Analysis Club

Restriction Theory

George Kinnear, 7 February 2011

The content of this note is based on [Tao10] and [Tao03].

1 Restriction and extension

Our setting is a smooth compact hypersurface S in $\mathbb{R}^d$ (e.g. the unit sphere S^2 in $\mathbb{R}^3$), with surface measure $d\sigma$.

Given $f \in L^1(\mathbb{R}^d)$, we have the Fourier transform

$$\hat{f}(x) = \int_{\mathbb{R}^d} e^{-2\pi i x.\xi} f(x)\, dx$$

which by Riemann-Lebesgue is a bounded, continuous function vanishing at infinity. Thus if we restrict $\hat{f}$ to S, we get a meaningful function which has finite L^q norm for every q.

However, starting with $f \in L^2(\mathbb{R}^d)$, we get $\hat{f} \in L^2(\mathbb{R}^d)$ by Plancherel. There is no meaningful way to restrict an arbitrary L^2 function to a set of measure zero such as the hypersurface S.

The question arises: what happens for $1 < p < 2$?

QUESTION 1 **RESTRICTION PROBLEM** $R_S(p \to q)$

For which p and q do we have

$$\left\| \hat{f} \right\|_{L^q(S,d\sigma)} \lesssim \| f \|_{L^p(\mathbb{R}^d)} \tag{1}$$

for all f $(\in \mathcal{S}$, say).
If p and q are a pair for which (1) holds, we say $R_S(p \to q)$ is true.

The earlier remarks show that $R_S(1 \to q)$ is true for all $1 \le q \le \infty$ while $R_S(2 \to q)$ is false for all $1 \le q \le \infty$.

PROPOSITION 2

If $R_S(p \to q)$ is true then $R_S(\tilde{p} \to \tilde{q})$ is true for all $\tilde{p} \le p$ and $\tilde{q} \le q$.

PROOF

Hölder and Sobolev inequalities. ■

Now if (1) holds for a certain p and q, we have

$$\sup_{\|f\|_p = 1} \left\| \hat{f} \right\|_{L^q(S,d\sigma)} \lesssim 1$$

which by duality means

$$\sup_{\|f\|_p=1} \sup_{\|g\|_{L^{q'}(S,d\sigma)}=1} \left| \int \hat{f}(\xi) g(\xi)\, d\sigma(\xi) \right| \lesssim 1.$$

Swapping the sup's and applying Parseval, followed by undoing the duality, we have

$$\begin{aligned} \sup_{\|g\|_{L^{q'}((S,d\sigma)}=1} \sup_{\|f\|_p=1} \left| \int f(\xi) \widehat{g d\sigma}(\xi)\, dx \right| &\lesssim 1 \\ \sup_{\|g\|_{L^{q'}((S,d\sigma)}=1} \left\| \widehat{g d\sigma} \right\|_{p'} &\lesssim 1 \\ \left\| \widehat{g d\sigma} \right\|_{p'} &\lesssim \|g\|_{q'} . \end{aligned} \tag{2}$$

In fact, these steps are reversible so (2) is equivalent to (1). This gives us an alternative question.

QUESTION 3 **EXTENSION PROBLEM** $R_S^*(q' \to p')$

For which p' and q' do we have

$$\left\| \widehat{F d\sigma} \right\|_{L^{p'}(\mathbb{R}^d)} \lesssim \|F\|_{L^{q'}(S,d\sigma)} \tag{3}$$

for all F ($\in \mathcal{S}$, say).
If p' and q' are a pair for which (3) holds, we say $R_S^*(q' \to p')$ is true.

Necessary conditions

Taking $F \equiv 1$ in (3) shows that we need $\widehat{d\sigma} \in L^{p'}$. For the sphere, it is well-known that

$$\widehat{d\sigma}(x) = O\left((1 + |x|)^{-(n-1)/2} \right)$$

so we must have $p' > \frac{2n}{n-1}$ (i.e. $p < \frac{2n}{n+1}$).

