ANALYSIS CLUB

Bochner-Riesz Multipliers

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To determine when we have the Fourier inversion formula,

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$$

we consider the partial sum operators S_R,

$$S_R f(x) = \int_{|\xi| \leq R} \widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$$

and then by some functional analysis we have that $\lim_{R\to\infty} S_R f = f$ in L^p if and only if

$$\|S_R f\|_p \lesssim \|f\|_p,$$

so we have turned the problem into studying the boundedness of certain operators on L^p spaces. In this case, we have a complete answer:

- for $n = 1$, we can write S_R in terms of the Hilbert transform, and obtain L^p-boundedness for $1 < p < \infty$.
- for $n \geq 2$, Fefferman [3] showed the S_R are only bounded on L^2.

We now consider the family of operators T_λ, $\lambda \geq 0$, defined on $\mathbb{R}^n$ by

$$\widehat{T_\lambda f}(\xi) = m_\lambda \hat{f}(\xi), \text{ where } m_\lambda(\xi) = (1 - |\xi|^2)_+^\lambda.$$

These are the **Bochner-Riesz multipliers**, which can be viewed as an attempt to smooth out the singularity of the disc multiplier to see if we can obtain boundedness on a wider range of L^p spaces. Note that when $\lambda = 0$ we obtain S_1, and that as λ increases, the multiplier m_λ becomes smoother hence more likely to produce a bounded operator.

The main references for the following discussion are [2, Ch 8, §5] and [1, pp143-157].

1 Useful tools

The following is a useful "duality" result, which allows us to consider only $p < 2$ or $p > 2$ as necessary.

Theorem 1. *[5, Ch IV, §3.1] If $\frac{1}{p} + \frac{1}{p'} = 1$, $1 \leq p \leq \infty$, then $\mathcal{M}_p = \mathcal{M}_{p'}$ with equality of norms.*

Proof. Let σ denote the involution $\sigma(f)(x) = \overline{f}(-x)$. We see that $\sigma^{-1}T_m\sigma = T_{\overline{m}}$, and since σ is an isometry of L^p this means that $\|m\|_{\mathcal{M}_p} = \|\overline{m}\|_{\mathcal{M}_p}$.

Now by Plancherel, and the definition of T_m,

$$\int T_m f \overline{g} = \int \widehat{T_m f} \hat{\overline{g}} = \int \hat{f} \overline{\widehat{T_{\overline{m}} g}} = \int f \overline{T_{\overline{m}} g},$$

so

$$\begin{aligned} \|m\|_{\mathcal{M}_p} &= \sup_{\|f\|_p = \|g\|_{p'} = 1} \left| \int T_m f \overline{g} \right| \\ &= \sup_{\|f\|_p = \|g\|_{p'} = 1} \left| \int f \overline{T_{\overline{m}} g} \right| = \|\overline{m}\|_{\mathcal{M}_{p'}}. \end{aligned}$$

Combining this with $\|m\|_{\mathcal{M}_p} = \|\overline{m}\|_{\mathcal{M}_p}$ we have $\|m\|_{\mathcal{M}_p} = \|m\|_{\mathcal{M}_{p'}}$. □

We will also need to make use of interpolation. Generally it will be enough to use Riesz-Thorin or Marcinkiewicz interpolation, but we note the following "complex interpolation" result due to Stein [4].

Theorem 2. *For a nice[1] family of operators T_z, $0 \leq \operatorname{Re} z \leq 1$, suppose*

$$\left\| T_{iy} f \right\|_{q_0} \lesssim \|f\|_{p_0} \quad \text{and} \quad \left\| T_{1+iy} f \right\|_{q_1} \lesssim \|f\|_{p_1}.$$

Then for $0 < \theta < 1$ we have

$$\left\| T_{\theta+iy} f \right\|_q \lesssim \|f\|_p$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

[1] see [4] for the full details, or [2, pp22-23] for a summary.

