′B′C′ be its polar triangle. Let K, L and M be the mid-points of BC, CA and AB respectively, and similarly define K′, L′ and M′. Let H be the common orthocentre, and let the altitude A′AH meet BC at D and meet B′C′ at D′, and define E, E′, F and F′ similarly. Let a, b and c be the sides, and A, B and C be the angles, of the first triangle. Then $a' = \pi - A$, $b' = \pi - B$, $c' = \pi - C$, are the sides of the polar triangle; also $A' = \pi - a$, $B' = \pi - b$, and $C' = \pi - c$ are the angles. It is often convenient to put $p = \cos a$, $q = \cos b$ and $r = \cos c$.



It is for the triangle A′B′C′ that we are going to prove our result.

The following coordinates are easily calculated.

Vertex A: (1, r, q) vertex B: (r, 1, p) vertex C: (q, p, 1)

A′: (1, 0, 0) B′: (0, 1, 0) C′: (0, 0, 1)

The altitude AA′HDD′ has equation: qy = rz

Foot D′ of altitude to B′C′: (0, r, q) = (0, cos c, cos b)

Mid-point K′ of B′C′: (0, sin c, sin b)

E′: (cos c, 0, cos a) F′: (cos b, cos a, 0)

L′: (sin c, 0, sin a) M′: (sin b, sin a, 0)

These six points are all on the conic:

$$x^2\sin 2a + y^2\sin 2b + z^2\sin 2c$$
$$= 2yz\sin(b+c) + 2zx\sin(c+a) + 2xy\sin(a+b).$$

The matrix of this quadratic form may be written as the symmetric part of the product

$$\begin{bmatrix} \cos a & 0 & 0 \\ 0 & \cos b & 0 \\ 0 & 0 & \cos c \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} \sin a & 0 & 0 \\ 0 & \sin b & 0 \\ 0 & 0 & \sin c \end{bmatrix}$$

New questions arise. Is the conic a circle? What is the centre of the conic? (A point P is called the centre of a conic Γ on a sphere when the polar of P with respect to Γ is the great circle orthogonal to P)

****

We need to look into the geometry of small circles on the sphere. In plane geometry with Cartesian coordinates it is easy to see if a quadratic equation represents a circle, but with trilinear coordinates it is more difficult; knowing the circular points is helpful.

What can we do in spherical geometry? Recall the matrix M (JCMN 69 p.7094), relating the normalized coordinates x, y, z, of any point with the coefficients of the unit vectors α, β, γ, that specify the vertices A, B, C, of the triangle of reference.

$$M^{-1} = \begin{bmatrix} 1 & r & q \\ r & 1 & p \\ q & p & 1 \end{bmatrix}$$

$$\det M^{-1} = 1 + 2pqr - p^2 - q^2 - r^2$$
$$= (1 - q^2)(1 - r^2) - (p - qr)^2$$
$$= \sin^2 b \sin^2 c - (\sin b \sin c \cos A)^2 = \Delta^2,$$

where Δ = sin b sin c sin A. This Δ is not twice the area of the triangle, but it is a related quantity, Δ/6 is the volume of the tetrahedron OABC. Also

$\Delta = 4 \sin\frac{a}{2} \sin\frac{b}{2} \sin\frac{c}{2} \cot R$ where R is the circumradius.

(There is the plane analogy, area = $\frac{abc}{4R}$ for plane triangles)

$$M = \begin{bmatrix} 1 - p^2 & pq - r & pr - q \\ pq - r & 1 - q^2 & qr - p \\ pr - q & qr - p & 1 - r^2 \end{bmatrix} \Delta^{-2}$$

A circle of radius ρ about the vertex A of the triangle of reference has the non-homogeneous equation x = cos ρ. To set up the homogeneous equation, let

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The circles with centre A form the pencil of conics

$$\mathbf{x}^T(\lambda M + \mu S)\,\mathbf{x} = 0.$$

The condition that such a conic be singular is (in general) a cubic in the ratio λ:μ, but two of the roots are the trivial λ = 0 repeated (corresponding to the great circle B′C′ with equation x = 0). The interesting root is given by λ + μ = 0, corresponding to the circle of zero radius, for which the quadratic equation has the matrix:

$$\begin{bmatrix} q^2 + r^2 - 2pqr & pq - r & pr - q \\ pq - r & 1 - q^2 & qr - p \\ pr - q & qr - p & 1 - r^2 \end{bmatrix}$$

