Integer Partitions

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Example

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4 &= 4 \\
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Example: Permutations of n objects

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Example (S_4)

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e		
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Conjugacy class rep.		Sum of cycle lengths
e	$(1)(2)(3)(4)$	$1+1+1+1$
$(1\ 2)$	$(1\ 2)(3)(4)$	$2+1+1$
$(1\ 2\ 3)$	$(1\ 2\ 3)(4)$	$3+1$
$(1\ 2)(3\ 4)$	$(1\ 2)(3\ 4)$	$2+2$
$(1\ 2\ 3\ 4)$	$(1\ 2\ 3\ 4)$	4

The partition function

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$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \left[\frac{d}{dx} \frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}\left(x - \frac{1}{24}\right)}\right)}{\sqrt{x - \frac{1}{24}}} \right]_{x=n}$$

Open problems

- Is $p(n)$ prime for infinitely many n?

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- Is the value of $p(n)$ even approximately half of the time?

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Rather than just $p(n)$, we can consider placing conditions on the kinds of partitions allowed.

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$p(4 \mid \text{odd parts}) = 2.$

Partition identities

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$$p(n \mid \text{condition A}) = p(n \mid \text{condition B})$$

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Proof.

odd parts distinct parts

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merge pairs of equal parts

odd parts → distinct parts

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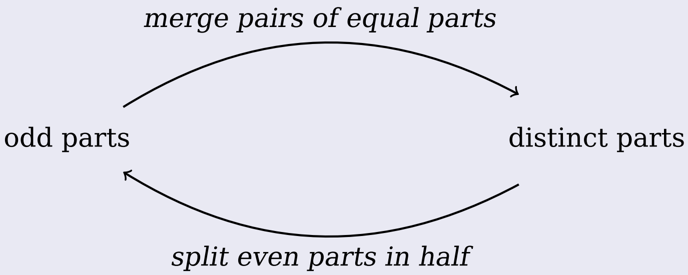

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Generating functions

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\begin{aligned}
(q^1 + q^3)(q^2 + q^4) &= q^{1+2} + q^{1+4} + q^{3+2} + q^{3+4} \\
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Example

$$\sum_{n=0}^{\infty} p(n \mid \text{distinct parts}\ )q^n = \prod_{n=1}^{\infty}(1+q^n)$$

$$\sum_{n=0}^{\infty} p(n \mid \text{odd parts}\ )q^n = \prod_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{1}{(1-q^n)}$$

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So Euler's identity is equivalent to

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The Rogers-Ramanujan identities

$$p(n \mid \text{parts} \equiv 1 \text{ or } 4 \pmod 5) = p(n \mid \text{2-distinct parts})$$

$$p(n \mid \text{parts} \equiv 2 \text{ or } 3 \pmod 5) = p(n \mid \text{2-distinct parts, each} \geqslant 2)$$

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In terms of generating functions:

$$1 + \sum_{m=1}^{\infty} \frac{q^{m^2}}{(1-q)(1-q^2)\cdots(1-q^m)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-4})(1-q^{5n-1})}$$

$$1 + \sum_{m=1}^{\infty} \frac{q^{m^2+m}}{(1-q)(1-q^2)\cdots(1-q^m)} = \prod_{n=1}^{\infty} \frac{1}{(1-q^{5n-3})(1-q^{5n-2})}$$

History

Discovered in 1894 by L. J. Rogers (1862-1933).

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Rediscovered in 1912 by S. Ramanujan (1887-1920), without proof.