2 Known results

Following [2, pp171-172], the kernel of T_λ is

$$K_\lambda(x) = \pi^{-\lambda}\Gamma(\lambda+1)|x|^{-\frac{n}{2}-\lambda}J_{\frac{n}{2}+\lambda}(2\pi|x|)$$

where J_μ is the Bessel function

$$J_\mu(t) = \frac{(\frac{t}{2})^\mu}{\Gamma(\mu+\frac{1}{2})\Gamma(\frac{1}{2})}\int_{-1}^{1} e^{its}(1-s^2)^{\mu-\frac{1}{2}}\,ds.$$

Applying the known behaviour $J_\mu = O(t^\mu)$ as $t \to 0$ and $J_\mu \approx t^{-1/2}$ as $t \to \infty$, we have

$$|K_\lambda(x)| \begin{cases} \leq C & \text{as } |x| \to 0 \\ \approx |x|^{-\left(\frac{n+1}{2}+\lambda\right)} & \text{as } |x| \to \infty. \end{cases}$$

Theorem 3. *If $\lambda > \frac{n-1}{2}$ then T_λ is bounded on all L^p.*

Proof. For $\lambda > \frac{n-1}{2}$, the bounds for K_λ above show that $K_\lambda \in L^1$. So by Young's inequality,

$$\|T_\lambda f\|_p = \|K_\lambda * f\|_p \leq \|K\|_1 \|f\|_p \lesssim \|f\|_p$$

i.e. T_λ is bounded on L^p. □

Theorem 4. *If $\left|\frac{1}{p}-\frac{1}{2}\right| \geq \frac{2\lambda+1}{2n}$ then T_λ is unbounded.*

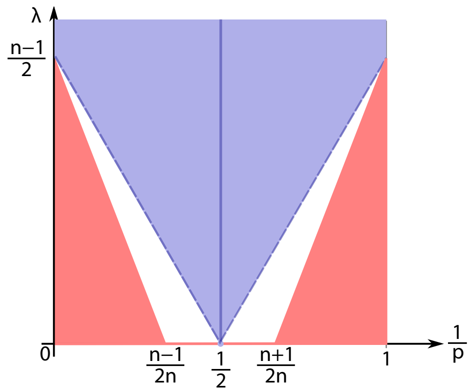
Proof. Now if $m \in \mathcal{M}_p$ has compact support, it follows[2] that $\hat{m} \in L^p$. This shows that a necessary condition for T_λ to be bounded on L^p is that $K_\lambda \in L^p$.

Again using the bounds on K_λ, we have $K_\lambda \in L^p$ only if $p\left(\frac{n+1}{2}+\lambda\right) > n$.

By duality, this becomes $\left|\frac{1}{p}-\frac{1}{2}\right| \geq \frac{2\lambda+1}{2n}$. □

Remark. We can interpolate the result for $\lambda > \frac{n-1}{2}$ with the disc multiplier result to get a whole region of boundedness, shaded in blue, as well as the region of unboundedness in red.

[2]Choose $f \in \mathcal{S}$ such that $\hat{f} = 1$ on the support of m. Then $f \in L^p$, so $T_m f \in L^p$ by assumption. But $\widehat{T_m f} = m\hat{f} = m$, so $\check{m} = T_m f \in L^p$ hence $\hat{m} \in L^p$.



We now essentially re-prove the result about boundedness; but the method of proof is interesting.

Theorem 5. *T_λ is bounded on L^p when $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{\lambda}{n-1}$.*

Proof. We follow the proof in [2, pp170-171], with some modifications.

Decompose T_λ on dyadic anulli as follows. Take a partition of unity ϕ_k subordinate to the open cover $(1 - 2^{-k+1}, 1 - 2^{-k-1})$ of $[0, 1]$ [3], so that

$$(1 - |\xi|^2)_+^\lambda = \sum_{k=0}^{\infty} (1 - |\xi|^2)^\lambda \phi_k(|\xi|).$$

Now, ϕ_k is supported near $1 - 2^{-k}$ (on an annulus of width $\sim 2^{-k}$), where we have

$$(1 - |\xi|^2)^\lambda = \Big((1 + |\xi|)(1 - |\xi|)\Big)^\lambda \sim (2 \times 2^{-k})^\lambda \sim 2^{-k\lambda}.$$

We define $\widetilde{\phi}_k(|\xi|) = 2^{k\lambda}(1 - |\xi|^2)^\lambda \phi_k(|\xi|)$ so that $\widetilde{\phi}_k \lesssim 1$, and then

$$T_\lambda f = \sum_{k=0}^{\infty} 2^{-k\lambda} T_k f$$

where T_k is the operator with multiplier $\widetilde{\phi}_k$.

We apply Minkowski's inequality to get

$$\|T_\lambda f\|_p \leq \sum_{k=0}^{\infty} 2^{-k\lambda} \|T_k f\|_p$$

and then estimating each of these norms, we will see that the series converges for the hypothesised range of λ and p.

