On polynomial sized representations of Hilbert's identity and moments tensors

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Abstract

Hilbert's identity is an important tool for the analysis of Hilbert's 17th problem. The identity asserts that the polynomial function $(x_1^2 + x_2^2 + \cdots + x_n^2)^d$ can be expressed as a sum of powered linear terms. However, in Hilbert's original construction, the number of powered linear terms is exponential in n. For $d = 2$, we present efficient algorithms to construct a sum of powered linear terms, where the number of terms is no more than $2n^4 + n$. Moreover, we prove that the number of required linear terms is at least $n(n+1)/2$ for $d = 2$. Our tools in the construction rely on a new notion called k-wise uncorrelated random variables.

Our results immediately imply that it is possible to find a $(2n^4 + n)$-point distribution whose fourth moments tensor is exactly the symmetrization of $Q \otimes Q$, where Q is a positive semidefinite matrix. Extensions of the results to complex tensors are discussed as well. As an application, we answer an open question to assert that the computation of the matrix $2 \mapsto 4$ norm is NP-hard in general. Finally we discuss a polynomial sized representation of Hilbert's identity for general degree d.

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Keywords: moments tensors, Hilbert's identity, uncorrelated random variables, matrix norm.