1st Year Report

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1 Introduction

The main object of this report is to state and prove the Littlewood-Paley theorem, which is a powerful tool for determining the "size" of a function in L^p by breaking it into pieces in a particular way; consequently it is useful for studying the boundedness of operators on L^p spaces. We then apply this to prove the Hörmander multiplier theorem. In order to do this, we first give some results about maximal functions, and singular integral operators.

This standard material can all be found in [3], although we often favour the more modern presentation in [1].

2 Maximal functions

Given a locally integrable function $f : \mathbb{R}^n \to \mathbb{R}$, its **Hardy-Littlewood maximal function** Mf is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy.$$

Theorem 1. *[3, Ch I, Thm 1]*

(a) The operator M is weak-type $(1,1)$, i.e.

$$|\{x : Mf(x) > \alpha\}| \leq \frac{A_n}{\alpha} \int_{\mathbb{R}^n} |f(x)| dx.$$

(b) For $1 < p \leq \infty$, the operator M is strong (p,p), i.e.

$$\|Mf\|_p \leq A_{p,n} \|f\|_p .$$

Proof. We begin with *(a)*, letting $E_\alpha = \{x : Mf(x) > \alpha\}$. For $x \in E_\alpha$, we can choose $r > 0$ so that $B_x = B(x,r)$ satisfies

$$\frac{1}{|B_x|}\int_{B_x} |f(y)|\,dy > \alpha, \tag{2.1}$$

otherwise we cannot have $Mf(x) > \alpha$. It follows that $|B_x| < \frac{1}{\alpha}\|f\|_1$ for $x \in E_\alpha$. Now $E_\alpha \subseteq \cup_{x\in E_\alpha} B_x$, and using a covering lemma (Appendix, Theorem 11) we can extract a sequence of mutually disjoint balls, $\{B_k\}$ satisfying

$$\sum_{k=0}^{\infty} |B_k| \geq C|E_\alpha|, \qquad \text{(e.g. } C = 3^{-n}\text{)}. \tag{2.2}$$

Applying (2.1) followed by (2.2), we have

$$\int_{\cup B_k} |f(y)|\,dy > \alpha \sum_k |B_k| \geq \alpha C |E_\alpha|.$$

Since $\|f\|_1 \geq \int_{\cup B_k} |f(y)|\,dy$, we obtain *(a)*.

To prove *(b)*, note that this is obvious when $p = \infty$ since

$$Mf(x) \leq \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} \|f\|_\infty\,dy = \|f\|_\infty.$$

So M is weak-type (∞,∞) by definition, since it is strong (∞,∞). This allows us to apply Marcinkiewicz interpolation (Appendix, Theorem 12), establishing *(b)* for $1 < p < \infty$. □

The following bound involving the maximal function will later prove to be useful.

Theorem 2. *[1, Prop 2.7] Let ϕ be a function which is positive, radial, decreasing and integrable. Then with $\phi_t(x) = t^{-n}\phi(x/t)$,*

$$\sup_{t>0} |\phi_t * f(x)| \leq \|\phi\|_1 Mf(x).$$

Proof. First we assume that ϕ is a simple function, i.e. $\phi(x) = \sum_j a_j \chi_{B(0,r_j)}(x)$ with $a_j > 0$ since ϕ is positive. Then

$$|\phi * f(x)| = \left|\sum_j a_j |B(0,r_j)| \frac{1}{B(0,r_j)} \chi_{B(0,r_j)} * f(x)\right| \leq \|\phi\|_1 Mf(x)$$

since $\|\phi\|_1 = \sum a_j |B(0,r_j)|$. An arbitrary function ϕ satisfying the hypotheses can be approximated by a sequence of simple functions which increase to it monotonically, so the estimate $|\phi * f(x)| \leq \|\phi\|_1 Mf(x)$ will hold. Since each ϕ_t is also positive, radial, decreasing and has the same L^1 norm as ϕ, we obtain the result. □

3 Singular integrals

3.1 The Calderón-Zygmund Theorem

Theorem 3. *[1, Thm 5.1] Let K be a tempered distribution on $\mathbb{R}^n$ which coincides with a locally integrable function on $\mathbb{R}^n \setminus \{0\}$, satisfying*

(i) $|\hat{K}(\xi)| \leq B$,

(ii) $\int_{|x|>2|y|} |K(x-y) - K(x)| dx \leq B, y \in \mathbb{R}^n$.

Then the operator $T : f \mapsto K * f$ is weak-type $(1,1)$ and strong (p,p), $1 < p < \infty$, i.e. we have

$$\|Tf\|_p \leq A \|f\|_p$$

where the constant A depends only on p, B and the dimension n.

Remark. Property (*ii*) is known as the **Hörmander condition**, and may be deduced from the stronger condition $|\nabla K(x)| \leq C|x|^{-n-1}$, via the mean value theorem.