[3] this only makes sense for $k \geq 1$, so we need to just add in the ϕ_0 manually

The estimate for these norms is produced in a Lemma in [2], but we get them via a slightly different method. There is the trivial L^2 boundedness, $\|T_k f\|_2 \lesssim_\lambda \|f\|_2$ using the fact that $\widetilde{\phi}_k \lesssim_\lambda 1$. Then by Young's inequality, for $q = 1, \infty$ we have

$$\|T_k f\|_q = \left\|\check{\widetilde{\phi}}_k * f\right\|_q \leq \left\|\check{\widetilde{\phi}}_k\right\|_1 \|f\|_q$$

so the problem reduces to estimating $\left\|\check{\widetilde{\phi}}_k\right\|_1$.

We do this by decomposing $\widetilde{\phi}_k$ smoothly in segments of the annulus; if the annulus is δ thick then all other dimensions of the segments are $\delta^{1/2}$. Each segment ν supports a piece ϕ_ν of $\widetilde{\phi}_k$.

- If ϕ_ν is one of the pieces, then $\|\check{\phi}_\nu\|_1 \lesssim 1$.

  Each piece has the same norm since they are all rotations of each other, so we may assume ϕ_ν is perpendicular to the ξ_1 axis; then ϕ_ν is a translate of $\Psi(\frac{\xi_1}{\delta}, \frac{\xi'}{\delta^{1/2}})$, for some fixed $\Psi \in \mathcal{S}$, hence

  $$\|\check{\phi}_\nu\|_1 = \|\hat{\phi}_\nu\|_1 = \left\|\widehat{\Psi(\frac{\xi_1}{\delta}, \frac{\xi'}{\delta^{1/2}})}\right\|_{L^1(\xi)} = \|\check{\Psi}\|_1 = C.$$

- Hence by the triangle inequality, $\left\|\check{\widetilde{\phi}}_k\right\|_1 \lesssim$ num. segments.

  Now each segment will have surface area $(\delta^{1/2})^{n-1}$, and since the radius is bounded by 1, the total surface area of the outside of the annulus is $O(1)$; hence there are $\approx \delta^{-(n-1)/2}$ segments, and $\delta \approx 2^{-k}$, giving $\left\|\check{\widetilde{\phi}}_k\right\|_1 \lesssim 2^{k(n-1)/2}$.

Finally, interpolation gives $\|T_k f\|_p \lesssim 2^{k\frac{n-1}{2}\left|\frac{2}{p}-1\right|} \|f\|_p$ and putting this into the summation we see that the geometric series converges if $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{\lambda}{n-1}$. □

There is another important known result, which uses restriction theory.

3 Using restriction estimates

There is an intimate connection[4] between estimates for Bochner-Riesz operators and estimates on the size of the Fourier transform of a function when

[4] See [8] for more on this.

restricted to a hypersurface (generally the sphere), i.e. "restriction theorems". This is illustrated in the proof of Theorem 7 below, which makes use of the following.

Theorem 6 (Tomas-Stein)**.** *For all* $1 \leq p \leq \frac{2n+2}{n+3}$ *we have*

$$\left(\int_{S^{n-1}} |\widehat{f}(\xi)|^q \, d\sigma(\xi) \right)^{1/q} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

where $q = \left(\frac{n-1}{n+1}\right) p'$*,* $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. See [6, p386] for the full proof. The following is a sketch proof of the "restricted" result (i.e. with $f = \chi_E$ for some set E) in the endpoint case $p = \frac{2n+2}{n+3}$, where $q = 2$ — it is this endpoint case which we will later make use of.

We write the L^2 norm as

$$\int \widehat{f}\overline{\widehat{f}} \, d\sigma = \int \overline{f}(\sigma^\vee * f) = \int \overline{f}(\sigma_1^\vee * f) + \int \overline{f}(\sigma_2^\vee * f)$$

where we have split $\sigma^\vee(\xi) = \sigma^\vee(\xi)\phi(\frac{\xi}{\lambda}) + \sigma^\vee(\xi)(1 - \phi(\frac{\xi}{\lambda}))$ with ϕ a standard bump. Thus $\sigma_1^\vee$ is supported on $|\xi| \lesssim \lambda$ and $\sigma_2^\vee$ on $|\xi| \gtrsim \lambda$. The appropriate choice of λ will be made later.

Using the fact that $|\sigma^\vee(\xi)| \lesssim |\xi|^{-(n-1)/2}$, we have

$$|\sigma_2^\vee(\xi)| \lesssim \lambda^{-\frac{n-1}{2}}.$$

On the other hand, $\sigma_1 = \sigma * \widehat{\phi_{1/\lambda}}$, so σ_1 is σ spread out on scale $\frac{1}{\lambda}$ maintaining mass 1. So $\|\sigma_1\|_\infty \sim \lambda$.

