Bochner-Riesz multipliers

... or, How to Get a PhD by Colouring in a Picture

George Kinnear

March 9, 2010

Operators

Definition

An **operator** is a function of functions.

Operators

Definition

An **operator** is a function of functions.

Examples

- Differentiation

$$D : f \mapsto f'$$

Operators

Definition

An **operator** is a function of functions.

Examples

- Differentiation
$$D : f \mapsto f'$$
- Fourier transform
$$\mathfrak{F} : f \mapsto \widehat{f}$$
$$\widehat{f}(\xi) = \int f(x) e^{-2\pi i x \cdot \xi} \, dx$$

Spaces of functions

Definition

- The **Lebesgue spaces**, $L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$.

$$f \in L^p(\mathbb{R}^n) \Leftrightarrow \|f\|_p = \left(\int |f(x)|^p \, dx \right)^{1/p} < \infty$$

(with $\|f\|_\infty = \sup |f(x)|$).

Spaces of functions

Definition

- The **Lebesgue spaces**, $L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$.

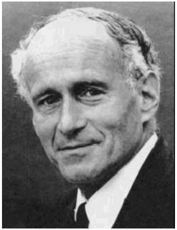
$$f \in L^p(\mathbb{R}^n) \Leftrightarrow \|f\|_p = \left(\int |f(x)|^p \, dx \right)^{1/p} < \infty$$

  (with $\|f\|_\infty = \sup |f(x)|$).

- The **Schwartz space**, $\mathcal{S}(\mathbb{R}^n)$, of smooth, rapidly decreasing functions.

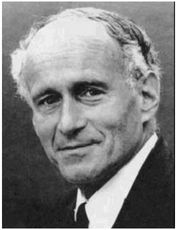
$$f \in C^\infty(\mathbb{R}^n) \qquad \sup |x^\alpha D^\beta f(x)| < \infty$$

Aside



Aside

Karl Hermann Amandus Schwarz
(1843-1921)



Laurent Schwartz
(1915-2002)

Boundedness of operators

Definition

We say the operator T is **bounded** from L^p to L^q if there is an absolute constant C such that

$$\|Tf\|_q \leqslant C \|f\|_p \quad \forall f \in L^p.$$

Boundedness of operators

Definition

We say the operator T is **bounded** from L^p to L^q if there is an absolute constant C such that

$$\|Tf\|_q \leqslant C\,\|f\|_p \quad \forall f \in L^p.$$

Examples

- $\mathcal{F}$ is bounded on L^2, since the Plancherel theorem says

$$\|\widehat{f}\|_2 = \|f\|_2\,.$$

Boundedness of operators

Definition

We say the operator T is **bounded** from L^p to L^q if there is an absolute constant C such that

$$\|Tf\|_q \leqslant C \|f\|_p \quad \forall f \in L^p.$$

Examples

- $\mathcal{F}$ is bounded on L^2, since the Plancherel theorem says
$$\|\widehat{f}\|_2 = \|f\|_2 .$$
- $\mathcal{F}$ is bounded from L^p to L^q, $1 \leqslant p \leqslant 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. This comes from the Hausdorff-Young inequality
$$\|\widehat{f}\|_q \leqslant \|f\|_p .$$

Interpolation

Theorem (Riesz-Thorin Interpolation)

If $\|Tf\|_{q_0} \lesssim \|f\|_{p_0}$ *and* $\|Tf\|_{q_1} \lesssim \|f\|_{p_1}$ *then*

$$\|Tf\|_q \lesssim \|f\|_p$$

Interpolation

Theorem (Riesz-Thorin Interpolation)

If $\|Tf\|_{q_0} \lesssim \|f\|_{p_0}$ *and* $\|Tf\|_{q_1} \lesssim \|f\|_{p_1}$ *then*

$$\|Tf\|_q \lesssim \|f\|_p$$

for p, q *such that for some* $0 < \theta < 1$

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

Interpolation

Theorem (Riesz-Thorin Interpolation)

If $\|Tf\|_{q_0} \lesssim \|f\|_{p_0}$ *and* $\|Tf\|_{q_1} \lesssim \|f\|_{p_1}$ *then*

$$\|Tf\|_q \lesssim \|f\|_p$$

for p, q *such that for some* $0 < \theta < 1$

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

Example

The Hausdorff-Young inequality comes from

$$\|\widehat{f}\|_\infty \leqslant \|f\|_1 \quad \text{and} \quad \|\widehat{f}\|_2 = \|f\|_2 .$$

Multipliers

Given $m \in L^\infty(\mathbb{R}^n)$ we can define an operator T_m by

$$\widehat{T_m f}(x) = m(x)\widehat{f}(x).$$

Multipliers

Given $m \in L^\infty(\mathbb{R}^n)$ we can define an operator T_m by

$$\widehat{T_m f}(x) = m(x)\widehat{f}(x).$$

We say m is an L^p **multiplier** if T_m is bounded on L^p.