Proof. That T is bounded on L^2 follows from property (*i*). We obtain the weak-type $(1,1)$ result using (*ii*) as described below, then apply Marcinkiewicz interpolation to get the strong (p,p) result for $1 < p < 2$. For $2 < p < \infty$, we use duality; the adjoint operator T^* has kernel $K(-x)$ which satisfies the conditions of the theorem.

To obtain the weak-type $(1,1)$ result, we form the Calderón-Zygmund decomposition at height α, giving $\mathbb{R}^n = F \cup \cup_{j=1}^{\infty} Q_j$ and $f = g + b$ (see Appendix, Theorem 13). The problem of showing $|\{x : |Tf(x)| > \alpha\}| \leq \frac{C}{\alpha} \|f\|_1$ then reduces to estimating each of Tg and Tb in this way.

For Tg, we use the properties of the Calderón-Zygmund decomposition to get $\|g\|_2^2 \leq C\alpha \|f\|_1$. Since T is bounded on L^2, we have the weak $(2,2)$ inequality hence

$$|\{x : |Tg(x)| > \alpha\}| \leq \frac{B^2}{\alpha^2} \|g\|_2^2 \leq \frac{C}{\alpha} \|f\|_1 .$$

For Tb, we consider the cubes Q_j^* having the same centre c_j as Q_j, but expanded $2n^{1/2}$ times. Setting $b_j = b\chi_{Q_j}$ we have $Tb(x) = \sum_j Tb_j(x)$ where

$$Tb_j(x) = \int_{Q_j} K(x-y) b_j(y)\, dy = \int_{Q_j} \left(K(x-y) - K(x-c_j)\right) b_j(y)\, dy$$

since by construction, $\int_{Q_j} b = 0$. Hence with $F^* = (\cup Q_j^*)^c$,

$$\begin{aligned}
\int_{F^*} |Tb(x)|\, dx &\leq \sum_j \int_{F^*} |Tb_j(x)|\, dx \\
&\leq \sum_j \int_{x \notin Q_j^*} |Tb_j(x)|\, dx \\
&\leq \sum_j \int_{x \notin Q_j^*} \int_{Q_j} \left|K(x-y) - K(x-c_j)\right| |b_j(y)|\, dy\, dx.
\end{aligned}$$

Now if $x \notin Q_j^*$, $y \in Q_j$, simple geometry shows $|x - c_j| \geq 2|y - c_j|$. Defining $x' = x - c_j$, $y' = y - c_j$, this gives

$$\int_{x \notin Q_j^*} \int_{Q_j} \left|K(x-y) - K(x-c_j)\right|\, dx \leq \int_{|x'|>2|y'|} |K(x'-y') - K(x')| dx' \leq B$$

due to (*ii*). Thus

$$\int_{F^*} |Tb(x)|\, dx \leq B \sum_j \int_{Q_j} |b(y)|\, dy \leq C \left\|f\right\|_1, \tag{3.1}$$

where the last inequality comes from using the properties of the Calderón-Zygmund decomposition. So

$$|\{x : |Tb(x)| > \tfrac{\alpha}{2}\}| \leq |\{x \in F^* : |Tb(x)| > \tfrac{\alpha}{2}\}| + |(F^*)^c| \leq \frac{c}{\alpha} \left\|f\right\|_1,$$

where the first term is bounded due to (3.1), and for the second term the bound comes from the definition of $(F^*)^c$ and a property of the Calderón-Zygmund decomposition.

Combining the bounds for Tg and Tb with the fact that $f = g + b$, we get the weak-type $(1,1)$ result for T. □

3.2 Truncated integrals

The **Hilbert transform** H is defined for $f \in \mathcal{S}(\mathbb{R})$ by

$$Hf(x) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|y|>\epsilon} \frac{f(x-y)}{y}\, dy.$$

This is not directly treated by the previous theorem, but we can replace the hypothesis $|\hat{K}(\xi)| \leq A$ in order to address this.

Theorem 4. *[3, Ch II, Thm 2] Suppose the kernel K satisfies the conditions* *(i)* $|K(x)| \le B|x|^{-n}, \quad |x| > 0,$

(ii) $\displaystyle\int_{R_1<|x|<R_2} K(x)\,dx = 0, \quad 0 < R_1 < R_2 < \infty,$

(iii) $\displaystyle\int_{|x|>2|y|} |K(x-y) - K(x)|\,dx \le B, \quad |y| > 0.$

For $f \in L^p(\mathbb{R}^n)$*,* $1 < p < \infty$*, and* $\epsilon > 0$*, let*

$$T_\epsilon(f)(x) = \int_{|y|>\epsilon} f(x-y)K(y)\,dy.$$

Then

$$\|T_\epsilon f\|_p \le A\,\|f\|_p$$

where the constant A *is independent of* f *and* ϵ*.*

Also, for each $f \in L^p(\mathbb{R}^n)$*,* $\lim_{\epsilon\to 0} T_\epsilon(f) = T(f)$ *exists in* L^p *norm; the operator* T *so defined is also strong* (p,p)*,* $1 < p < \infty$*.*