This matrix is singular (top row + r × second + q × third

= 0) and so the quadratic factorises. This circle of zero radius consists of the two great circles:

$$x(r-pq+iq\Delta) - y(1-q^2) + z(p-qr-i\Delta) = 0, \text{ and}$$
$$x(r-pq-iq\Delta) - y(1-q^2) + z(p-qr+i\Delta) = 0.$$

The point $\left(1, -1, \frac{q-p-i\Delta}{1-r}\right)$ is on the great circles

$$x(r-pq-iq\Delta) + y(q^2-1) + z(p-qr-i\Delta) = 0 \quad \text{and}$$
$$x(q-pr-i\Delta) + y(p-qr+ir\Delta) + z(r^2-1) = 0, \quad \text{but not on}$$
$$x(p^2-1) + y(r-pq-i\Delta) + z(q-pr+ip\Delta) = 0.$$

We seem forced to the conclusion that there is nothing on the sphere analogous to the circular points of the projective plane.

ELEMENTARY NUMBER THEORY

Take any numbers b and n, and look at the prime factors p of $b^n + 1$. In a lot of cases it may be observed that

$$p \equiv 1 \pmod{2n}.$$

Can anyone offer an explanation?

Here is numerical evidence. In each table b is fixed, n is in the top row, and the prime factors p of $b^n + 1$ are given in the third and fourth rows, separated into those that are $\not\equiv 1$ and those $\equiv 1 \pmod{2n}$.

b = 2

n		1	2	3	4	5	6	7	8	9
$b^n + 1$		3	5	9	17	33	65	129	257	513
p	$\not\equiv 1$			3, 3		3	5	3		3, 3, 3
	$\equiv 1$	3	5		17	11	13	43	257	19

n		10	11	12	13	14	15
$2^n + 1$		1025	2049	4097	8193	16385	32769
p	$\not\equiv 1$	5, 5	3	17	3	5	3, 3, 11
	$\equiv 1$	41	683	241	2731	29, 113	331

n		16	17	18	19	20
$2^n + 1$		65537	131073	262145	524289	1048577
p	$\not\equiv 1$		3	5, 13	3	17
	$\equiv 1$	65537	43691	37, 109	174763	61681

n		21	22	23	24
$2^n + 1$		2097153	4194305	8388609	16777217
p	$\not\equiv 1$	3, 3	5	3	257
	$\equiv 1$	43, 5419	397, 2113	2796203	97, 673

n		25	26	27
$2^n + 1$		33554433	67108865	134217729
p	$\not\equiv 1$	3, 11	5	3, 3, 3, 3, 19
	$\equiv 1$	251, 4051	53, 157, 1613	87211

n		28	29	30
$2^n + 1$		268435457	536870913	1073741825
p	$\not\equiv 1$	17	3	5, 5, 13, 41
	$\equiv 1$	15790321	59, 3033169	61, 1321

n		31	32	33
$2^n + 1$		2147483649	4294967297	8589934593
p	$\not\equiv 1$	3		3, 3, 683
	$\equiv 1$	715827883	641, 6700417	67, 20857

The tables above illustrate the case of b = 2, now consider the case of b = 3, the tables are drawn up in the same way.

b = 3

n		1	2	3	4	5	6	7
$3^n + 1$		4	10	28	82	244	730	2188
p	$\not\equiv 1$	2, 2	2	2, 2	2	2, 2	2, 5	2, 2
	$\equiv 1$		5	7	41	61	73	547

n		8	9	10	11	12
$3^n + 1$		6562	19684	59050	177148	531442
p	$\not\equiv 1$	2	2, 2, 7	2, 5, 5	2, 2	2, 41
	$\equiv 1$	17, 193	19, 37	1181	67, 661	6481

n		13	14	15	16
$3^n + 1$		1594324	4782970	14348908	43046722
p	$\not\equiv 1$	2, 2	2, 5	2, 2, 7	2
	$\equiv 1$	398581	29, 16493	31, 61, 271	21523361

n		17	18	19
$3^n + 1$		129140164	387420490	11622261468
p	$\not\equiv 1$	2, 2	2, 5	2, 2
	$\equiv 1$	103, 307, 1021	73, 53713	2851, 101917

n		20	21	22
$3^n + 1$		3486784402	1046035204	31381059610
p	$\not\equiv 1$	2	2, 2, 7, 7	2, 5
	$\equiv 1$	41, 42521761	43, 547, 2269	5501, 570461

There is not much point in showing a table for b = 4, because it is essentially included in the data for b = 2. Therefore we continue with a table for b = 5.