So applying Hölder and Plancherel to the first term, and Hölder and Young on the second, we get

$$\begin{aligned} \int |\widehat{f}|^2 \, d\sigma &= \int \overline{f}(\sigma_1^\vee * f) + \int \overline{f}(\sigma_2^\vee * f) \\ &\leq \|f\|_2 \, \|\sigma_1\|_\infty \, \|f\|_2 + \|f\|_1 \, \|\sigma_2^\vee\|_\infty \, \|f\|_1 \\ &\leq C \, \|f\|_2^2 \, \lambda + \|f\|_1^2 \, \lambda^{-\frac{n-1}{2}}. \end{aligned}$$

Taking $f = \chi_E$ we have

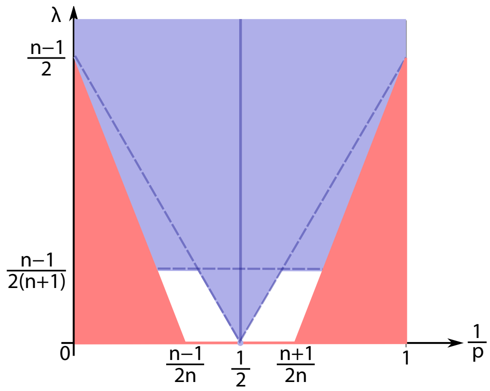
$$\int |\widehat{f}|^2 \, d\sigma \lesssim |E|\lambda + |E|^2 \lambda^{-\frac{n-1}{2}}$$

and choosing λ so that both terms are the same (i.e. taking $\lambda = |E|^{2/(n+1)}$) we get

$$\int |\widehat{f}|^2 \, d\sigma \lesssim |E|^{\frac{n+3}{n+1}} = \|f\|^2_{\frac{2(n+1)}{n+3}}$$

i.e. that the restriction estimate holds from $p = \frac{2(n+1)}{n+3}$ to $L^2(S^{n-1})$. □

Theorem 7. *If $\lambda > \frac{n-1}{2(n+1)}$ then T_λ is bounded on L^p when $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{2\lambda+1}{2n}$.*



Proof. We again decompose T_λ on annuli, putting

$$K_\lambda = \sum_{k=0}^{\infty} K_k \qquad \text{where } K_k(x) = K_\lambda(x)\eta(2^{-k}x) \quad (k \geq 1),$$

with $\eta \in C^\infty$ radial and supported on $\frac{1}{4} \leq |x| \leq 1$, so that each K_k is supported on $|x| \in [2^{k-2}, 2^k]$.

We want $\|T_\lambda f\|_p \lesssim \|f\|_p$ for suitable p, λ. If we can show

$$\|f * K_k\|_{p_0} \leq A 2^{-\varepsilon k} \|f\|_{p_0} \quad (\exists \varepsilon > 0) \tag{3.1}$$

for some $p_0 \leq 2$ then with the triangle inequality we get

$$\|T_\lambda f\|_{p_0} = \left\| \sum_{k=0}^{\infty} f * K_k \right\|_{p_0} \leq \sum_{k=0}^{\infty} \|f * K_k\|_{p_0} \overset{(3.1)}{\leq} \sum_{k=0}^{\infty} A 2^{-\varepsilon k} \|f\|_{p_0} \lesssim \|f\|_{p_0}$$

because the geometric series is summable. Then by interpolation with the obvious L^2 estimate, we have that T_λ is bounded on L^p for $p_0 \leq p \leq 2$.

Now the multiplier corresponding to K_k is

$$\begin{aligned} m_k(\xi) = \widehat{K_k}(\xi) &= (\widehat{K_\lambda} * \widehat{\eta(2^{-k}\cdot)})(\xi) \\ &= \int m_\lambda(\xi - y) 2^{nk} \hat{\eta}(2^k y)\, dy \end{aligned}$$

and we can show[5], using the compact support and smoothness of $(1 - |\xi|^2)_+^\lambda$ away from $|\xi| = 1$, that for $|1 - |\xi|| > \frac{1}{2}$,

$$|m_k(\xi)| \leq A_N 2^{-Nk}(1 + |\xi|)^{-N} \qquad \forall N \geq 0.$$

[5]We exploit the fact that $\int \xi^\alpha \hat{\eta} = (D^\alpha \eta)(0) = 0\ \forall \alpha$ to introduce extra terms in the integrand. For large $|\xi|$ we replace $m_\lambda(\xi - y)$ with $(m_\lambda(\xi - y) - m_\lambda(\xi))$ and use the mean value theorem, while for small $|\xi|$ we subtract the degree N Taylor approximation of m_λ. Then use the fact that $\eta \in \mathcal{S}$.