Proof. Defining $K_\epsilon(x) = K(x)\chi_{\{|x|\ge\epsilon\}}$, we have $K_\epsilon \in L^2(\mathbb{R}^n)$ and it can be checked that K_ϵ also satisfies conditions (*i*)-(*iii*), with bounds not greater than $C_n B$. Using these properties, it can be shown [3, II §3.3] that for $\epsilon > 0$,

$$\sup_y |\hat{K}_\epsilon(y)| \le C_n B.$$

Thus we can apply Theorem 3 to get the strong (p,p) result for T_ϵ.

Now write $f \in L^p$ as $f = f_1 + f_2$; with $f_1 \in C^1$ having compact support, we can take $\|f_2\|_p$ as small as we please. We have that $\{T_\epsilon f_1\}_{\epsilon>0}$ is Cauchy in L^p since (if $\epsilon < \epsilon' < 1$)

$$T_\epsilon f_1(x) - T_{\epsilon'} f_1(x) = \int_{\epsilon\le|y|\le\epsilon'} K(y)(f_1(x-y) - f_1(x))\,dy$$

where the extra $f_1(x)$ is introduced thanks to (*ii*). Note that since f_1 has compact support, this is supported on a fixed compact set S. Now by the fact that f_1 is differentiable, and applying (*i*), we have that on S,

$$\int_{\epsilon\le|y|\le\epsilon'} |K(y)| \left|\frac{f_1(x-y) - f_1(x)}{y}\right| |y|\,dy \le AB \int_{\epsilon\le|y|\le\epsilon'} |y|^{-n+1}\,dy = C_{\epsilon,\epsilon'}$$

where $C_{\epsilon,\epsilon'} \to 0$ as $\epsilon, \epsilon' \to 0$. Hence

$$\|T_\epsilon f_1 - T_{\epsilon'} f_1\|_p \le \left(\int_S C^p_{\epsilon,\epsilon'}\right)^{1/p} \to 0 \quad \text{as } \epsilon, \epsilon' \to 0.$$

Finally, because $\|T_\epsilon f_2\|_p \le A\,\|f_2\|_p$ is as small as we please, we get that $\{T_\epsilon f\}_{\epsilon>0}$ is Cauchy in L^p; we call the limit of this sequence Tf, and it is clear that $\|Tf\|_p \le A\,\|f\|_p$. □

Remark. We see that this applies to the Hilbert transform, by considering the kernel $K(x) = \frac{1}{\pi x}$ for $x \in \mathbb{R}^1$. We have:

(i) $|K(x)| = \frac{1}{\pi|x|} = B|x|^{-1}$,

(ii) $|x| \geq 2|y| \implies |x-y| \geq \frac{1}{2}|x|$, so

$$\int_{|x|\geq 2|y|} \frac{1}{\pi}\left|\frac{1}{x-y} - \frac{1}{x}\right| dx \leq 2\frac{|y|}{\pi}\int_{|x|\geq 2|y|} |x|^{-2}\, dx \leq B$$

(iii) $\int_{R_1<|x|<R_2} \frac{1}{\pi x}\, dx = 0$ since the integrand is odd.

3.3 Homogeneous kernels

By virtue of the convolution in their definition, the operators which we have been considering commute with translations. We now look at operators which also commute with dilations; i.e. setting $\tau_\epsilon f(x) = f(\epsilon x)$, we require $\tau_{\epsilon^{-1}} T \tau_\epsilon = T$. In terms of the kernel K corresponding to T, this requirement becomes $K(\epsilon x) = \epsilon^{-n} K(x)$ for $\epsilon > 0$. Such K are said to be **homogeneous of degree** $-n$, and may be written as

$$K(x) = \frac{\Omega(x)}{|x|^n}$$

with Ω homogeneous of degree 0, i.e. $\Omega(\epsilon x) = \Omega(x)$, so that Ω is determined by its restriction to the unit sphere S^{n-1}.

Theorem 5. *[3, Ch II, Thm 3] Let Ω be homogeneous of degree 0, and suppose*

(i) $\displaystyle\int_{S^{n-1}} \Omega(x)\, d\sigma = 0,$

(ii) $\displaystyle\int_0^1 \frac{\omega(\delta)}{\delta}\, d\delta < \infty, \quad$ *where* $\quad \displaystyle\omega(\delta) = \sup_{\substack{|x-x'|\leq\delta \\ |x|=|x'|=1}} |\Omega(x) - \Omega(x')|.$

For $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, and $\epsilon > 0$, let

$$T_\epsilon(f)(x) = \int_{|y|>\epsilon} \frac{\Omega(y)}{|y|^n} f(x-y)\, dy.$$

Then

(a) $\|T_\epsilon f\|_p \leq A_p \|f\|_p$, *for some constant A_p independent of f or ϵ.*

(b) $\lim_{\epsilon\to 0} T_\epsilon f = Tf$ *exists in* L^p *norm, and* $\|Tf\|_p \leq A_p \|f\|_p$.