b = 5

n		1	2	3	4	5	6
$5^n + 1$		6	26	126	626	3126	15626
p	$\not\equiv 1$	2	2	2, 3, 3	2	2, 3	2
	$\equiv 1$	3	13	7	313	521	13, 601

n	7	8	9	10
$5^n + 1$	78126	390626	1953126	9765626
p ≢ 1	2, 3	2	2, 3, 3, 3, 7	2, 13
p ≡ 1	29, 449	17, 11489	51467	41, 9161

n	11	12	13
$5^n + 1$	48828126	244140626	1220703126
p ≢ 1	2, 3	2	2, 3
p ≡ 1	23, 67, 5281	313, 390001	5227, 39823

n	14	15	16
$5^n + 1$	6103515626	30517578126	152587890626
p ≢ 1	2, 13	2, 3, 3, 7, 521	2
p ≡ 1	234750601	61, 7621	2593, 29423041

Finally, a few figures for b = 6, to show that it makes no difference if b should be a prime or a composite number.

n	7	8	9	10
$6^n + 1$	279937	1679617	10077697	60466177
p ≢ 1	7, 7		7, 31	37
p ≡ 1	29, 197	1679617	46441	241, 6781

n	11	12	13
$6^n + 1$	362797057	2176782337	13060694017
p ≢ 1	7		7
p ≡ 1	51828151	1297, 1678321	53, 937, 37571

PARADOX

This is about generalized functions. We start with a set of "test functions" such that every test function f(x) is very smooth and very small at infinity, and has a derivative f′(x) and a Fourier transform F(x), which are also test functions. An inner product $\langle f|g\rangle$ of test functions is the integral of the product in the sense of elementary calculus, there is no trouble about integrability or convergence. There is "weak convergence", with $f_n \to f$ when $\langle f_n - f|g\rangle \to 0$ for all test functions g.

The space of generalized functions is a completion under weak convergence. If f(x) is a generalized function then xf(x), f′(x), f(ax+b), and the Fourier transform F(x) may be defined in the obvious way, and they are all generalized functions (each is defined by the rule that gives its inner product with any test function). An "ordinary function" is any f(x), absolutely integrable in every finite interval, and small enough at infinity that its product with any test function is absolutely integrable. Every ordinary function f (it may be proved) can be regarded as a generalized function, its inner product with any test function being the integral of the product. But there are generalized functions not expressible as ordinary functions, for example Dirac's delta function $\delta(x)$. Its inner product with any test function f is easily described as f(0), but it is not the integral of anything times f(x) in the sense of ordinary integration.

Now for the difficulties. Consider $f(x) = \log|x|$, it is an ordinary function, therefore a generalized function, so that it has a Fourier transform $F(x)$.

Because $\int_{-\infty}^{\infty} \log|y|\, e^{-2\pi ixy}\, dy = 2\int_0^{\infty} \log y \cos 2\pi xy\, dy$

(taking $x > 0$) (putting $2\pi xy = \theta$) $= \frac{-1}{\pi x}\int_0^{\infty} \frac{\sin\theta}{\theta}\, d\theta = \frac{-1}{2x}$

it is tempting to think that $F(x) = -1/|2x|$. But there are objections, (a) we have used a non-absolute integral, (b) the function that we have found is not integrable in any interval containing the origin, and (c) the conclusion does not fit the rule that $f(x/k)$ (any $k > 0$) has transform $kF(kx)$.

On the other hand, there is something to be said for the suggested $-1/|2x|$, when it is multiplied by $2i\pi x$ it gives the function $-i\pi\,\mathrm{sgn}(x)$ which we know to be the transform of $1/x$. And $1/x$ is the derivative of $\log|x|$. The rule is that if f has transform F then $f'(x)$ has transform $2\pi ixF(x)$. If this $-1/|2x|$ means anything at all then it is partly right in the sense that when multiplied by x it gives the right answer. Could we put things right by adding a term that will give zero when multiplied by x? But the only generalized functions that give zero when multiplied by x are the multiples of $\delta(x)$. And $F(x) = -1/|2x| - C\delta(x)$ cannot be right because it fails the condition that $f(x/2)$ should have transform $2F(2x)$.

The use of the integral expression for the transform, as above, is not correct, though it often gives the right answer. The proper way is to use the integral expression only to

define the transform of a test function, and then the transform F of any generalized function f is defined by $\langle F|g\rangle = \langle f|G\rangle$ for any test function g with Fourier transform G.