Multipliers

Given $m \in L^\infty(\mathbb{R}^n)$ we can define an operator T_m by

$$\widehat{T_m f}(x) = m(x)\widehat{f}(x).$$

We say m is an L^p **multiplier** if T_m is bounded on L^p.

Theorem

An L^p multiplier is automatically an $L^{p'}$ multiplier, where $\frac{1}{p} + \frac{1}{p'} = 1$, $1 \leqslant p \leqslant \infty$.

Spherical summation multipliers

$$S_R f(x) = \int_{|\xi| \leqslant R} \widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$$

Spherical summation multipliers

$$S_R f(x) = \int_{|\xi| \leqslant R} \widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$$

$$\lim_{R \to \infty} S_R f = f \quad \Leftrightarrow \quad \|S_R f\|_p \lesssim \|f\|_p$$

Spherical summation multipliers

$$S_R f(x) = \int_{|\xi| \leqslant R} \widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$$

$$\lim_{R \to \infty} S_R f = f \quad \Leftrightarrow \quad \|S_R f\|_p \lesssim \|f\|_p$$

S_R is the operator corresponding to the **disc multiplier**;

$$\widehat{S_R f}(\xi) = \chi_{\{|\xi| \leqslant R\}} \widehat{f}(\xi).$$

Spherical summation multipliers

$$S_R f(x) = \int_{|\xi| \leqslant R} \widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$$

$$\lim_{R \to \infty} S_R f = f \quad \Leftrightarrow \quad \|S_R f\|_p \lesssim \|f\|_p$$

S_R is the operator corresponding to the **disc multiplier**;

$$\widehat{S_R f}(\xi) = \chi_{\{|\xi| \leqslant R\}} \widehat{f}(\xi).$$

- $n = 1$: bounded on $L^p(\mathbb{R})$, $1 < p < \infty$.

Spherical summation multipliers

$$S_R f(x) = \int_{|\xi| \leqslant R} \widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$$

$$\lim_{R \to \infty} S_R f = f \quad \Leftrightarrow \quad \|S_R f\|_p \lesssim \|f\|_p$$

S_R is the operator corresponding to the **disc multiplier**;

$$\widehat{S_R f}(\xi) = \chi_{\{|\xi| \leqslant R\}} \widehat{f}(\xi).$$

- $n = 1$: bounded on $L^p(\mathbb{R})$, $1 < p < \infty$.
- $n \geqslant 2$
  - trivially bounded on $L^2(\mathbb{R}^n)$ by Plancherel.

Spherical summation multipliers

$$S_R f(x) = \int_{|\xi| \leqslant R} \widehat{f}(\xi) e^{2\pi i \xi \cdot x} d\xi$$

$$\lim_{R \to \infty} S_R f = f \quad \Leftrightarrow \quad \|S_R f\|_p \lesssim \|f\|_p$$

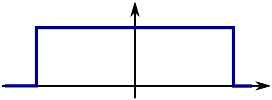
S_R is the operator corresponding to the **disc multiplier**;

$$\widehat{S_R f}(\xi) = \chi_{\{|\xi| \leqslant R\}} \widehat{f}(\xi).$$

- $n = 1$: bounded on $L^p(\mathbb{R})$, $1 < p < \infty$.
- $n \geqslant 2$
  - trivially bounded on $L^2(\mathbb{R}^n)$ by Plancherel.
  - not bounded on $L^p(\mathbb{R}^n)$ if $p \neq 2$.