(c) If $f \in L^2(\mathbb{R}^n)$ *then* $\widehat{(Tf)}(x) = m(x)\hat{f}(x)$, *with* m *homogeneous of degree* 0, *and given for* $|x| = 1$ *by*

$$m(x) = \int_{S^{n-1}} \left(\frac{\pi i}{2} \operatorname{sgn}(x \cdot y) + \log\left(\frac{1}{|x \cdot y|} \right) \right) \Omega(y)\, d\sigma(y). \tag{3.2}$$

Proof. Conclusions (*a*) and (*b*) follow immediately from Theorem 4 once we show that the conditions on Ω translate into the proper conditions on $K(x) = \Omega(x)/|x|^n$; for details, see [3, Ch II, §4.2].

Part (*c*) comes from some detailed calculations; see [3, Ch II, §4.3]. □

Example. The **Riesz transforms** R_j are given by the kernels

$$K_j(x) = \frac{\Omega_j(x)}{|x|^n}, \quad \text{where } \Omega_j(x) = c_n \frac{x_j}{|x|}, \quad j = 1, \ldots, n.$$

The Ω_j clearly satisfy the conditions of Theorem 5, so the Riesz transforms are bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

3.4 Vector-valued analogues

We remark briefly that the preceding results can be generalised to deal with functions which take values in a Hilbert space; it is this form of the theory which will be put to use in the following section. For details, see [3, Ch II §5].

4 Littlewood-Paley Theory

Theorem 6. *[1, Thm 8.6] Given* $\psi \in \mathcal{S}(\mathbb{R}^n)$ *with* $\psi(0) = 0$, *let* S_j *be the operator defined by* $\widehat{(S_j f)}(\xi) = \psi_j(\xi)\hat{f}(\xi)$ *where* $\psi_j(\xi) = \psi(2^{-j}\xi)$. *Then for* $1 < p < \infty$,

$$\left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_p \leq C_p \|f\|_p .$$

Furthermore, if for all $\xi \neq 0$ *we have* $\sum_j |\psi(2^{-j}\xi)|^2 = C$, *then also*

$$\|f\|_p \leq C'_p \left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_p .$$

Proof. We apply a vector-valued analogue of the singular integral theory to the operator $f \mapsto \{S_j f\}$. It is bounded from L^2 to $L^2(\ell^2)$ since

$$\left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_2^2 = \int_{\mathbb{R}^n} \sum_j |\psi_j(\xi)|^2 |\hat{f}(\xi)|^2 \, d\xi \leq C \, \|f\|_2^2 \tag{4.1}$$

by Plancherel and the fact that $\sum_j |\psi_j(\xi)|^2 \leq C$ since $\psi \in \mathcal{S}$ and $\psi(0) = 0$.

Boundedness on other L^p comes from checking that the Hörmander condition is satisfied, for which it suffices to show $\left\| \nabla \check{\psi}_j(x) \right\|_{\ell^2} \leq C|x|^{-n-1}$. This can be achieved using the fact that $\psi \in \mathcal{S}$.

Hence, for $1 < p < \infty$, we have the first part of the theorem,

$$\left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_p \leq C_p \, \|f\|_p \, .$$

For the second part of the theorem, note that if we have $\sum_j |\psi(2^{-j}\xi)|^2 = C$ then (4.1) in fact improves to the equality

$$\left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_2 = \sqrt{C} \, \|f\|_2 \, .$$

It follows by polarization that

$$\int_{\mathbb{R}^n} \sum_j S_j f \overline{S_j g} = \sqrt{C} \int_{\mathbb{R}^n} f \overline{g}$$

and taking $f \in L^p$, $g \in L^{p'}$ we apply Cauchy-Schwarz to the summation, followed by Hölder's inequality, to get

$$\begin{aligned} \left| \int f\overline{g} \right| &\leq C' \int \left(\sum_j |S_j f|^2 \right)^{1/2} \left(\sum_j |S_j g|^2 \right)^{1/2} \\ &\leq C' \left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_p \left\| \left(\sum_j |S_j g|^2 \right)^{1/2} \right\|_{p'} \\ &\leq C' \left\| \left(\sum_j |S_j f|^2 \right)^{1/2} \right\|_p \|g\|_{p'} \end{aligned}$$

by the first part of the theorem, applied to g. Now taking the supremum over $g \in L^{p'}$ with $\|g\|_{p'} \leq 1$ gives the desired result. □

5 Multipliers

Given $m \in L^\infty(\mathbb{R}^n)$, we define the linear transformation T_m on $L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ by

$$\widehat{(T_m f)}(x) = m(x)\hat{f}(x).$$

We say that m is a **multiplier for** L^p if, for all $f \in L^2 \cap L^p$, $T_m f$ is in L^p ($T_m f \in L^2$ is automatic) and T_m is bounded, i.e.

$$\|T_m f\|_p \leq A \|f\|_p . \tag{5.1}$$

In this case, T_m has a unique bounded extension to L^p, which we also call T_m. The class of all multipliers for L^p is denoted $\mathcal{M}_p$, with the norm of $m \in \mathcal{M}_p$ being the smallest possible A in (5.1).