There are general rules, if f is a generalized function then its derivative f', its transform F, and $xf(x)$ are defined by $\langle f'|g\rangle = -\langle f|g'\rangle$, $\langle F|g\rangle = \langle f|G\rangle$, and $\langle xf(x)|g\rangle = \langle f|xg(x)\rangle$, for any test function g. The Fourier transform of f' is $2\pi ixF(x)$, and of $xf(x)$ is $iF'(x)/(2\pi)$. Recall the theorem mentioned above about "ordinary" functions, it applies to $\log|x|$ but not to $1/x$ (which is not integrable in any interval containing the origin).

Now the paradox may be expressed a little differently. Let f be the generalized function given by the ordinary function $\log|x|$. It has a transform F.

The derivative $f'(x)$ is $1/x$, (as in M. J. Lighthill's 1958 book *An introduction to Fourier analysis and generalised functions*, this equation is taken as the definition of the generalized function $1/x$). Then $xf'(x) = 1$, which has transform $\delta(x)$. But f' has transform $2\pi ixF(x)$, and so $xf'(x)$ has transform $= -(d/dx)(xF(x))$. Therefore $(d/dx)(xF(x)) = -\delta(x)$. Because f and therefore F are even functions, $xF(x)$ is an odd function, and the only odd function with derivative $-\delta(x)$ is $xF(x) = (-1/2)\mathrm{sgn}(x)$. Solving this for F, we get

$$F(x) = (-1/2)/|x| + \text{some multiple of } \delta(x).$$

(Here the generalized function $1/|x|$ is defined as the derivative of the ordinary function sgn(x)log|x|)

Thus we find as before

$$F(x) = -\frac{1}{2|x|} - C\delta(x)$$

and the unknown constant C can be found numerically by taking the inner product with some convenient test function. Choose $S(x) = \exp(-\pi x^2)$ which is its own transform.

$$\langle f|S\rangle = \langle F|S\rangle = -\frac{1}{2}\langle 1/|x|\,|S(x)\rangle - C\langle\delta|S\rangle$$

$$= \frac{1}{2}\langle \log|x|\operatorname{sgn}(x)|S'(x)\rangle - C.$$

The inner product of an ordinary function with a test function is expressible as an integral, so that:

$$C = -\int_{-\infty}^{\infty}\log|x|S(x)dx + \frac{1}{2}\int_{-\infty}^{\infty}\log|x|\operatorname{sgn}(x)S'(x)\,dx$$

$$= -2\int_0^{\infty}\log x\ S(x)\ dx + \int_0^{\infty}\log x\ S'(x)\ dx$$

These integrals obviously exist, but evaluation is a little tricky (try them on your third year calculus class!). It helps to establish first the lemma:

$\int_0^{\infty}\log x\ e^{-x}\ dx = -\gamma$, where γ is Euler's constant, ·5772157.

The integrals can be found to be:

$$2\int_0^{\infty}\log x \exp(-\pi x^2)\ dx = -\gamma/2 - \frac{1}{2}\log(4\pi) = -1{\cdot}55412,$$

$$\int_0^{\infty}\log x\ (-2\pi x)\exp(-\pi x^2)\ dx = \gamma/2 + \frac{1}{2}\log\pi = {\cdot}860973,$$

and so $C = \gamma + \log(2\pi)$, and for our Fourier transform we get

$$F(x) = -\frac{1}{2|x|} - (\log(2\pi) + \gamma)\,\delta(x).$$

There remains the paradox that if k > 0 and f has Fourier transform F then f(x/k) should have transform kF(kx). In our case kF(kx) = F(x) but f(x/k) = f(x) - log k.

Now, the happy ending — Here is the explanation of the difficulties, thanks to T. C. S. Tao.

The trouble is with the 1/|x|. Is there any such thing? It is not an ordinary function, for it is not integrable at the origin. We may of course define it to be anything that we choose, but if we define it to be the derivative of the ordinary function (log x)(sgn x), then it means something rather strange. With this definition, we do not find the expected result that if k > 0 then (1/k)(1/|x|) = 1/|kx|. In fact 1/|kx| = (1/k)(1/|x|) - 2 log k δ(x). The definition of 1/x as the derivative of log|x| does not run into this kind of difficulty.

The best that we seem able to produce for the Fourier transform of log|x| is

$$-\frac{d}{dx}\frac{\log|x|\ \operatorname{sgn} x}{2} - (\log(2\pi) + \gamma)\ \delta(x).$$

Clumsy, probably useless, but true!

Note that M. J. Lighthill in his book chooses (*Definition 15*, p.39) to make 1/|x| a many-valued function; this avoids the paradox, but at the cost of introducing other difficulties. He gives (Example 21, p.39) the Fourier transform of log|x| to be equal to -1/(2|x|), but notes that this is true for only one of the possible values of 1/|x|.