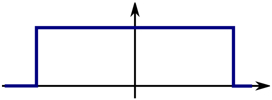
Spherical summation multipliers

The multiplier $\chi_{\{|\xi|\leqslant R\}}$ has a jump:

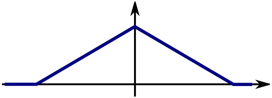


Spherical summation multipliers

The multiplier $\chi_{\{|\xi|\leqslant R\}}$ has a jump:

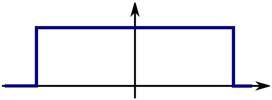


What if we smooth it out? e.g. $\left(1-\frac{|\xi|}{R}\right)_+$

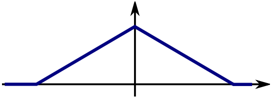


Spherical summation multipliers

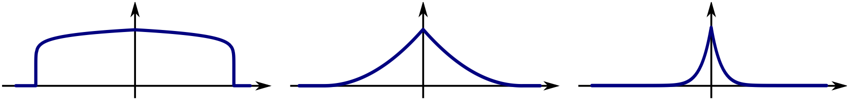
The multiplier $\chi_{\{|\xi|\leqslant R\}}$ has a jump:



What if we smooth it out? e.g. $\left(1-\frac{|\xi|}{R}\right)_+$



Or have various amounts of smoothing out? e.g. $\left(1-\frac{|\xi|}{R}\right)_+^\lambda$



Bochner-Riesz multipliers

Instead of $\left(1-\frac{|\xi|}{R}\right)_+^\lambda$ we consider a closely related family.

Definition

The **Bochner-Riesz multipliers** are defined for $\lambda > 0$ by

$$m_\lambda(\xi) = \left(1-|\xi|^2\right)_+^\lambda .$$

The corresponding operators are

$$\widehat{T_\lambda f}(\xi) = m_\lambda(\xi)\widehat{f}(\xi).$$

Bochner-Riesz multipliers

Instead of $\left(1 - \frac{|\xi|}{R}\right)_+^\lambda$ we consider a closely related family.

Definition

The **Bochner-Riesz multipliers** are defined for $\lambda > 0$ by

$$m_\lambda(\xi) = \left(1 - |\xi|^2\right)_+^\lambda .$$

The corresponding operators are

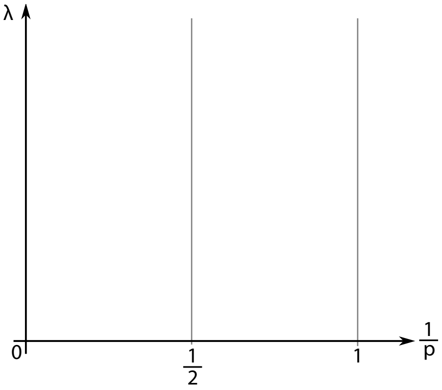
$$\widehat{T_\lambda f}(\xi) = m_\lambda(\xi)\widehat{f}(\xi).$$

Question

For which combinations of λ and p is T_λ bounded on $L^p(\mathbb{R}^n)$?

Boundedness diagram

Shade in where T_λ is bounded on $L^p(\mathbb{R}^n)$.



- symmetrical about $\frac{1}{p} = \frac{1}{2}$ (i.e. $p = 2$)
- any two shaded-in points can be joined by a line

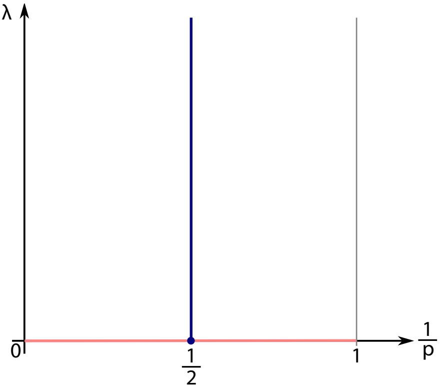
Boundedness diagram

Boundedness on L^2 is easy by Plancherel.

The $\lambda = 0$ case is the disc multiplier, so only bounded on L^2.

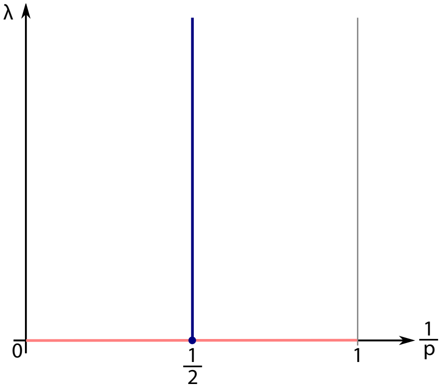
Boundedness diagram

Boundedness on L^2 is easy by Plancherel.
The $\lambda = 0$ case is the disc multiplier, so only bounded on L^2.