Example. $\mathcal{M}_2 = L^\infty$. It is clear that every L^∞ function is a multiplier for L^2. For the converse, if $m \in \mathcal{M}_2$ then $\|T_m f\|_2 \leq A \|f\|_2$ for all $f \in L^2$. Applying Plancherel,

$$\int |m\hat{f}|^2 \leq \int |A\hat{f}|^2.$$

This implies $\|m\|_\infty \leq A$, for if $|m(x)| > A$ on a compact set E of positive measure, then taking $\hat{f} = \chi_E$ gives a contradiction.

Theorem 7. *[3, Ch IV, §3.1] If $\frac{1}{p} + \frac{1}{p'} = 1$, $1 \leq p \leq \infty$, then $\mathcal{M}_p = \mathcal{M}_{p'}$ with equality of norms.*

Proof. Let σ denote the involution $\sigma(f)(x) = \overline{f}(-x)$. We see that $\sigma^{-1} T_m \sigma = T_{\overline{m}}$, and since σ is an isometry of L^p this means that $\|m\|_{\mathcal{M}_p} = \|\overline{m}\|_{\mathcal{M}_p}$.

Now by Plancherel,

$$\int T_m f \overline{g} = \int \widehat{T_m f}\,\overline{\hat{g}} = \int \hat{f}\,\overline{\widehat{T_{\overline{m}} g}} = \int f \overline{T_{\overline{m}} g},$$

so

$$\begin{aligned} \|m\|_{\mathcal{M}_p} &= \sup_{\|f\|_p = \|g\|_{p'} = 1} \left| \int T_m f \overline{g} \right| \\ &= \sup_{\|f\|_p = \|g\|_{p'} = 1} \left| \int f \overline{T_{\overline{m}} g} \right| = \|\overline{m}\|_{\mathcal{M}_{p'}} . \end{aligned}$$

Combining this with $\|m\|_{\mathcal{M}_p} = \|\overline{m}\|_{\mathcal{M}_p}$ we have $\|m\|_{\mathcal{M}_p} = \|m\|_{\mathcal{M}_{p'}}$. □

For $1 \leq p < \infty$ and k a non-negative integer, the **Sobolev space** $L^p_k(\mathbb{R}^n)$ is defined as the space of functions $f \in L^p$ such that for all $|\alpha| \leq k$, $D^\alpha f$ exists in the weak sense, and lies in L^p. This can be made a normed space, with the norm

$$\|f\|_{L^p_k} = \sum_{|\alpha| \le k} \|D^\alpha f\|_p .$$

An alternative definition of L^2_a for general $a > 0$ is

$$L^2_a = \{g \in L^2 \, : \, (1 + |\xi|^2)^{a/2} \hat{g}(\xi) \in L^2\},$$

where the norm is $\|g\|_{L^2_a} = \left\|(1 + |\cdot|^2)^{a/2} \hat{g}\right\|_2$. For integer values of a, this is equivalent to the previous definition with $p = 2$ since

$$\begin{aligned}
\|f\|^2_{L^2_k} &\approx \sum_{|\alpha| \le k} \|D^\alpha f\|^2_2 \\
&= C \sum_{|\alpha| \le k} \left\|\xi^\alpha \hat{f}\right\|^2_2 \\
&= C \int_{\mathbb{R}^n} \left(\sum_{|\alpha| \le k} |\xi^{2\alpha}| \right) |\hat{f}(\xi)|^2 \, d\xi \\
&\approx \int_{\mathbb{R}^n} (1 + |\xi|^2)^k |\hat{f}(\xi)|^2 \, d\xi = \left\|(1 + |\cdot|^2)^{k/2} \hat{f}\right\|^2_2 .
\end{aligned}$$

The relevance of Sobolev spaces to the theory of multipliers is demonstrated by the following theorem.