There is another choice of F, known as the Knapp example, which is essentially the characterisitic function of a small "cap" on the sphere. Plugging this in to (3) shows that we must also have

$$q \leq \frac{n-1}{n+1} p'.$$

2 The restriction conjecture

CONJECTURE 4 **RESTRICTION CONJECTURE**

These necessary conditions are sufficient, i.e. if

$$p < \frac{2n}{n+1} \qquad \text{and} \qquad q \leq \frac{n-1}{n+1} p'$$

then $R_S(p \to q)$ is true.

The following is a classic partial result in this direction.

THEOREM 5 TOMAS-STEIN (1975)

$R_S(p \to q)$ is true for all $1 \le p \le \frac{2n+2}{n+3}$, with $q = \left(\frac{n-1}{n+1}\right) p'$.

Note that Proposition 2 means that we can replace $q = \cdots$ with $q \le \left(\frac{n-1}{n+1}\right) p'$.

PROOF

See [Ste93, p386] for the full proof. The following is a sketch proof of the "restricted" result (i.e. with $f = \chi_E$ for some set E) in the endpoint case $p = \frac{2n+2}{n+3}$, where $q = 2$.

We want to show

$$\left(\int_{S^{n-1}} |\widehat{f}(\xi)|^2 \, d\sigma(\xi)\right)^{1/2} \lesssim \|f\|_{L^{\frac{2n+2}{n+3}}(\mathbb{R}^n)}$$

We write the L^2 norm as

$$\int \widehat{f}\overline{\widehat{f}} \, d\sigma = \int \overline{f}(\sigma^\vee * f) = \int \overline{f}(\sigma_1^\vee * f) + \int \overline{f}(\sigma_2^\vee * f)$$

where we have split $\sigma^\vee(\xi) = \sigma^\vee(\xi)\phi(\frac{\xi}{\lambda}) + \sigma^\vee(\xi)(1 - \phi(\frac{\xi}{\lambda}))$ with ϕ a standard bump. Thus $\sigma_1^\vee$ is supported on $|\xi| \lesssim \lambda$ and $\sigma_2^\vee$ on $|\xi| \gtrsim \lambda$. The appropriate choice of λ will be made later.

Using the fact that $|\sigma^\vee(\xi)| \lesssim |\xi|^{-(n-1)/2}$, we have

$$|\sigma_2^\vee(\xi)| \lesssim \lambda^{-\frac{n-1}{2}}.$$

On the other hand, $\sigma_1 = \sigma * \widehat{\phi_{1/\lambda}}$, so σ_1 is σ spread out on scale $\frac{1}{\lambda}$ maintaining mass 1. So $\|\sigma_1\|_\infty \sim \lambda$.

So applying Hölder and Plancherel to the first term, and Hölder and Young on the second, we get

$$\begin{aligned}
\int |\widehat{f}|^2 \, d\sigma &= \int \overline{f}(\sigma_1^\vee * f) + \int \overline{f}(\sigma_2^\vee * f) \\
&\le \|f\|_2 \|\sigma_1\|_\infty \|f\|_2 + \|f\|_1 \|\sigma_2^\vee\|_\infty \|f\|_1 \\
&\le C \|f\|_2^2 \lambda + \|f\|_1^2 \lambda^{-\frac{n-1}{2}}.
\end{aligned}$$

Taking $f = \chi_E$ we have

$$\int |\widehat{f}|^2 \, d\sigma \lesssim |E|\lambda + |E|^2 \lambda^{-\frac{n-1}{2}}$$

and choosing λ so that both terms are the same (i.e. taking $\lambda = |E|^{2/(n+1)}$) we get

$$\int |\widehat{f}|^2 \, d\sigma \lesssim |E|^{\frac{n+3}{n+1}} = \|f\|^2_{\frac{2(n+1)}{n+3}}$$

i.e. that the restriction estimate holds from $p = \frac{2(n+1)}{n+3}$ to $L^2(S^{n-1})$. ■

The most recent results on this problem have used "bilinear" or "multilinear" techniques, which we will now look at.