Now since K_k is supported in the ball of radius 2^k, we only need to consider f with this support[6] in (3.1). But for such f we can perform a neat trick with Hölder's inequality (since $p_0 \le 2$):

$$\begin{aligned}\|f * K_k\|_{p_0} &= \left\|\chi_{B(0,2^k)}|f * K_k|^{p_0}\right\|_1^{1/p_0} \\ &\le \left\|\chi_{B(0,2^k)}\right\|_{\frac{2}{2-p_0}}^{1/p_0} \left\||f * K_k|^{p_0}\right\|_{\frac{2}{p_0}}^{1/p_0} \\ &= C2^{nk\left(\frac{1}{p_0}-\frac{1}{2}\right)} \|f * K_k\|_2\end{aligned}$$

and now we can use Plancherel to write

$$\|f * K_k\|_2^2 = \int_{|1-|\xi||\le\frac{1}{2}} |\widehat{f}(\xi)|^2|m_k(\xi)|^2\, d\xi + \int_{|1-|\xi||>\frac{1}{2}} |\widehat{f}(\xi)|^2|m_k(\xi)|^2\, d\xi$$

We use the estimate on $|m_k(\xi)|$ as well as the fact that (by Hölder),

$$|\widehat{f}(\xi)| \le \int |f(x)|\, dx \le \left\|\chi_{B(0,2^k)}\right\|_{\frac{p_0}{p_0-1}} \|f\|_{p_0} = 2^{C_{n,p_0}k} \|f\|_{p_0}$$

to get the second term

$$\lesssim \|f\|_{p_0}^2 \frac{2^{Ck}}{2^{2Nk}} \int_{|1-|\xi||>\frac{1}{2}} \frac{1}{(1+|\xi|)^{2N}}\, d\xi \lesssim 2^{-k(1+2\lambda)} \|f\|_{p_0}^2$$

by taking N large enough.

For the first term, we use polar coordinates and then apply the L^{p_0}-$L^2(S^{n-1})$ restriction theorem:

$$\begin{aligned}&\int_{1/2}^{3/2} \int_{S^{n-1}} |\widehat{f}(ru)|^2|m_k(r)|^2 r^{n-1}\, du\, dr \\ &\le \left(\sup_{\frac{1}{2}\le r\le\frac{3}{2}} \int_{S^{n-1}} |\widehat{f}(ru)|^2\, du\right) \int_{1/2}^{3/2} |m_k(r)|^2 r^{n-1}\, dr \\ &\lesssim \left(\|f\|_{p_0}^2\right) \|K_k\|_2^2 \\ &\lesssim 2^{-k(1+2\lambda)} \|f\|_{p_0}^2\end{aligned}$$

where $p_0 = \frac{2n+2}{n+3}$ here (see [6, IX§2.1]). Putting this together, we have (3.1) when with $\varepsilon = \lambda + \frac{1}{2} - n\left(\frac{1}{p_0} - \frac{1}{2}\right)$, and since we require $\varepsilon > 0$ to ensure convergence of T_λ, this means we need

$$\frac{1}{p_0} - \frac{1}{2} < \frac{2\lambda+1}{2n}$$

[6]This is a general principle; see Lemma 1.6 in Lecture 3 of [7]. It is basically because the convolution with K_k will kill any contribution from outside the ball anyway.

in order to get boundedness of T_λ on L^p with $p_0 \leq p \leq 2$. Since we must have $p_0 = \frac{2n+2}{n+3}$ from the use of the restriction theorem, this implies

$$\lambda > \frac{n-1}{2(n+1)}.$$

So for λ in this range, we have boundedness of T_λ on L^p if

$$\frac{2n}{2\lambda + n + 1} < p_0 \leq p \leq 2$$

and by duality this means for p such that $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{2\lambda+1}{2n}$. □

4 More recent progress

The main reference here is Tao's set of notes, [8, Lecture 5].

Lee has shown that bilinear restriction estimates can imply Bochner-Riesz. Combining this with bilinear restriction estimates due to Tao, we have that the Bochner-Riesz conjecture is true for

$$p \geq \frac{2(n+2)}{n} \quad \text{and} \quad p \leq \frac{2(n+2)}{n+4},$$

so the range $\lambda > \frac{n-1}{2(n+1)}$ of the previous theorem has been improved to $\lambda > \frac{n-2}{2(n+2)}$.

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