Theorem 8. *[1, Prop 8.8] If $m \in L^2_a$ with $a > \frac{n}{2}$ then $m \in \mathcal{M}_p$ for $1 \le p \le \infty$.*

Proof. From the definition of T_m, we deduce $T_m f = K * f$ where $K = \check{m}$. So by Young's inequality,

$$\|T_m f\|_p \le \|K\|_1 \|f\|_p = \|\check{m}\|_1 \|f\|_p .$$

Since $m \in L^2_a$ we have $(1 + |\xi|^2)^{a/2} \hat{m}(\xi) = h(\xi) \in L^2$, so

$$\|\check{m}\|_1 = \|\hat{m}\|_1 = \left\| \frac{h(\xi)}{(1 + |\xi|^2)^{a/2}} \right\|_1 \le \|h\|_2 \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^{-a} \, d\xi \right) .$$

Since $a > \frac{n}{2}$, the integral is finite. We also have $\|h\|_2 = \|m\|_{L^2_a} < \infty$, so overall $\|\check{m}\|_1 < \infty$. Combining this with the previous inequality, we have $m \in \mathcal{M}_p$. □

5.1 The Hörmander multiplier theorem

Theorem 9. *[3, IV Thm 3] Suppose $m \in C^k(\mathbb{R}^n \setminus \{0\})$ for some integer $k > \frac{n}{2}$. If*

$$|D^\alpha m(x)| \le C|x|^{-|\alpha|} \qquad \text{for } |\alpha| \le k$$

then $m \in \mathcal{M}_p$ for all $1 < p < \infty$.

This is a consequence of the following result.

Theorem 10. *[1, Thm 8.10] Let $\psi \in C^\infty$ be a radial function supported on $\frac{1}{2} \leq |x| \leq 2$ satisfying*

$$\sum_{j=-\infty}^{\infty} |\psi(2^{-j}x)|^2 = 1, \quad x \neq 0.$$

If m is such that, for some $k > \frac{n}{2}$,

$$\sup_j \left\| m(2^j \cdot)\psi \right\|_{L^2_k} < \infty$$

then $m \in \mathcal{M}_p$ for all $1 < p < \infty$.

Remark. Letting $m_j = m(2^j \cdot)\psi$, we note that m_j is a dilate of $m\psi(2^{-j}\cdot)$, which is a "piece" of m supported where $|x| \approx 2^j$. In order to have $m \in \mathcal{M}_p$, it is not enough to have each $m_j \in \mathcal{M}_p$, but this result shows that the additional knowledge that each $m_j \in L^2_k$ (with norms uniformly bounded in j) is sufficient.

Proof. Defining the operators S_j by $\widehat{(S_j f)}(\xi) = \psi(2^{-j}\xi)\hat{f}(\xi)$, we can use Littlewood-Paley (Theorem 6) to get

$$\|Tf\|_p \leq C_p \left\| \left(\sum_j |S_j T f|^2 \right)^{1/2} \right\|_p . \tag{5.2}$$

Given another C^∞ function, $\tilde{\psi}$, supported on $\frac{1}{4} \leq |\xi| \leq 4$ and equal to 1 on $\operatorname{supp} \psi$, we define the operators $\tilde{S}_j$ by $\widehat{(\tilde{S}_j f)}(\xi) = \tilde{\psi}(2^{-j}\xi)\hat{f}(\xi)$ and again apply Theorem 6 to get

$$\left\| \left(\sum_j |\tilde{S}_j f|^2 \right)^{1/2} \right\|_p \leq C_p \|f\|_p . \tag{5.3}$$

Now since $S_j T \tilde{S}_j = S_j T$, (5.2) becomes

$$\|Tf\|_p \leq C_p \left\| \left(\sum_j |S_j T \tilde{S}_j f|^2 \right)^{1/2} \right\|_p . \tag{5.4}$$

The multiplier associated with $S_j T$ is $m_j = \psi(2^{-j}\cdot)m(\cdot)$, and by hypothesis there is a $k > \frac{n}{2}$ for which this multiplier is an L^2_k function for each j. Thus

$$\begin{aligned}
\int_{\mathbb{R}^n} |S_j T f|^2 u &= \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \check{m}_j(x-y)(1+|x-y|^2)^{k/2} \frac{f(y)}{(1+|x-y|^2)^{k/2}}\, dy \right|^2 u(x)\, dx \\
&\leq \|m_j\|^2_{L^2_k} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} u(x) \frac{|f(y)|^2}{(1+|x-y|^2)^k}\, dy\, dx \\
&\leq C_k \int_{\mathbb{R}^n} |f(y)|^2 Mu(y)\, dy \qquad (5.5)
\end{aligned}$$

where we have used the Cauchy-Schwarz inequality and the definition of the L^2_k norm, followed by an application of Theorem 2 with $\phi(x) = (1 + |x|^2)^{-k}$. Note that the bound obtained is independent of j. Now for $p > 2$,

$$\left\| \left(\sum_j |S_j T f_j|^2 \right)^{1/2} \right\|_p^2 = \left\| \sum_j |S_j T f_j|^2 \right\|_{p/2} = \int_{\mathbb{R}^n} \sum_j |S_j T f_j|^2 u$$