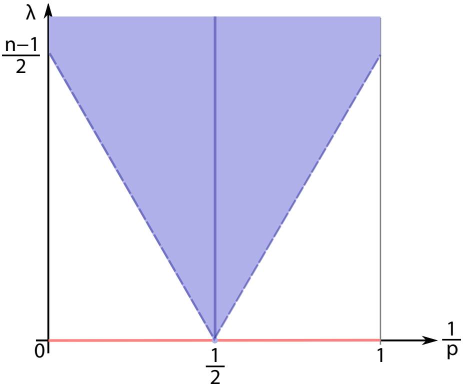
Boundedness diagram

Above the critical index $\lambda = \frac{n-1}{2}$, T_λ is bounded.



Boundedness diagram

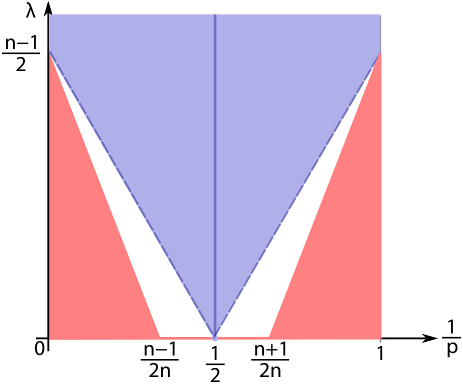
Above the critical index $\lambda = \frac{n-1}{2}$, T_λ is bounded.



First proved by E. M. Stein in 1956, using interpolation.

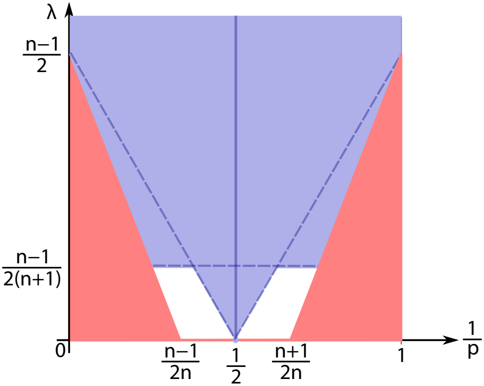
Boundedness diagram

An argument involving Bessel functions shows where T_λ is definitely not bounded.



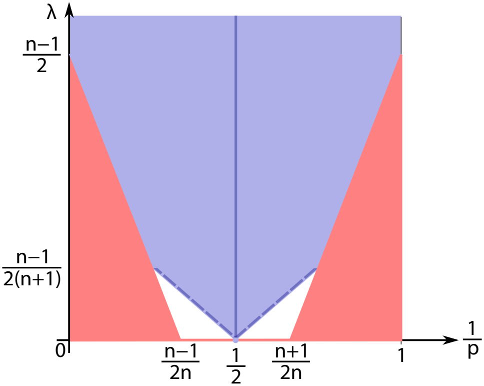
Boundedness diagram

Using Fourier restriction estimates gets us a little further (Fefferman, 1970).



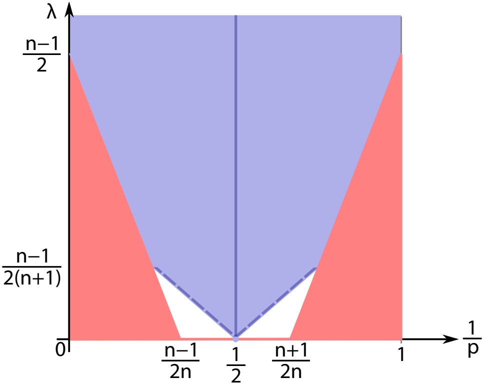
Boundedness diagram

Using Fourier restriction estimates gets us a little further (Fefferman, 1970).



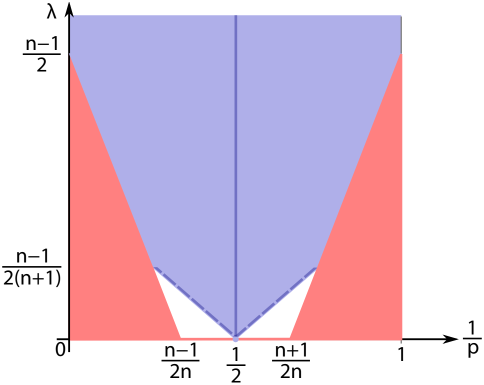
We complete the picture by interpolation.

Boundedness diagram



- when $n = 2$, we have boundedness in the white region (Carleson and Sjölin, 1972).

Boundedness diagram



- when $n = 2$, we have boundedness in the white region (Carleson and Sjölin, 1972).
- **unknown** for higher dimensions (the conjecture is that T_λ *is* bounded there).