3 Bilinear estimates

Note that when p' is an even integer, $\left\|\widehat{Fd\sigma}\right\|_{p'}$ in the extension problem (3) can be expanded out using Plancherel; for instance

$$\left\|\widehat{Fd\sigma}\right\|_4 = \left\|\widehat{Fd\sigma}\widehat{Fd\sigma}\right\|_2^{1/2} = \|Fd\sigma * Fd\sigma\|_2^{1/2}.$$

This means $R^*_S(q' \to 4)$ is equivalent to

$$\|Fd\sigma * Fd\sigma\|_2 \lesssim \|F\|^2_{L^{q'}(S,d\sigma)}$$

which involves no Fourier transforms, so could be proven by more direct methods.

The idea can be partially extended to p' which are not even integers, since the squaring step can still be carried out on $R^*_S(q' \to p')$ to give

$$\left\|\widehat{Fd\sigma} * \widehat{Fd\sigma}\right\|_{p'/2} \lesssim \|F\|^2_{L^{q'}(S,d\sigma)}.$$

This leads us to consider more general estimates of the form

$$\left\|\widehat{F_1 d\sigma_1} * \widehat{F_2 d\sigma_2}\right\|_{p'/2} \lesssim \|F_1\|_{L^{q'}(S_1,d\sigma_1)} \|F_2\|_{L^{q'}(S_2,d\sigma_2)} \tag{4}$$

for arbitrary pairs of smooth compact hypersurfaces S_1, S_2 with respective surface measures $d\sigma_1, d\sigma_2$, and F_1, F_2 supported on S_1, S_2.

DEFINITION

If (4) holds, we say $R^*_{S_1,S_2}(q' \times q' \to p'/2)$ is true.

Such bilinear estimates are more general than the linear ones, but if we restrict to $S_1 = S_2 = S$, we have equivalence:

PROPOSITION 6

$$R^*_{S,S}(q' \times q' \to p'/2) \iff R^*_S(q' \to p')$$

PROOF

This is because $R^*_{S,S}(q' \times q' \to p'/2)$ says

$$\left\|\widehat{F_1 d\sigma} * \widehat{F_2 d\sigma}\right\|_{p'/2} \lesssim \|F_1\|_{L^{q'}(S,d\sigma)} \|F_2\|_{L^{q'}(S,d\sigma)}$$

for all F_1, F_2 supported on S, which by polarisation is equivalent to

$$\left\|\widehat{Fd\sigma} * \widehat{Fd\sigma}\right\|_{p'/2} \lesssim \|F\|^2_{L^{q'}(S,d\sigma)},$$

i.e. $R^*_S(q' \to p')$. ■

Rather than putting $S_1 = S_2 = S$, we might instead put some conditions on S_1, S_2 and hope to prove bilinear estimates in that setting. We will see something along those lines later.

Multilinear restriction

There is no reason to limit ourselves to squaring out the norm to get a bilinear estimate; in d dimensions we can consider a d-linear estimate

$$\left\|\prod_{j=1}^{d} \widehat{F_j d\sigma_j}\right\|_{p'/d} \lesssim \prod_{j=1}^{d} \|F_j\|_{L^{q'}(S_j,d\sigma_j)}.$$

For instance, in [BCT06] the following is obtained:

THEOREM 7 NEAR-OPTIMAL MULTILINEAR RESTRICTION

For $\varepsilon > 0$, $p' \geq \frac{2d}{d-1}$, $q \leq p' \frac{d-1}{d}$,

$$\left\| \prod_{j=1}^{d} \widehat{F_j d\sigma_j} \right\|_{L^{p'/d}(B(0,R))} \lesssim R^{\varepsilon} \prod_{j=1}^{d} \|F_j\|_{L^{q'}(S_j, d\sigma_j)}$$

assuming the S_j are smooth enough, and are **transverse**.

The assumption that the S_j are transverse (i.e. their normals are never coplanar) is quite restrictive – in particular, we certainly can't make them all S and use Proposition 6 to make progress on the restriction conjecture.

In [Tao10] Tao discusses a new result of Bourgain and Guth, which he says "interpolates" between the Theorem of [BCT06] and a result of Córdoba to obtain progress on the restriction conjecture.

Bibliography

[BCT06] Jonathan Bennett, Anthony Carbery, and Terence Tao. On the multilinear restriction and Kakeya conjectures. *Acta Math.*, 196(2):261–302, 2006.

[Ste93] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[Tao03] Terence Tao. Recent progress on the restriction conjecture, 2003.

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