for some $u \in L^{(p/2)'}$ with norm 1, by Riesz Representation (e.g. [2, §6.4.8]). Applying (5.5) to this, followed by Hölder's inequality,

$$\begin{aligned}
\left\| \left(\sum_j |S_j T f_j|^2 \right)^{1/2} \right\|_p^2 &\leq C \int_{\mathbb{R}^n} \sum_j |f_j|^2 Mu \\
&\leq C \left\| \sum_j |f_j|^2 \right\|_{p/2} \|Mu\|_{(p/2)'} \\
&\leq C' \left\| \sum_j |f_j|^2 \right\|_{p/2} \qquad (5.6)
\end{aligned}$$

since (by Theorem 1) M is strong (p, p) for $1 < p < \infty$ and $(p/2)'$ lies in this range; also, $\|u\|_{(p/2)'} = 1$.

Finally, we return to (5.4) and apply (5.6) followed by (5.3), giving

$$\|Tf\|_p \leq C' \left\| \left(\sum_j |\tilde{S}_j f|^2 \right)^{1/2} \right\|_p \leq C_p \|f\|_p$$

for $p > 2$. For $p < 2$ the result follows by duality since the adjoint T^* is associated with the multiplier $m(-\cdot)$ and hence is bounded as above; the result for $p = 2$ follows by interpolation. □

It remains to see how Theorem 9 is deduced from this result.

Proof of Theorem 9. By the definition of the Sobolev norm, we have

$$\left\| m(2^j\cdot)\psi \right\|_{L^2_k} = \sum_{|\beta|\le k} \left\| D^\beta(m(2^j\cdot)\psi) \right\|_2 .$$

Using Leibniz's rule to expand the D^β terms,

$$D^\beta(m(2^j\cdot)\psi) = \sum_{|\gamma|\le|\beta|} C_{\gamma,\beta} D^\gamma m(2^j\cdot) D^{\beta-\gamma}\psi$$

hence

$$\left\| D^\beta(m(2^j\cdot)\psi) \right\|_2 \le \sum_{|\gamma|\le|\beta|} C'_{\gamma,\beta} \left\| D^\gamma m(2^j\cdot)\chi_{\{\frac{1}{2}<|x|<2\}} \right\|_2$$

since the definition of ψ gives $|D^{\beta-\gamma}\psi| \le C$ on $\frac{1}{2} < |x| < 2$ and $= 0$ otherwise. Now using the chain rule, and the hypothesis of Theorem 9,

$$|(D^\gamma m(2^j\cdot))(x)| = 2^{j|\gamma|}|(D^\gamma m)(2^j x)| \le C|x|^{-|\gamma|}$$

hence $\left\| D^\gamma m(2^j\cdot)\chi_{\{\frac{1}{2}<|x|<2\}} \right\|_2 \le C_{|\gamma|}$. Putting this together,

$$\left\| m(2^j\cdot)\psi \right\|_{L^2_k} \le \sum_{|\beta|\le k} \sum_{|\gamma|\le|\beta|} C'_{\gamma,\beta} C_{|\gamma|} = C_k$$

and since this is independent of j we have $\sup_j \left\| m(2^j\cdot)\psi \right\|_{L^2_k} < \infty$ so that Theorem 10 can be applied to give the result. □

6 Appendix

Theorem 11 (Covering Lemma)**.** *[3, Ch I, §1.6] Suppose the measurable set $E \subseteq \mathbb{R}^n$ is covered by the union of a family of balls $\{B_j\}$ whose diameters are bounded. Then we can select a disjoint subsequence $B_1, B_2, \ldots$ (finite or infinite) so that*

$$\sum_k |B_k| \ge C|E|$$

with C a positive constant depending only on the dimension n; e.g. $C = 3^{-n}$ suffices.

Proof. Choose B_1 as large as possible, i.e. $\operatorname{diam} B_1 \ge \frac{1}{2}\sup\{\operatorname{diam} B_j\}$, and proceed like this; if $B_1, \ldots, B_k$ are chosen, then choose B_{k+1} from those B_j disjoint from $B_1, \ldots, B_k$, and with $\operatorname{diam} B_{k+1} \ge \frac{1}{2}\sup\{\operatorname{diam} B_j\}$.

If $\sum |B_k| = \infty$ then we are done. Otherwise, we claim

$$E \subset \cup_k 3B_k, \qquad 3B_k = \text{concentric with } B_k\text{, but } 3\times \text{ diameter.}$$

We need to see that for each j, $B_j \subset \cup_k 3B_k$. This is certainly true of the B_k, so suppose B_j is not one of them.

Since $\sum |B_k| < \infty$, diam $B_k \to 0$ so we can find the smallest k s.t. diam $B_{k+1} < \frac{1}{2}$ diam B_j. Then B_j must intersect one of $B_1, \ldots, B_k$, otherwise it would have been picked as B_{k+1}. Say it intersects B_{j_0} where $1 \leq j_0 \leq k$; then by simple geometry, $B_j \subset 3B_{j_0}$. This shows the claim, and the result follows. □

Theorem 12 (Marcinkiewicz interpolation)**.** *[1, p29] Let (X, μ), (Y, ν) be measure spaces, and $1 \leq p_0 < p_1 \leq \infty$.*

Let T be a mapping from $L^{p_0} + L^{p_1}$ to the measurable functions on Y which is sublinear, i.e.

$$|T(f+g)(x)| \leq |Tf(x)| + |Tg(x)|, \quad |T(\lambda f)(x)| \leq |\lambda||Tf(x)|.$$

If T is weak (p_0, p_0) and (p_1, p_1), then T is strong (p, p) for all $p_0 < p < p_1$.

Proof. Given $f \in L^p$ and $\lambda > 0$, put $f_0 = f\chi_{|f|>c\lambda}$, $f_1 = f\chi_{|f|\leq c\lambda}$ for some constant c to be fixed below. We have $f_0 \in L^{p_0}$, $f_1 \in L^{p_1}$, and since $|Tf(x)| \leq |Tf_o(x)| + |Tf_1(x)|$,

$$|\{x : |Tf(x)| > \lambda\}| \leq |\{x : |Tf_0(x)| > \tfrac{\lambda}{2}\}| + |\{x : |Tf_1(x)| > \tfrac{\lambda}{2}\}|. \quad (6.1)$$

Now suppose $p_1 = \infty$. We have $\|Tg\|_\infty \leq A_1 \|g\|_\infty$. Choose $c = \frac{1}{2A_1}$, so $\|f_1\|_\infty \leq c\lambda \leq \frac{\lambda}{2A_1}$. Since $|Tf_1(x)| \leq \|Tf_1\|_\infty \leq A_1 \|f\|_\infty \leq \frac{\lambda}{2}$, we have $|\{x : |Tf_1(x)| > \frac{\lambda}{2}\}| = 0$. Putting this, and the weak (p_0, p_0) inequality, into (6.1) gives

$$\{x : |Tf(x)| > \lambda\}| \leq \left(\frac{2A_0}{\lambda} \|f_0\|_{p_0}\right)^{p_0}.$$

Hence

$$\begin{aligned}
\|Tf\|_p^p &= p \int_0^\infty \lambda^{p-1} |\{x : |Tf(x)| > \lambda\}| \, d\lambda \\
&\leq p(2A_0)^{p_0} \int_0^\infty \lambda^{p-1-p_0} \int_{|f|>c\lambda} |f(x)|^{p_0} \, dx \, d\lambda \\
&= p(2A_0)^{p_0} \int_X |f(x)|^{p_0} \int_0^{|f|/c} \lambda^{p-1-p_0} \, d\lambda \, dx \\
&= p(2A_0)^{p_0} \int_X |f(x)|^{p_0} \left(\frac{|f(x)|}{c}\right)^{p-p_0} \frac{1}{p-p_0} \, dx \\
&= \frac{p}{p-p_0} (2A_0)^{p_0} (2A_1)^{p-p_0} \|f\|_p^p.
\end{aligned}$$

For $p_1 < \infty$, we use a similar argument but with (6.1) bounded using both weak-type inequalities, i.e.

$$|\{x : |Tf(x)| > \lambda\}| \leq \left(\frac{2A_0}{\lambda} \|f_0\|_{p_0}\right)^{p_0} + \left(\frac{2A_1}{\lambda} \|f_1\|_{p_1}\right)^{p_1}.$$

□

Theorem 13 (Calderón-Zygmund decomposition)**.** *[3, Ch I, §3.2-4] Let f be a non-negative integrable function on $\mathbb{R}^n$, and let α be a positive constant. Then there exists a decomposition of $\mathbb{R}^n$ so that*

(i) $\mathbb{R}^n = F \cup \Omega$, $F \cap \Omega = \emptyset$,

(ii) $f(x) \leq \alpha$ *for almost all* $x \in F$,

(iii) $\Omega = \cup_k Q_k$ *where the* Q_k *are cubes with disjoint interiors,*

(iv) $|\Omega| \leq \frac{A}{\alpha} \|f\|_1$,

(v) *for each cube* Q_k, $\frac{1}{|Q_k|} \int_{Q_k} f(x)\, dx \leq B\alpha$,

where A, B *are constants depending only on* n.

We can then decompose f *as* $f(x) = g(x) + b(x)$ *where* g *is defined almost everywhere by*

$$g(x) = \begin{cases} f(x) & x \in F, \\ \frac{1}{|Q_j|} \int_{Q_j} f(x)\, dx & x \in Q_j^0. \end{cases}$$

This gives $b(x) = 0$ *for* $x \in F$, *and* $\int_{Q_j} b(x)\, dx = 0$ *for each cube* Q_j.

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